Dynamical foundations of nonextensive statistical mechanics

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Abstract

We construct classes of stochastic differential equations with fluctuating friction forces that generate a dynamics correctly described by Tsallis statistics and nonextensive statistical mechanics. These systems generalize the way in which ordinary Langevin equations underly ordinary statistical mechanics to the more general nonextensive case. As a main example, we construct a dynamical model of velocity fluctuations in a turbulent flow, which generates probability densities that very well fit experimentally measured probability densities in Eulerian and Lagrangian turbulence. Our approach provides a dynamical reason why many physical systems with fluctuations in temperature or energy dissipation rate are correctly described by Tsallis statistics.

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Recently there has been considerable interest in the formalism of nonextensive statistical mechanics (NESM) as introduced by Tsallis [1] and further developed by many others (e.g. [2, 3]). In the mean time there is growing evidence that the formalism, rather than being just a theoretical construction, is of relevance to many complex physical systems. Applications in various areas have been reported, mainly for systems with either long-range interactions [4]–[6], multifractal behaviour [7, 8], or fluctuations of temperature or energy dissipation rate [9]–[13]. A recent interesting application of the formalism is that to fully developed turbulence [8, 10, 11]. Precision measurements of probability density functions (pdfs) of longitudinal velocity differences in high-Reynolds number turbulent Couette-Taylor flows are found to agree quite perfectly with analytic formulas of pdfs as predicted by NESM [11].

Despite this apparent success of the nonextensive approach, still the question remains *why* in many cases (such as the above turbulent flow) NESM works so well. To answer this question, let us first go back to ordinary statistical mechanics and just consider a very simple well known example, the Brownian particle [14]. Its velocity u satisfies the linear Langevin equation

$$\dot{u} = -\gamma u + \sigma L(t), \tag{1}$$

where $L(t)$ is Gaussian white noise, $\gamma > 0$ is a friction constant, and σ describes the strength of the noise. The stationary probability density of u is Gaussian with average 0 and variance β^{-1}, where where $\beta = \frac{\gamma}{\sigma^2}$ can be identified with the inverse temperature of ordinary statistical mechanics (we assume that the Brownian particle has mass 1).

The above simple situation completely changes if one allows the parameters γ and σ in the stochastic differential equation (SDE) to fluctuate as well. To be specific, let us assume that either γ or σ or both fluctuate in such a way that $\beta = \gamma/\sigma^2$ is χ^2 distributed with degree n. This means the probability density of β is given by

$$f(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n\beta}{2\beta_0}\right\} \tag{2}$$

The χ^2 distribution is a typical distribution that naturally arises in many circumstances. For example, consider n independent Gaussian random variables X_i, $i = 1, \ldots, n$ with average 0, then the sum

$$\beta := \sum_{i=1}^{n} X_i^2 \tag{3}$$

is a random variable with probability density (2). The average is given by

$$\langle \beta \rangle = n \langle X^2 \rangle = \int_0^\infty \beta f(\beta) d\beta = \beta_0 \tag{4}$$

and the variance by

$$\langle \beta^2 \rangle - \beta_0^2 = \frac{2}{n} \beta_0^2 \tag{5}$$

(see also [9]).

Now assume that the time scale on which β fluctuates is much larger than the typical time scale of order γ^{-1} that the Langevin system (1) needs to reach equilibrium. In this case one obtains for the conditional probability $p(u|\beta)$ (i.e. the probability of u given some value of β)

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp\left\{ -\frac{1}{2} \beta u^2 \right\}, \tag{6}$$

for the joint probability $p(u, \beta)$ (i.e. the probability to observe both a certain value of u and a certain value of β)

$$p(u, \beta) = p(u|\beta) f(\beta) \tag{7}$$

and for the marginal probability $p(u)$ (i.e. the probability to observe a certain value of u no matter what β is)

$$p(u) = \int p(u|\beta) f(\beta) d\beta. \tag{8}$$

The integral (8) is easily evaluated and one obtains

$$p(u) = \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\beta_0}{\pi n} \right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{\beta_0}{n} u^2\right)^{\frac{n}{2} + \frac{1}{2}}} \tag{9}$$

