On the small-scale statistics of Lagrangian turbulence

Christian Beck[1]

Isaac Newton Institute for Mathematical Sciences, University of Cambridge, 20 Clarkson Road, Cambridge CB3 0EH, UK

Abstract

We provide evidence that the small-scale statistics of the acceleration of a test particle in high-Reynolds number Lagrangian turbulence is correctly described by Tsallis statistics with entropic index $q = \frac{3}{2}$. We present theoretical arguments why Tsallis statistics can naturally arise in Lagrangian turbulence and why at the smallest scales $q = \frac{3}{2}$ is relevant. A generalized Heisenberg-Yaglom formula is derived from the nonextensive model.

[1] permanent address: School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS.

Recently, methods from nonextensive statistical mechanics [1, 2, 3] have been successfully applied to fully developed turbulent flows [4, 5, 6, 7]. As a driven noneqilibrium system a turbulent flow cannot extremize the Boltzmann-Gibbs entropy — it is obvious that ordinary statistical mechanics fails to provide a correct description of turbulence. But there is experimental and theoretical evidence that the statistics of velocity differences is well described if one assumes that the flow extremizes the 'second best' information measures available. These are the Tsallis entropies [1] defined by

$$S_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right). \tag{1}$$

$q \neq 1$ is the entropic index and p_i are the probabilities associated with the microstates of the physical system. The Tsallis entropies, closely related to (but different from) the Rényi information measures [8, 9], are convex and distinguished by generalized Khinchin axioms [10]. For $q \to 1$ they reduce to the ordinary Boltzmann-Gibbs entropy. Their importance has been demonstrated in numerous recent papers (see [11] for a detailed listing).

Whereas previous papers on turbulence and Tsallis statistics mainly dealt with the inertial range [4, 6], in this Letter we will concentrate on the small-scale characteristics of fully developed turbulent flows. Here one is still far away from a complete theory, though many empirical facts of the small-scale statistics are well known and have been experimentally verified (see e.g. [12] for a review). Most turbulence measurements have been conducted in the Eulerian frame, *i.e.*, by measuring the spatial fluctuations of the velocities using the Taylor hypothesis. Recently, experimental progress has been made in investigating the Lagrangian properties of fully developed turbulence, by tracking test particles that are advected by the flow [13, 14]. Examples of measured histograms of the acceleration of a test particle as measured by Bodenschatz et al. are shown in Fig. 1. The distribution has been rescaled to variance 1. The acceleration has been extracted by parabolic fits over $0.75\tau_\eta$, where $\tau_\eta = (\nu/\epsilon)^{\frac{1}{2}}$ is the Kolmogorov time, ν is the kinematic viscosity and ϵ is the average energy dissipation rate. The figure also shows as a solid line the normalized probability density function

$$p(x) = \frac{2}{\pi}(1 + 2x^2 + x^4)^{-1} = \frac{2}{\pi}(1 + x^2)^{-2}, \tag{2}$$

which has variance 1. Apparently, the experimental data and the above distribution function agree quite well. We will now present theoretical arguments why this type of distribution is relevant.

The acceleration a of the Lagrangian test particle is a strongly fluctuating random variable. It can be regarded as a velocity difference on a small time scale τ, i.e. $a \approx (v(t) - v(t+\tau))/\tau := u/\tau$, where $\tau := \kappa\tau_\eta$ is of the order of the Kolmogorov time. It is not clear whether the limit $\tau \to 0$ exists in a mathematically rigorous way. For example, already for an ordinary Brownian particle described by the Ornstein Uhlenbeck process [15], the velocity exists but the acceleration is singular. Hence it seems to make sense to statistically describe the movement of the Lagrangian test particle using a small effective finite time scale τ. One can now develop the formalism of nonextensive statistical mechanics for the fluctuating temporal changes in velocity. Following the ideas of [4] we define formal energy levels $E(u)$ by the kinetic energy of the velocity differences

$$E(u) := \frac{1}{2}a^2\tau^2 \approx \frac{1}{2}(v(t) - v(t+\tau))^2 = \frac{1}{2}u^2. \tag{3}$$

Moreover, a formal temperature β_0^{-1} is introduced as

$$\beta_0^{-1} := \epsilon\tau = \epsilon\kappa\tau_\eta = \kappa\epsilon^{1/2}\nu^{1/2} \tag{4}$$

