# KÄHLER-RICCI SOLITONS ON COMPACT COMPLEX MANIFOLDS WITH $c_1(M) > 0$

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ABSTRACT. In this paper, we discuss the relation between the existence of Kähler-Ricci solitons and certain functional associated to some complex Monge-Ampère equation on compact complex manifolds with positive first Chern class. In particular, we obtain a strong inequality of Moser-Trudinger type on a compact complex manifold admitting a Kähler-Ricci soliton. Our result also improves the one obtained in [Ti] and [TZ1].

## 0. Introduction.

In this paper, we study the existence of Kähler-Ricci solitons by using properness of certain functionals. Our approach is similar to that of [Ti] for Kähler-Einstein metrics. A Kähler metric $g$ on a compact complex manifold $M$ with first Chern class $c_1(M) > 0$ is called a (homothetically shrinking) Kähler-Ricci soliton if there is a holomorphic vector field $X$ on $M$ such that the Kähler form $\omega_g$ satisfies

$$\mathrm{Ric}(\omega_g) - \omega_g = L_X \omega_g,$$

where $\mathrm{Ric}(\omega_g)$ denotes the Ricci form of $\omega_g$ and $L_X$ is the Lie derivative operator along $X$. In particular, if $X = 0$, such a $g$ is a Kähler-Einstein metric. So Kähler-Ricci solitons can be regarded as a generalization of Kähler-Einstein metrics of positive scalar curvature. Ricci solitons have been studied extensively in the recent years (cf. [Ko], [Ha], [C1], [C2], [Ti], [TZ2], [TZ3], [CH], etc.). One motivation is that they are very closely related to the limiting singular behavior of solutions of certain PDEs which arise from geometric analysis, such as Hamilton's Ricci flow ([Ha]) and certain complex Monge-Ampère equations associated to Kähler-Einstein metrics ([Ti]). Kähler-Ricci solitons are special Ricci solitons. It was proved recently in ([TZ3], [TZ4]) that there exists at most one Kähler-Ricci soliton on any compact Kähler manifold with positive first Chern class, modulo holomorphic automorphisms. This extends Bando and Mabuchi's theorem on the uniqueness of Kähler-Einstein metrics with positive scalar curvature ([BM]).

Let $g$ be a Kähler metric on $M$ with its Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

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representing $c_1(M)$. Then, by the Hodge theory, there is a smooth real-valued function $h_g$ such that

$$\mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} h_g.$$

Suppose that $X$ is a holomorphic vector field on $M$ so that the integral curve of $\mathrm{Im}(X)$ consists of isometries of $g$. By the Hodge Decomposition Theorem, there is a unique smooth real-valued function $\theta_X = \theta_X(\omega_g)$ on $M$ such that

$$\begin{cases} i_X\omega_g = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\theta_X, \\ \int_M e^{\theta_X}\omega_g^n = \int_M \omega_g^n = V. \end{cases}$$

Set

$$\mathcal{M}_X(\omega_g) = \{\psi \in C^\infty(M) |\ \omega_\psi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi > 0,\ \mathrm{Im}(X)(\psi) = 0\}.$$

The following functional on $\mathcal{M}_X(\omega_g)$ was introduced in [TZ4],

$$\tilde{F}_{\omega_g}(\psi) = \tilde{J}_{\omega_g}(\psi) - \frac{1}{V}\int_M \psi e^{\theta_X}\omega_g^n - \log(\frac{1}{V}\int_M e^{h_g-\psi}\omega_g^n),$$

where

$$\omega_\psi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi,$$

and

$$\begin{aligned} &\tilde{J}_{\omega_g}(\psi) \\ &= \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M (\int_0^1\int_0^1 t(st)^k(1-st)^{n-1-k}e^{\theta_X+stX(\psi)}dtds) \qquad (0.1)\\ &\times \partial\psi\wedge\overline{\partial}\psi\wedge\omega_\psi^k\wedge\omega_g^{n-1-k}. \end{aligned}$$

Let $K_0$ be a maximal compact subgroup of the automorphism group of $M$ such that $\sigma\cdot\eta = \eta\cdot\sigma$ for any $\eta \in K_0$ and any isometry $\sigma$ in the subgroup $K_X$ generated by $\mathrm{Im}(X)$. Then a Kähler-Ricci soliton with respect to $X$ on $M$ must be $K_0$-invariant by the uniqueness theorem in [TZ3]. We introduce

**Definition 0.1.** *The functional $\tilde{F}_{\omega_g}$ is said to be proper with respect to some holomorphic vector field $X$ on $M$ if there is a monotone increasing function $\mu(t)$ : $(a, +\infty) \to R$ such that*

$$lim_{t\to+\infty}\mu(t) = +\infty,$$

*and*

$$\tilde{F}_{\omega_g}(\psi) \geq \mu(I_{\omega_g}(\psi)),$$

*for any $K_0$-invariant function $\psi \in \mathcal{M}_X(\omega_g)$, where*

$$I_{\omega_g}(\psi) = \frac{1}{V}\int_M \psi(\omega_g^n - \omega_\psi^n).$$

It was essentially proved in [TZ4] that the properness of $\tilde{F}_{\omega_g}$ is a sufficient condition for the existence of Kähler-Ricci soliton, that is,

**Theorem 0.1.** *Suppose that $\tilde{F}_\omega$ is proper with respect to a holomorphic vector field $X$ on $M$. Then there is a Kähler-Ricci soliton with respect to $X$ on $M$.*

The main purpose of this paper is to show that the properness of $\tilde{F}_{\omega_g}$ is also a necessary condition for the existence of Kähler-Ricci solitons (cf. Section 5). Suppose that $M$ admits a Kähler-Ricci soliton $\omega_{KS}$ with respect to some holomorphic vector field $X$. We define a weighted inner product on $C^\infty(M)$ by

$$(\phi, \psi) = \int_M \phi\psi e^{\theta_X(\omega_{KS})}\omega_{KS}^n,$$

and denote by

$$\Lambda_1(M, \omega_{KS}) = \{u \in C^\infty(M) |\ \triangle_{\omega_{KS}} u + X(u) = -u\}.$$

**Theorem 0.2.** *Let $M$ be a compact complex manifold admitting a Kähler-Ricci soliton $\omega_{KS}$ with respect to some holomorphic vector field $X$ and $G(\subseteq K_0)$ a compact subgroup of $Aut(M)$ with $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\sigma \in K_X$ and $\eta \in G$. Suppose that any $G$-invariant smooth function on $M$ is perpendicular to the space $\Lambda_1(M, \omega_{KS})$ with respect to the weighted inner product defined above. Then there are two positive numbers $c$ and $C$ such that for any $G$-invariant $\psi$,*

$$\tilde{F}_{\omega_{KS}}(\psi) \geq cI_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C. \tag{0.2}$$

*In particular, $\tilde{F}_{\omega_{KS}}$ is proper under the same assumption.*

The inequality (0.2) is equivalent to the following non-linear inequality of Moser-Trudinger type,

$$\int_M e^{-\psi}\omega_{KS}^n \leq C\exp\{\tilde{J}_{\omega_{KS}}(\psi) - c\tilde{J}_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - \frac{1}{V}\int_M \psi\omega_{KS}^n\}. \tag{0.3}$$

The idea of our proof of Theorem 0.2 is inspired by [Ti] and [TZ1], where the authors proved a fully non-linear inequality for Kähler-Einstein manifolds. The inequality (0.3) generalizes such an inequality in [Ti] and [TZ1]. A weak version of (0.2) was obtained in [TZ4].

The organization of this paper is as follows. In Section 1, we give a proof of Theorem 0.1 following [TZ4]. In Section 2, we give an estimate for certain heat kernel on manifolds with modified positive Ricci curvature. In Section 3, a $C^0$-estimate for certain complex Monge-Ampère equations is obtained. In Section 4, we prove a smoothing lemma by using Hamilton's Ricci flow. Theorem 0.2 will be proved in Section 5.