Hence the SDE (1) with χ^2-distributed $\beta = \gamma / \sigma^2$ generates the generalized canonical distributions of NESM [1]

$$p(u) \sim \frac{1}{\left(1 + \frac{1}{2} \tilde{\beta} (q-1) u^2\right)^{\frac{1}{q-1}}} \tag{10}$$

provided the following identifications are made.

$$\frac{1}{q-1} = \frac{n}{2} + \frac{1}{2} \iff q = 1 + \frac{2}{n+1} \tag{11}$$

$$\frac{1}{2}(q-1)\tilde{\beta} = \frac{\beta_0}{n} \iff \tilde{\beta} = \frac{2}{3-q}\beta_0. \tag{12}$$

We already see from this simple example that the physical inverse temperature $\beta_0 = \langle \beta \rangle$ not necessarily coincides with the inverse temperature parameter $\tilde{\beta}$ used in the nonextensive formalism (see also [15] for related results).

More generally, we may also consider nonlinear Langevin equations of the form

$$\dot{u} = -\gamma F(u) + \sigma L(t) \tag{13}$$

where $F(u) = -\frac{\partial}{\partial u} V(u)$ is a nonlinear forcing. To be specific, let us assume that $V(u) = C|u|^{2\alpha}$ is a power-law potential. The SDE (13) then generates the conditional pdf

$$p(u|\beta) = \frac{\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)} (C\beta)^{\frac{1}{2\alpha}} \exp\left\{-\beta C |u|^{2\alpha}\right\} \tag{14}$$

and for the marginal distributions $p(u) = \int p(u|\beta) f(\beta) d\beta$ we obtain after a short calculation

$$p(u) = \frac{1}{Z_q} \frac{1}{(1 + (q-1)\tilde{\beta} C |u|^{2\alpha})^{\frac{1}{q-1}}}, \tag{15}$$

where

$$Z_q^{-1} = \alpha \left(C(q-1)\tilde{\beta} \right)^{\frac{1}{2\alpha}} \cdot \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\frac{1}{q-1} - \frac{1}{2\alpha}\right)} \tag{16}$$

and

$$q = 1 + \frac{2\alpha}{\alpha n + 1} \tag{17}$$

$$\tilde{\beta} = \frac{2\alpha}{1 + 2\alpha - q}\beta_0. \tag{18}$$

To generalize to N particles in d space dimensions, we may consider coupled systems of SDEs with fluctuating friction forces, as given by

$$\dot{\vec{u}}_i = -\gamma_i \vec{F}_i(\vec{u}_1, \ldots, \vec{u}_N) + \sigma_i \vec{L}_i(t) \qquad i = 1, \ldots, N \tag{19}$$

Suppose that a potential $V(\vec{u_1}, \dots, \vec{u_N})$ exists for this problem such that $\vec{F_i} = \frac{\partial}{\partial \vec{u}_i} V$. Moreover, assume that all $\beta_i = \frac{\gamma_i}{\sigma_i^2}$ fluctuate in the same way, i.e. are given by the same fluctuating χ^2 distributed random variable $\beta_i = \beta$. One then has for the conditional probability

$$p(\vec{u}_1, \dots, \vec{u}_N|\beta) = \frac{1}{Z(\beta)} \exp\{-\beta V(\vec{u}_1, \dots, \vec{u}_N)\}, \tag{20}$$

where $Z(\beta) = \int d\vec{u}_1 \dots d\vec{u}_N \, e^{-\beta V}$ is the ordinary partition function. Suppose that $Z(\beta)$ is of the form

$$Z(\beta) \sim \beta^x e^{-y\beta}, \tag{21}$$

then integration over the fluctuating β leads to marginal distributions of the form

$$p(\vec{u_1}, \dots, \vec{u_N}) \sim \frac{1}{(1 + \tilde{\beta}(q-1)V(\vec{u_1}, \dots, \vec{u}_N))^{\frac{1}{q-1}}}, \tag{22}$$

i.e. the generalized canonical distributions of NESM with

$$q = 1 + \frac{2}{n - 2x} \tag{23}$$

and

$$\tilde{\beta} = \frac{\beta_0}{1 + (q-1)(x - \beta_0 y)}. \tag{24}$$

Eq. (22) is correct if all particles see the same fluctuating β at the same time—an assumption that can only be true for a very dense and concentrated system of particles. In many physical applications, the various particles will be dilute and only weakly interacting. Hence in this case β is expected to fluctuate spatially in such a way that the local inverse temperature β_i surrounding one particle i is almost independent from the local β_j surrounding another particle j. Moreover, the potential is approximately just a sum of single-particle potentials $V(\vec{u}_1, \dots, \vec{u}_N) = \sum_{i=1}^N V_s(\vec{u}_i)$. In this case integration over all β_i leads to marginal densities of the form

