REPRESENTATIONS OF GRADED HECKE ALGEBRAS

CATHY KRILOFF AND ARUN RAM

Abstract. Representations of affine and graded Hecke algebras associated to Weyl groups play an important role in the Langlands correspondence for the admissible representations of a reductive p-adic group. We work in the general setting of a graded Hecke algebra associated to any real reflection group with arbitrary parameters. In this setting we provide a classification of all irreducible representations of graded Hecke algebras associated to dihedral groups. Dimensions of generalized weight spaces, Langlands parameters, and a Springer-type correspondence are included in the classification. We also give an explicit construction of all irreducible calibrated representations (those possessing a simultaneous eigenbasis for the commutative subalgebra) of a general graded Hecke algebra. While most of the techniques used have appeared previously in various contexts, we include a complete and streamlined exposition of all necessary results, including the Langlands classification of irreducible representations and the irreducibility criterion for principal series representations.

1. Introduction

The affine Hecke algebra is tightly connected to the geometry and representation theory of a semisimple Lie group. In fact, the representation theory of affine Hecke algebras provides a large piece of the Langlands correspondence for the admissible representation theory of a reductive p-adic group [Bo, KL]. The affine Hecke algebra is also present in the geometry of a semisimple group via the equivariant K-theory of the Steinberg variety. This connection plays an important role in the Springer correspondence and the Langlands classification. Recent conjectures of Lusztig tie the representation theory of the affine Hecke algebra to the modular representation theory of semisimple Lie algebras in positive characteristic. So there are many good reasons to study the representations of affine Hecke algebras.

With appropriate definitions, the graded Hecke algebra is the associated graded algebra of the affine Hecke algebra. Lusztig [Lu3] has shown that the representation theory of graded Hecke algebras of Weyl groups is essentially equivalent to the representation theory of affine Hecke algebras. In the same way that the affine Hecke algebra is connected to equivariant K-theory [KL, CG] the graded Hecke algebra is connected to equivariant cohomology [Lu3].

This paper is a study of the combinatorial representation theory of graded Hecke algebras associated to finite real reflection groups (including the noncrystallographic cases). The geometric representation theory of these algebras has been studied in [Lu1, Lu2, Lu3] and fundamental results have appeared in [HO, Op]. However, a wealth of information can be obtained with purely combinatorial techniques. Here we develop the combinatorial theory from elementary principles. Most of the techniques we use are known in the affine Hecke algebra setting but they are spread over various parts of the literature, and in several cases the generalization to the graded Hecke algebras for the crystallographic case is nontrivial. We have collected these results, streamlined them, proved them in the general setting that includes noncrystallographic graded Hecke algebras and made an effort to produce an up-to-date presentation. This paper includes

(a) the Langlands classification of irreducible representations,

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(b) the theory of principal series representations (including the irreducibility criterion),
(c) the theory of intertwining operators,
(d) the classification of all irreducible representations for rank two algebras (including all dihedral cases $I_2(n)$),
(e) the classification of irreducible calibrated representations, and
(f) proofs of two conjectures from [Ra3].

The Langlands classification for graded Hecke algebras is due to Evens [Ev]. We have shortened his proof but the shorter proof does not differ in any essential ideas. Our proof of the irreducibility criterion for principal series modules is a graded Hecke algebra analogue of the proof given by Kato [Ka] for affine Hecke algebras. Proofs of this criterion for graded Hecke algebras have appeared in [Ch1, Kr2] but our proof is more constructive and gives detailed information about the spherical vectors in the principal series modules.

To our knowledge, the theory of intertwining operators originates from the study of affine Hecke algebra representations in Matsumoto [Ma]. In recent years this theory has played an important role in the theory of orthogonal polynomials, in particular, the study of Macdonald polynomials [Ch2, Op, KS]. In this paper we do not view these operators as intertwiners between principal series representations but rather as local operators on the weight spaces of any representation (τ-operators). This generalized approach is increasingly common in the theory of Macdonald polynomials [Mac]. Though we do not know of a reference for this theory in its application to representations of graded Hecke algebras, certainly all of these techniques are now standard in the orthogonal polynomial literature.

The full classification of all irreducible representations for rank two graded Hecke algebras is given in Section 3. We include detailed analysis of the structure (dimensions of generalized weight spaces) for these representations and their Langlands parameters. This analysis extends and completes the work on representations of rank two graded Hecke algebras included as part of [Kr1, HO]. In [Kr1] only one-parameter algebras were included and the classification was only complete for n odd; we now include the two-parameter case that arises when n is even and treat nonregular central characters. In [HO], general graded Hecke algebras were considered but the representations classified were spherical and tempered. An important consequence of our rank two construction is that it establishes a "Springer correspondence" for all dihedral groups. This correspondence is given in the final part of Section 3. As in [Ra2], we express the hope that the irreducible representations in the rank two case will provide the foundation for a combinatorial construction of all irreducible representations.

In Section 4 we classify the irreducible calibrated representations (those with a simultaneous eigenbasis for a large commutative subalgebra) of graded Hecke algebras. These results are graded Hecke algebra analogues of the results in [Ra1]. In addition to the classification, we give an elementary combinatorial construction of all irreducible calibrated representations of graded Hecke algebras. This construction is a generalization of the (seminormal) construction of the irreducible representations of the symmetric group given by Alfred Young [Yg]. In our construction the local regions and their chambers take the role that partitions and standard tableaux play in the symmetric group construction. Otherwise the formulas used in the construction of the irreducible calibrated modules are exactly the same as those used by Young.

In Section 5, we give proofs of two conjectures from [Ra3] which describe the combinatorial structure of the weights of graded Hecke algebra modules. One of these conjectures was proved by Losonczy [Lo] and we present a slightly simplified version of his proof here. We then prove the other conjecture with a short reduction to the statement proved by Losonczy and exploit the reduction procedure to obtain new information about the combinatorial weight structure. The conjectures in [Ra3] were only stated for the case when the reflection group W is crystallographic and our proofs only hold for this case. We give examples that show analagous statements do not hold in the noncrystallographic case.

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2. Preliminaries

2.1. **The graded Hecke algebra.** Let W be a finite reflection group, defined by its action on its reflection representation $\mathfrak{h}_{\mathbb{R}}^*$. For each reflection $s_\alpha \in W$ fix a *root* α in the -1 eigenspace of s_α. The roots α are chosen so that the set R of roots is W-invariant. Then s_α fixes a hyperplane

$$H_\alpha = (+1 \text{ eigenspace of } s_\alpha) = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \alpha^\vee(x) = 0\},$$

where we fix the linear function $\alpha^\vee \in \mathfrak{h}_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}^*, \mathbb{R})$ so that $\alpha^\vee(\alpha) = 2$. By fixing a nondegenerate symmetric W-invariant bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$ we may identify $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$. Then

$$s_\alpha x = x - \langle x, \alpha^\vee \rangle \alpha, \qquad \text{for all } x \in \mathfrak{h}_{\mathbb{R}}^*. \tag{2.1}$$

Fix simple roots $\alpha_1, \ldots, \alpha_n$ in the root system for W and let $s_i = s_{\alpha_i}$ be the corresponding reflections.

By extension of scalars W acts on the complexification $\mathfrak{h}_{\mathbb{C}}^* = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$ and, in terms of its action on $\mathfrak{h}_{\mathbb{C}}^*$, W is a complex reflection group. Then W acts on the symmetric algebra $S(\mathfrak{h}_{\mathbb{C}}^*)$ which is naturally identified with the algebra of polynomial functions on the vector space $\mathfrak{h}_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}^*, \mathbb{C})$.

Fix *parameters* $c_\alpha \in \mathbb{C}$, $c_\alpha \neq 0$, labeled by the roots, such that

$$c_\alpha = c_{w\alpha}, \qquad \text{for } w \in W.$$

This amounts to the choice of one or two values, depending on whether there are one or two orbits of roots under the action of W. The group algebra of W is

$$\mathbb{C}W = \mathbb{C}\text{-span}\{t_w \mid w \in W\} \qquad \text{with multiplication} \qquad t_w t_{w'} = t_{ww'}.$$

The *graded Hecke algebra* is

$$\mathbb{H} = \mathbb{C}W \otimes S(\mathfrak{h}_{\mathbb{C}}^*)$$

with multiplication determined by the multiplication in $S(\mathfrak{h}_{\mathbb{C}}^*)$ and the multiplication in $\mathbb{C}W$ and the relations

$$x t_{s_i} = t_{s_i} s_i(x) + c_{\alpha_i} \langle x, \alpha_i^\vee \rangle, \qquad \text{for } x \in \mathfrak{h}_{\mathbb{C}}^*, \tag{2.2}$$

where $\alpha_1^\vee, \ldots, \alpha_n^\vee \in \mathfrak{h}_{\mathbb{R}}$ are the simple co-roots. More generally, it follows that for any $p \in S(\mathfrak{h}_{\mathbb{C}}^*)$,

$$p t_{s_i} = t_{s_i} s_i(p) + c_{\alpha_i} \Delta_i(p) \qquad \text{and} \qquad t_{s_i} p = s_i(p) t_{s_i} + c_{\alpha_i} \Delta_i(p),$$

where $\Delta_i : S(\mathfrak{h}_{\mathbb{C}}^*) \to S(\mathfrak{h}_{\mathbb{C}}^*)$ is the *BGG-operator* given by

$$\Delta_i(p) = \frac{p - s_i(p)}{\alpha_i} \qquad \text{for } p \in S(\mathfrak{h}_{\mathbb{C}}^*).$$

Proposition 2.1. [Lu1, Theorem 6.5] *The center of the graded Hecke algebra $\mathbb{H}$ is $Z(\mathbb{H}) = S(\mathfrak{h}_{\mathbb{C}}^*)^W$, the ring of W-invariant polynomials on $\mathfrak{h}_{\mathbb{C}}$.*

Proof. If $p \in S(\mathfrak{h}_{\mathbb{C}}^*)^W$, then

$$p t_{s_i} = t_{s_i} s_i(p) + c_{\alpha_i} \frac{p - s_i(p)}{\alpha_i} = t_{s_i} p + 0 = t_{s_i} p,$$

and so p commutes with t_{s_i}. Therefore $S(\mathfrak{h}_{\mathbb{C}}^*)^W \subseteq Z(\mathbb{H})$.

Let $p \in Z(\mathbb{H})$ and write $p = \sum_{w \in W} p_w t_w$. Fix v of maximal length such that p_v has maximal degree. Let $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ be *regular*, meaning that the stabilizer W_μ is trivial. Then

$$\mu p = \sum_{w \in W} \mu p_w t_w \quad \text{equals} \quad p\mu = \sum_{w \in W} p_w t_w \mu = \sum_{w \in W} p_w \left((w\mu) t_w + \sum_{u < w} c^{\mu}_{u,w} t_u \right),$$

where $c^{\mu}_{u,w} \in \mathbb{C}$. Comparing coefficients of t_v yields

$$\mu p_v = p_v \cdot (v\mu).$$

So $\mu = (v\mu)$ and thus $v = 1$ since μ is regular. So $p \in S(\mathfrak{h}_{\mathbb{C}}^*)$. Comparing coefficients of t_{s_i} in

$$p t_{s_i} = s_i(p) t_{s_i} + c_{\alpha_i} \frac{p - s_i(p)}{\alpha_i}$$

shows that $p = s_i(p)$ for all $1 \le i \le n$. So $p \in S(\mathfrak{h}_{\mathbb{C}}^*)^W$. Thus $Z(\mathbb{H}) = S(\mathfrak{h}_{\mathbb{C}}^*)^W$. □

2.2. **Harmonic polynomials.** Let us briefly review the relationship between $S(\mathfrak{h}_{\mathbb{C}}^*)$, $S(\mathfrak{h}_{\mathbb{C}}^*)^W$, and harmonic polynomials [CG, § 6.3]. Let $x_1, x_2, \ldots, x_n$ be an orthonormal basis of $\mathfrak{h}_{\mathbb{C}}$ and define a symmetric bilinear form $\langle\, , \rangle$ on $S(\mathfrak{h}_{\mathbb{C}}^*)$ by

$$\langle P, Q \rangle = (P(\partial) Q)\big|_{x_i = 0}, \qquad \text{for } P, Q \in S(\mathfrak{h}_{\mathbb{C}}^*),$$

where $P(\partial) = P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ and $\big|_{x_i=0}$ denotes specializing the variables to 0 (or, equivalently, taking the constant term). The monomials are an orthogonal basis of $S(\mathfrak{h}_{\mathbb{C}}^*)$,

$$\begin{aligned} \langle x_1^{\lambda_1} \cdots x_n^{\lambda_n}, x_1^{\mu_1} \cdots x_n^{\mu_n} \rangle &= \left(\frac{\partial}{\partial x_1} \right)^{\lambda_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\lambda_n} x_1^{\mu_1} \cdots x_n^{\mu_n} \\ &= \delta_{\lambda_1 \mu_1} \cdots \delta_{\lambda_n \mu_n} (\lambda_1! \lambda_2! \cdots \lambda_n!), \end{aligned}$$

and so the bilinear form $\langle\, , \rangle$ is nondegenerate. The vector space $\mathcal{H}$ of *harmonic polynomials* is the set of polynomials orthogonal to the ideal of $S(\mathfrak{h}_{\mathbb{C}}^*)$ generated by W-invariants in $S(\mathfrak{h}_{\mathbb{C}}^*)$ with constant term 0,

$$\mathcal{H} = (\langle f \in S(\mathfrak{h}_{\mathbb{C}}^*) \mid f(0) = 0 \rangle)^{\perp}, \qquad \text{and} \qquad S(\mathfrak{h}_{\mathbb{C}}^*) = S(\mathfrak{h}_{\mathbb{C}}^*)^W \otimes \mathcal{H},$$

as vector spaces. More precisely, if $\{h_w\}$ is a $\mathbb{C}$-basis of $\mathcal{H}$ then any $f \in S(\mathfrak{h}_{\mathbb{C}}^*)$ can be written uniquely in the form

$$f = \sum_w p_w h_w, \qquad p_w \in S(\mathfrak{h}_{\mathbb{C}}^*)^W.$$

If the basis $\{h_w\}$ consists of homogeneous polynomials then the number and the degrees of these polynomials are determined by the Poincaré polynomial of W,

$$P_W(t) = \sum_{k \ge 0} \dim(\mathcal{H}^k) t^k = \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t} = \sum_{w \in W} t^{\ell(w)}, \tag{2.3}$$

where $d_1, \ldots, d_n$ are the degrees of a set $f_1, \ldots, f_n$ of homogeneous generators of $S(\mathfrak{h}_{\mathbb{C}}^*)^W = \mathbb{C}[f_1, \ldots, f_n]$ and $\mathcal{H}^k$ is the k^{th} homogeneous component of $\mathcal{H}$. In particular, $\dim(\mathcal{H}) = \text{Card}(\{h_w\}) = P_W(1) = |W|$ and $S(\mathfrak{h}_{\mathbb{C}}^*)$ is a free module over $S(\mathfrak{h}_{\mathbb{C}}^*)^W$ of rank $|W|$.

2.3. **Weights and calibrated representations.** The group W acts on

$$\mathfrak{h}_{\mathbb{C}} = \mathrm{Hom}(\mathfrak{h}_{\mathbb{C}}^*, \mathbb{C}) \qquad \text{by} \qquad (w\gamma)(x) = \gamma(w^{-1}x),$$

for $w \in W$, $\gamma \in \mathfrak{h}_{\mathbb{C}}$ and $x \in \mathfrak{h}_{\mathbb{C}}^*$.

The *inversion set* of an element $w \in W$ is

$$R(w) = \{\alpha > 0 \mid w\alpha < 0\} \tag{2.4}$$

The choice of the simple roots $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}_{\mathbb{R}}^*$ determines a *fundamental chamber*

$$C = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \alpha_i, x\rangle > 0, 1 \le i \le n\} \tag{2.5}$$

for the action of W on $\mathfrak{h}_{\mathbb{R}}^*$. For a root $\alpha \in R$, the *positive side* of the hyperplane H_α is the side towards C, i.e. $\{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha\rangle > 0\}$, and the *negative side* of H_α is the side away from C. There is a bijection

$$\begin{array}{ccc} W & \longleftrightarrow & \{\text{fundamental chambers for } W \text{ acting on } \mathfrak{h}_{\mathbb{R}}^*\} \\ w & \longmapsto & w^{-1}C \end{array} \tag{2.6}$$

and the chamber $w^{-1}C$ is the unique chamber which is on the positive side of H_α for $\alpha \notin R(w)$ and on the negative side of H_α for $\alpha \in R(w)$.

If s_α is a reflection in W which fixes $\gamma \in \mathfrak{h}_{\mathbb{C}}$ then $\langle \gamma, \alpha^\vee\rangle = 0$. By [St, Theorem 1.5], [Bou, Ch. V §5 Ex. 8] the stabilizer W_γ of γ under the W-action is generated by the reflections which stabilize γ and so

$$W_\gamma = \langle s_\alpha \mid \alpha \in Z(\gamma)\rangle \qquad \text{where} \qquad Z(\gamma) = \{\alpha \mid \gamma(\alpha) = 0\}.$$

The orbit $W\gamma$ can be viewed in several different ways via the bijections

$$\begin{aligned} W\gamma \;\longleftrightarrow\; W/W_\gamma \;&\longleftrightarrow\; \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset\} \\ &\longleftrightarrow\; \left\{\begin{matrix}\text{chambers on the positive} \\ \text{side of } H_\alpha \text{ for } \alpha \in Z(\gamma)\end{matrix}\right\}, \end{aligned} \tag{2.7}$$

where the last bijection is the restriction of the map in (2.6). If γ is real and dominant (i.e. $\gamma(\alpha) \in \mathbb{R}_{\ge 0}$ for all $\alpha \in R$) then W_γ is a parabolic subgroup of W and $\{w \in W \mid R(w) \cap Z(\gamma) = \emptyset\}$ is the set of minimal length coset representatives of the cosets in W/W_γ.