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## 1. An Analytic Criterion for Kähler-Ricci Solitons.

Let $(M, g)$ be an $n$-dimensional compact Kähler manifold with the first Chern class $c_1(M) > 0$. Denote by $\mathrm{Aut}^\circ(M)$ the connected component of the automorphism group of $M$. Let $K$ be a maximal compact subgroup of $\mathrm{Aut}^\circ(M)$, then the Chevalley decomposition allows us to write $\mathrm{Aut}^\circ(M)$ as a semi-direct decomposition ([FM]),

$$\mathrm{Aut}^\circ(M) = \mathrm{Aut}_r(M) \propto R_u,$$

where $\mathrm{Aut}_r(M) \subset \mathrm{Aut}^\circ(M)$ is a reductive algebraic subgroup and the complexification of $K$, and $R_u$ is the unipotent radical of $\mathrm{Aut}^\circ(M)$. Let $\eta_r(M)$ be the Lie subalgebra of $\mathrm{Aut}_r(M)$.

Let $X \in \eta_r(M)$ such that the one-parameter subgroup $K_X$ generated by $\mathrm{Im}(X)$ is contained in $K$, where $\mathrm{Im}(X)$ denotes the imaginary part of $X$. Choose a $K_X$-invariant Kähler metric $g$ on $M$ with the Kähler form $\omega_g$. Then by the Hodge Decomposition Theorem, there is a smooth real-valued function $\theta_X = \theta_X(\omega_g)$ of $M$ such that $i_X\omega_g = \frac{\sqrt{-1}}{2\pi}\bar\partial\theta_X$, and consequently,

$$L_X\omega_g = di_X(\omega_g) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\theta_X.$$

We consider the following complex Monge-Ampère equations with parameter $t \in [0,1]$:

$$\begin{cases} \det(g_{i\bar j} + \phi_{i\bar j}) = \det(g_{i\bar j})\exp\{h - \theta_X - X(\phi) - t\phi\} \\ (g_{i\bar j} + \phi_{i\bar j}) > 0, \end{cases} \tag{1.1}$$

where $h = h_{\omega_g}$ is a smooth real-valued function on $M$ defined by

$$\begin{cases} \mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial h \\ \int_M e^h\omega_g^n = \int_M \omega_g^n = V. \end{cases}$$

Here $\omega_g^n = \omega_g\wedge...\wedge\omega_g$. Then one can check that $\omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\phi$ is a Kähler-Ricci soliton with respect to $X$, i.e., $\omega_\phi$ satisfies

$$\mathrm{Ric}(\omega_\phi) - \omega_\phi = L_X\omega_\phi,$$

if and only if $\phi$ is a solution of equations $(1.1)_t$ for $t = 1$. In fact, $(1.1)_t$ is equivalent to

$$\mathrm{Ric}(\omega_{\phi_t}) - L_X(\omega_{\phi_t}) = t\omega_{\phi_t} + (1-t)\omega_g, \tag{1.2}$$

where $\phi_t$ is a solution of $(1.1)_t$. Furthermore, we can normalize $\theta_X$ by

$$\int_M e^{\theta_X}\omega_g^n = \int_M \omega_g^n = V. \tag{1.3}$$

Since

$$\frac{d}{dt}\left(\int_M e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n\right) = \int_M (\triangle'\dot\phi_t + X(\dot\phi_t))e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n = 0,$$

then by (1.3), we have

$$\int_M e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n = \int_M e^{\theta_X}\omega_g^n = V.$$

Integrating $(1.1)_t$ after multiplied by $e^{\theta_X+X(\phi)}$, it follows

$$\int_M e^{h-t\phi_t}\omega_g^n = V.$$

Therefore, differentiating the above identities, we get

$$\int_M \dot{\phi}_t e^{h-t\phi_t}\omega_g^n = -\frac{1}{t}\int_M \phi_t e^{h-t\phi_t}\omega_g^n. \tag{1.4}$$

Set

$$\mathcal{M}(\omega_g) = \{\phi \in C^\infty(M) | \ \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0\}.$$

and

$$\mathcal{M}_X(\omega_g) = \{\phi \in \mathcal{M}(\omega_g)) | \ \mathrm{Im}(X)(\phi) = 0\}.$$

In [TZ3], the following two functionals on $\mathcal{M}_X(\omega_g)$ were introduced:

$$\tilde{I}_{\omega_g}(\phi) = \frac{1}{V}\int_M \phi(e^{\theta_X}\omega_g^n - e^{\theta_X + X(\phi)}\omega_\phi^n)$$

and

$$\tilde{J}_{\omega_g}(\phi) = \frac{1}{V}\int_0^1\int_M \dot{\phi}_s(e^{\theta_X}\omega_g^n - e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n)\wedge ds, \tag{1.5}$$

where $\phi_s$ is a path from 0 to $\phi$ in $\mathcal{M}_X(\omega_g)$. It is proved in [TZ3] that there are two uniform positive constants $c_1$ and $c_2$ such that

$$c_1 I_\omega(\phi) \le \tilde{I}_\omega(\phi) - \tilde{J}_\omega(\phi) \le c_2 I_{\omega_g}(\phi), \tag{1.6}$$

where $I_{\omega_g}(\phi) = \frac{1}{V}\int_M \phi(\omega_g^n - \omega_\phi^n)$.

Let

$$\begin{aligned}\hat{F}_{\omega_g}(\phi) &= \tilde{J}_{\omega_g}(\phi) - \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n \\ &= -\frac{1}{V}\int_0^1\int_M \dot{\phi}_s e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n \wedge ds.\end{aligned}$$

By simple computations, one can show that for any two functions $\phi$ and $\psi$ in $\mathcal{M}_X(\omega_g)$, the following co-cycle condition is satisfied,

$$\hat{F}_{\omega_g}(\psi) = \hat{F}_{\omega_g}(\phi) + \hat{F}_\phi(\psi),$$

where $\hat{F}_\phi(\psi) = -\frac{1}{V}\int_0^1\int_M \dot{\phi}_s e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n \wedge ds$ and $\phi_s$ is a path from $\phi$ to $\psi$ in $\mathcal{M}_X(\omega_g)$.

**Proposition 1.1.** *Let $\phi_t$ be a solution of $(1.1)_s$ for $s = t$. Then*

$$\hat{F}_{\omega_g}(\phi_t) = -\frac{1}{t}\int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)ds.$$

*Proof.* Differentiating $(1.1)_t$ with respect to $t$, we have

$$\triangle'\dot{\phi}_t + X(\dot{\phi}_t) = -(t\dot{\phi}_t + \phi_t). \tag{1.7}$$

Then by using (1.7) and (1.4), we get

$$\begin{aligned}
&\frac{d}{dt}(\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\
&= -\frac{1}{V}\int_M \phi_t \frac{d}{dt}(e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n) \\
&= -\frac{1}{V}\int_M \phi_t(\triangle'\dot{\phi}_t + X(\dot{\phi}_t))e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n \\
&= \frac{1}{V}\int_M \phi_t(t\dot{\phi}_t + \phi_t)e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n \\
&= \frac{1}{V}\int_M \phi_t(t\dot{\phi}_t + \phi_t)e^{f - t\phi_t}\omega_g^n \\
&= \frac{1}{V}\frac{d}{dt}(\int_M (-\phi_t)e^{f-t\phi_t}\omega_g^n) + \frac{1}{V}\int_M \dot{\phi}_t e^{f-t\phi_t}\omega_g^n) \\
&= \frac{1}{tV}\frac{d}{dt}(\int_M t(-\phi_t)e^{f-t\phi_t}\omega_g^n) \\
&= \frac{1}{tV}\frac{d}{dt}(\int_M t(-\phi_t)e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n).
\end{aligned}$$