The multiplication with a time scale τ is necessary for dimensionality reasons, since ϵ has dimension $length^2/time^3$. Extremizing the Tsallis entropies one obtains the following generalized version of a canonical distribution [4]

$$p(u) = \frac{1}{Z_q}\left(1 + \frac{1}{2}(q-1)\beta_0 u^2\right)^{-\frac{1}{1-q}}. \tag{5}$$

For $q \to 1$ the above probability density reduces to the ordinary Boltzmann factor $p(u) \sim e^{-\frac{1}{2}\beta_0 u^2}$. The value of the entropic index $q > 1$ depends on the Reynolds number and the scale (see [19] for precision measurements in Eulerian turbulence). The normalization constant Z_q is given by

$$Z_q = \left(\frac{2\pi}{(q-1)\beta_0}\right)^{1/2} \frac{\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)}. \tag{6}$$

For $q = \frac{3}{2}$ and if rescaled to variance 1, the distribution (5) is identical with the distribution (2), which apparently is in very good agreement with the experimental data.

All moments of the generalized canonical distribution (5) can be evaluated analytically. In particular, one obtains for the second moment

$$\langle u^2 \rangle = \int_{-\infty}^{+\infty} u^2 p(u) du = \frac{1}{\beta_0} \frac{2}{5-3q} \tag{7}$$

This yields for the second moment of the acceleration $a = u/\tau$

$$\langle a^2 \rangle = \frac{1}{\beta_0 \tau^2} \frac{2}{5-3q} = \epsilon^{3/2} \nu^{-1/2} \frac{1}{\kappa} \frac{2}{5-3q} \tag{8}$$

Thus we obtain from the nonextensive model the Heisenberg-Yaglom relation $\langle a^2 \rangle = a_0 \epsilon^{3/2} \nu^{-1/2}$, identifying the constant a_0 with

$$a_0 = \frac{1}{\kappa} \frac{2}{5-3q}. \tag{9}$$

Gaussian statistics ($q = 1$) and $\kappa \approx 1$ would imply $a_0 \approx 1$. On the other hand, the true turbulent small scale statistics is much better described by $q = 3/2$, which yields $a_0 \approx 4$, in agreement with direct numerical simulations for large Reynolds numbers [16]. The precise numerical value of a_0 also depends on the ratio $\kappa = \tau/\tau_\eta$ which enters into the formal thermodynamic description via eq. (4). Bodenschatz et al. [14] measure distributions with $q \approx 1.5$ and $a_0 \approx 5.3$ for large R_λ, which corresponds to $\kappa \approx 0.75$. Generally, a measured Reynolds number dependence of a_0 (as presented in [14]) can be translated into a measurement of the entropic index q in Lagrangian turbulence, by solving eq. (9) for q.

Let us now argue on theoretical grounds a) why Tsallis statistics can naturally arise in Lagrangian turbulent flows and b) why the entropic index is $q = \frac{3}{2}$ at the smallest scales.

Generally, Tsallis statistics with $q > 1$ can arise from ordinary statistical mechanics (with ordinary Boltzmann factors $e^{-\beta E(u)}$) if one assumes that the formal temperature β^{-1} is locally fluctuating (see [17] for similar ideas). In our application to a turbulent flow, β^{-1} is identified with the product $\epsilon\tau$ of local energy dissipation and the typical time scale τ during which energy is transferred. Both quantities can fluctuate. β^{-1} is a formal variance parameter describing the fluctuating environment of the Lagrangian test particle, measured relative to the movement of the particle. Using the integral represention of the gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{10}$$

and substituting

$$t = \beta\left(E(u) + \frac{1}{(q-1)\beta_0}\right) \tag{11}$$

$$z = \frac{1}{q-1} \tag{12}$$

one may write

$$(1+(q-1)\beta_0 E(u))^{-\frac{1}{q-1}} = \int_0^\infty e^{-\beta E(u)} f(\beta) d\beta \tag{13}$$

with

$$f(\beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \left\{\frac{1}{(q-1)\beta_0}\right\}^{\frac{1}{q-1}} \beta^{\frac{1}{q-1}-1} \exp\left\{-\frac{\beta}{(q-1)\beta_0}\right\} \tag{14}$$

being the χ^2 distribution. The physical interpretation of eq. (13) is that due to fluctuations of β with probability density $f(\beta)$ the Boltzmann factor $e^{-\beta E(u)}$ of ordinary statistical mechanics has to be replaced by the generalized Boltzmann factor $(1+(q-1)\beta_0 E(u))^{-\frac{1}{q-1}}$ of nonextensive statistical mechanics. The Tsallis distribution with fixed variance parameter β_0 effectively arises by integrating over all possible fluctuating variance parameters β. This illustrates why the nonextensive formalism can be relevant to nonequilibrium systems (formally described by a fluctuating β) if there is a quasi-stationary state in probability space.