Let M be a simple $\mathbb{H}$-module. Dixmier's version of Schur's lemma (see [Wa]) implies that $Z(\mathbb{H})$ acts on M by scalars. Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ be such that

$$pm = \gamma(p)m, \qquad \text{for all } m \in M,\ p \in S(\mathfrak{h}_{\mathbb{C}}^*)^W.$$

The element γ is only determined up to the action of W since $\gamma(p) = w\gamma(p)$ for all $w \in W$. Because of this, any element of the orbit $W\gamma$ is referred to as the *central character* of M.

Let M be a finite dimensional $\mathbb{H}$-module and let $\gamma \in \mathfrak{h}_{\mathbb{C}}$. The γ*-weight space* and the *generalized* γ*-weight space* of M are

$$M_\gamma = \{m \in M \mid xm = \gamma(x)m \text{ for all } x \in \mathfrak{h}_{\mathbb{C}}^*\}, \tag{2.8}$$

$$M_\gamma^{\mathrm{gen}} = \{m \in M \mid \text{for all } x \in \mathfrak{h}_{\mathbb{C}}^*, (x - \gamma(x))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\}. \tag{2.9}$$

Then

$$M = \bigoplus_{\gamma \in \mathfrak{h}_{\mathbb{C}}} M_\gamma^{\mathrm{gen}},$$

and we say that γ is a *weight* of M if $M_\gamma^{\mathrm{gen}} \neq 0$. Note that $M_\gamma^{\mathrm{gen}} \neq 0$ if and only if $M_\gamma \neq 0$. A finite dimensional $\mathbb{H}$-module

$$M \text{ is } \textit{calibrated} \text{ if } M_\gamma^{\mathrm{gen}} = M_\gamma, \text{ for all } \gamma \in \mathfrak{h}_{\mathbb{C}}. \tag{2.10}$$

2.4. **Tempered representations and the Langlands classification.** Any $\lambda \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}^*, \mathbb{C})$ is determined by its values $\langle \lambda, \alpha_i \rangle$ on the simple roots. Define $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ in $\mathfrak{h}_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}^*, \mathbb{R})$ by $\langle \mathrm{Re}(\lambda), \alpha_i \rangle = \mathrm{Re}(\langle \lambda, \alpha_i \rangle)$ and $\langle \mathrm{Im}(\lambda), \alpha_i \rangle = \mathrm{Im}(\langle \lambda, \alpha_i \rangle)$, and write

$$\lambda = \mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda).$$

For any simple reflection s_j, we have $s_j\lambda = \mathrm{Re}(\lambda) - \mathrm{Re}(\langle \lambda, \alpha_j^\vee \rangle)\alpha_j + i\,\mathrm{Im}(\lambda) - i\,\mathrm{Im}(\langle \lambda, \alpha_j^\vee \rangle)\alpha_j = s_j\mathrm{Re}(\lambda) + i\, s_j\mathrm{Im}(\lambda)$, and so

$$\mathrm{Re}(w\lambda) = w\mathrm{Re}(\lambda), \qquad \text{for all } w \in W.$$

Let $\omega_i^\vee$ be the dual basis to $\alpha_i^\vee$ in $\mathfrak{h}_{\mathbb{R}}$ defined by $\langle \omega_i^\vee, \alpha_j^\vee \rangle = \delta_{ij}$ and let $\overline{C}$ be the closure of the fundamental chamber $C \subseteq \mathfrak{h}_{\mathbb{R}}$ defined in (2.5). For $\lambda \in \mathfrak{h}_{\mathbb{C}}$ let λ_0 be the point of $\overline{C}$ which is closest to $\mathrm{Re}(\lambda)$. This point is uniquely defined because of the convexity of the region C. Since $\lambda_0 \in \overline{C}$ and the $\omega_i^\vee$ are on the boundary of $\overline{C}$ there is a uniquely determined set I such that

$$\lambda_0 = \sum_{j \notin I} c_j \omega_i^\vee, \qquad \text{with} \qquad c_j > 0,$$

and we say that the weight λ is *I-tempered.* For each I the set $\{\omega_j^\vee, \alpha_i^\vee \mid j \notin I, i \in I\}$ is a basis of $\mathfrak{h}_{\mathbb{R}}$ and λ_0 and I can, alternatively, be determined by the unique expansion

$$\mathrm{Re}(\lambda) = \sum_{j \notin I} c_j \omega_i^\vee + \sum_{i \in I} d_i \alpha_i^\vee, \qquad \text{with } c_j > 0 \text{ and } d_i \leq 0. \tag{2.11}$$

Proposition 2.2. [Kn, Lemma 8.59] *Let $\lambda \geq \mu$ denote the dominance ordering on $\mathfrak{h}_{\mathbb{R}}$. If $\lambda, \mu \in \mathfrak{h}_{\mathbb{R}}$ such that $\lambda \geq \mu$ then $\lambda_0 \geq \mu_0$.*

For any subset $I \subseteq \{1, \ldots, n\}$, let $\mathbb{H}_I$ be the subalgebra of $\mathbb{H}$ generated by t_{s_i}, $i \in I$, and all $x \in \mathfrak{h}_{\mathbb{C}}^*$. An $\mathbb{H}_I$-module M is *tempered* if all weights of M are I-tempered.

Theorem 2.3. *Let L be a simple $\mathbb{H}$-module.*

(a) *There is a subset $I \subseteq \{1, 2, \ldots, n\}$ and a tempered $\mathbb{H}_I$-module U such that L is the unique simple quotient of $\mathbb{H} \otimes_{\mathbb{H}_I} U$.*

(b) *If I and I' are subsets of $\{1, 2, \ldots, n\}$ and U and U' are tempered $\mathbb{H}_I$ and $\mathbb{H}_{I'}$-modules, respectively, such that L is a quotient of both $\mathbb{H} \otimes_{\mathbb{H}_I} U$ and $\mathbb{H} \otimes_{\mathbb{H}_{I'}} U'$ then $I = I'$ and $U \cong U'$ as $\mathbb{H}_I$-modules.*

Proof. Let L be a simple $\mathbb{H}$-module. Let λ be a weight of L such that

$$\lambda_0 \text{ is a maximal element of } \quad \{\mu_0 \mid \mu \text{ is a weight of } L\} \tag{2.12}$$

with respect to the dominance ordering on $\mathfrak{h}_{\mathbb{R}}$. Let $I \subseteq \{1, 2, \ldots, n\}$ be determined by

$$\lambda_0 = \sum_{j \notin I} c_j \omega_j^\vee$$

and let V be the $\mathbb{H}_I$-submodule of L generated by a nonzero vector m_λ in L_λ. Let W_I be the subgroup of W generated by s_i, $i \in I$. The weights of V are of the form $w\lambda$ with $w \in W_I$. If $w \in W_I$ then

$$\mathrm{Re}(w\lambda) = \sum_{j \notin I} c_j \omega_j^\vee + \sum_{a_i \leq 0, i \in I} a_i \alpha_i^\vee + \sum_{a_i > 0, i \in I} a_i \alpha_i^\vee \geq \sum_{j \notin I} c_j \omega_j^\vee + \sum_{a_i \leq 0, i \in I} a_i \alpha_i^\vee,$$

since $\mathrm{Re}(\lambda)$ is as in (2.11). So, by Proposition 2.2,

$$(w\lambda)_0 \geq \left(\sum_{j \notin I} c_j \omega_j^\vee + \sum_{a_i \leq 0} a_i \alpha_i^\vee \right)_0 = \sum_{j \notin I} c_j \omega_j^\vee = \lambda_0.$$

Thus, by the maximality of λ_0, $\mu_0 = \lambda_0$ for all weights μ of V. So V is tempered.

Let U be a simple $\mathbb{H}_I$-submodule of V. All weights of $\mathbb{H}\otimes_{\mathbb{H}_I} U$ are of the form $w\mu$ with $w\in W$ and μ a weight of U. Let W^I denote the set of minimal length coset representatives of cosets in W/W_I. If $w\mu$ is a weight and $w=w_1w_2$ with $w_1\in W^I$ and $w_2\in W_I$ then by the argument just given $w_2\mu$ is I-tempered and so

$$\mathrm{Re}(w_2\mu)=\sum_{j\notin I} c_j\omega_i^\vee+\sum_{i\in I} a_i\alpha_i^\vee \qquad \text{with} \qquad c_j>0, a_i\le 0.$$

If $w_1\neq 1$ then

$$\text{(2.13)} \qquad \mathrm{Re}(w_1w_2\mu)<\mathrm{Re}(w_2\mu) \quad \text{since} \quad \begin{array}{ll} w_1\omega_j^\vee\le\omega_j^\vee, & \text{for } j\notin I,\\ w_1\alpha_i^\vee\ge\alpha_i^\vee, & \text{for } i\in I,\\ w_1\omega_j^\vee<\omega_j^\vee, & \text{for some } j\notin I.\end{array}$$

Let ν be a weight of U such that $\mathrm{Re}(\nu)$ is maximal among weights of U. If N is an $\mathbb{H}$-submodule of $\mathbb{H}\otimes_{\mathbb{H}_I} U$ such that $N_\nu\neq 0$ then, by (2.13), $N_\nu\subseteq U_\nu$ and so $N\cap U\neq 0$. Since U is simple as an $\mathbb{H}_I$-module, any vector of U generates all of $\mathbb{H}\otimes_{\mathbb{H}_I} U$ and so $N=\mathbb{H}\otimes_{\mathbb{H}_I} U$. This shows that if

$$M_{\max}=\begin{pmatrix}\text{sum of all } \mathbb{H}\text{-submodules } N \text{ of } \mathbb{H}\otimes_{\mathbb{H}_I} U\\ \text{such that } N_\nu=0\end{pmatrix}$$

then $M_{\max}$ is equal to the sum of all proper submodules of $\mathbb{H}\otimes_{\mathbb{H}_I} U$ and is the (unique) maximal proper submodule of $\mathbb{H}\otimes_{\mathbb{H}_I} U$. So $\mathbb{H}\otimes_{\mathbb{H}_I} U$ has a unique simple quotient.

Since U is an $\mathbb{H}_I$-submodule of L and induction is the adjoint functor to restriction, there is an $\mathbb{H}$-module homomorphism

$$\begin{array}{ccc} \mathbb{H}\otimes_{\mathbb{H}_I} U & \longrightarrow & L\\ u & \longmapsto & u \qquad \text{for } u\in U.\end{array}$$

Thus, since L is simple, $L\cong(\mathbb{H}\otimes_{\mathbb{H}_I} U)/M_{\max}$. This proves (a) and shows that for any tempered $\mathbb{H}_I$-module U the module $\mathbb{H}\otimes_{\mathbb{H}_I} U$ has a unique simple quotient.

To prove (b) let us analyze the freedom of the choices that are made in the above construction of $\mathbb{H}\otimes_{\mathbb{H}_I} U$. Equation (2.13) and Proposition 2.2 show that $\nu_0\le\lambda_0$ for all weights ν of $\mathbb{H}\otimes_{\mathbb{H}_I} U$. In particular, all weights ν of L satisfy $\nu_0\le\lambda_0$ and so λ_0 is the same for all weights λ of L which satisfy (2.12). This shows that there is a unique choice of I in the construction of $\mathbb{H}\otimes_{\mathbb{H}_I} U$. If U' is another simple $\mathbb{H}_I$-submodule of V then either $U\cap U'=0$ or $U=U'$. The case $U\cap U'=0$ is impossible since it would imply that $U\oplus U'$ is a tempered submodule of L, and there would be a surjective homomorphism from $\mathbb{H}\otimes_{\mathbb{H}_I}(U\oplus U')\cong(\mathbb{H}\otimes_{\mathbb{H}_I} U)\oplus(\mathbb{H}\otimes_{\mathbb{H}_I} U')$ to L which is nonzero on both components. This is impossible since L is simple. □

2.5. τ **operators.** The following proposition defines maps $\tau_i: M_\gamma^{\mathrm{gen}}\to M_{s_i\gamma}^{\mathrm{gen}}$ on generalized weight spaces of finite-dimensional $\mathbb{H}$-modules M. These are "local operators" and are only defined on weight spaces M_γ^{gen} such that $\gamma(\alpha_i)\neq 0$. In general, τ_i does not extend to an operator on all of M.

Proposition 2.4. *Let M be a finite dimensional $\mathbb{H}$-module. Fix i, let $\gamma\in\mathfrak{h}_{\mathbb{C}}$ be such that $\gamma(\alpha_i)\neq 0$ and define*

$$\begin{array}{cccc}\tau_i: & M_\gamma^{\mathrm{gen}} & \longrightarrow & M_{s_i\gamma}^{\mathrm{gen}}\\ & m & \mapsto & \left(t_{s_i}-\frac{c_{\alpha_i}}{\alpha_i}\right)m.\end{array}$$

(a) *The map $\tau_i: M_\gamma^{\mathrm{gen}}\to M_{s_i\gamma}^{\mathrm{gen}}$ is well defined.*
(b) *As operators on M_γ^{gen}, $x\tau_i=\tau_i s_i(x)$ for all $x\in S(\mathfrak{h}_{\mathbb{C}}^*)$.*
(c) *As operators on M_γ^{gen}, $\tau_i\tau_i=\dfrac{(c_{\alpha_i}+\alpha_i)(c_{\alpha_i}-\alpha_i)}{(\alpha_i)(-\alpha_i)}$.*
(d) *Both maps $\tau_i: M_\gamma^{\mathrm{gen}}\to M_{s_i\gamma}^{\mathrm{gen}}$ and $\tau_i: M_{s_i\gamma}^{\mathrm{gen}}\to M_\gamma^{\mathrm{gen}}$ are invertible if and only if $\gamma(\alpha_i)\neq\pm c_{\alpha_i}$.*

(e) *If* $1 \le i,j \le n, i \ne j$, *let* m_{ij} *be the order of* $s_i s_j$ *in* W. *Then*

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}\ \text{factors}} = \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}\ \text{factors}},$$

whenever both sides are well defined operators on M_γ^{gen}.

Proof. Since α_i acts on M_γ^{gen} by $\gamma(\alpha_i)$ times a unipotent transformation, the operator α_i on M_γ^{gen} has nonzero determinant and is invertible. Since c_{α_i}/α_i is not an element of $S(\mathfrak{h}_\mathbb{C}^*)$ or $\mathbb{H}$ it will be viewed only as an operator on M_γ^{gen} in the following calculations.

If $x \in \mathfrak{h}_\mathbb{C}^*$ and $m \in M_\gamma^{\text{gen}}$ then

$$\begin{aligned} x\tau_i m &= x\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)m = \left(t_{s_i}s_i(x) + c_{\alpha_i}\langle x, \alpha_i^\vee\rangle - c_{\alpha_i}\frac{x}{\alpha_i}\right)m \\ &= \left(t_{s_i}s_i(x) - c_{\alpha_i}\frac{x - \langle x,\alpha_i^\vee\rangle\alpha_i}{\alpha_i}\right)m = \left(t_{s_i}s_i(x) - c_{\alpha_i}\frac{s_i(x)}{\alpha_i}\right)m \\ &= \left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)s_i(x)m = \tau_i s_i(x) m. \end{aligned}$$

This proves (a) and (b).

$$\begin{aligned} \tau_i\tau_i m &= \left(t_{s_i}^2 - \frac{c_{\alpha_i}}{\alpha_i}t_{s_i} - t_{s_i}\frac{c_{\alpha_i}}{\alpha_i} + \frac{c_{\alpha_i}^2}{\alpha_i^2}\right)m \\ &= \left(1 - \frac{c_{\alpha_i}}{\alpha_i}t_{s_i} - \frac{c_{\alpha_i}}{-\alpha_i}t_{s_i} - c_{\alpha_i}\frac{\left(\frac{c_{\alpha_i}}{\alpha_i} - \frac{c_{\alpha_i}}{-\alpha_i}\right)}{\alpha_i} + \frac{c_{\alpha_i}^2}{\alpha_i^2}\right)m \\ &= \left(1 + \frac{c_{\alpha_i}^2}{(\alpha_i)(-\alpha_i)}\right)m = \left(\frac{(c_{\alpha_i}+\alpha_i)(c_{\alpha_i}-\alpha_i)}{(\alpha_i)(-\alpha_i)}\right)m, \end{aligned}$$

proving (c).

(d) Since α_i acts on M_γ^{gen} by $\gamma(\alpha_i)$ times a unipotent transformation, $\det((c_{\alpha_i}+\alpha_i)(c_{\alpha_i}-\alpha_i)) = 0$ if and only if $\gamma(\alpha_i) = \pm c_{\alpha_i}$. Thus $\tau_i\tau_i$, and each factor in this composition, is invertible if and only if $\gamma(\alpha_i) \ne \pm c_{\alpha_i}$.

(e) We may assume that $\mathbb{H}$ is the graded Hecke algebra corresponding to a rank two root system R_{ij} generated by simple roots α_i and α_j. Let w_0 be the longest element of the corresponding rank 2 reflection group W. Every element $w \in W$, $w \ne w_0$ has a unique minimal length expression as a product of generators of s_i and s_j. Let t_w be the corresponding product of the t_{s_i}'s and t_{s_j}'s. Expanding both sides of the relation in (e) in terms of the t_{s_i} and using the defining relation (2.2) for $\mathbb{H}$ yields

$$\underbrace{\cdots\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)\left(t_{s_j} - \frac{c_{\alpha_j}}{\alpha_j}\right)\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)}_{m_{ij}\ \text{factors}} = \underbrace{\cdots t_{s_i}t_{s_j}t_{s_i}}_{m_{ij}\ \text{factors}} + \sum_{w<w_0} t_w P_w, \tag{2.14}$$

and

$$\underbrace{\cdots\left(t_{s_j} - \frac{c_{\alpha_j}}{\alpha_j}\right)\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)\left(t_{s_j} - \frac{c_{\alpha_j}}{\alpha_j}\right)}_{m_{ij}\ \text{factors}} = \underbrace{\cdots t_{s_j}t_{s_i}t_{s_j}}_{m_{ij}\ \text{factors}} + \sum_{w<w_0} t_w Q_w, \tag{2.15}$$

where both sums are in fact over all $w \in W$, $w \ne w_0$ and P_w and Q_w are rational functions of the $\alpha \in R_{ij}$. We will show that $P_w = Q_w$.