It follows

$$\begin{aligned}
&\frac{d}{dt}(t(\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t))) - (\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\
&= \frac{1}{V}\frac{d}{dt}(\int_M t(-\phi_t)e^{\theta_X(\phi_t)}\omega_{\phi_t}^n),
\end{aligned} \tag{1.8}$$

and consequently,

$$\begin{aligned}
\hat{F}_{\omega_g}(\phi_t) &= -\frac{1}{V}\int_M \phi e^{\theta_X + X(\phi_t)}\omega_t^n - (\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\
&= -\frac{1}{t}\int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)ds.
\end{aligned}$$

□

Recall that the functional $\tilde{F}_{\omega_g}$ is defined as

$$\begin{aligned}
\tilde{F}_{\omega_g}(\psi) &= \hat{F}_{\omega_g}(\psi) - \log(\frac{1}{V}\int_M e^{h-\psi}\omega_g^n) \\
&= \tilde{J}_{\omega_g}(\psi) - \frac{1}{V}\int_M \psi e^{\theta_X}\omega_g^n - \log(\frac{1}{V}\int_M e^{h-\psi}\omega_g^n),
\end{aligned}$$

and $\tilde{F}_{\omega_g}$ is called proper with respect to some holomorphic vector field $X$ on $M$ if there is a monotone increasing function $\mu(t) : (a, +\infty) \to R$ such that

$$\lim_{t\to+\infty}\mu(t) = +\infty,$$

and

$$\tilde{F}_{\omega_g}(\psi) \geq C_1\mu(I_{\omega_g}(\psi)) - C_2$$

for any $K_0$-invariant function $\psi$, where $K_0(\supseteq K_X)$ is a maximal compact subgroup of automorphism group $\mathrm{Aut}(M)$ such that $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\sigma \in K_X$ and $\eta \in K_0$.

*Proof of Theorem 0.1.* Assuming the functional $\tilde{F}_{\omega_g}$ is proper, we shall prove the exsitence of a Kähler-Ricci soliton with respect to $X$ on $M$. This is equivelant to prove that there is a solution of $(1.1)_t$ for $t = 1$. It suffices to prove that $I_{\omega_g}(\phi_t)$ is uniformly bounded for any solution of $(1.1)_t$ for $0 \leq t \leq 1$. This is because $C^3$-norm of $\phi_t$ can be uniformly bounded by $I_{\omega_g}(\phi_t)$ (cf. [TZ3], [Ya]). By the Implicit Function Theorem, one can show the solution of $(1.1)_t$ varies smoothly with $t < 1$. Without the loss of generality, we may assume that the Kähler form $\omega_g$ is $G_0$-invariant, where $G_0(\supseteq K_0)$ is a maximal compact subgroup of $\mathrm{Aut}(M)$. Then solutions $\phi_t$ are all $K_0$-invariant.

By Proposition 1.1, we have

$$\begin{aligned}
&\tilde{F}_{\omega_g}(\phi_t) \\
&= -\frac{1}{t}\int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s))ds - \log(\frac{1}{V}\int_M e^{h_{\omega_g}-\phi_t}\omega_g^n) \\
&\leq -\log(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n).
\end{aligned} \tag{1.9}$$

On the other hand, by using $(1.1)_t$ and concavity of the logarithmic function, one can deduce

$$\begin{aligned}
-\log(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n) &= \frac{1-t}{V}\int_M \phi_t e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n \\
&\leq \frac{1-t}{V}\int_M \phi_t e^{h-t\phi_t}\omega_g^n \leq C.
\end{aligned} \tag{1.10}$$

Combining (1.9) and (1.10), we get

$$\tilde{F}_{\omega_g}(\phi_t) \leq C.$$

Therefore, the assumption of properness on $\tilde{F}_{\omega_g}$ implies

$$I_{\omega_g}(\phi_t) \leq C,$$

and consequently, there is a Kähler-Ricci soliton metric on $M$ with respect to $X$. □

## 2. A Heat Kernel Estimate.

In this section, we give an estimate on the heat kernel of the linear elliptic operator $P = P_\omega = \triangle + \mathrm{Re}(X(.))$ associated to a Kähler form $\omega$ and a a holomorphic vector field $X$ on $M$. As a consequence, we derive a lower bound of the Green function of $P$. The method here follows that of T. Mabuchi in [M1] with modifications which in turn is inspired by Li-Yau [LY]. Note that $P$ is a self-adjoint elliptic operator acting on $C^\infty(M)$ with respect to the inner product,

$$(\phi, \psi) = \int_M \phi\psi e^{\theta_X}\omega^n.$$

**Lemma 2.1.** *Let $\omega$ be a Kähler form on $M$ and $X$ a holomorphic vector field on $M$ with*

$$Ric(\omega) - L_X\omega \geq 0,$$

*and*

$$\triangle\theta_X \leq k,$$

*for some positive number $k$. Let $v(x,t)$ be a positive smooth solution on $M \times (0,\infty)$ of equation $(P - \frac{\partial}{\partial t})v = 0$. Suppose that*

$$\lim_{t\to 0} \sup_{M\times\{t\}} t(v^{-2} < \overline{\partial}v, \overline{\partial}v > -v^{-1}v_t) \leq 2n,$$

*where $v_t = \frac{\partial v}{\partial t}$. Then there is a positive number $C$ depending only on*

$$m_1 = -\max_M \theta_X \text{ and } m_2 = -\min_M \theta_X$$

*such that*

$$v(x,t_1) \leq v(y,t_2)(\frac{t_2}{t_1})^{\frac{n}{C}} exp\{(t_2-t_1)^{-1}r(x,y)^2/2 + C^{-1}k(t_2-t_1)\}, \tag{2.1}$$

*where $r(x,y)$ denotes the distance between $x$ and $y$ associate to metric $\omega$.*

*Proof.* Let $f = \ln v$ and $\hat{F} = t(< \overline{\partial}f, \overline{\partial}f > -f_t)$. Then

$$\begin{aligned} &P\hat{F} + tPf_t \\ &= t(< \mathrm{tr}_\omega(\overline{\nabla\nabla})\partial f, \overline{\partial}f > + < \mathrm{tr}_\omega(\overline{\nabla}\nabla)\partial f, \partial f > \\ &+ \mathrm{Re}(X(< \overline{\partial}f, \overline{\partial}f >)) + \|\nabla\overline{\nabla}f\|^2 + \|\nabla\nabla f\|^2). \end{aligned} \tag{2.2}$$

For simplicity, we choose a local holomorphic coordinate system $(z_1, ..., z_n)$ near each point $p$ such that $g_{i\bar{j}}(p) = \delta_{i\bar{j}}$ and $f_{i\bar{j}}(p) = \delta_{i\bar{j}}f_{i\bar{i}}(p)$. By a direct computation, one sees,

$$f_{\bar{i}}f_{ij\bar{j}} = f_{\bar{i}}(f_{j\bar{j}i} + f_l R_{i\bar{l}}),$$
$$f_i f_{\bar{i}j\bar{j}} = f_i f_{\bar{j}j\bar{i}},$$

and

$$\begin{aligned} &X_{\bar{i}}(f_j f_{\bar{j}})_i + f_{\bar{i}}f_j X_{\bar{j}i} \\ &= f_i(X_{\bar{j}}f_j)_{\bar{i}} + f_{\bar{i}}(X_{\bar{j}}f_j)_i. \end{aligned}$$

Inserting the above indentities into (2.2), we obtain

$$\begin{aligned} P\hat{F} &\geq t(< \partial f, \partial(Pf) > + < \partial(Pf), \partial f >) - tPf_t + \frac{t}{n}(\triangle f)^2 \\ &= 2t\mathrm{Re} < \partial(Pf), \partial f > -tPf_t + \frac{t}{n}(\triangle f)^2 \end{aligned} \tag{2.3}$$