$$p(\vec{u}_1, \dots, \vec{u}_N) \sim \prod_{i=1}^N \frac{1}{(1 + \tilde{\beta}(q-1)V_s(\vec{u}_i))^{\frac{1}{q-1}}}, \tag{25}$$

i.e. the N-particle nonextensive system reduces to products of 1-particle nonextensive systems (this type of factorization was successfully used in [13]).

The truth of what the correct nonextensive thermodynamic description is will often lie inbetween the two extreme cases (22) and (25).

Let us now come to our main physical example, namely fully developed turbulence. Let u in eq. (13) represent a local velocity difference in a fully developed turbulent flow as measured on a certain scale r. We define

$$\beta = \epsilon\tau \tag{26}$$

where ϵ is the (fluctuating) energy dissipation rate (averaged over r^3) and τ is a typical time scale during which energy is transferred. Both ϵ *and* τ can fluctuate, and we assume that $\epsilon\tau$ is χ^2 distributed. For power-law friction forces the SDE (13) generates the stationary pdf (15). In Fig. 1 this theoretical distribution is compared with experimental measurements in two turbulence experiments performed on two very different scales. All distributions have been rescaled to variance 1. Apparently, there is very good coincidence between experimental and theoretical curves, thus indicating that our simple model assumptions are a good approximation of the true turbulent statistics.

Generally, in turbulent systems the entropic index q is observed to decrease with increasing r (see [11] for precision measurements of $q(r)$). At the smallest scale we expect from the definition of the energy dissipation rate

$$\epsilon = 5\nu \left\{ \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\} \tag{27}$$

(ν: kinematic viscosity, v: velocity) that the independent Gaussian random variables X_i in eq. (3) are given by

$$X_i = \sqrt{5\nu\tau}\frac{\partial v}{\partial x_i}, \tag{28}$$

and that there are indeed 3 of them, due to the 3 spatial dimensions. This means $n = 3$ or, using eq. (17), $q \approx \frac{3}{2}$ if $\alpha \approx 1$. This is indeed confirmed by the fit of the small-scale data of the Bodenschatz group in Fig. 1, yielding $q \approx 1.5$

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Figure captions

Fig. 1 Histogram of longitudinal velocity differences as measured by Swinney et al. [11, 16] in a turbulent Couette Taylor flow with Reynolds number $R_\lambda = 262$ at scale $r = 116\eta$ (solid line), where η is the Kolmogorov length. The experimental data are very well fitted by the analytic formula (15) with $q = 1.1$ and $\alpha = 0.9$ (dashed line). The data points + are a histogram of the acceleration (= velocity difference on a very small time scale) of a Lagrangian test particle as measured by Bodenschatz et al. for $R_\lambda = 200$ [17]. These data are well fitted by (15) with $q = 1.48$ and $\alpha = 0.9$ (dotted line).

References

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988)

[2] S. Abe, Phys. Lett. **271A**, 74 (2000)

[3] C. Tsallis, R.S. Mendes and A.R. Plastino, Physica **261A**, 534 (1998)

[4] A.R. Plastino and A. Plastino, Phys. Lett. **174A**, 384 (1993)

[5] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati and C. Tsallis, Astrophys. Lett. and Comm. **35**, 449 (1998)

[6] M. Antoni and S. Ruffo, Phys. Rev. **52E**, 2361 (1995)

[7] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998)

[8] T. Arimitsu and N. Arimitsu, J. Phys. **33A**, L235 (2000)

[9] G. Wilk and Z. Wlodarczyk, Phys. Rev. Lett. **84**, 2770 (2000)

[10] C. Beck, Physica **277A**, 115 (2000)

[11] C. Beck, G.S. Lewis and H.L. Swinney, Phys. Rev. **63E**, 035303(R) (2001)

[12] I. Bediaga, E.M.F. Curado, and J. Miranda, Physica **286A**, 156 (2000)