Choose generic $\gamma \in \mathfrak{h}_{\mathbb{C}}$ (the exact condition is that $P(\gamma)=\emptyset$ and $Z(\gamma)=\emptyset$, where $P(\gamma)$ and $Z(\gamma)$ are as defined in (2.19) below) and let

$$M(\gamma) = \mathrm{Ind}_{S(\mathfrak{h}_{\mathbb{C}}^*)}^{\mathbb{H}}(\mathbb{C}v_\gamma) = \mathbb{H} \otimes_{S(\mathfrak{h}_{\mathbb{C}}^*)} \mathbb{C}v_\gamma$$

where $\mathbb{C}v_\gamma$ is the one dimensional $S(\mathfrak{h}_{\mathbb{C}}^*)$-module defined by $xv_\gamma = \gamma(x)v_\gamma$ for $x \in \mathfrak{h}_{\mathbb{C}}^*$. The module $M(\gamma)$ has basis $\{t_w \otimes v_\gamma \mid w \in W\}$ and, by the defining relations for $\mathbb{H}$, for $x \in \mathfrak{h}_{\mathbb{C}}^*$, $w \in W$,

$$xt_w v_\gamma = (w\gamma)(x) t_w \otimes v_\gamma + \sum_{z<w} c_{zw}(x) t_z \otimes v_\gamma, \qquad \text{with } c_{zw}(x) \in \mathbb{C}.$$

Since γ is generic, all the $w\gamma$ are distinct and

$$M(\gamma) = \bigoplus_{w\in W} M_{w\gamma} \qquad \text{with} \qquad \dim(M_{w\gamma}) = 1.$$

Thus, there is a unique basis $\{v_{w\gamma} \mid w \in W\}$ of $M(\gamma)$ determined by

(2.16) $$xv_{w\gamma} = (w\gamma)(x)v_{w\gamma}, \qquad \text{for all } w \in W \text{ and } x \in \mathfrak{h}_{\mathbb{C}}^*,$$

(2.17) $$v_{w\gamma} = t_w \otimes v_\gamma + \sum_{u<w} a_{wu}(\gamma)(t_u \otimes v_\gamma), \qquad \text{where } a_{wu}(\gamma) \in \mathbb{C}.$$

Alternatively,

(2.18) $$v_{w\gamma} = \tau_w v_\gamma$$

where $\tau_w = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}$ for a reduced word $w = s_{i_1}\cdots s_{i_p}$ of w. The uniqueness of the element $v_{w\gamma}$ given by the conditions (2.16) and (2.17) shows that $v_{w\gamma} = \tau_w v_\gamma$ does not depend on the reduced decomposition which is chosen for w. Thus we have

$$v_{w_0\gamma} = \underbrace{\cdots\tau_i\tau_j\tau_i}_{m_{ij}\ \text{factors}} v_\gamma = \underbrace{\cdots t_{s_i}t_{s_j}t_{s_i}}_{m_{ij}\ \text{factors}} v_\gamma + \sum_{w<w_0} t_w P_w v_\gamma = t_{w_0}\otimes v_\gamma + \sum_{w<w_0}\gamma(P_w)t_w\otimes v_\gamma,$$

$$v_{w_0\gamma} = \underbrace{\cdots\tau_j\tau_i\tau_j}_{m_{ij}\ \text{factors}} v_\gamma = \underbrace{\cdots t_{s_j}t_{s_i}t_{s_j}}_{m_{ij}\ \text{factors}} v_\gamma + \sum_{w<w_0} t_w Q_w v_\gamma = t_{w_0}\otimes v_\gamma + \sum_{w<w_0}\gamma(Q_w)t_w\otimes v_\gamma.$$

where P_w and Q_w are as in (2.14) and (2.15). It follows from (2.17) that $\gamma(P_w) = a_{w_0w}(\gamma) = \gamma(Q_w)$ for all $w \in W$, $w \neq w_0$.

We have shown that, for each $w \in W$, $\gamma(P_w) = \gamma(Q_w)$ for all generic $\gamma \in \mathfrak{h}_{\mathbb{C}}$. Since P_w and Q_w are rational functions that agree on all generic points, it follows that

$$P_w = Q_w \qquad \text{for all } w \in W.$$

Thus,

$$\begin{aligned}\underbrace{\cdots\tau_i\tau_j\tau_i}_{m_{ij}\ \text{factors}} &= \underbrace{\cdots\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)\left(t_{s_j} - \frac{c_{\alpha_j}}{\alpha_j}\right)\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)}_{m_{ij}\ \text{factors}} \\ &= \underbrace{\cdots\left(t_{s_j} - \frac{c_{\alpha_j}}{\alpha_j}\right)\left(t_{s_i} - \frac{c_{\alpha_i}}{\alpha_i}\right)\left(t_{s_j} - \frac{c_{\alpha_j}}{\alpha_j}\right)}_{m_{ij}\ \text{factors}} = \underbrace{\cdots\tau_j\tau_i\tau_j}_{m_{ij}\ \text{factors}},\end{aligned}$$

whenever both sides are well defined operators on M_γ. □

Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ and define

(2.19) $$Z(\gamma) = \{\alpha > 0 \mid \gamma(\alpha) = 0\} \qquad \text{and} \qquad P(\gamma) = \{\alpha > 0 \mid \gamma(\alpha) = \pm c_\alpha\}.$$

If $J \subseteq P(\gamma)$, define

$$\mathcal{F}^{(\gamma,J)} = \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset \text{ and } R(w) \cap P(\gamma) = J\}. \tag{2.20}$$

A *local region* is a pair (γ, J) such that $\gamma \in \mathfrak{h}_{\mathbb{C}}$, $J \subseteq P(\gamma)$, and $\mathcal{F}^{(\gamma,J)} \neq \emptyset$. Under the bijection (2.6) the set $\mathcal{F}^{(\gamma,J)}$ maps to the set of points $x \in \mathfrak{h}_{\mathbb{R}}^*$ which are

(a) on the positive side of the hyperplanes H_α for $\alpha \in Z(\gamma)$,
(b) on the positive side of the hyperplanes H_α for $\alpha \in P(\gamma) \backslash J$, and
(c) on the negative side of the hyperplanes H_α for $\alpha \in J$.

In this way the local region (γ, J) really does correspond to a region in $\mathfrak{h}_{\mathbb{R}}^*$. This is a connected convex region in $\mathfrak{h}_{\mathbb{R}}^*$ since it is cut out by half spaces in $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^n$. The elements $w \in \mathcal{F}^{(\gamma,J)}$ index the *chambers $w^{-1}C$ in the local region.* and the sets $\mathcal{F}^{(\gamma,J)}$ form a partition of the set $\{w \in W \mid R(w) \cap Z(\gamma) = \emptyset\}$ (which, by (2.7), indexes the cosets in W/W_γ).

Corollary 2.5. *Let M be a finite dimensional $\mathbb{H}$-module. Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ and let $J \subseteq P(\gamma)$. Then*

$$\dim(M_{w\gamma}^{\mathrm{gen}}) = \dim(M_{w'\gamma}^{\mathrm{gen}}) \qquad \text{for } w, w' \in \mathcal{F}^{(\gamma,J)},$$

where $\mathcal{F}^{(\gamma,J)}$ is given by (2.20).

Proof. If $w, s_i w \in \mathcal{F}^{(\gamma,J)}$ then $(w\gamma)(\alpha_i) \neq \pm c_{\alpha_i}$ and $(s_i w\gamma)(\alpha_i) \neq \pm c_{\alpha_i}$. Thus, by Proposition 2.4(d), the map $\tau_i : M_{w\gamma}^{\mathrm{gen}} \to M_{s_i w\gamma}^{\mathrm{gen}}$ is invertible. It remains to note that if $w, w' \in \mathcal{F}^{(\gamma,J)}$, then $w' = s_{i_1} \cdots s_{i_\ell} w$ where $s_{i_k} \cdots s_{i_\ell} w \in \mathcal{F}^{(\gamma,J)}$ for all $1 \leq k \leq \ell$. This follows from the fact that (γ, J) corresponds to a connected convex region in $\mathfrak{h}_{\mathbb{R}}$. □

The following lemma will be used in the classification in Section 3 to analyze weight spaces for representations with nonregular central character.

Lemma 2.6. *Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ such that $\gamma(\alpha_i) = 0$. Let M be an $\mathbb{H}$-module such that $M_\gamma^{\mathrm{gen}} \neq 0$ and let $w \in \mathcal{F}^{(\gamma,\emptyset)}$. Then*

(a) $\dim M_{w\gamma}^{\mathrm{gen}} \geq 2$ and
(b) if $M_{s_j w\gamma}^{\mathrm{gen}} = 0$, then $(w\gamma)(\alpha_j) = \pm c_{\alpha_j}$ and $\langle w^{-1}\alpha_j, \alpha_i^\vee \rangle = 0$.

Proof. Let $\mathbb{H}A_1$ be the subalgebra of $\mathbb{H}$ generated by t_{s_i} and all $x \in S(\mathfrak{h}_{\mathbb{C}}^*)$. Let $\mathbb{C}v_\gamma$ be the one dimensional representation of $S(\mathfrak{h}_{\mathbb{C}}^*)$ defined by $xv_\gamma = \gamma(x)v_\gamma$ and let $M(\gamma) = \mathrm{Ind}_{S(\mathfrak{h}_{\mathbb{C}}^*)}^{\mathbb{H}A_1}(\mathbb{C}v_\gamma) = \mathbb{H}A_1 \otimes_{S(\mathfrak{h}_{\mathbb{C}}^*)} \mathbb{C}v_\gamma$. This module is irreducible and has basis $\{v_\gamma, t_{s_i}v_\gamma\}$ and, with respect to this basis, the action of $x \in \mathfrak{h}_{\mathbb{C}}^*$ on $M(\gamma)$ is given by the matrix

$$\rho_\gamma(x) = \gamma(x) \begin{pmatrix} 1 & c_{\alpha_i}\langle x, \alpha_i^\vee \rangle \\ 0 & 1 \end{pmatrix}. \tag{2.21}$$

Let n_γ be a nonzero vector in M_γ. As an $S(\mathfrak{h}_{\mathbb{C}}^*)$-module $\mathbb{C}n_\gamma \cong \mathbb{C}v_\gamma$ and, since induction is the adjoint functor to restriction, there is a unique $\mathbb{H}A_1$-module homomorphism given by

$$\begin{array}{ccc} M(\gamma) & \longrightarrow & M \\ v_\gamma & \longmapsto & n_\gamma \end{array}$$

Since $M(\gamma)$ is irreducible, this homomorphism is injective, and the vectors $n_\gamma, t_{s_i}n_\gamma$ span a two-dimensional subspace of M_γ^{gen} on which the action of $x \in \mathfrak{h}_{\mathbb{C}}^*$ is given by the matrix in (2.21).

Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced word for w. Proposition 2.4(d) and the assumption that $w \in \mathcal{F}^{(\gamma,\emptyset)}$ guarantee that the map

$$\tau_w = \tau_{i_1} \cdots \tau_{i_l} : M_\gamma^{\mathrm{gen}} \to M_{w\gamma}^{\mathrm{gen}}$$

is well-defined and bijective. Thus $\tau_w n_\gamma$ and $\tau_w t_{s_i} n_\gamma$ span a two-dimensional subspace of $M^{\text{gen}}_{w\gamma}$ and, by Proposition 2.4(b), the $\mathbb{H}A_1$ action of $x \in X$ on this subspace is given by

$$\rho_{w\gamma}(x) = \gamma(w^{-1}x)\begin{pmatrix} 1 & c_{\alpha_i}\langle w^{-1}x, \alpha_i^\vee\rangle \\ 0 & 1 \end{pmatrix}.$$

This proves (a).

Using α_j for x and inverting the above matrix yields

$$\rho_{w\gamma}\left(\frac{1}{\alpha_j}\right) = \frac{1}{\gamma(w^{-1}\alpha_j)}\begin{pmatrix} 1 & -c_{\alpha_i}\langle w^{-1}\alpha_j, \alpha_i^\vee\rangle \\ 0 & 1 \end{pmatrix}.$$

If $M^{\text{gen}}_{s_j w\gamma} = 0$ then $\tau_j : M^{\text{gen}}_{w\gamma} \to M^{\text{gen}}_{s_j w\gamma}$ is the zero map and

$$\rho_{w\gamma}(t_{s_j}) = \rho_{w\gamma}\left(\frac{c_{\alpha_j}}{\alpha_j}\right) = \frac{c_{\alpha_j}}{\gamma(w^{-1}\alpha_j)}\begin{pmatrix} 1 & -c_{\alpha_i}\langle w^{-1}\alpha_j, \alpha_i^\vee\rangle \\ 0 & 1 \end{pmatrix}.$$

Since $t_{s_j}^2 - 1 = (t_{s_j} - 1)(t_{s_j} + 1) = 0$, $\rho_{w\gamma}(t_{s_j})$ must have Jordan blocks of size 1 and eigenvalues ± 1. Since $c_{\alpha_i} \neq 0$, it follows that $\gamma(w^{-1}\alpha_j) = \pm c_{\alpha_j}$ and $\langle w^{-1}\alpha_j, \alpha_i^\vee\rangle = 0$. □

2.6. **Principal series modules.** For $\gamma \in \mathfrak{h}_{\mathbb{C}}$ let $\mathbb{C}v_\gamma$ be the one dimensional $S(\mathfrak{h}^*_{\mathbb{C}})$-module given by

$$xv_\gamma = \gamma(x)v_\gamma, \qquad \text{for } x \in \mathfrak{h}^*_{\mathbb{C}}.$$

The *principal series representation* $M(\gamma)$ is the $\mathbb{H}$-module defined by

$$M(\gamma) = \mathbb{H} \otimes_{S(\mathfrak{h}^*_{\mathbb{C}})} \mathbb{C}v_\gamma = \operatorname{Ind}^{\mathbb{H}}_{S(\mathfrak{h}^*_{\mathbb{C}})}(\mathbb{C}v_\gamma). \tag{2.22}$$

The module $M(\gamma)$ has basis $\{t_w \otimes v_\gamma \mid w \in W\}$ with $\mathbb{C}W$ acting by left multiplication.

These modules are very useful for studying the combinatorics of representations of $\mathbb{H}$. In fact, we have already used this module in the proofs of Proposition 2.4(e) and Lemma 2.6.

Part (a) of the following proposition implies that the dimension of every irreducible $\mathbb{H}$-module is less than $|W|$. In combination, part (a) and part (b) show that every irreducible $\mathbb{H}$-module with regular central character is calibrated. Part (c) is a graded Hecke analogue of a result of Rogawski [Ro, Proposition 2.3].

Proposition 2.7.

(a) *If M is an irreducible finite dimensional $\mathbb{H}$-module with $M^{\text{gen}}_\gamma \neq 0$, then M is a quotient of $M(\gamma)$.*
(b) *If $\gamma \in \mathfrak{h}_{\mathbb{C}}$ is regular then $M(\gamma)$ is calibrated.*
(c) *For fixed $\gamma \in \mathfrak{h}_{\mathbb{C}}$ and any $w \in W$, $M(\gamma)$ and $M(w\gamma)$ have the same composition factors.*

Proof. (a) Since $S(\mathfrak{h}^*_{\mathbb{C}})$ is commutative, an irreducible $S(\mathfrak{h}^*_{\mathbb{C}})$ submodule must be one-dimensional. Thus there exists a nonzero vector m_γ in M_γ and, as an $S(\mathfrak{h}^*_{\mathbb{C}})$-module, $\mathbb{C}m_\gamma \cong \mathbb{C}v_\gamma$. Since induction is the adjoint functor to restriction there is a unique $\mathbb{H}$-module homomorphism given by

$$\begin{array}{ccc} M(\gamma) & \longrightarrow & M \\ v_\gamma & \longmapsto & m_\gamma \end{array}$$

and, since M is irreducible, this homomorphism is surjective. Thus M is a quotient of $M(\gamma)$.

(b) Since γ is regular, $W_\gamma = \{1\}$,

$$M(\gamma) = \bigoplus_{w \in W} M_{w\gamma} \qquad \text{and} \qquad \dim(M(\gamma)_{w\gamma}) = 1$$

for all $w \in W$. Since $M(\gamma)_{w\gamma}$ is nonzero whenever $M(\gamma)^{\text{gen}}_{w\gamma}$ is nonzero and $\dim(M(\gamma)^{\text{gen}}_{w\gamma}) = 1$, $M(\gamma)_{w\gamma} = M(\gamma)^{\text{gen}}_{w\gamma}$ for all $w \in W$.