Since

$$Pf - f_t = - < \overline{\partial}f, \overline{\partial}f >,$$

we have

$$\hat{F} = -tPf,$$

and

$$\frac{\partial}{\partial t}\hat{F} - \frac{1}{t}\hat{F} = -tPf_t.$$

Hence by (2.3), we get

$$\begin{aligned} &(P - \frac{\partial}{\partial t})\hat{F} \\ &\geq -2\text{Re} < \partial\hat{F}, \partial f > -\frac{1}{t}\hat{F} + \frac{t}{n}(\frac{1}{t}\hat{F} + \text{Re}(Xf))^2. \end{aligned} \tag{2.4}$$

As in [M1], we define a monotone-increasing function $\eta$ on $[m_1, m_2]$ by

$$\eta(s) = \exp\int_m^s \frac{1}{b_0 e^{-y} - 1} dy,$$

where $b_0 = e^{m_2}(1+n)$. Then $\eta$ is a solution of the ODE on $[m_1, m_2]$,

$$\frac{\eta''}{\eta} - \frac{\eta'}{\eta} = 2(\frac{\eta'}{\eta})^2.$$

Moreover, one can check that the number $C$ defined by

$$C = \min_{s\in[m_1,m_2]} \eta(s)^{-1}(1 - (1 - \eta(s)^{-1}\eta(s)'n)^2)$$

is positive, and

$$0 < \frac{\eta'}{\eta} \leq \frac{1}{n}.$$

Let $F = \eta(-\theta_X)\hat{F}$. Then by (2.4), one can show that

$$\begin{aligned} &(P - \frac{\partial}{\partial t})F \\ &\geq -2\text{Re} < \partial F, \partial f > -\frac{1}{t}F \\ &+ (\frac{\eta''}{\eta} - \frac{\eta'}{\eta} - 2(\frac{\eta'}{\eta})^2)\|X\|^2 F \\ &+ \eta^{-1}\eta'(-2\text{Re}(XF) - F\triangle\theta_X) \\ &+ n^{-1}t^{-1}F^2\eta^{-1}(1 - (1 - \eta^{-1}\eta' n)^2) \\ &+ \frac{t}{n}\eta(-\frac{1}{t}\eta^{-1}F(1 - n\eta^{-1}\eta')^2 - \text{Re}(Xf))^2. \end{aligned}$$

By the assumption in the lemma, we have

$$\begin{aligned} (P - \frac{\partial}{\partial t})F \geq & -2\text{Re} < \partial F, \partial f > -2\eta^{-1}\eta'\text{Re}(X(F)) \\ & - (\frac{1}{t} + \frac{k}{n})F + \frac{C}{nt}F^2. \end{aligned} \tag{2.5}$$

Applying the Maximal Principle for the function $F(x,t)$ on $M \times (0,T]$, we get from (2.5),

$$F(x,t) \leq C^{-1}(2n + kt),$$

for any $(x,t) \in M \times (0,T]$, and consequently,

$$v(x,t)^{-2} < \overline{\partial} v(x,t), \overline{\partial} v(x,t) > -v(x,t)^{-1} v_t(x,t) \leq C^{-1}(\frac{2n}{t} + k),$$

for any $(x,t) \in M \times (0,\infty)$. Now by integrating the above estimate as in [LY], we can immediately obtain (2.1). □

Let $H = H(x,y,t)$ be a fundamental solution on $M \times M \times [0,\infty)$ of equation

$$(P - \frac{\partial}{\partial t}) v(x,y,t) = 0, \tag{2.6}$$

i.e. $H$ is a smooth solution of (2.6) satisfying

$$\begin{cases} H(x,y.t) = H(y,x,t), \\ H(x,y,t) = \int_M H(x,z,t-s) H(z,y.s) e^{\theta_X} \omega^n, \\ \lim_{t\to 0} H(x,y,t) = \delta_x(y). \end{cases}$$

By using the asymptotic behaviour of $H$, for each fixed $x \in M$, one sees

$$\lim_{t\to 0} \sup_{M\times\{t\}} t(H^{-2} < \overline{\partial} H(x,.,t), \overline{\partial} H(x,.,t) > -H^{-1} H_t) \leq 2n.$$

Moreover, by using an argument in [LY] and Lemma 3.2, we can deduce

**Lemma 2.2.** *Let $Z_1$ and $Z_2$ be two measurable subset of $M$. Let $T, \delta, \tau$ be three positive numbers with $\tau < (1+2\delta)T$. For each $x, y \in M$ and $0 < t \leq \tau$, denote*

$$F_{x,T}(y,t) = \int_{Z_1} H(y,.,t) H(x,.,T) e^{\theta_X} \omega^n.$$

*Then*

$$\int_{Z_2} F_{x,T}(.,t)^2 e^{\theta_X} \omega^n \leq \exp\{\frac{-r(x,Z_1)^2}{2(1+2\delta)T} + \frac{R(x,Z_2)^2}{2(1+2\delta)T - 2t}\} F_{x,T}(x,T), \tag{2.7}$$

*where $r(x,Z_1) = \inf_{z\in Z_1} r(x,z)$ and $R(x,Z_2) = \sup_{z\in Z_2} r(x,z)$.*

**Proposition 2.1.** *Let $H(x,y,t)$ be a fundamental solution on $M \times M \times [0,\infty)$ of equation (2.6). Suppose*

$$Ric(\omega) - L_X \omega \geq 0,$$

*and*

$$\triangle \theta_x \leq k,$$

*for some positive number $k$. Then we have*

$$\mathit{vol}(B_x(\sqrt{t}))^{\frac{1}{2}}\mathit{vol}(B_y(\sqrt{t}))^{\frac{1}{2}}H(x,y,t)$$
$$\leq (1+\delta)^{4n/C}\mathit{exp}\{-\frac{\{r(x,y)-\sqrt{t}\}_+^2}{4t(1+3\delta+2\delta^2)}+t\delta(2+\delta)C^{-1}k+\frac{3}{4\delta(1+\delta)}+\frac{1}{2\delta}\},$$

*where* $vol(B_y(\sqrt{t}))=\int_{B_y(\sqrt{t})}e^{\theta_X}\omega^n$, *and* $\{r(x,y)-\sqrt{t}\}_+=\max\{0,r(x,y)-\sqrt{t}\}$.

*Proof.* First applying Lemma 2.1 to the function $F_{x,T}(y,t)$ with $(t_1,t_2)=(T,\tau=(1+\delta)T)$, we have

$$F_{x,T}(x,T)\leq F_{x,T}(y,\tau)(1+\delta)^{2n/C}\exp\{T^{-1}\delta^{-1}r(x,y)^2/2+C^{-1}kT\delta\}. \quad (2.8)$$

Let $Z_1=B_y(\sqrt{t})$ and $Z_2=B_x(\sqrt{t})$. Then integrating the square of the above inequality over all $y\in Z_2$ and using (2.8) and Lemma 2.2, it follows

$$\mathrm{vol}(B_x(\sqrt{t}))F_{x,T}(x,T)^2$$
$$\leq (1+\delta)^{4n/C}\exp\{\frac{-r(x,Z_1)^2}{2(1+2\delta)T}+2C^{-1}kT\delta+\frac{3t}{2T\delta}\}F_{x,T}(x,T),$$

and consequently,

$$\begin{aligned}&\mathrm{vol}(B_x(\sqrt{t}))\int_{Z_1}H(x,.,T)^2e^{\theta_X}\omega^n\\&=\mathrm{vol}(B_x(\sqrt{t}))F_{x,T}(x,T)\\&\leq (1+\delta)^{4n/C}\exp\{\frac{-r(x,Z_1)^2}{2(1+2\delta)T}+2C^{-1}kT\delta+\frac{3t}{2T\delta}\}.\end{aligned} \quad (2.9)$$