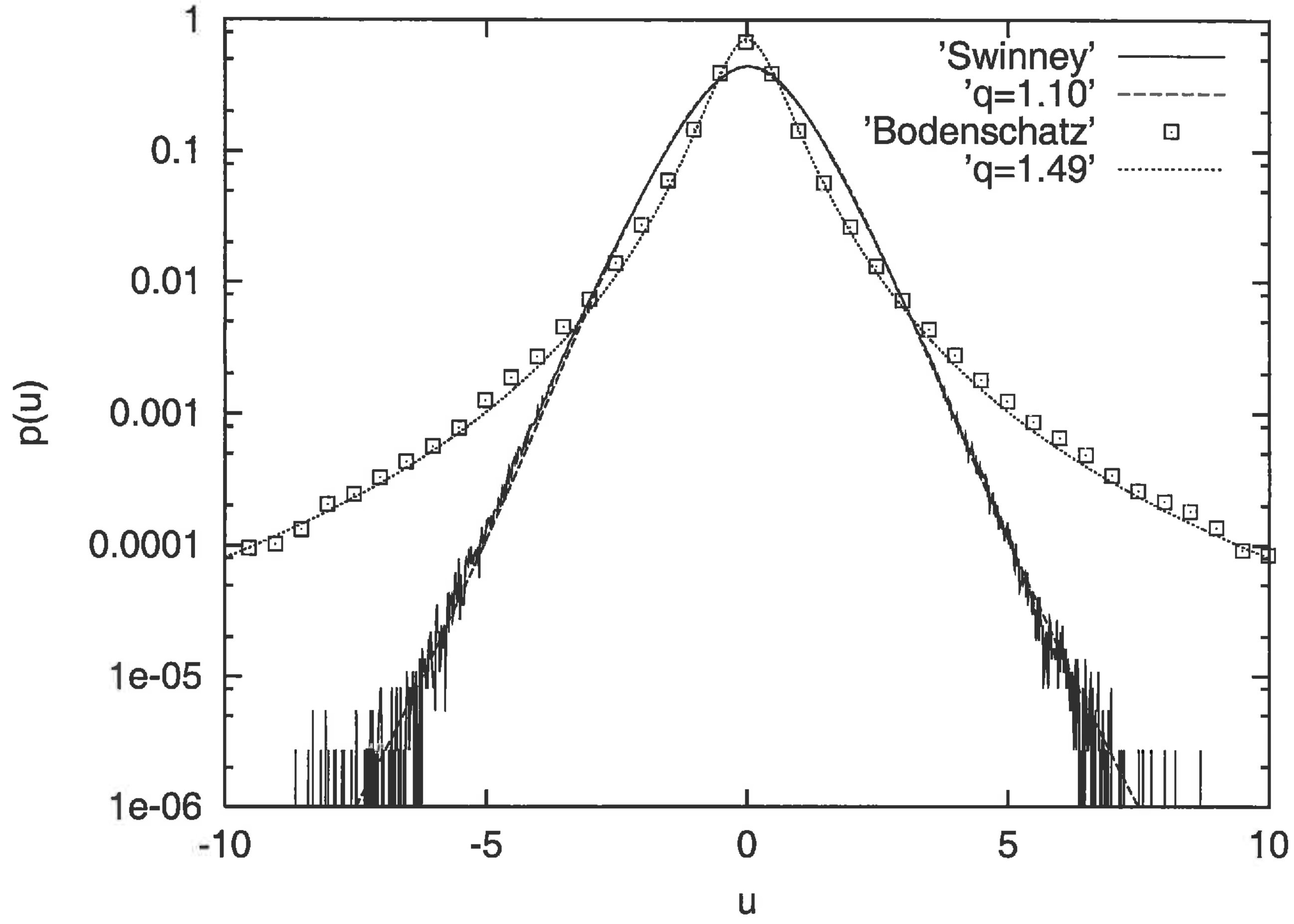
[13] C. Beck, Physica **286A**, 164 (2000)

[14] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam (1981)

[15] S. Abe, in print, Phys. Lett. **A** (2001)

[16] G.S. Lewis and H.L. Swinney, Phys. Rev. **59E**, 5457 (1999)

[17] A. La Porta, G.A. Voth, A.M. Crawford, J. Alexander, and E. Bodenschatz, Nature **409**, 1017 (2001)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-10
-5
0
5
10
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