(c) Let s_i be a simple reflection such that $s_i\gamma \neq \gamma$. Then $\gamma(\alpha_i) \neq 0$ and the operator τ_i is well defined on $M(s_i\gamma)^{\text{gen}}_{s_i\gamma}$. The vector $v_{s_i\gamma}$ is a weight vector in $M(s_i\gamma)_{s_i\gamma}$ and, by Proposition 2.4(b),

$\tau_i v_{s_i\gamma}$ is a weight vector of weight γ (it is nonzero since $t_{s_i}v_{s_i\gamma}$ and $(s_i\gamma)(c_{\alpha_i}/\alpha_i)v_{s_i\gamma}$ are linearly independent in $M(s_i\gamma)$). Thus, there is an $\mathbb{H}$-module homomorphism

$$\begin{array}{rccl} A(s_i,\gamma): & M(\gamma) & \longrightarrow & M(s_i\gamma) \\ & hv_\gamma & \longmapsto & h\tau_i v_{s_i\gamma}, \qquad h\in\mathbb{H}. \end{array}$$

The modules $M(\gamma)$ and $M(s_i\gamma)$ have bases

$$\{t_w(t_{s_i}+1)v_\gamma,\ \ t_w(t_{s_i}-1)v_\gamma\}_{s_iw>w} \qquad \text{and} \qquad \{t_w(t_{s_i}+1)v_{s_i\gamma},\ \ t_w(t_{s_i}-1)v_{s_i\gamma}\}_{s_iw>w}, \tag{2.23}$$

respectively. Since $(t_{s_i}+1)t_{s_i}=t_{s_i}+1$ and $(t_{s_i}-1)t_{s_i}=-(t_{s_i}-1)$,

$$\begin{aligned} A(s_i,\gamma)(t_w(t_{s_i}+1)v_\gamma) &= t_w(t_{s_i}+1)\left(t_{s_i}-\frac{c_{\alpha_i}}{\alpha_i}\right)v_{s_i\gamma} = t_w(t_{s_i}+1)\left(1-\frac{c_{\alpha_i}}{\alpha_i}\right)v_{s_i\gamma} \\ &= \left(s_i\gamma\left(\frac{\alpha_i-c_{\alpha_i}}{\alpha_i}\right)\right)t_w(t_{s_i}+1)v_{s_i\gamma} \\ A(s_i,\gamma)(t_w(t_{s_i}-1)v_\gamma) &= t_w(t_{s_i}-1)\left(t_{s_i}-\frac{c_{\alpha_i}}{\alpha_i}\right)v_{s_i\gamma} = t_w(t_{s_i}-1)\left(-1-\frac{c_{\alpha_i}}{\alpha_i}\right)v_{s_i\gamma} \\ &= \left(s_i\gamma\left(\frac{\alpha_i+c_{\alpha_i}}{-\alpha_i}\right)\right)t_w(t_{s_i}-1)v_{s_i\gamma} \end{aligned}$$

and so the matrix of $A(s_i,\gamma)$ with respect to the bases in (2.23) is diagonal with $|W|/2$ diagonal entries equal to $(s_i\gamma)((\alpha_i-c_{\alpha_i})/\alpha_i)$ and $|W|/2$ diagonal entries equal to $(s_i\gamma)((\alpha_i+c_{\alpha_i})/(-\alpha_i))$. If $\gamma(\alpha_i)\neq\pm c_{\alpha_i}$ then $A(s_i,\gamma)$ is an isomorphism and so $M(\gamma)$ and $M(s_i\gamma)$ have the same composition factors. If $\gamma(\alpha_i)=\pm c_{\alpha_i}$ then $\dim(\ker A(s_i,\gamma))=|W|/2$. In this case $A(s_i,s_i\gamma)A(s_i,\gamma)=0$ and so the sequence

$$M(\gamma)\stackrel{A(s_i,\gamma)}{\longrightarrow}M(s_i\gamma)\stackrel{A(s_i,s_i\gamma)}{\longrightarrow}M(\gamma)$$

is exact. Since $\dim(M(\gamma))=|W|$ and $\dim(\ker A(s_i,\gamma))=|W|/2$, $M(\gamma)$ and $M(s_i\gamma)$ have the same composition factors. □

Our next goal is to prove Theorem 2.10 which determines exactly when the principal series module $M(\gamma)$ is irreducible. For this we shall need the following lemma.

Lemma 2.8. *Let $\{b_w\}_{w\in W}$ be a basis for the vector space of $\mathcal{H}$ of harmonic polynomials and let X be the $|W|\times|W|$ matrix given by*

$$X=(z^{-1}b_w)_{z,w\in W}. \qquad \textit{Then} \qquad \det X=\xi\cdot\Big(\prod_{\alpha>0}\alpha\Big)^{|W|/2},$$

where ξ is a nonzero constant in $\mathbb{C}$.

Proof. Note that if b'_w is another basis of $\mathcal{H}$ and we write

$$b'_w=\sum_{v\in W}c_{vw}b_v, \qquad c_{vw}\in\mathbb{C}, \qquad \text{then}$$

$$X'=(z^{-1}b'_w)_{z,w\in W} = \big(z^{-1}b_v\big)\big(c_{vw}\big) \qquad \text{and so} \qquad \det X'=\xi\det X,$$

for some nonzero constant $\xi=\det((c_{vw}))$. Thus, by changing basis if necessary, we may assume that the b_w are homogeneous.

Subtract row $z^{-1}b_w$ from row $s_\alpha z^{-1}b_w$. Then this row is divisible by α. By doing this subtraction for each of the $|W|/2$ pairs $\{z^{-1},s_\alpha z^{-1}\}$ we conclude that $\det(X)$ is divisible by $\alpha^{|W|/2}$. Thus, since the roots are co-prime as elements of the polynomial ring $S(\mathfrak{h}^*_\mathbb{C})$,

$$\det(X) \qquad \text{is divisible by} \qquad \Big(\prod_{\alpha>0}\alpha\Big)^{|W|/2}.$$

The degree of $\prod_{\alpha>0}\alpha^{|W|/2}$ is $(|W|/2)\mathrm{Card}(R^+)$ and, using (2.3), the degree of $\det(X)$ is

$$\begin{aligned}\prod_{w\in W}\deg(b_w) &= \sum_k k\dim(\mathcal{H}^k) = \left(\frac{d}{dt}P_W(t)\right)\Big|_{t=1} = \sum_{w\in W}\ell(w)\\ &= \sum_{w\in W}\mathrm{Card}(R(w)) = \sum_{\alpha\in R^+}(|W|/2) = (|W|/2)\mathrm{Card}(R^+).\end{aligned}$$

Since these two polynomials are homogeneous of the same degree it follows that the quotient $\det(X)/(\prod_{\alpha>0}\alpha)^{|W|/2}$ is a constant. If $\det(X)=0$ then the columns of X are linearly dependent. In particular, there exist constants $c_w\in\mathbb{C}$, not all zero, such that $\sum_w c_w b_w = 0$. But this is a contradiction to the assumption that $\{b_w\}$ is a basis of $\mathcal{H}$. So $\det(X)\neq 0$. □

Let $\gamma\in\mathfrak{h}_\mathbb{C}$ and let $M(\gamma)=\mathbb{H}\otimes_{S(\mathfrak{h}_\mathbb{C}^*)}\mathbb{C}v_\gamma$ be the corresponding principal series module for $\mathbb{H}$. The *spherical vector* in $M(\gamma)$ is

$$\mathbf{1}_\gamma = \sum_{w\in W} t_w v_\gamma. \tag{2.24}$$

Up to multiplication by constants this is the unique vector in $M(\gamma)$ such that $t_w\mathbf{1}_\gamma=\mathbf{1}_\gamma$ for all $w\in W$. The following proposition provides a graded Hecke analogue of the results in [Ka, Proposition 1.20] and [Ka, Lemma 2.3]. Mention of this analogue was made in [Op].

Proposition 2.9.

(a) *If γ is a generic element of $\mathfrak{h}_\mathbb{C}$ and $v_{w\gamma}$, $w\in W$, is the basis of $M(\gamma)$ defined in* (2.18) *then*

$$\mathbf{1}_\gamma = \sum_{z\in W}\gamma(c_z)v_{z\gamma}, \qquad \text{where} \qquad c_z = \prod_{\alpha\in R(w_0 z)}\frac{\alpha+c_\alpha}{\alpha}.$$

(b) *The spherical vector $\mathbf{1}_\gamma$ generates $M(\gamma)$ if and only if $\prod_{\alpha>0}(\gamma(\alpha)+c_\alpha)\neq 0$.*
(c) *For $\gamma\in\mathfrak{h}_\mathbb{C}$, the principal series module $M(\gamma)$ is irreducible if and only if $\mathbf{1}_{w\gamma}$ generates $M(w\gamma)$ for all $w\in W$.*

Proof. (a) Suppose that $\xi_z\in\mathbb{C}$ are constants such that

$$\mathbf{1}_\gamma = \left(\sum_{w\in W}t_w\right)v_\gamma = \sum_{z\in W}\xi_z v_{z\gamma}.$$

We shall prove that the ξ_z are given by the formula in the statement of the proposition. Since $t_{s_i}\left(\sum_{w\in W}t_w\right)=\sum_{w\in W}t_w$,

$$\begin{aligned}\mathbf{1}_\gamma = t_{s_i}\mathbf{1}_\gamma &= \left(\tau_i+\frac{c_{\alpha_i}}{\alpha_i}\right)\sum_{z\in W}\xi_z v_{z\gamma} = \left(\tau_i+\frac{c_{\alpha_i}}{\alpha_i}\right)\sum_{s_iz>z}(\xi_z v_{z\gamma}+\xi_{s_iz}v_{s_iz\gamma})\\ &= \sum_{s_iz>z}\left(\xi_z v_{s_iz\gamma}+\xi_z\frac{c_{\alpha_i}}{\gamma(z^{-1}\alpha_i)}v_{z\gamma}+\xi_{s_iz\gamma}\tau_i^2 v_{z\gamma}+\xi_{s_iz}\frac{c_{\alpha_i}}{\gamma(-z^{-1}\alpha_i)}v_{s_iz\gamma}\right).\end{aligned}$$

Comparing coefficients of $v_{s_iz\gamma}$ on each side of this expression gives

$$\xi_{s_iz}=\xi_z+\xi_{s_iz}\frac{c_{\alpha_i}}{\gamma(-z^{-1}\alpha_i)}, \qquad \text{and so} \qquad \frac{\xi_z}{\xi_{s_iz}}=\gamma\left(\frac{z^{-1}\alpha_i+c_{\alpha_i}}{z^{-1}\alpha_i}\right), \qquad \text{if } s_iz>z.$$

Using this formula inductively gives

$$\xi_w = \xi_{s_{i_1}\cdots s_{i_p}} = \gamma\left(\frac{s_{i_p}\cdots s_{i_2}\alpha_{i_1}}{s_{i_p}\cdots s_{i_2}\alpha_{i_1}+c_{\alpha_i}}\right)\cdots\gamma\left(\frac{\alpha_{i_p}}{\alpha_{i_p}+c_{\alpha_{i_p}}}\right)\xi_1$$
$$= \gamma\left(\prod_{\alpha\in R(w)}\frac{\alpha}{\alpha+c_\alpha}\right)\xi_1.$$

Since the transition matrix between the basis $\{t_w v_\gamma\}$ and the basis $\{v_{w\gamma}\}$ is upper unitriangular with respect to Bruhat order, $\xi_{w_0}=1$. Thus, the last equation implies that

$$\xi_1 = \gamma\left(\prod_{\alpha>0}\frac{\alpha+c_\alpha}{\alpha}\right) \qquad \text{and} \qquad \xi_w = \gamma\left(\prod_{\alpha\in R(w)}\frac{\alpha}{\alpha+c_\alpha}\right)\cdot\xi_1 = \gamma\left(\prod_{\alpha\in R(w_0w)}\frac{\alpha+c_\alpha}{\alpha}\right).$$

(b) By expanding $v_{z\gamma} = \tau_z v_\gamma = \tau_{i_1}\cdots\tau_{i_p}v_\gamma$ for a reduced word $s_{i_1}\cdots s_{i_p}=z$ it follows that there exist rational functions m_{uz} such that

$$v_{z\gamma} = \sum_{u\in W}\gamma(m_{uz})t_u v_\gamma,$$

for all generic $\gamma\in\mathfrak{h}_\mathbb{C}$. Furthermore the matrix $M=(m_{uz})_{u,z\in W}$ with these rational functions as entries is upper unitriangular.

Let b_w, $w\in W$, be a basis of harmonic polynomials and define polynomials $q_{uy}\in S(\mathfrak{h}^*_\mathbb{C})$, $u,y\in W$, by

$$b_y\left(\sum_{w\in W}t_w\right) = \sum_{u\in W}t_u q_{uy}, \qquad y\in W,$$

where these equations are equalities in $\mathbb{H}$. Then,

$$b_y\mathbf{1}_\gamma = b_y\left(\sum_{w\in W}t_w\right) = \sum_{u\in W}\gamma(q_{uy})\,(t_u\otimes v_\gamma),$$

and part (a) implies that if γ is generic then

$$b_y\mathbf{1}_\gamma = b_y\sum_{z\in W}\gamma(c_z)v_{z\gamma} = \sum_{z\in W}\gamma(c_z(z^{-1}b_y))\,v_{z\gamma} = \sum_{z,u\in W}\gamma(c_z(z^{-1}b_y)m_{uz})\,(t_u\otimes v_\gamma).$$

Since these two expressions are equal for all generic $\gamma\in\mathfrak{h}_\mathbb{C}$ it follows that

$$q_{uy} = \sum_{z\in W}m_{uz}\cdot c_z\cdot(z^{-1}b_y), \qquad u,y\in W, \tag{2.25}$$

as rational functions (in fact both sides are polynomials).

Since t_w, $w\in W$, and $p\in Z(\mathbb{H}) = S(\mathfrak{h}^*_\mathbb{C})^W$ act on $\mathbf{1}_\gamma$ by constants, the $\mathbb{H}$-module $M(\gamma)$ is generated by $\mathbf{1}_\gamma$ if and only if there exist constants $p_{yw}\in\mathbb{C}$ such that

$$t_w\otimes v_\gamma = \sum_{y\in W}p_{yw}b_y\mathbf{1}_\gamma, \qquad \text{for each } w\in W.$$

If these constants exist then, for each $w\in W$,

$$t_w\otimes v_\gamma = \sum_{y\in W}p_{yw}b_y\mathbf{1}_\gamma = \sum_{y,z,u\in W}\gamma(m_{uz}c_z(z^{-1}b_y)p_{yw})t_u\otimes v_\gamma,$$

where, by (2.25), there is no restriction that γ be generic. If

$$M=(m_{uz})_{u,z\in W}, \qquad C=\mathrm{diag}(c_z)_{z\in W}, \qquad X=\left(z^{-1}b_y\right)_{z,y\in W} \qquad P=(p_{yw})_{y,w\in W},$$

then $P = (\gamma(MCX))^{-1}$ and so P exists if and only if $\det(\gamma(MCX)) \neq 0$. Now $\det(M) = 1$, and, by Lemma 2.8 and part (a),

$$\det(X) = \xi \cdot \prod_{\alpha>0} \alpha^{|W|/2} \qquad \text{and} \qquad \det(C) = \prod_{z\in W} \prod_{\alpha\in R(w_0 z)} \frac{\alpha + c_\alpha}{\alpha} = \left(\prod_{\alpha>0} \frac{\alpha + c_\alpha}{\alpha}\right)^{|W|/2},$$

where $\xi \in \mathbb{C}$ is nonzero. Thus P exists if and only if $\prod_{\alpha>0}(\gamma(\alpha) + c_\alpha) \neq 0$.

(c) $\Longrightarrow$: If $M(\gamma)$ is irreducible then, by Proposition 2.7(c), $M(w\gamma)$ is irreducible for all $w \in W$. Hence $M(w\gamma)$ is generated by $\mathbf{1}_{w\gamma}$.

$\Longleftarrow$: Suppose that $\mathbf{1}_{w\gamma}$ generates $M(w\gamma)$ for all $w \in W$. Let E be a nonzero irreducible submodule of $M(\gamma)$ and let $w \in W$ be such that the weight space $E_{w\gamma}$ is nonzero. Then, by Proposition 2.7(a), there is a nonzero surjective $\mathbb{H}$-module homomorphism $\varphi\colon M(w\gamma) \to E$. Since $\mathbf{1}_{w\gamma}$ generates $M(w\gamma)$, $\varphi(\mathbf{1}_{w\gamma})$ is a nonzero vector in E such that $t_v\varphi(\mathbf{1}_{w\gamma}) = \varphi(\mathbf{1}_{w\gamma})$ for all $v \in W$. Since there is a unique, up to constant multiples, spherical vector in $M(\gamma)$ $\phi(\mathbf{1}_{w\gamma})$ is a multiple of $\mathbf{1}_\gamma$ and $\mathbf{1}_\gamma$ is nonzero. This implies that $E = M(\gamma)$ since $\mathbf{1}_\gamma$ generates $M(\gamma)$. □

Together the three parts of Proposition 2.9 prove the following graded Hecke algebra analogue of [Ka, Theorem 2.1].

Theorem 2.10. *Let $\gamma \in \mathfrak{h}_\mathbb{C}$ and let $P(\gamma) = \{\alpha > 0 \mid \gamma(\alpha) = \pm c_\alpha\}$. The principal series $\mathbb{H}$-module*

$$M(\gamma) \textit{ is irreducible if and only if } P(\gamma) = \emptyset.$$

3. Classification of Irreducible Representations for Rank 2

3.1. **The root system.** The reflection group $I_2(n)$ is the dihedral group of order $2n$. Let $\varepsilon_1, \varepsilon_2$ be an orthonormal basis of $\mathfrak{h}^*_\mathbb{R} = \mathbb{R}^2$ and define

$$\beta_k = \cos(k\theta)\varepsilon_1 + \sin(k\theta)\varepsilon_2, \qquad \text{where } \theta = \pi/n.$$

Fix the roots, positive roots and simple roots for the reflection group $I_2(n)$ by

$$\begin{aligned} R &= \{\beta_k \mid 0 \le k \le 2n-1\}, \\ R^+ &= \{\beta_k \mid 0 \le k \le n-1\}, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \alpha_1 &= \beta_0, \\ \alpha_2 &= \beta_{n-1}. \end{aligned}$$

For $0 \le k \le n-1$, $-\beta_k = \beta_{n+k}$, $s_1\beta_k = \beta_{n-k}$ and $s_2\beta_k = \beta_{n-2-k}$, and when n is even there are two orbits of roots, $\{\pm\beta_{2k} \mid 0 \le k < n/2\}$ and $\{\pm\beta_{2k+1} \mid 0 \le k < n/2\}$. Let $c_k = c_{\beta_k}$ be a choice of parameters for the graded Hecke algebra $\mathbb{H}$. When n is odd all of the c_k are equal and, when n is even, there are two, possibly unequal, parameters $c_0 = c_{2k}$ and $c_1 = c_{2k+1}$. Figure 1 displays the roots β_k and hyperplanes $H_{\beta_k} = \{x \in \mathbb{R}^2 \mid \langle \beta_k, x\rangle = 0\}$ for $I_2(7)$ and $I_2(8)$. When n is even each root β_k lies on the hyperplane $H_{\beta_{k+n/2}}$ and this is why, in the picture of hyperplanes and roots for $I_2(8)$ there are multiple labels on each line.