On the other hand, applying Lemma 2.1 to the function $H(x,z,t)$ in $z$ with $(t_1,t_2)=(t,T=(1+\delta)t)$, we have for any $x,y,z\in M$,

$$H(x,y,t)^2\leq (1+\delta)^{4n/C}H(x,z,T)^2\exp\{T^{-1}\delta^{-1}r(y,z)^2+2C^{-1}kT\delta\}.$$

Integrating this inequality over all $z\in Z_1$, and using (2.9), we can get (2.7). □

**Theorem 2.1.** *Let* $\phi\in\mathcal{M}_X(\omega)$. *Suppose that*

$$Ric(\omega_\phi)-L_X\omega_\phi\geq\lambda\omega_\phi, \quad (2.10)$$

*and*

$$\triangle\theta_X(\omega_\phi)\leq k,$$

*for some positive numbers* $\lambda$ *and* $k$. *Then there is a uniform constant* $C$ *depending only on* $\lambda$ *and* $k$ *such that*

$$\sup_M(-\phi)\leq\frac{1}{V}\int_M(-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n+C. \quad (2.11)$$

*Proof.* Let $\mu_i(\mu_0 = 0), i = 0, 1, ..,$ be the increasing sequence of eingenvalues of operator $-P$ associated to metric $\omega_\phi$. Then by using the standard Bochner technique, one can obtain $\mu_1 \geq \lambda$ (cf. [TZ3]). Let $G(x, y)$ be the Green function with $\int_M G(x, .)e^{\theta_X(\omega_\phi)}\omega_\phi^n = 0$ associated to operator $P$. Then

$$G(x, y) = \int_0^\infty (H(x, y, t) - \frac{1}{V})dt.$$

Since

$$H_0(x, y, t) = H(x, y, t) - \frac{1}{V} = \sum_{i=1}^{\infty} e^{-\mu_i t} f_i(x) f_i(y),$$

we have

$$H_0(x, x, t + t_0) \leq e^{-\mu_1 t} H_0(x, x, t_0), \tag{2.12}$$

for any $t_0, t > 0$, where $f_i(x)$ denote the eigenfunctions of $\mu_i$.

In [M2], it was proved under the condition (2.10) that there is a uniform constant $C_1$ such that

$$\mathrm{Diam}(M, \omega_\phi) \leq \frac{C_1}{\sqrt{\lambda}}.$$

Choose $t_0 = \frac{1}{4}\mathrm{Diam}(M, \omega_\phi)^2$. Then by Proposition 2.1 we have

$$H_0(x, x, t_0) \leq C_2, \tag{2.13}$$

for some uniform constant $C_2$ depending only on $\lambda$ and $k$.

By using (2.12) and (2.13), we get

$$\begin{aligned} G(x, y) &\geq - \int_0^{t_0} \frac{1}{V} dt - \int_{t_0}^\infty e^{-\mu_1(t - t_0)} (H_0(x, x, t_0) H_0(y, y, t_0))^{1/2} dt \\ &\geq -C_3. \end{aligned}$$

Therefore applying the Green formula to function $-\phi$, together with the fact (cf. [Zh]),

$$\sup_{\mathcal{M}_X(\omega_g)} |X(\phi)| \leq c \tag{2.14}$$

for some uniform constant $c = c(\omega_g, X)$, we prove

$$\begin{aligned} \sup_M(-\phi) &= \frac{1}{V}\int_M (-\phi) e^{\theta_X(\omega_\phi)} \omega_\phi^n \\ &\quad + \sup_M \int_M P(\phi(.)) G(x, .) e^{\theta_X(\omega_\phi)} \omega_\phi^n \\ &\leq \frac{1}{V}\int_M (-\phi) e^{\theta_X(\omega_\phi)} \omega_\phi^n + V(n + \sup_{x \in M} |X(\phi)(x)|) C_3 \\ &\leq \frac{1}{V}\int_M (-\phi) e^{\theta_X(\omega_\phi)} \omega_\phi^n + C. \end{aligned}$$

□

## 3. An $C^0$-Estimate for Solutions of $(1.1)_t$.

**Lemma 3.1.** *There are two positive numbers $c_1$ and $c_2 < 1$ such that for any $\phi \in \mathcal{M}_X(\omega_g)$,*

$$c_1 \tilde{I}_{\omega_g}(\phi) \leq \tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \leq c_2 \tilde{I}_{\omega_g}(\phi). \tag{3.1}$$

*Proof.* Let $\omega_{s\phi} = \omega + s\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$. Then one can compute

$$\begin{aligned}
&\tilde{I}_{\omega_g}(\phi)\\
&= \frac{1}{V}\int_M \phi \int_0^1 \frac{d}{ds}(e^{\theta_X + sX(\phi)}\omega_{s\phi}^n) \wedge ds\\
&= \frac{n\sqrt{-1}}{2\pi V}\int_0^1 ds \int_M \partial\phi \wedge \overline{\partial}\phi e^{\theta_X + sX(\phi)} \wedge \omega_{s\phi}^{n-1}\\
&= \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M (\int_0^1 s^k(1-s)^{n-1-k}e^{\theta_X + sX(\phi)}ds)\\
&\times \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}\\
&\leq \frac{n\sqrt{-1}}{2\pi V}C_1 \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}.
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\tilde{J}_{\omega_g}(\phi)\\
&= \frac{n\sqrt{-1}}{2\pi V}\int_0^1 dt \int_0^1 ds \int_M t\partial\phi \wedge \overline{\partial}\phi e^{\theta_X + stX(\phi)} \wedge \omega_{st\phi}^{n-1}\\
&= \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M (\int_0^1 \int_0^1 t(st)^k(1-st)^{n-1-k}e^{\theta_X + stX(\phi)}dt \wedge ds)\\
&\times \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}\\
&\geq \frac{n\sqrt{-1}}{2\pi V}C_1' \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}.
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\tilde{J}_\omega(\phi) \geq \frac{C_1'}{C_1}\tilde{I}_\omega(\phi),$$

and consequently, prove the second inequality of (3.1).

On the other hand, we have (cf. [TZ3]),

$$\begin{aligned}
&\tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi)\\
&= \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M (\int_0^1 s^{k+1}(1-s)^{n-1-k}e^{\theta_X + sX(\phi)}ds)\\
&\partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}\\
&\geq \frac{n\sqrt{-1}}{2\pi V}C_2 \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}.
\end{aligned} \tag{3.4}$$

Hence, combining (3.2) and (3.4), we also prove the first inequality of (3.1) . □

**Proposition 3.1.** *Let $\phi = \phi_t (t \geq t_0)$ be a solution of equation $(1.1)_t$. Then there are two uniform constants $C_1$ and $C_2$ depending only on $X$ and $t_0$ such that*

$$osc_M \phi \leq C_1 \int_M \phi(\omega_g^n - \omega_\phi^n) + C_2. \tag{3.5}$$

*Proof.* Let $\theta'_X = \theta_X(\omega_\phi)$. First we note

$$\triangle \theta'_X = -\theta'_X - X(h_{\omega_\phi}) + c,$$

for some constant $c$. Clearly $c \leq \mathrm{osc}_M |\theta'_X|$, since $\theta'_X$ changes the sign. By using the fact (cf. (1.2)),

$$h_{\omega_\phi} = \theta'_X - (1-t)\phi + \text{const.},$$

we have

$$\begin{aligned} \triangle \theta'_X &= -\theta'_X - \|X\|_{\omega_\phi} + (1-t)X(\phi) + c \\ &\leq 2\mathrm{osc}_M |\theta_X| + 3|X(\phi)| \leq C'_1, \end{aligned}$$

for some uniform constant $C'_1$. Applying Theorem 2.1, we see that there is some uniform constant $C'_2$ depending only on $X$ and $t_0$ such that

$$\sup_M (-\phi) \leq \frac{1}{V} \int_M (-\phi) e^{\theta'_X} \omega_\phi^n + C'_2. \tag{3.6}$$