The χ^2 distribution is well known to occur in many very common circumstances (see e.g. [18]). For example, if one has n Gaussian random variables X_i then the sum of the squares of their deviations from their mean $\chi^2 := \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is χ^2 distributed with $\frac{2}{q-1} = n-1$. Or, if $T = \beta^{-1}$ obeys a linear Langevin equation with a constant source term and a damping that fluctuates stochastically, one also obtains a χ^2 distribution for β [17].

The average of β is given by $\overline{\beta} = \int_0^\infty \beta f(\beta) d\beta = \beta_0$ and the variance by $\overline{\beta^2} - \overline{\beta}^2 = (q-1)\beta_0^2$. From this one obtains a physical interpretation of the entropic index q in terms of the variance of β, namely

$$q = \frac{\overline{\beta^2}}{\overline{\beta}^2}. \tag{15}$$

If there are no fluctuations of β at all, as in ordinary statistical mechanics, eq. (15) just reduces to $q = 1$, as expected.

In a turbulent flow, the variance parameter β surrounding the Lagrangian test particle fluctuates, and hence Tsallis statistics can naturally arise in this context. Since the fluctuations of β become smaller if the volume r^3 over which the energy dissipation is averaged increases, q must be a montonously decreasing function of the scale r. In fact, at largest scales r one observes approximately Gaussian behaviour ($q \approx 1$), in the inertial range $q = 1.1....1.2$ gives good fits of the experimental data of Eulerian turbulence [4, 19], and Fig. 1 indicates that $q \approx 1.5$ at the smallest time scales of Lagrangian turbulence.

Let us now provide a theoretical argument why $q = \frac{3}{2}$ at the smallest scales. The observation is that for large $|u|$ the Tsallis distributions (5) (also called student or t- distributions in the statistics textbooks) decay as $|u|^{-w}$, where $w = \frac{2}{q-1}$. Hence only moments $\langle |u|^m \rangle$ with $m < w - 1$ exist. If $q = \frac{3}{2}$ at the smallest scale, this means $w = 4$ and hence the third moment would just cease to exist at the smallest scale. If q is precisely $\frac{3}{2}$ the third moment is logarithmically divergent, if $q = \frac{3}{2} - 0^+$ it just exists. Since generally the third moment is the most important moment in turbulence, related to average energy dissipation, the existence of this moment is necessary for turbulence to make sense. Since q is monotonically decreasing with scale r, one ends up with the largest allowed value of q at the smallest possible scale. This is just $q = \frac{3}{2} - 0^+$.

There is further experimental evidence for the above conjecture on the small scale statistics. In [19] systematic measurements of the exponent $w(r)$ were performed for Eulerian turbulence (for a turbulent Taylor Couette flow). The measurements were performed at distances r much larger than the Kolmogorov scale η. Over a large range of scales r the measured exponents $w(r)$ were very well fitted by a power law of the form

$$w(r) = 4 \left(\frac{r}{\eta} \right)^{\delta} \tag{16}$$

with $\delta \approx 0.3$. Extrapolating this down to the Kolmogorov scale η one obtains $w = 4$ at $r = \eta$, which again supports our hypothesis. Although for very small r deviations of this power law (pointing towards smaller values of q) were observed in [19], these deviations can be explained by the disturbing effects of noise, which naturally shifts the entropic index to lower q values, since Gaussian white noise implies $q = 1$.

Our conjecture on the small scale statistics is also consistent with a large amount of other experimental data. In [12] the Reynolds number dependence of the third and fourth moment of u at the smallest possible scales was analysed. By averaging the data of many experiments it emerged that the 4th moment increases roughly like $\sim R_\lambda^{1/3}$, whereas the 3rd moment stays almost constant or increases much less rapidly with R_λ than the 4th moment. This means that at the smallest scale the 4th moment is expected to diverge for $R_\lambda \to \infty$ and the 3rd moment may either just exist or may weakly diverge. All these experimentally observed features are correctly reproduced by the Tsallis distribution (5) with $q = \frac{3}{2}$.

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Figure captions

Fig. 1 Experimentally measured probability density of the acceleration of a test particle in Lagrangian turbulence for Reynolds number $R_\lambda = 200, 690, 970$, respectively, and comparison with the distribution (2).