Figure 2 displays, using thin and thick lines, the hyperplanes

$$H_{\beta_k} = \{x \in \mathbb{R}^2 \mid \langle \beta_k, x\rangle = 0\} \qquad \text{and} \qquad H_{\beta_k \pm \delta} = \{x \in \mathbb{R}^2 \mid \langle \beta_k, x\rangle = \pm c_k\}$$

for $I_2(7)$ and $I_2(8)$ (and a particular choice of the parameters c_k).

3.2. **The central characters.** Using the orthonormal basis $\varepsilon_1, \varepsilon_2$ we can identify $\mathfrak{h}_\mathbb{R}$ with $\mathbb{R}^2$ and $\mathfrak{h}_\mathbb{C}$ with $\mathbb{C}^2$. If $\gamma \in \mathfrak{h}_\mathbb{C}$ then

$$Z(\gamma) = \{\beta_k \in R^+ \mid \langle \gamma, \beta_k\rangle = 0\} \qquad \text{and} \qquad P(\gamma) = \{\beta_k \in R^+ \mid \langle \gamma, \beta_k\rangle = \pm c_k\}.$$

In terms of the pictures in Figure 2, if γ is a point in $\mathbb{R}^2$ then the elements of $Z(\gamma)$ label the H_{β_k} (thin lines) that γ is on and the elements of $P(\gamma)$ label the set of $H_{\beta_k \pm \delta}$ (thick lines) that γ is on.

Let us analyze the possibilities for $Z(\gamma)$ and $P(\gamma)$. For the purpose of analyzing representations of $\mathbb{H}$, γ labels a central character. Since a central character is really a W-orbit we may replace γ by any more convenient element in the orbit $W\gamma$. If $\gamma(\alpha) = c_\alpha$ then $(1/c_\alpha)\gamma(\alpha) = 1$ and so we may,

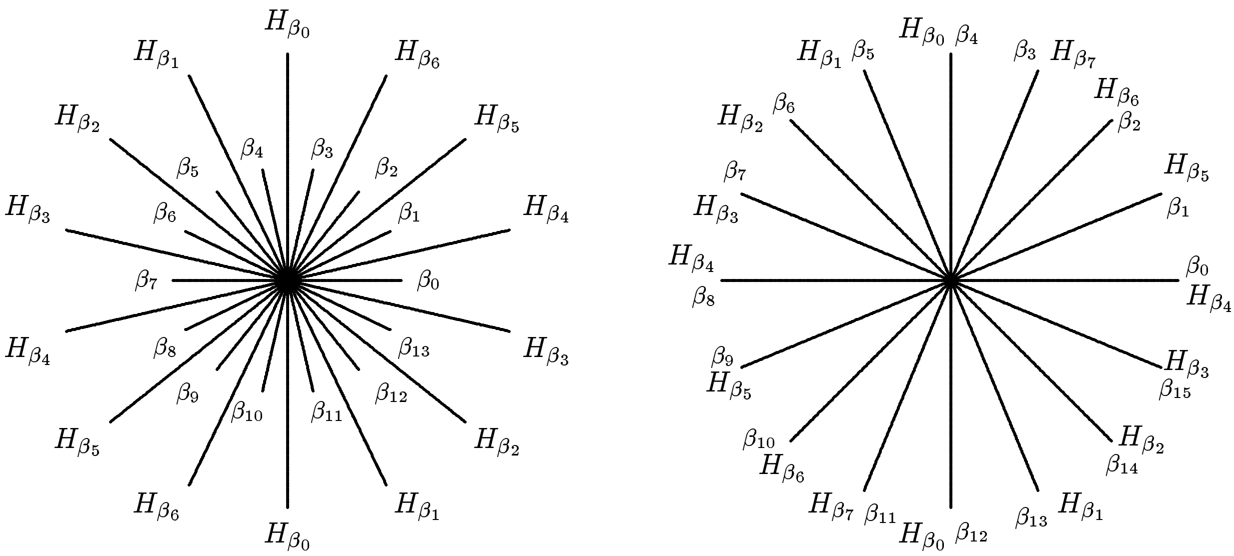


FIGURE 1. Hyperplanes and roots for $I_2(7)$ and $I_2(8)$

without loss of generality, assume that $c_k = 1$ for all k when n is odd, and $c_{2k} = 1$ and $c_{2k+1} = c$ when n is even.

(a) If $Z(\gamma)$ contains 2 roots or more then $\gamma = 0$, since any two distinct positive roots are linearly independent. This is the central character γ_0 in Table 1.

(b) If $Z(\gamma)$ contains one root then, by replacing γ with another element of $W\gamma$, we may assume that $Z(\gamma) = \{\beta_0\}$. When n is even, we may also have to use the automorphism of the root system which switches $\alpha_1 = \beta_0$ and $\alpha_2 = \beta_{n-1}$ to get $Z(\gamma) = \{\beta_0\}$. Applying this automorphism changes the central character but the representations of $\mathbb{H}$ with the new central character will have exactly the same structure as the representations of central character γ.

(b$'$) If $Z(\gamma) = \{\beta_0\}$ and $\beta_k \in P(\gamma)$ then the equations $0 = \gamma(\beta_0) = \gamma(\varepsilon_1)$ and

$$c_k = \gamma(\beta_k) = \gamma(\cos(k\theta)\varepsilon_1 + \sin(k\theta)\varepsilon_2) = \sin(k\theta)\gamma(\varepsilon_2) \tag{3.1}$$

uniquely determine γ. Since $\sin(k\theta) = \sin((n-k)\theta)$, β_{n-k} must also be in $P(\gamma)$. This happens for the central characters $\gamma_{b,k}$, $\gamma_{b,n/2}$ and γ_q in Table 1.

(b$''$) If $Z(\gamma) = \{\beta_0\}$, $\beta_k, \beta_\ell \in P(\gamma)$ and $\ell \neq n-k$ then equation (3.1) for k and ℓ forces $c_k \neq c_\ell$ which forces n even and k and ℓ to be of different parity. Furthermore the parameters must satisfy $c_k/c_\ell = \sin(k\theta)/\sin(\ell\theta)$ and, when this happens, it happens for a unique choice of the 4-tuple $(k, \ell, n-k, n-\ell)$. Thus, the only possible option is $P(\gamma) = \{\beta_k, \beta_{n-k}, \beta_\ell, \beta_{n-\ell}\}$ (if $\ell = n/2$ then $P(\gamma) = \{\beta_{n/2}, \beta_k, \beta_{n-k}\}$). This is the central character γ_q in Table 1.

(c) If $Z(\gamma) = \emptyset$ and $\beta_k, \beta_\ell \in P(\gamma)$ such that $c_k = c_\ell = c$ then γ is uniquely determined by the equations $c = \cos(k\theta)\gamma(\varepsilon_1) + \sin(k\theta)\gamma(\varepsilon_2) = \cos(\ell\theta)\gamma(\varepsilon_1) + \sin(\ell\theta)\gamma(\varepsilon_2)$. These equations force $\beta_{(n+k+\ell)/2} \in Z(\gamma)$ if $(n+k+\ell)$ is even (the easiest way to see this is to look at the pictures in Figure 2). Since we assumed $Z(\gamma) = \emptyset$ it follows that $n+k+\ell$ is odd. If $P(\gamma)$ contains 3 elements then at least two of them would satisfy $n+k+\ell$ even, and so it follows that $P(\gamma)$ contains a maximum of two elements. By replacing γ by an appropriate element of the orbit $W\gamma$ we can assume that $P(\gamma) = \{\beta_{k-1}, \beta_{n-k}\}$ for some $1 \leq k \leq n/2$. This case corresponds to the central character $\gamma_{c,k}$ in Table 1.

This analysis shows that Table 1 covers all $(P(\gamma), Z(\gamma))$ possibilities.

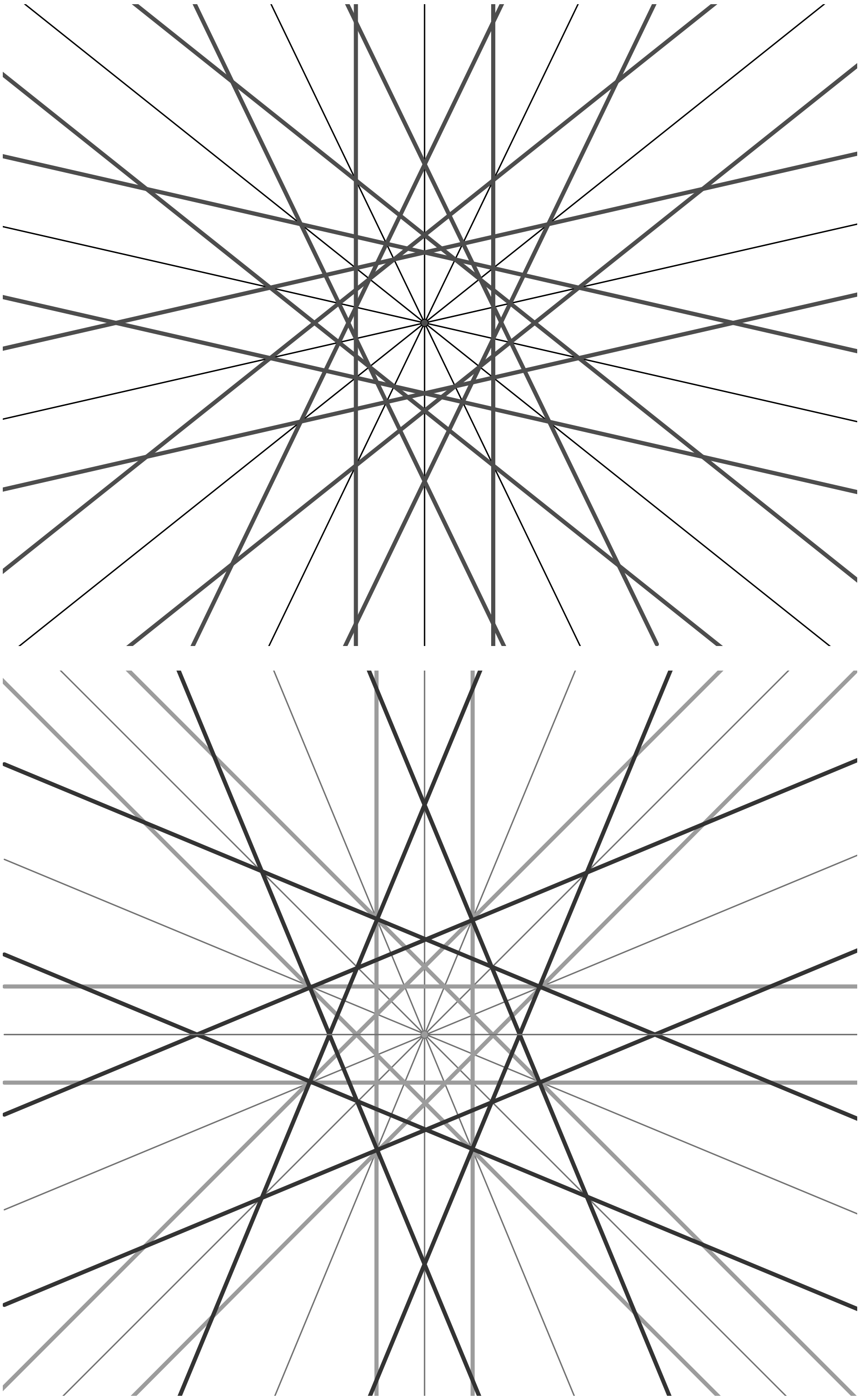


FIGURE 2. Hyperplanes for $I_2(7)$ and $I_2(8)$.

3.3. **The irreducible representations.** The following analysis determines the structure of each of the irreducible $\mathbb{H}$-modules: the dimensions of each generalized weight space and the Langlands parameters. The results are summarized in Table 1. An irreducible representation that is calibrated (see (2.10)) has all its weights of the form $w\gamma$ with $w \in \mathcal{F}^{(\gamma,J)}$ for a unique J, and this is the set which is displayed in the fourth column of Table 1. The notation 'nc' indicates that the representation is not calibrated.

The derivation of the irreducible representations below proceeds by considering, separately, each central character γ. In each case we have included a picture showing the local regions (γ, J). In these pictures the solid lines correspond to hyperplanes H_α for $\alpha \in Z(\gamma)$ and the dotted lines correspond to hyperplanes H_α for $\alpha \in P(\gamma)$. Each local region is labeled by the corresponding set J of roots which determines its location in the picture (see the discussion before Corollary 2.5).

The Langlands parameters of an irreducible $\mathbb{H}$-module M are determined by the real parts of weights of M. This means that, according to the labeling of the simple modules as in Table 1, the Langlands parameters *can* depend on the choice of the parameters c_k. In our calculations of Langlands parameters, and in the Langlands data displayed in Table 1, we assume that all $c_k \in \mathbb{R}_{>0}$ (this assumption is used *only* in the analysis of Langlands parameters). When $I \subseteq \{1,2\}$ contains only one element, a tempered $\mathbb{H}_I$-module is determined by its maximal weight. Thus, in Table 1, we specify Langlands parameters in the form (λ, I) where λ indicates the maximal weight of a tempered $\mathbb{H}_I$-module.

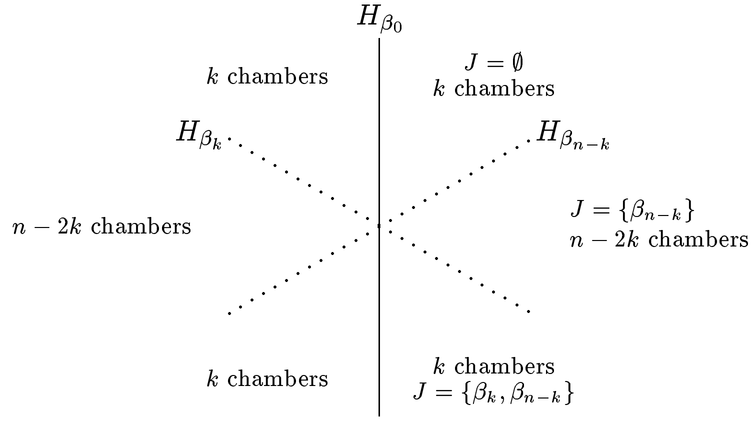
In the case when n is even not all roots are in the orbit of $\alpha_1 = \beta_0$ and one should really consider central characters γ which have $Z(\gamma) = \{\beta_{n-1}\} = \{\alpha_2\}$. These central characters γ'_a, $\gamma'_{b,k}$, $\gamma'_{c,k}$ are the images of the central characters γ_a, $\gamma_{b,k}$ and $\gamma_{c,k}$ under the automorphism of the root system which switches α_1 and α_2. This automorphism extends to an automorphism of $\mathbb{H}$ and thus it follows that the modules with central characters γ'_a, $\gamma'_{b,k}$, $\gamma'_{c,k}$ have exactly the same structures as the modules with central characters γ_a, γ_b and $\gamma_{c,k}$, respectively.

Central character γ_a*:* $Z(\gamma_a) = \emptyset$, $P(\gamma_a) = \emptyset$.

By Theorem 2.10 the principal series module $M(\gamma_a)$ is irreducible and, by Proposition 2.7(a), this is the unique irreducible module with central character γ_a. Since γ_a is regular $M(\gamma_a)$ is calibrated.

Central character $\gamma_{b,k}$*:* $Z(\gamma_{b,k}) = \{\beta_0\}$, $\quad P(\gamma_{b,k}) = \{\beta_k, \beta_{n-k}\}$, $\quad 1 \le k \le (n-1)/2$.

The weight $\gamma_{b,k}$ is uniquely determined by the fact that $\gamma_{b,k}(\beta_0) = \gamma(\varepsilon_1) = 0$ and $c_k = \gamma(\beta_k) = \sin(k\theta)\gamma(\varepsilon_2)$, where $\theta = \pi/n$.



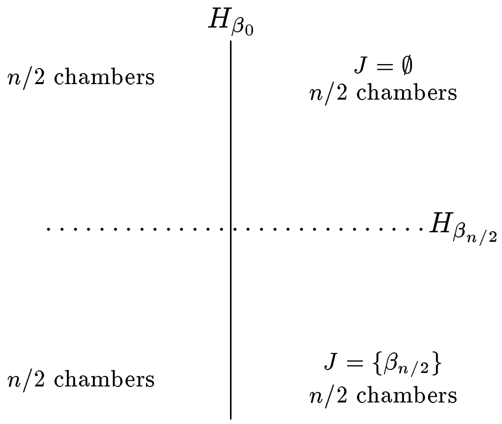
Use Lemma 2.6 to decompose the principal series module $M(\gamma_{b,k})$ and conclude that there are two irreducible modules M and N with central character $\gamma_{b,k}$ and

$$\dim(M^{\mathrm{gen}}_{w\gamma_{b,k}}) = 2 \text{ for } w \in \mathcal{F}^{(\gamma_{b,k},\emptyset)}, \qquad \dim(M^{\mathrm{gen}}_{w\gamma_{b,k}}) = 1 \text{ for } w \in \mathcal{F}^{(\gamma_{b,k},\{\beta_{n-k}\})},$$
$$\dim(N^{\mathrm{gen}}_{w\gamma_{b,k}}) = 1, \text{ for } w \in \mathcal{F}^{(\gamma_{b,k},\{\beta_{n-k}\})} \qquad \dim(N^{\mathrm{gen}}_{w\gamma_{b,k}}) = 2, \text{ for } w \in \mathcal{F}^{(\gamma_{b,k},\{\beta_k,\beta_{n-k}\})},$$

and all other weight spaces of M and N are 0. Neither of the two irreducible modules M and N with central character $\gamma_{b,k}$ are calibrated.