On the other hand, by using the Green formula, we have (cf. [TZ4]),

$$\sup_M \phi \leq \frac{1}{V} \int_M \phi e^{\theta_X} \omega^n + C'_3, \tag{3.7}$$

for some uniform constant $C'_3$. Hence, combing (3.6) and (3.7), we get

$$\mathrm{osc}_M \phi \leq \frac{1}{V} \int_M \phi(e^{\theta_X}\omega^n - e^{\theta'_X}\omega_\phi^n) + C'_2 + C'_3.$$

By using (1.6) and (3.1), one can prove (3.5). □

**Remark 3.1.** *Proposition 3.1 improves our previous result about $C^0$-estimate for equations $(1.1)_t$ obtained in [TZ3].*

## 4. A Smoothing Lemma.

In this section, we prove a smoothing lemma by using Hamilton's Ricci flow. This lemma will be used in the proof of Theorem 0.2. Let $\omega$ be any Kähler form in $c_1(M) > 0$ such that

$$\begin{cases} \mathrm{Ric}(\omega) - L_X \omega \geq (1-\epsilon)\omega \\ |X(h_\omega - \theta_X(\omega))| \leq \epsilon c_1, \end{cases} \tag{4.1}$$

for some constant $c_1$ and $0 < \epsilon < 1$. We consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \log(\frac{(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u)^n}{\omega^n}) + u - h_\omega + \theta_X(\omega), \\ u|_{t=0} = 0. \end{cases} \tag{4.2}$$

Note that Eq. (4.2) is the scalar version of the modified Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t + L_X\omega_t.$$

Here $\omega_t = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_t$, and $u_t = u(x,\cdot)$. Denote by $h_t = h_{\omega_t}$ and $\theta_t = \theta_X(\omega_t)$, then it follows from the above equation and maximum principle that

$$h_t - \theta_t = -\frac{\partial u}{\partial t} + \tilde{c}_t$$

where $\tilde{c}_t$ depends on $t$ only. Also, $u_0 = 0$ and hence $\tilde{c}_0 = 0$.

We list a few basic estimates for the solution $u(x,t)$. Differentiating (4.2), we get

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = (\Delta + X)(\frac{\partial u}{\partial t}) + (\frac{\partial u}{\partial t}). \tag{4.3}$$

Applying the maximum principle, we have

**Lemma 4.1.** *Let $u_t$ be a solution of (4.2). Then*

$$\|u_t\|_{C^0} \leq e^t\|h_\omega - \theta_X(\omega)\|_{C^0},$$

*and*

$$\|\frac{\partial u}{\partial t}\|_{C^0} \leq e^t\|h_\omega - \theta_X(\omega)\|_{C^0}.$$

**Lemma 4.2.**

$$(\Delta + X)(h_t - \theta_t) \geq -(c_1 + n)\epsilon e^t.$$

*Proof.* From Eq.(4.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}((\Delta + X)(\frac{\partial u}{\partial t})) &= (\Delta + X)^2(\frac{\partial u}{\partial t}) + (\Delta + X)(\frac{\partial u}{\partial t}) - |\nabla\bar{\nabla}(\frac{\partial u}{\partial t})|^2 \\ &\leq (\Delta + X)^2(\frac{\partial u}{\partial t}) + (\Delta + X)(\frac{\partial u}{\partial t}).\end{aligned}$$

It follows from the maximum principle that

$$(\Delta + X)(h_t - \theta_t) \geq e^t \inf_M(\Delta + X)(h_\omega - \theta_X(\omega)).$$

On the other hand, (4.1) implies that, at $t = 0$,

$$(\Delta + X)(h_\omega - \theta_X(\omega)) \geq -(c_1 + n)\epsilon.$$

Then the lemma follows directly. □

**Lemma 4.3.**

$$\|\frac{\partial}{\partial t}u\|^2_{C^0} + t\|\nabla(\frac{\partial}{\partial t}u)\|^2_{C^0} \leq e^{2t}\|h_\omega - \theta_X(\omega)\|^2_{C^0}.$$

*Proof.* By direct computations, we have

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t})^2 = (\Delta + X)(\frac{\partial u}{\partial t})^2 - |\nabla(\frac{\partial u}{\partial t})|^2 + 2(\frac{\partial u}{\partial t})^2,$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}(|\nabla(\frac{\partial u}{\partial t})|^2)\\ &= (\Delta + X)(|\nabla(\frac{\partial u}{\partial t})|^2) - |\nabla\nabla(\frac{\partial u}{\partial t})|^2 - |\nabla\bar{\nabla}(\frac{\partial u}{\partial t})|^2 + |\nabla(\frac{\partial u}{\partial t})|^2.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t}(\frac{\partial u}{\partial t})^2 + t|\nabla(\frac{\partial u}{\partial t})|^2)\\ &\leq (\Delta + X)(\frac{\partial u}{\partial t})^2 + t|\nabla(\frac{\partial u}{\partial t})|^2) + 2((\frac{\partial u}{\partial t})^2 + t|\nabla(\frac{\partial u}{\partial t})|^2).\end{aligned}$$

Lemma 4.3 follows from the maximum principle again. □

Set

$$v = h_1 - \theta_1 - \frac{1}{V}\int_M (h_1 - \theta_1)e^{\theta_1}{\omega_1}^n.$$

**Lemma 4.4.**

$$\|v\|^2_{L^2} \leq \frac{2(c_1+n)e^2V}{\lambda_1}\epsilon\|h_\omega - \theta_X(\omega)\|_{C^0}$$

*Proof.* By Lemma 4.2, we have

$$(\Delta + X)v + (c_1+n)e\epsilon \geq 0.$$

It follows

$$\begin{aligned}\int_M |(\Delta + X)v + (c_1+n)e\epsilon|e^{\theta_1}{\omega_1}^n &= \int_M ((\Delta + X)v + (c_1+n)e\epsilon)e^{\theta_1}{\omega_1}^n\\ &= (c_1+n)e\epsilon V.\end{aligned}$$

Hence, by applying the Poincaré inequality and Lemma 4.1, we have

$$
\begin{aligned}
\frac{\lambda_1}{V}\int_M |v|^2 e^{\theta_1}{\omega_1}^n &\le \frac{1}{V}\int_M |\nabla v|^2 e^{\theta_1}\omega_1^n \\
&= \frac{1}{V}\int_M (-v)(\Delta + X)v e^{\theta_1}{\omega_1}^n \\
&= \frac{1}{V}\int_M (-v)[(\Delta + X)v + c_1 e\epsilon]e^{\theta_1}{\omega_1}^n \\
&\le \frac{1}{V}\|v\|_{C^0}\int_M ((\Delta + X)v + (c_1+n)e\epsilon)e^{\theta_1}{\omega_1}^n \\
&\le 2e^2(c_1+n)\epsilon\|h_\omega - \theta_X(\omega)\|_{C^0}.
\end{aligned}
$$

This shows the lemma is true. □

**Lemma 4.5.** *We have*

$$
\|v\|_{C^0} \le C(n, c_1 a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{2(n+1)}}, \tag{4.4}
$$

*provided that the following condition holds: there exists a constant $a > 0$ such that for any $x_0 \in M$ and $0 < r < 1$,*

$$
vol(B_r(x_0)) \ge ar^{2n} \tag{4.5}
$$

*with respect to the metric $\omega_1$.*

*Proof.* Pick $r = \epsilon^{\frac{1}{2(n+1)}}$ and cover $M$ by geodesic balls of radius r. For any $x \in M$, we have $x \in B_r(x_0)$ for some $x_0 \in M$. Now

$$
\begin{aligned}
\inf_{B_r(x_0)} |v|^2\epsilon^{\frac{n}{n+1}} &\le \frac{1}{a}\int_{B_r(x_0)} |v|^2 e^{\theta_1}{\omega_1}^n \\
&\le \frac{2(c_1+n)e^2V^2}{a\lambda_1}\epsilon\|h_\omega - \theta_X(\omega)\|_{C^0}.
\end{aligned}
$$

Hence

$$
\inf_{B_r(x_0)} |v| \le C(n, c_1, a, \lambda_1)\epsilon^{\frac{1}{2(n+1)}}\|h_\omega - \theta_X(\omega)\|^{\frac{1}{2}}_{C^0}.
$$

Assuming $\inf_{B_r(x_0)}|v| = v(x_0')$, then

$$
\begin{aligned}
|v(x)| &\le |v(x) - v(x_0')| + v(x_0') \le r \sup_{B_r(x_0)} |\nabla v| + v(x_0') \\
&\le e\|h_\omega - \theta_X(\omega)\|_{C^0}\epsilon^{\frac{1}{2(n+1)}} + C\epsilon^{\frac{1}{2(n+1)}}\|h_\omega - \theta_X(\omega)\|^{\frac{1}{2}}_{C^0} \\
&\le C'(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{2(n+1)}}.
\end{aligned}
$$

This finishes the proof of Lemma 4.5. □

**Proposition 4.1 (Smoothing Lemma).** *Let $\omega \in c_1(M) > 0$ be any Kähler metric satisfying (4.1). Then there is another Käher form $\omega' = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$ such that*

*(1)* $\|u\|_{C^0} \le e\|h_\omega - \theta_X(\omega)\|$

*(2)* $\|h'-\theta'\|_{C^{\frac{1}{2}}} \le C(n,c_1,a,\lambda_1)(1+\|h_\omega-\theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{4(n+1)}}$, *where* $C(n,c_1,a,\lambda_1)$ *is a constant depending only on the dimension* $n$*, the Poincare constant* $\lambda_1$*, constants* $c_1$ *and* $a$ *appeared in (4.5).*

*Proof.* We shall prove that $\omega_1$ satisfies the above two conditions of the proposition under the assumption (4.5). By Lemma 4.1, it suffices to check the second condition only.

Since

$$\frac{1}{V}\int_M e^{h'-\theta'}e^{\theta_1}{\omega_1}^n = 1,$$

by (4.4) in Lemma 4.5, we have

$$\|h'-\theta'\|_{C^0} \le C(n,c_1,a,\lambda_1)(1+\|h_\omega-\theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{2(n+1)}}. \tag{4.6}$$

For any two points $x, y$ in $M$, if the distance $d(x,y) \le \epsilon^{\frac{1}{2(n+1)}}$, then Lemma 4.3 implies that $|\nabla(h'-\theta')| \le e\|h_\omega-\theta_X(\omega)\|_{C^0}$ and hence

$$\frac{|(h'-\theta')(x)-(h'-\theta')(y)|}{\sqrt{d(x,y)}} \le e\|h_\omega-\theta_X(\omega)\|_{C^0}\epsilon^{\frac{1}{4(n+1)}}.$$

On the other hand, if $d(x,y) \ge \epsilon^{\frac{1}{2(n+1)}}$ then (4.6) implies that

$$\frac{|(h'-\theta')(x)-(h'-\theta')(y)|}{\sqrt{d(x,y)}} \le C(n,c_1,a,\lambda_1)(1+\|h_\omega-\theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{4(n+1)}}.$$

This completes the proof of Proposition 4.1. □

## 5. Properness of $\tilde{F}_\omega(\psi)$.

We are now ready to prove Theorem 0.2 stated in the introduction.

*Proof of Theorem 0.2.* Without the loss of generality, we may assume that the Kähler-Ricci soliton $g_{KS}$ is $K_0(\supseteq K)$-invariant ([TZ3]). Let $\omega_{KS}$ be the Kähler form of $g_{KS}$ and $\omega_g = \omega_\psi = \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\psi$. We consider the complex Monge-Ampère equations with parameter $t \in [0,1]$:

$$\begin{cases} \det(g_{i\bar j}+\varphi_{i\bar j}) = \det(g_{i\bar j})\exp\{h_g-\theta_X-X(\varphi)-t\varphi\} \\ (g_{i\bar j}+\varphi_{i\bar j}) > 0, \end{cases} \tag{5.1}$$

Clearly, $-\psi + const.$ is a solution of $(5.1)_t$ for $t=1$. Since $\psi$ is $G$-invariant, by the Implicit Function Theorem, there are $G$-invariant solutions of $(5.1)_t$ for $t$ sufficiently close to 1. In fact, in [TZ3], it is proved that there are $G$-invariant solutions of $(5.1)_t$ for any $t\in[0,1]$. This is because $\tilde{I}_\omega(\varphi_t)-\tilde{J}_\omega(\varphi_t)$ is nondecreasing in $t$, and consequently the $C^3$-norm of $\varphi_t$ can be uniformly bounded.

Put $\omega_t = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_t$. Then $\omega_1 = \omega_{KS}$. Moreover, by (1.2), we have

$$\begin{cases} h_{\omega_t}-\theta_X(\omega_t) = -(1-t)\varphi_t + c_t, \\ \mathrm{Ric}(\omega_t) - L_X(\omega_t) = t\omega_t + (1-t)\omega \ge t\omega_t, \end{cases}$$

where $c_t$ is determined by

$$\int_M e^{-(1-t)\varphi_t + c_t}\omega_t^n = V.$$

In particular,

$$|c_t| \le (1-t)\|\varphi_t\|_{C^0}, \ \|h_{\omega_t} - \theta_X(\omega_t)\|_{C^0} \le 2(1-t)\|\varphi_t\|_{C^0},$$

and

$$\begin{aligned} &|X(h_{\omega_t} - \theta_X(\omega_t))| = |(1-t)X(\varphi_t)| \\ &\le (1-t)(|X(\varphi_t - \varphi_1)| + |X(\psi)|) \le (1-t)c_1(\omega_{KS}, X) \end{aligned}$$

Hence by applying Proposition 4.1 to each $\omega_t$, we obtain a Kähler form $\omega_t' = \omega_t + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} u_t$ satisfying

$$\|u_t\|_{C^0} \le 2e(1-t)\|\varphi_t\|_{C^0},$$

$$\|h_{\omega_t'}\|_{C^{\frac{1}{2}}} \le C(n, c_1, a, \lambda_1)(1 + \|(1-t)\varphi_t\|_{C^0})(1-t)^{\frac{1}{4(n+1)}}.$$

As before, there are $\psi_t$ such that $\omega_{KS} = \omega_t' + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi_t$ and

$$\omega_{KS}^n = (\omega_t')^n e^{h_{\omega_t'} - \theta_X(\omega_t') - X(\psi_t) - \psi_t}$$

It follows from the maximum principle that

$$\phi_t = \phi_1 - \psi_t + \mu_t, \tag{5.2}$$

where $\mu_t$ are constants with

$$|\mu_t| \le 2(e+1)(1-t)\|\phi_t\|_{C^0} + c_1(\omega_{KS}, X). \tag{5.3}$$

Hence, $\phi_t$ is uniformly equivalent to $\phi_1$ as long as $\psi_t$ is uniformly bounded. Consider the operator $\Phi_t : C^{2,\frac{1}{2}}(M) \to C^{0,\frac{1}{2}}$ by

$$\Phi_t(\psi) = \log\left(\frac{\omega_{KS} - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi)^n}{\omega_{KS}^n}\right) + h_{\omega_t'} - \theta_X(\omega_t') - X(\psi) - \psi.$$

Its linearization at $t = 0$ is $(-\triangle - 1 - X(\cdot))$, so it is invertiable in the space of $G$-invariant functions by the assumption of Theorem 0.2. Then by the Implicit Function Theorem, there is a $\delta > 0$, such that if the Hölder norm $\|h_{\omega_t'}\|_{C^{\frac{1}{2}}(\omega_{KS})}$ with respect to $\omega_{KS}$ is less than $\delta$, then there is a unique $\psi$ such that $\Phi_t(\psi) = 0$ and $\|\psi\|_{C^{2,\frac{1}{2}}} \le C(\delta)$.