The maximal weight of M is $\gamma_{b,k}$ which is dominant and on the hyperplane H_{α_1}. The Langlands set for this weight is $I = \{1\}$. The maximal weight of N is on the hyperplane H_{β_k} if k is even, and on the hyperplane $H_{\beta_{n-(k+1)}}$ if k is odd. This observation determines the set I in the Langlands decomposition of the (real part) of the maximal weight of N (equation (2.11)).

Central character $\gamma_{b,n/2}$*:* n even, $Z(\gamma_{b,n/2}) = \{\beta_0\}$, $\quad P(\gamma_{b,n/2}) = \{\beta_{n/2}\}$.



Use Lemma 2.6 to decompose the principal series module $M(\gamma_{b,n/2})$ and conclude that there are two irreducible modules M and N with central character $\gamma_{b,n/2}$ with

$$\begin{aligned} \dim(M^{\text{gen}}_{w\gamma_{b,n/2}}) = 2, &\quad \text{for } w \in \mathcal{F}^{(\gamma_{b,n/2},\emptyset)}, \qquad \text{and} \\ \dim(N^{\text{gen}}_{w\gamma_{b,n/2}}) = 2, &\quad \text{for } w \in \mathcal{F}^{(\gamma_{b,k},\{\beta_{n/2}\})}. \end{aligned}$$

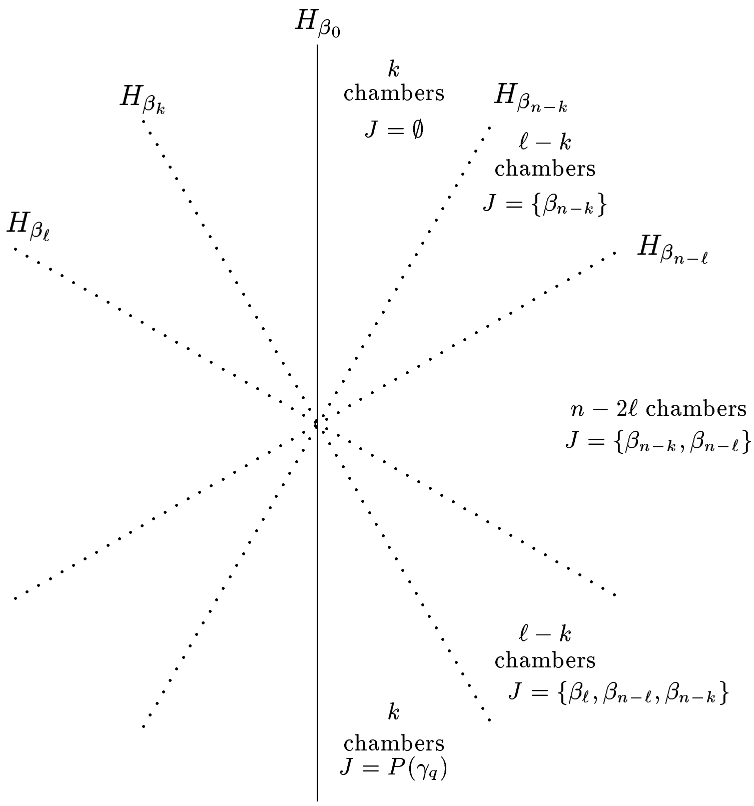
All other weight spaces of M and N are 0. Neither of the two irreducible modules M and N with central character $\gamma_{b,n/2}$ are calibrated.

The maximal weight of M is $\gamma_{b,n/2}$ which is dominant and on the hyperplane H_{α_1}. The Langlands set for this weight is $I = \{1\}$. The module N is tempered with maximal weight $\underbrace{\cdots s_1 s_2}_{n/2 \text{ factors}} \gamma_{b,n/2}$.

Central character γ_q*:* $Z(\gamma_q) = \{\beta_0\}$, $P(\gamma_q) = \{\beta_k, \beta_{n-k}, \beta_\ell, \beta_{n-\ell}\}$.

It may be that $\ell = n/2 = n - \ell$ so that the hyperplanes H_{β_ℓ} and $H_{\beta_{n-\ell}}$ are the same and $P(\gamma)$ contains only 3 roots. We do not have to consider this situation separately.

In some sense, the special central character γ_q occurs when the parameters are exactly right so that the central characters $\gamma_{b,k}$ and $\gamma_{b,\ell}$ "coalesce". This occurs only if n is even, k and ℓ are of different parity, and the parameters satisfy $c_k/c_\ell = \sin(k\theta)/\sin(\ell\theta)$. For a fixed choice of parameters, there is at most one choice of the quadruple $(k, \ell, n-k, n-\ell)$.



There are five nonisomorphic irreducible $\mathbb{H}$-modules L, M, N, P and Q with central character γ_q, unless $\ell = n/2$, in which case there are only four (N has dimension 0).

$$\begin{aligned}
\dim(L^{\mathrm{gen}}_{w\gamma_q}) &= 2, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \emptyset)}, \\
\dim(L^{\mathrm{gen}}_{w\gamma_q}) &= 1, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \{\beta_{n-k}\})},
\end{aligned}$$

$$\dim(M^{\mathrm{gen}}_{w\gamma_q}) = 1, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \{\beta_{n-k}\})},$$

$$\dim(N^{\mathrm{gen}}_{w\gamma_q}) = 1, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \{\beta_{n-k}, \beta_{n-\ell}\})},$$

$$\dim(P^{\mathrm{gen}}_{w\gamma_q}) = 1, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \{\beta_\ell, \beta_{n-k}, \beta_{n-\ell}\})},$$

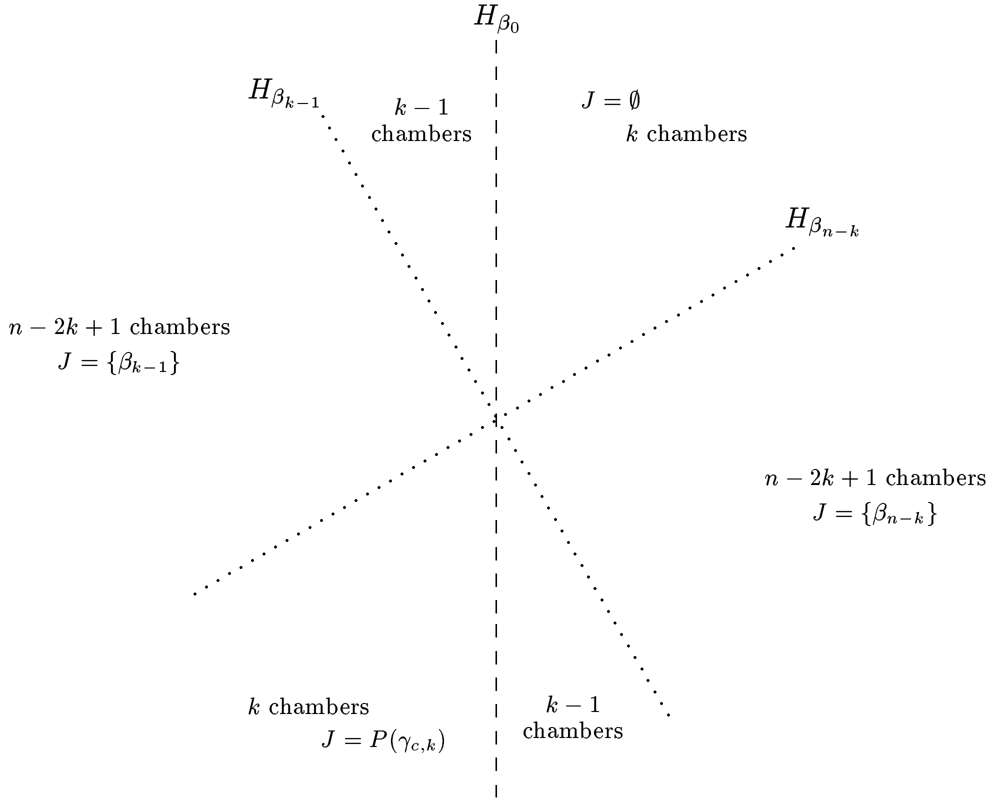
$$\begin{aligned}
\dim(Q^{\mathrm{gen}}_{w\gamma_q}) &= 1, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \{\beta_\ell, \beta_{n-k}, \beta_{n-\ell}\})}, \\
\dim(Q^{\mathrm{gen}}_{w\gamma_q}) &= 2, \quad \text{for } w \in \mathcal{F}^{(\gamma_q, \{\beta_k, \beta_\ell, \beta_{n-k}, \beta_{n-\ell}\})},
\end{aligned}$$

and all other weight spaces of these modules are 0.

Both modules P and Q are tempered and have the same maximal weight $\underbrace{\cdots\ s_1 s_2}_{n-\ell \text{ factors}} \gamma_q$.

Central character $\gamma_{c,k}$*:* $Z(\gamma_{c,k}) = \emptyset, \quad P(\gamma_{c,k}) = \{\beta_{k-1}, \beta_{n-k}\}, \quad 1 \le k \le (n-1)/2.$

The weight $\gamma_{c,k}$ is uniquely determined by $\gamma(\beta_{k-1}) = c_{k-1}$ and $\gamma(\beta_{n-k}) = c_{n-k}$.



The dashed line in this picture is for reference only, it does not correspond to a root in $Z(\gamma)$ or $P(\gamma)$.

Since $\gamma_{c,k}$ is regular the irreducible $\mathbb{H}$-modules with central character $\gamma_{c,k}$ are calibrated and can be indexed by the sets J. The irreducible calibrated module $\mathbb{H}^{(\gamma_{c,k},J)}$ indexed by the set J has

$$\dim(\mathbb{H}^{(\gamma_{c,k},J)})_{w\gamma_{c,k}} = 1 \text{ for } w \in \mathcal{F}^{(\gamma_{c,k},J)}$$

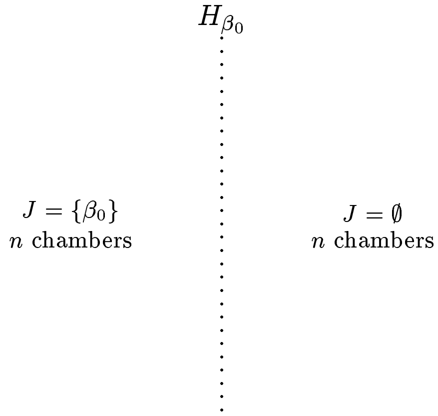
and all other weight spaces 0. A construction of $\mathbb{H}^{(\gamma_{c,k},J)}$ is given in Theorem 4.5.

To compute the Langlands parameters of these modules we first assume that n is odd and $m = \frac{n-1}{2}$. If $J = \{\beta_{k-1}\}$ the maximal weight of the module $\mathbb{H}^{(\gamma_{c,k},J)}$ is in the same chamber as β_{m-k} if k is even, and in the same chamber as β_{m+k} if k is odd. If $J = \{\beta_{n-k}\}$ the maximal weight of $\mathbb{H}^{(\gamma_{c,k},J)}$ is in the same chamber as β_{m-k} if k is odd, and in the same chamber as β_{m+k} if k is even. In each case this information determines the set I in the Langlands parameters. If $J = \{\beta_{k-1}, \beta_{n-k}\}$ the module $\mathbb{H}^{(\gamma_{c,k},J)}$ is tempered with maximal weights

$$\underbrace{\cdots s_2 s_1}_{n-k+1 \text{ factors}} \gamma_{c,k}, \qquad \text{and} \qquad \underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_{c,k}.$$

If n is even and all parameters c_k are equal then the Langlands parameters are as in the previous paragraph. In the case that n is even and $c_{2k} \neq c_{2k+1}$ then it may happen that $\gamma_{c,k}$ is not in the dominant chamber. The structure of the modules with central character $\gamma_{c,k}$ does not change *but the Langlands parameters of the representations may change significantly.* One of the four irreducibles with central character $\gamma_{c,k}$ will always be tempered, but which one (and thus the dimension of the tempered module with this central character) depends on the values of the parameters c_{2k} and c_{2k+1}.

Central character γ_d: $Z(\gamma_d) = \emptyset, \quad P(\gamma_d) = \{\beta_0\}$



Since γ_d is regular the irreducible modules with central character γ_d are calibrated and can be indexed by the sets J. The module $\mathbb{H}^{(\gamma_d, J)}$ has

$$\dim(\mathbb{H}^{(\gamma_{c,k}, J)})_{w\gamma_{c,k}} = 1 \text{ for } w \in \mathcal{F}^{(\gamma_{c,k}, J)}$$

and all other weight spaces 0. A construction of $\mathbb{H}^{(\gamma_{d,k}, J)}$ is given in Theorem 4.5.

The Langlands parameters given in Table 1 for irreducible representations with central character γ_d assume that $\gamma_d \notin W\gamma_{d'}$ where n is odd and $\gamma_{d'} = \xi \cdot \beta_{(n-1)/2}$, $\xi \in \mathbb{R}_{>0}$. In the particular case n odd and $\gamma_d \in W\gamma_{d'}$ the irreducible module indexed by the set $J = \{\beta_0\}$ is tempered.

3.4. **Tempered representations and the Springer correspondence.** The Springer correspondence for Weyl groups (see [BM1, p.34]) associates to each tempered representation M of $\mathbb{H}$ with real central character, the unique "maximal" irreducible W-module which is contained in M. For Weyl groups (crystallographic reflection groups) this is a one-to-one correspondence between tempered representations of $\mathbb{H}$ and irreducible representations of W. Using our classification of $\mathbb{H}$-modules in Table 1, we can establish a similar correspondence for the noncrystallographic groups $I_2(n)$.

If n is odd then the group $I_2(n)$ has 2 one-dimensional irreducible representations and $(n-1)/2$ two-dimensional irreducible representations. The trivial (resp. sign) representation of $I_2(n)$ corresponds to the tempered irreducible $\mathbb{H}$-module with central character γ_0 (resp. $\gamma_{c,1}$). The two-dimensional representations of $I_2(n)$ correspond to the tempered $\mathbb{H}$-modules with central characters $\gamma_d \in W\gamma_d'$ and $\gamma_{c,k}$, $1 \le k \le (n-1)/2$. Note that γ_0, γ_d and $\gamma_{c,k}$, $1 \le k \le (n-1)/2$, can all be taken to be multiples of the root $\beta_{(n-1)/2}$ and in the dominant chamber. In this normalization the 1-dimensional representations correspond to the two extreme elements of this chain of weights.

If n is even and the parameters c_k are all equal the trivial (resp. sign) representation of $I_2(n)$ corresponds to the tempered irreducible $\mathbb{H}$-module with central character γ_0 (resp. $\gamma_{c,1}$) and the other two 1-dimensional representations of $I_2(n)$ correspond to the tempered $\mathbb{H}$-modules with central characters $\gamma_{b,n/2}$ and $\gamma_{b',n/2}$. The 2-dimensional $I_2(n)$-modules correspond to the tempered $\mathbb{H}$-modules with central characters $\gamma_{c,k}$, $2 \le k \le n/2$. As in the case n odd, the central characters γ_0 and $\gamma_{c,k}$, $1 \le k \le (n-1)/2$, can be taken to be in the dominant chamber and on the line through the origin and the point $\beta_{n/2} + \beta_{n/2-1}$. In this normalization the trivial and the sign representations correspond to the two extreme elements of this chain of weights. In the case when the parameters are unequal two of the points on this chain may coalesce in the weight γ_q and "become" the two tempered representations of $\mathbb{H}$ with central character γ_q. The case where $P(\gamma_q)$ contains only 3 roots comes from one of the central characters $\gamma_{b,n/2}$ or $\gamma_{b',n/2}$ coalescing with one of the $\gamma_{c,k}$.

This analysis establishes the "Springer correspondence" for all dihedral groups and all choices of the parameters c_k of $\mathbb{H}$ with $c_k \in \mathbb{R}_{>0}$.

TABLE 1. Irreducible representations of $\mathbb{H}I_2(n)$

Character	$Z(\gamma), P(\gamma)$	Dimension	J	Langlands Parameters
$\gamma_0 = 0$	$R^+, \emptyset$	$2n$	nc	tempered
γ_a	$\{\beta_0\}, \emptyset$	$2n$	nc	$(\gamma_a, \{1\})$
$\gamma_{b,k}$	$\{\beta_0\}, \{\beta_k, \beta_{n-k}\}$	n	nc	$(\gamma_{b,k}, \{1\})$
$1 \le k < n/2$		n	nc	$(\underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_{b,k}, \{1\})$ k even
				$(\underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_{b,k}, \{2\})$, k odd
$\gamma_{b,n/2}$	$\{\beta_0\}, \{\beta_{n/2}\}$	n	nc	$(\gamma_{b,n/2}, \{1\})$
(n even)		n	nc	tempered
γ_q	$\{\beta_0\}$,	$\ell + k$	nc	$(\gamma_q, \{1\})$
(n even)	$\{\beta_k, \beta_{n-k}, \beta_\ell, \beta_{n-\ell}\}$	$\ell - k$	$\{\beta_{n-k}\}$	$(\underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_q, \{1\})$, k even
$0 < k < \ell \le n/2$				$(\underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_q, \{2\})$, k odd
		$n - 2\ell$	$\{\beta_{n-k}, \beta_{n-\ell}\}$	$(\underbrace{\cdots s_1 s_2}_{\ell \text{ factors}} \gamma_q, \{1\})$, ℓ even
				$(\underbrace{\cdots s_1 s_2}_{\ell \text{ factors}} \gamma_q, \{2\})$, ℓ odd
		$\ell - k$	$\{\beta_{n-k}, \beta_{n-\ell}, \beta_\ell\}$	tempered
		$\ell + k$	nc	tempered
$\gamma_{c,k}$	$\emptyset, \{\beta_{k-1}, \beta_{n-k}\}$	$2k - 1$	$\emptyset$	$(\gamma_{c,k}, \emptyset)$
$1 \le k \le n/2$		$n - 2k + 1$	$\{\beta_{k-1}\}$	$(\underbrace{\cdots s_2 s_1}_{k \text{ factors}} \gamma_{c,k}, \{1\})$, k odd
				$(\underbrace{\cdots s_2 s_1}_{k \text{ factors}} \gamma_{c,k}, \{2\})$, k even
		$n - 2k + 1$	$\{\beta_{n-k}\}$	$(\underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_{c,k}, \{1\})$, k even
				$(\underbrace{\cdots s_1 s_2}_{k \text{ factors}} \gamma_{c,k}, \{2\})$, k odd
		$2k - 1$	$\{\beta_{k-1}, \beta_{n-k}\}$	tempered
γ_d	$\emptyset, \{\beta_0\}$	n	$\emptyset$	$(\gamma_d, \emptyset)$
		n	$\{\beta_0\}$	$(s_1 \gamma_d, \{1\})^\dagger$
γ_{gen}	$\emptyset, \emptyset$	$2n$	$\emptyset$	$(\gamma_{\text{gen}}, \emptyset)$

† This module is tempered if n is odd and $\gamma_d \in W\gamma'_d$, with $\gamma'_d = \xi \cdot \beta_{(n-1)/2}$, $\xi \in \mathbb{R}_{>0}$.

4. Classification of Calibrated Representations

4.1. **Structural results.** We first examine some properties which hold for irreducible modules that are calibrated, i.e., can be decomposed into a direct sum of weight spaces (see (2.10)). This section follows closely the similar results for affine Hecke algebras in [Ra1].

Lemma 4.1. *Let M be an irreducible calibrated module. Then, for all $\gamma \in \mathfrak{h}_{\mathbb{C}}$ such that $M_\gamma \neq 0$,*

(a) $\gamma(\alpha_i) \neq 0$ *for all* $1 \leq i \leq n$, *and*
(b) $\dim(M_\gamma) = 1$.

Proof. (a) The proof is by contradiction. Assume $\gamma(\alpha_i) = 0$. Let $\mathbb{H}A_1$ be the subalgebra of $\mathbb{H}$ generated by t_{s_i} and all $x \in \mathfrak{h}_{\mathbb{C}}^*$. Then the two-dimensional $\mathbb{H}A_1$ principal series module $M(\gamma)$ is irreducible and there is an $\mathbb{H}A_1$-module homomorphism given by

$$\begin{array}{ccc} M(\gamma) & \longrightarrow & M \\ v_\gamma & \longmapsto & m_\gamma \end{array}$$

where m_γ is a nonzero element of M_γ. Since $M(\gamma)$ is simple this is an injection and thus, M is not calibrated since $M(\gamma)$ is not calibrated. Thus $\gamma(\alpha_i) \neq 0$.

(b) The proof is by contradiction. Assume $\gamma \in \mathfrak{h}_{\mathbb{C}}$ is such that $\dim(M_\gamma) > 1$. Let m_γ be a nonzero element of M_γ. Since M is calibrated τ_i acts on m_γ as a linear combination of the action of t_{s_i} and a multiple of the identity. Since M is irreducible it follows from Proposition 2.4(b) that the action of the τ-operators must generate all of M. Thus, since $\dim(M_\gamma) > 1$, there is a sequence of τ-operators such that

$$n_\gamma = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}m_\gamma$$

is a nonzero vector in M_γ which is not a multiple of m_γ.

Assume that the sequence $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}$ is chosen so that p is minimal. Since the τ-operators in this sequence are all well defined the elements $s_{i_k}\cdots s_{i_p}\gamma$, $1 \leq k \leq p$, in the orbit $W\gamma$ correspond (under the bijection in (2.7)) to a sequence of chambers in $\mathfrak{h}_{\mathbb{R}}^*$ on the positive side of all H_α, $\alpha \in Z(\gamma)$. Each chamber in this sequence shares a face with the next chamber in the sequence. Since both n_γ and m_γ are in M_γ this is a sequence which begins and ends at the chamber C. Since the chambers are in bijection with the elements of W it follows that $s_{i_1}\cdots s_{i_p} = 1$ in W.

This means that there is some $1 < k \leq p$ such that $s_{i_1}\cdots s_{i_k}$ is not reduced and we can use the braid relations to rewrite this word as $s_{i'_1}\cdots s_{i'_{k-2}}s_{i_k}s_{i_k}$. By Proposition 2.4(e) the τ-operators also satisfy the braid relations and so

$$n_\gamma = \tau_{i'_1}\tau_{i'_2}\cdots\tau_{i'_{k-2}}\tau_{i_k}\tau_{i_k}\cdots\tau_{i_p}m_\gamma.$$

By Proposition 2.4(c), the operator $\tau_{i_k}\tau_{i_k}$ in this expression will act (on $\tau_{i_{k+1}}\cdots\tau_{i_p}m_\gamma$) by a constant $\xi \in \mathbb{C}$ and so

$$n_\gamma = \xi \cdot \tau_{i'_1}\tau_{i'_2}\cdots\tau_{i'_{k-2}}\tau_{i_{k+1}}\cdots\tau_{i_p}m_\gamma,$$

where the constant ξ is nonzero since n_γ is nonzero. But the expression

$$\xi^{-1}n_\gamma = \tau_{i'_1}\tau_{i'_2}\cdots\tau_{i'_{k-2}}\tau_{i_{k+1}}\cdots\tau_{i_p}m_\gamma,$$

is shorter than the original expression of n_γ and this contradicts the minimality of p. It follows that $\dim(M_\gamma) \leq 1$. □

Lemma 4.2. *Let M be an irreducible calibrated module. Suppose that M_γ and $M_{s_i\gamma}$ are both nonzero. Then the map $\tau_i : M_\gamma \to M_{s_i\gamma}$ is a bijection.*

Proof. By Proposition 4.1(b), $\dim(M_\gamma) = \dim(M_{s_i\gamma}) = 1$, and thus it is sufficient to show that τ_i is not the zero map. Let v_γ be a nonzero vector in M_γ. Since M is irreducible there must be a sequence of τ-operators such that

$$v_{s_i\gamma} = \tau_{i_1}\cdots\tau_{i_p}v_\gamma$$

is a nonzero element of $M_{s_i\gamma}$. Let p be minimal such that this is the case. Since $\tau_i\tau_{i_1}\cdots\tau_{i_p}v_\gamma \in M_\gamma$, it follows, as in the second paragraph of the proof of Lemma 4.1(b), that $s_i s_{i_1}\cdots s_{i_p} = 1$ in W. For notational convenience let $i_0 = i$. Let $0 \le k < p$ be maximal such that $s_{i_k}s_{i_{k+1}}\cdots s_{i_p}$ is not reduced. If $k \neq 0$ then we can use the braid relations to get

$$v_{s_i\gamma} = \tau_{i_1}\cdots\tau_{i_k}\tau_{i_k}\tau_{i'_{k+2}}\cdots\tau_{i'_p}v_\gamma.$$

Since $\tau_{i_k}\tau_{i_k}$ acts on $\tau_{i'_{k+2}}\cdots\tau_{i'_p}v_\gamma$ by a constant $\xi \in \mathbb{C}$,

$$v_{s_i\gamma} = \xi\cdot\tau_{i_1}\cdots\tau_{i_{k-1}}\tau_{i'_{k+2}}\cdots\tau_{i'_p}v_\gamma,$$

and $\xi \neq 0$ since $v_{s_i\gamma}$ is not 0. But this contradicts the minimality of p. Thus we must have that $k = 0$, $p = 1$ and

$$v_{s_i\gamma} = \tau_i v_\gamma.$$

Thus, since $v_{s_i\gamma} \neq 0$, $\tau_i \neq 0$. □

For simple roots α_i and α_j in R, let R_{ij} be the rank two root subsystem of R generated by α_i and α_j. A weight $\mu \in \mathfrak{h}_\mathbb{C}$ is *skew* if

(a) for all simple roots α_i, $1 \le i \le n$, $\mu(\alpha_i) \neq 0$, and
(b) for all pairs of simple roots α_i, α_j such that $\{\alpha \in R_{ij} \mid \mu(\alpha) = 0\} \neq \emptyset$, the set $\{\alpha \in R_{ij} \mid \mu(\alpha) = \pm c_\alpha\}$ contains more than two elements.

Condition (a) says that μ is regular for all rank 1 subsystems of R generated by simple roots. Condition (b) is an "almost regular" condition on μ with respect to rank 2 subsystems generated by simple roots. By the analysis in Section 3, the weights which appear in calibrated modules for graded Hecke algebras corresponding to rank two root systems are skew.

Recall from Section 2.3 that a pair (γ, J) is a local region if the set

$$\mathcal{F}^{(\gamma,J)} = \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset \text{ and } R(w) \cap P(\gamma) = J\}$$

is nonempty. A local region (γ, J) is *skew* if, for all $w \in \mathcal{F}^{(\gamma,J)}$, the weight $w\gamma$ is skew for all pairs α_i, α_j of simple roots in R.

The following Theorem specifies the weight space structure of an irreducible calibrated $\mathbb{H}$-module.

Theorem 4.3. *If M is an irreducible calibrated $\mathbb{H}$-module with central character $\gamma \in \mathfrak{h}_\mathbb{C}$ then there is a unique skew local region (γ, J) such that*

$$\dim(M_{w\gamma}) = \begin{cases} 1, & \text{for all } w \in \mathcal{F}^{(\gamma,J)}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.1 all nonzero generalized weight spaces of M have dimension 1 and by Lemma 4.2 all τ-operators between these weight spaces are bijections. This already guarantees that there is a unique local region (γ, J) which satisfies the condition. It only remains to show that this local region is skew.

Let $\mathbb{H}_{ij}$ be the subalgebra of $\mathbb{H}$ generated by t_{s_i}, t_{s_j} and $S(\mathfrak{h}_\mathbb{C}^*)$. Since M is calibrated as an $\mathbb{H}$-module it is calibrated as an $\mathbb{H}_{ij}$-module and so all factors of a composition series of M as an $\mathbb{H}_{ij}$-module are calibrated. Thus, by the classification in Section 3, the weights of M are skew. So (γ, J) is a skew local region. □

4.2. **Construction.** The following Proposition shows that the weight structure of calibrated representations as determined in Theorem 4.3 essentially forces the $\mathbb{H}$-action on a weight basis.

Proposition 4.4. *Let M be a calibrated $\mathbb{H}$-module and for all $\gamma \in \mathfrak{h}_\mathbb{C}$ such that $M_\gamma \neq 0$, assume that*

$$(A1) \quad \gamma(\alpha_i) \neq 0 \text{ for all } 1 \le i \le n, \quad \text{and} \quad (A2) \quad \dim(M_\gamma) = 1.$$

For each $b \in \mathfrak{h}_{\mathbb{C}}$ such that $M_b \neq 0$ let v_b be a nonzero vector in M_b. The vectors $\{v_b\}$ form a basis of M. Let $(t_{s_i})_{cb} \in \mathbb{C}$ and $b(x) \in \mathbb{C}$ be given by

$$t_{s_i} v_b = \sum_c (t_{s_i})_{cb} v_c \quad \text{and} \quad x v_b = b(x) v_b, \quad \text{for } x \in \mathfrak{h}_{\mathbb{C}}^*.$$

Then

(a) $(t_{s_i})_{bb} = \dfrac{c_{\alpha_i}}{b(\alpha_i)}$ *for all v_b in the basis,*
(b) *if $(t_{s_i})_{cb} \neq 0$ then $c = s_i b$,*
(c) $(t_{s_i})_{b,s_ib}(t_{s_i})_{s_ib,b} = 1 - (t_{s_i})_{bb}^2 = (1 + (t_{s_i})_{bb})(1 + (t_{s_i})_{s_ib,s_ib})$.

Proof. The relation

$$x t_{s_i} - t_{s_i} s_i(x) = c_{\alpha_i} \frac{x - s_i(x)}{\alpha_i}$$

forces

$$\sum_c (c(x)(t_{s_i})_{cb} - (t_{s_i})_{cb} b(s_i x)) v_c = c_{\alpha_i} \frac{b(x) - b(s_i x)}{b(\alpha_i)} v_b.$$

Comparing coefficients yields

$$\begin{aligned} c(x)(t_{s_i})_{cb} - (t_{s_i})_{cb} b(s_i x) &= 0, \qquad \text{if } b \neq c, \text{ and} \\ b(x)(t_{s_i})_{bb} - (t_{s_i})_{bb} b(s_i x) &= c_{\alpha_i} \frac{b(x) - b(s_i x)}{b(\alpha_i)}. \end{aligned}$$

These equations imply that

$$\begin{aligned} \text{if} \quad (t_{s_i})_{cb} \neq 0 \quad &\text{then} \quad b(s_i x) = c(x) \quad \text{for all } x \in \mathfrak{h}_{\mathbb{C}}^*, \text{ and} \\ (t_{s_i})_{bb} = \frac{c_{\alpha_i}}{b(\alpha_i)} \quad &\text{if } b(\alpha_i) \neq 0 \text{ and } b(x) \neq b(s_i x) \text{ for some } x \in \mathfrak{h}_{\mathbb{C}}^*. \end{aligned}$$

Thus

$$t_{s_i} v_b = (t_{s_i})_{bb} v_b + (t_{s_i})_{s_ib,b} v_{s_ib} \quad \text{with} \quad (t_{s_i})_{bb} = \frac{c_{\alpha_i}}{b(\alpha_i)}.$$

This completes the proof of (a) and (b). The relation $t_{s_i}^2 = 1$ in $\mathbb{H}$ implies that

$$\begin{aligned} v_b = t_{s_i}^2 v_b &= \left[(t_{s_i})_{bb}^2 + (t_{s_i})_{b,s_ib}(t_{s_i})_{s_ib,b}\right] v_b + \left[(t_{s_i})_{bb} + (t_{s_i})_{s_ib,s_ib}\right] (t_{s_i})_{s_ib,b} v_{s_ib} \\ &= \left[(t_{s_i})_{bb}^2 + (t_{s_i})_{b,s_ib}(t_{s_i})_{s_ib,b}\right] v_b, \end{aligned}$$

since $(t_{s_i})_{bb} + (t_{s_i})_{s_ib,s_ib} = 0$. Thus

$$(t_{s_i})_{b,s_ib}(t_{s_i})_{s_ib,b} = 1 - (t_{s_i})_{bb}^2 = (1 + (t_{s_i})_{bb})(1 + (t_{s_i})_{s_ib,s_ib}).$$

□

Theorem 4.5. *Let (γ, J) be skew and let $\mathcal{F}^{(\gamma,J)}$ index the chambers in the local region (γ, J). Define*

$$\mathbb{H}^{(\gamma,J)} = \mathbb{C}\text{-span}\{v_w \mid w \in \mathcal{F}^{(\gamma,J)}\},$$

so that the symbols v_w are a labeled basis of the vector space $\mathbb{H}^{(\gamma,J)}$. Then the following formulas make $\mathbb{H}^{(\gamma,J)}$ into an irreducible $\mathbb{H}$-module. For each $w \in \mathcal{F}^{(\gamma,J)}$,

$$\begin{aligned} x v_w &= (w\gamma)(x) v_w, && \text{for } x \in \mathfrak{h}_{\mathbb{C}}^*, \text{ and} \\ t_{s_i} v_w &= \frac{c_{\alpha_i}}{w\gamma(\alpha_i)} v_w + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right) v_{s_iw}, && \text{for } 1 \leq i \leq n, \end{aligned}$$

where we set $v_{s_iw} = 0$ if $s_iw \notin \mathcal{F}^{(\gamma,J)}$.

Proof. Since (γ, J) is skew, $(w\gamma)(\alpha_i) \neq 0$ for all $w \in \mathcal{F}^{(\gamma,J)}$ and all simple roots α_i. This implies that the coefficients in $t_{s_i}v_w$ are well defined for all i and $w \in \mathcal{F}^{(\gamma,J)}$.

By construction, the nonzero weight spaces of $\mathbb{H}^{(\gamma,J)}$ are $(\mathbb{H}^{(\gamma,J)})^{\text{gen}}_{w\gamma} = (\mathbb{H}^{(\gamma,J)})_{w\gamma}$ where $w \in \mathcal{F}^{(\gamma,J)}$. Since $\dim((\mathbb{H}^{(\gamma,J)})_{u\gamma}) = 1$ for $u \in \mathcal{F}^{(\gamma,J)}$, any proper submodule N of $\mathbb{H}^{(\gamma,J)}$ must have $N_{w\gamma} \neq 0$ and $N_{w'\gamma} = 0$ for some $w \neq w'$, with $w, w' \in \mathcal{F}^{(\gamma,J)}$. This is a contradiction to Corollary 2.5. So $\mathbb{H}^{(\gamma,J)}$ is irreducible if it is an $\mathbb{H}$-module.