We observe that

$$\lambda_{1,\omega'} \ge 2^{-n-1}\lambda_{1,\omega_{KS}}, \ a \ge \frac{1}{2^{2n}}a_0,$$

whenever $\frac{1}{2}\omega_{KS} \le \omega' \le 2\omega_{KS}$, where $a$ is a constant appeared in (4.5) and $a_0$ is a constant such that

$$a_0 r^{2n} \le \mathrm{vol}_{\omega_{KS}}(B_r(x)), \ \forall x \in M.$$

Now we choose $t_0$ such that $(1-t_0) \leq (\frac{\delta}{4C_0})^{4(n+1)}$ and

$$(1-t_0)\|\varphi_t\|_{C^0}(1-t_0)^{\frac{1}{4(n+1)}} = \frac{\delta}{4C_0}, \tag{5.4}$$

where $C_0 = C_0(n, c_1, \frac{1}{2^{2n}}a_0, 2^{-n-1}\lambda_{1,\omega_{KS}})$. Then by Proposition 4.1, one can prove that for any $t \geq t_0$, we have

$$\|u_t\|_{C^0} \leq 2e(1-t)\|\varphi_t\|_{C^0}, \ \|\psi_t\|_{C^0} \leq \frac{1}{4}.$$

Therefore, by using (5.2) and (5.3), we get

$$\|\varphi_1 - \varphi_t\|_{C^0} \leq 6e(1-t)\|\varphi_t\|_{C^0} + 2c_1, \tag{5.5}$$

and

$$\frac{1}{2}\|\varphi_1\|_{C^0} - c_2 \leq \|\varphi_t\|_{C^0} \leq 2\|\phi_1\|_{C^0} + c_2, \tag{5.6}$$

for some uniform constants $c_1$ and $c_2$, as long as $1-t \leq \min\{\frac{1}{12e}, 1-t_0\}$.

Since $\tilde{I}(\varphi_t) - \tilde{J}(\varphi_t)$ is nondecreasing, by (5.5), we have

$$\begin{aligned}
&\tilde{F}_{\omega_{KS}}(\psi) = -\tilde{F}_{\omega_g}(\varphi_1) \\
&= \int_0^1 (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t))dt \\
&\geq (1-t)(\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) \\
&\geq C_1(1-t)I_{\omega_g}(\varphi_t) \\
&\geq C_1(1-t)J_{\omega_g}(\varphi_t) \\
&\geq C_1(1-t)J_{\omega_g}(\varphi_1) - C_1(1-t)\mathrm{osc}_M(\varphi_t - \varphi_1) \\
&\geq C_1\frac{1-t}{n}I_{\omega_g}(\varphi_1) - 8eC_1(1-t)^2\|\varphi_1\|_{C^0} - C_2 \\
&= C_1\frac{1-t}{n}I_{\omega_{KS}}(\psi) - 8eC_1(1-t)^2\mathrm{osc}_M\psi - C_2.
\end{aligned} \tag{5.7}$$

In case

$$\mathrm{osc}_M\psi \leq C'(1 + I_{\omega_{KE}}(\psi))$$

for some uniform constant $C'$, then by using (5.4) and (5.6), we see that there are two positive numbers $C_1'$ and $C_2'$ such that

$$\tilde{F}_{\omega_{KS}}(\psi) \geq C_1' I_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C_2', \tag{5.8}$$

and consequently this would prove the theorem.

In the general case, we shall use a trick in [TZ1]. First by Proposition 3.1, we have for any $t \geq \frac{1}{2}$,

$$\mathrm{osc}_M(\phi_t - \phi_1) \leq C''(1 + I_{\omega_{KS}}(\phi_t - \phi_1)).$$

Set $\psi' = \phi_t - \phi_1$. Then by (5.8), we get

$$\tilde{F}_{\omega_g}(\phi_t) - \tilde{F}_{\omega_g}(\phi_1) = \tilde{F}_{\omega_{KS}}(\psi') \geq C_3 I_{\omega_{KS}}(\psi')^{\frac{1}{4n+5}} - C_4. \tag{5.9}$$

On the other hand, by integrating (1.6) from $t$ to 1, we have

$$\begin{aligned}&\hat{F}_{\omega_g}(\phi_1) - \hat{F}_{\omega_g}(\phi_t)\\ &\geq \tilde{J}_{\omega_g}(\phi_1) - \frac{1}{V}\int_M \phi_1 e^{\theta_X}\omega_g^n - t(\tilde{J}_{\omega_g}(\phi_t) - \frac{1}{V}\int_M \phi_t e^{\theta_X}\omega_g^n)\\ &\geq -(1-t)(\tilde{I}_{\omega_g}(\phi_1) - \tilde{J}_{\omega_g}(\phi_1))\\ &\geq -C_5(1-t)I_{\omega_g}(\phi_1) = -C_5(1-t)I_{\omega_{KS}}(\psi).\end{aligned} \tag{5.10}$$

By using the concavity of the logarithmic function and (3.6), we also have,

$$\begin{aligned}&-\log(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n) \leq \frac{1-t}{V}\int_M \phi_t e^{\theta'_X}\omega_{\phi_t}^n\\ &\leq -\frac{1-t}{V}\sup_M(-\phi_t) + C_6 \leq C_6.\end{aligned} \tag{5.11}$$

Hence combining (5.10) and (5.11), we get

$$\tilde{F}_{\omega_g}(\phi_t) - \tilde{F}_{\omega_g}(\phi_1) \leq C_5(1-t)I_{\omega_{KS}}(\psi) + C_6. \tag{5.12}$$

¿From (5.9) and (5.12), we deduce

$$(1-t)I_{\omega_{KS}}(\psi) \geq c_3 \mathrm{osc}_M(\phi_t - \phi_1)^{\frac{1}{4n+5}} - c_4.$$

Then as in (5.7), we prove (cf. [TZ1]),

$$\begin{aligned}\tilde{F}_{\omega_{KS}}(\psi) &\geq C_1\frac{1-t}{n}I_{\omega_{KS}}(\psi) - C_1(1-t)\mathrm{osc}_M(\varphi_t - \varphi_1)\\ &\geq C_1\frac{1-t}{n}I_{\omega_{KS}}(\psi) - C_1(1-t)(c_3^{-1})^{4n+5}((1-t)I_{\omega_{KS}}(\psi) + c_4)^{4n+5}\\ &\geq cI_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C,\end{aligned} \tag{5.13}$$

for some small positive number $c$ and large number $C$. Thus Theorem 0.2 is proved. □

**Remark 5.1.** *By (1.6) and (3.1), we see that the inequality (5.13) is equivalent to the following non-linear inequality of Moser-Trudinger type,*

$$\int_M e^{-\psi}\omega_{KS}^n \leq C exp\{\tilde{J}_{\omega_{KS}}(\psi) - c\tilde{J}_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - \frac{1}{V}\int_M \psi\omega_{KS}^n\}$$

*for some positive numbers $c$ and $C$. The inequality of this type was first obtained in [Ti] and [TZ1] on Kähler-Einstein manifolds with positive scalar curvature. Inequality (5.13) improves the result in those two papers. A weak version of (5.13) was obtained in [TZ4].*

**Remark 5.2.** *It seems that Theorem 0.2 can be improved as follows:*

$$\tilde{F}_{\omega_{KS}}(\psi) \geq cI_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C$$

*holds for any* $\psi \in \Lambda_1(M, \omega_{KS})^{\perp}$*. In the proof of Theorem 0.2, we used a technical assumption on subgroup* $G(\subseteq K_0)$ *in order to apply the Implicit Theorem. We also notice from Theorem 0.2 that (5.13) holds for any almost plurisubharmonic function on a Kähler-Einstein manifold without any nontrivial holomorphic vector field.*

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