It remains to show that the defining relations for $\mathbb{H}$ are satisfied. Let $w \in \mathcal{F}^{(\gamma,J)}$. Then

$$\begin{aligned}\left(s_i(x)t_{s_i} + c_{\alpha_i}\frac{x - s_ix}{\alpha_i}\right)v_w &= s_ix\left[\frac{c_{\alpha_i}}{w\gamma(\alpha_i)}v_w + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)v_{s_iw}\right] \\ &\quad + c_{\alpha_i}\frac{w\gamma(x) - w\gamma(s_ix)}{w\gamma(\alpha_i)}v_w \\ &= \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}w\gamma(x)v_w + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)s_iw\gamma(s_ix)v_{s_iw} \\ &= t_{s_i}xv_w.\end{aligned}$$

Let $w \in \mathcal{F}^{(\gamma,J)}$. Then

$$\begin{aligned}t_{s_i}^2v_w &= t_{s_i}\left[\frac{c_{\alpha_i}}{w\gamma(\alpha_i)}v_w + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)v_{s_iw}\right] \\ &= \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\left[\frac{c_{\alpha_i}}{w\gamma(\alpha_i)}v_w + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)v_{s_iw}\right] \\ &\quad + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)\left[\frac{c_{\alpha_i}}{s_iw\gamma(\alpha_i)}v_{s_iw} + \left(1 + \frac{c_{\alpha_i}}{s_iw\gamma(\alpha_i)}\right)v_w\right] \\ &= \left(\frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)^2v_w + \left(1 + \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)\left(1 - \frac{c_{\alpha_i}}{w\gamma(\alpha_i)}\right)v_w + 0 \\ &= v_w.\end{aligned}$$

Now let us check the braid relations. Write $t_{s_i} = \tau_i + d_i$ where

$$\tau_iv_w = \left(1 + \frac{c_{\alpha_i}}{(w\gamma)(\alpha_i)}\right)v_{s_iw} \qquad \text{and} \qquad d_iv_w = \frac{c_{\alpha_i}}{(w\gamma)(\alpha_i)}v_w,$$

for $w \in \mathcal{F}^{(\gamma,J)}$. Then d_i is a diagonal matrix and τ_i is a pseudo-permutation matrix, in the sense that each row and each column contains at most one nonzero entry. For a sequence $j_1, \ldots, j_p$ define a diagonal matrix $d_i^{j_1,\ldots,j_p}$ by the relation

$$d_i\tau_{j_1}\cdots\tau_{j_p} = \tau_{j_1}\cdots\tau_{j_p}d_i^{j_1,\ldots,j_p}. \tag{4.1}$$

If γ is generic then, for all $w \in W$,

$$d_i^{j_1,\ldots,j_p}v_w = \left(\frac{c_{\alpha_i}}{(s_{j_p}\cdots s_{j_1}w\gamma)(\alpha_i)}\right)v_w,$$

and all diagonal entries are nonzero, but, in general, some diagonal entries of $d_i^{j_1,\ldots,j_p}$ may be 0. Use this method to expand the expression

$$\underbrace{t_{s_i}t_{s_j}t_{s_i}\cdots}_{m_{ij}\text{ factors}} = \underbrace{(\tau_i + d_i)(\tau_j + d_j)(\tau_i + d_i)\cdots}_{m_{ij}\text{ factors}} = \sum_{z\in W}\tau_zp_z,$$

and move all the diagonal operators d_i to the right of the τ_i and obtain diagonal operators p_z. The operators τ_w are pseudo-permutation operators that may have some rows and columns without a nonzero entry. By replacing some diagonal entries of the p_z operators by 0, we may "fix the τ_z"

and replace the τ_z with operators τ_z' which have exactly one nonzero entry in each row and each column. This yields an expression

$$\underbrace{t_{s_i}t_{s_j}t_{s_i}\cdots}_{m_{ij}\ \text{factors}} = \sum_{z\in W} \tau_z' p_z'. \tag{4.2}$$

If γ is generic then the diagonal entries $(p_z')_{ww}$ of p_z' are nonzero and $(p_z')_{ww} = w\gamma(P_z')$, $w \in W$, where P_z' is a rational function in the α_i. A similar expansion gives

$$\underbrace{t_{s_j}t_{s_i}t_{s_j}\cdots}_{m_{ij}\ \text{factors}} = \sum_{z\in W} \tau_z' q_z', \tag{4.3}$$

where the q_z' are diagonal operators which, for generic γ, have diagonal entries $(q_z')_{ww} = w\gamma(Q_z')$, where Q_z' is a rational function of the α_i. As in the proof of Proposition 2.4(e), $\gamma(P_z') = \gamma(Q_z')$ for all generic γ, and so it follows that $P_z' = Q_z'$ as rational functions.

When γ is not generic the operators p_z' and q_z' may have some diagonal entries equal to zero. From the classification of representations of rank two graded Hecke algebras we know that there exists a calibrated representation of $\mathbb{H}_{ij}$ when (γ, J) is skew. This representation has a unique, up to constant multiples, basis of simultaneous eigenvectors for the action of $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, and Proposition 4.4 shows that the action on this basis is forced except for the values of the off diagonal elements of the t_{s_i}. These values depend on the normalization of the basis. Because we know that this representation exists we know that there are choices of the nonzero entries in the τ_z' such that (4.2) and (4.3) are equal. If a diagonal entry $(p_z')_{ww}$ of p_z' is nonzero then it is equal to $(w\gamma)(P_z')$ and $(p_z')_{ww} = (w\gamma)(P_z') = (w\gamma)(Q_z') = (q_z')_{ww}$, since (as shown above) $P_z' = Q_z'$. Thus it follows that nonzero contributions from the terms $\tau_z'p_z'$ and $\tau_z'q_z'$ are equal and that $t_{s_i}t_{s_j}t_{s_i}\cdots v_w$ is equal to $t_{s_j}t_{s_i}t_{s_j}\cdots v_w$. □

Remark 4.6. The action of $\mathbb{H}$ on a weight basis of $\mathbb{H}^{(\gamma,J)}$ is forced up to the freedom in Proposition 4.4(c). Our choice $(t_{s_i})_{s_ib,b} = 1 + (t_{s_i})_{bb}$ in Theorem 4.5 and the alternative choice $(t_{s_i})_{s_ib,b} = 1 + (t_{s_i})_{s_ib,s_ib}$ yield isomorphic modules. The change of basis $v_b' = \dfrac{1}{(1+(t_{s_i})_{bb})}v_b$ provides the isomorphism.

We summarize the results of this section with the following corollary of Theorem 4.3 and the construction in Theorem 4.5.

Theorem 4.7. *Let M be an irreducible calibrated $\mathbb{H}$-module. Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ be (a fixed choice of) the central character of M and let $J = R(w) \cap P(\gamma)$ for any $w \in W$ such that $M_{w\gamma} \neq 0$. Then (γ, J) is skew and $M \simeq \mathbb{H}^{(\gamma,J)}$, where $\mathbb{H}^{(\gamma,J)}$ is the module defined in Theorem 4.5.*

5. Combinatorics of Local Regions

When W is a crystallographic reflection group two conjectures were stated in [Ra3, (1.3) and (1.11)], the first giving necessary and sufficient conditions for $\mathcal{F}^{(\gamma,J)}$ (as defined in (2.20)) to be nonempty when γ is dominant and the second determining the form of $\mathcal{F}^{(\gamma,J)}$ as an interval in the weak Bruhat order when γ is dominant and integral. Loszoncy [Lo] proved the second conjecture (Theorem 5.2 below). His theorem implies the nonemptiness conjecture of [Ra3] under the additional assumption that γ is integral. Here we review Loszoncy's proof and prove the nonemptiness conjecture in full generality. We give an example (Example 5.4) to show that integrality is necessary in Theorem 5.2. Finally, we provide Example 5.7, which shows that one cannot expect analogous statements to hold when W is noncrystallographic.

Let R be the root system of a finite real reflection group W and fix a set $R^+ = \{\alpha > 0\}$ of positive roots in R. A set of positive roots S is *closed* if it satisfies the condition

$$\text{If } \alpha, \beta \in S \text{ and } a, b > 0 \text{ are such that } a\alpha + b\beta \in R^+ \text{ then } a\alpha + b\beta \in S.$$

The following theorem characterizes the sets which appear as inversion sets of elements of W. Recall that $R(w)$ denotes the inversion set of w, see equation (2.4). This result is in [Bj, Proposition 2], but is stated there without proof and we are not aware of a published proof. The following proof was shown to us by J. Stembridge and appears in the thesis of D. Waugh [Wg].

Theorem 5.1. *Let W be a real reflection group. A set of positive roots S is equal to $R(w)$ for some element $w \in W$ if and only if S is closed and $S^c = R^+ \backslash S$ is closed.*

Proof. $\Longrightarrow$: Let $w \in W$ and suppose that $\alpha, \beta \in R(w)$ and $a\alpha + b\beta$ is a positive root. Then $w(a\alpha + b\beta) = a(w\alpha) + b(w\beta)$ is a negative root since $w\alpha$ and $w\beta$ are both negative roots. So $R(w)$ is closed. Similarly one shows that $R(w)^c$ is closed.
$\Longleftarrow$: Assume that S is closed and that S^c is closed. We will construct w such that $R(w) = S$ by finding a reduced word $w = s_{i_1} \cdots s_{i_k}$ for w. This is done by induction on the size of S, with the induction step being the combination of the two steps below.

Step 1: S contains a simple root.

Let α be a root of minimal height in S and assume that $\alpha = \sum_i c_{\alpha_i}\alpha_i$, $c_{\alpha_i} \in \mathbb{R}_{\geq 0}$, is not simple. Then

$$\langle \alpha, \alpha_i \rangle > 0 \quad \text{for some } i, \qquad \text{since} \qquad 0 < \langle \alpha, \alpha \rangle = \sum_{i=1}^{n} c_{\alpha_i} \langle \alpha, \alpha_i \rangle.$$

Since α is not simple, $\alpha \neq \alpha_i$, and so both $s_{\alpha_i}\alpha$ and α_i are positive roots. Since $s_{\alpha_i}\alpha = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$ and α_i both have lower height than α they must both be in S^c. But then the equation

$$\alpha = s_{\alpha_i}\alpha + \langle \alpha, \alpha_i^\vee \rangle \alpha_i$$

contradicts the assumption that S^c is closed. So α is simple.

Step 2: Let α_{i_1} be a simple root in S and let $S_1 = s_{i_1}(S \setminus \{\alpha_{i_1}\})$.
Claim: S_1 is closed and S_1^c is closed.

Let $\alpha, \beta \in S_1$ and assume that $a\alpha + b\beta$ is a positive root. Then

$$\begin{aligned} s_{i_1}(a\alpha + b\beta) = as_{i_1}\alpha + bs_{i_1}\beta \in S \quad &\text{and } a\alpha + b\beta \in S_1, \text{ or} \\ as_{i_1}\alpha + bs_{i_1}\beta = \alpha_{i_1} \quad &\text{and } a\alpha + b\beta = -\alpha_{i_1}. \end{aligned}$$

The second is impossible since $s_{i_1}\alpha_{i_1}$ is not a positive root. So $a\alpha + b\beta \in S_1$ and S_1 is closed.

Let $\alpha, \beta \in S_1^c$ and suppose that $a\alpha + b\beta$ is a positive root. Since $s_{i_1}\alpha$ and $s_{i_1}\beta$ are not in S, $s_{i_1}(a\alpha + b\beta) \notin S$. So $a\alpha + b\beta \notin S_1$. Thus S_1^c is closed. □

An element $\gamma \in \mathfrak{h}_{\mathbb{C}}$ is *dominant* (resp. *integral*) if $\gamma(\alpha_i) \in \mathbb{R}_{\geq 0}$ (resp. $\gamma(\alpha_i) \in \mathbb{Z}$) for all simple roots α_i. The *closure* $\overline{S}$ of a set of positive roots S is the smallest closed set of positive roots containing S.

Theorem 5.2. *Let W be a crystallographic reflection group and let R be the crystallographic root system of W. Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ be dominant and integral and set*

$$Z(\gamma) = \{\alpha > 0 \mid \langle \gamma, \alpha \rangle = 0\} \qquad \textit{and} \qquad P(\gamma) = \{\alpha > 0 \mid \langle \gamma, \alpha \rangle = 1\}.$$

Let $J \subseteq P(\gamma)$ be such that

$$\textit{if } \beta \in J, \ \alpha \in Z(\gamma) \textit{ and } \beta - \alpha \in R^+ \quad \textit{then} \quad \beta - \alpha \in J,$$

and set

$$\mathcal{F}^{(\gamma, J)} = \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset, \ R(w) \cap P(\gamma) = J\}.$$

Then there exist elements $w_{\min}, w_{\max} \in W$ *such that*

$$R(w_{\min}) = \overline{J}, \qquad R(w_{\max}) = \overline{(P(\gamma)\backslash J) \cup Z(\gamma)}^c, \qquad \text{and} \qquad \mathcal{F}^{(\gamma,J)} = [w_{\min}, w_{\max}],$$

where K^c *denotes the complement of* K *in* R^+ *and* $[w_{\min}, w_{\max}]$ *denotes the interval between* $w_{\min}$ *and* $w_{\max}$ *in the weak Bruhat order.*

Proof. By Theorem 5.1, the element $w_{\min} \in W$ will exist if $\bar{J}^c$ is closed. Assume that $\beta = \beta_1 + \beta_2$ where $\beta \in \bar{J}$, $\beta_1, \beta_2 \in R^+$. We must show that $\beta_1 \in \bar{J}$ or $\beta_2 \in \bar{J}$. Since $\beta \in \bar{J}$,

$$\beta = \delta_1 + \cdots + \delta_m, \qquad \text{with} \qquad \delta_i \in J.$$

We will decompose $\beta = \delta_1 + \cdots + \delta_m$ into two pieces $\beta_1 = \delta_1 + \cdots + \delta_k + \eta_1$ and $\beta_2 = \eta_2 + \delta_{k+2} + \cdots + \delta_m$, via the following inductive procedure. Since

$$0 < \langle \beta_1 + \beta_2, \beta_1 + \beta_2 \rangle = \sum_i \langle \beta_1 + \beta_2, \delta_i \rangle, \quad \text{then} \quad \langle \beta_1 + \beta_2, \delta_j \rangle > 0 \quad \text{for some } j.$$

By reindexing the δ_i we can assume that $j = 1$. Thus $\langle \beta_1, \delta_1 \rangle > 0$ or $\langle \beta_2, \delta_1 \rangle > 0$ and we may assume that $\langle \beta_1, \delta_1 \rangle > 0$. Since $s_{\delta_1}\beta_1 = \beta_1 - \langle \beta_1, \delta_1^\vee \rangle \delta_1$ is a root and R is crystallographic, $\beta_1 - \delta_1$ is also a root. If $\beta_1 - \delta_1$ is a negative root then

$$\beta_1 = \beta_1 \qquad \text{and} \qquad \beta = (\delta_1 - \beta_1) + \delta_2 + \cdots + \delta_m,$$

gives the desired decomposition. If $\beta_1 - \delta_1 \in R^+$ then

$$\beta_1 + \beta_2 = \delta_1 + ((\beta_1 - \delta_1) + \beta_2) \qquad \text{and} \qquad (\beta_1 - \delta_1) + \beta_2 = \delta_2 + \cdots + \delta_m,$$

and so we may inductively apply this decomposition procedure on $\beta' = (\beta_1 - \delta_1) + \beta_2 = \delta_2 + \ldots + \delta_m$.

In this way we conclude that, after possible reindexing of the δ_i, either

$$\beta_1 = \delta_1 + \cdots + \delta_k \qquad \text{and} \qquad \beta_2 = \delta_{k+1} + \cdots + \delta_m,$$

or

$$\beta_1 = \delta_1 + \cdots + \delta_k + \eta_1 \qquad \text{and} \qquad \beta_2 = \eta_2 + \delta_{k+2} + \cdots + \delta_m,$$

where η_1 and η_2 are positive roots such that $\eta_1 + \eta_2 = \delta_{k+1}$. In the first case it is immediate that $\beta_1, \beta_2 \in \bar{J}$. In the second case $\langle \gamma, \delta_{k+1} \rangle = \langle \gamma, \eta_1 + \eta_2 \rangle = 1$, and so $\langle \gamma, \eta_1 \rangle \leq 1$ and $\langle \gamma, \eta_2 \rangle \leq 1$. Thus, since γ is dominant and integral, one of η_1, η_2 is in $Z(\gamma)$ and the other is in $P(\gamma)$. If $\eta_1 \in Z(\gamma)$, $\eta_2 = \delta_{k+1} - \eta_1$ and the condition on J implies that $\eta_2 \in J$. Similarly, if $\eta_2 \in Z(\gamma)$ then $\eta_1 \in J$. Thus $\beta_1 \in \bar{J}$ or $\beta_2 \in \bar{J}$. So $\bar{J}^c$ is closed. Since $\bar{J}$ is closed and $\bar{J}^c$ is closed, Theorem 5.1 shows that there is an element $w_{\min} \in W$ such that $R(w_{\min}) = \bar{J}$.

The same method can be used to establish the existence of $w_{\max}$: one must show that $\overline{(P(\gamma)\backslash J) \cup Z(\gamma)}^c$ is closed and this is accomplished by similar arguments.

By the definition of $\mathcal{F}^{(\gamma,J)}$ an element $w \in W$ is in $\mathcal{F}^{(\gamma,J)}$ if

$$\overline{J} \subseteq R(w) \subseteq \overline{(P(\gamma)\backslash J) \cup Z(\gamma)}^c.$$

Since the weak Bruhat order is the order determined by inclusions of $R(w)$ [Bj, Proposition 3] the result is a consequence of the existence of the elements $w_{\min}$ and $w_{\max}$. □

Remark 5.3. An alternative way to establish the existence of $w_{\max}$ in the proof of Theorem 5.2 is to use the *conjugation* involution

$$(5.1) \qquad \begin{matrix} \mathcal{F}^{(\gamma,J)} & \overset{1-1}{\longleftrightarrow} & \mathcal{F}^{(\gamma,J)'} \\ w & \longleftrightarrow & wu^{-1} \end{matrix} \qquad \text{where} \qquad (\gamma, J)' = (-u\gamma, -u(P(\gamma)\backslash J)),$$

where u is the minimal length coset representative of $w_0 W_\gamma$ and w_0 is the longest element of W. The fact that this is a well defined involution is proved in [Ra3, (1.7)]. This involution takes $w_{\max}$ for $\mathcal{F}^{(\gamma,J)}$ to $w_{\min}$ for $\mathcal{F}^{(\gamma,J)'}$. In terms of the weak Bruhat order, the structure of the interval $\mathcal{F}^{(\gamma,J)'}$ is the same as the structure of the interval $\mathcal{F}^{(\gamma,J)}$ but with all relations reversed.

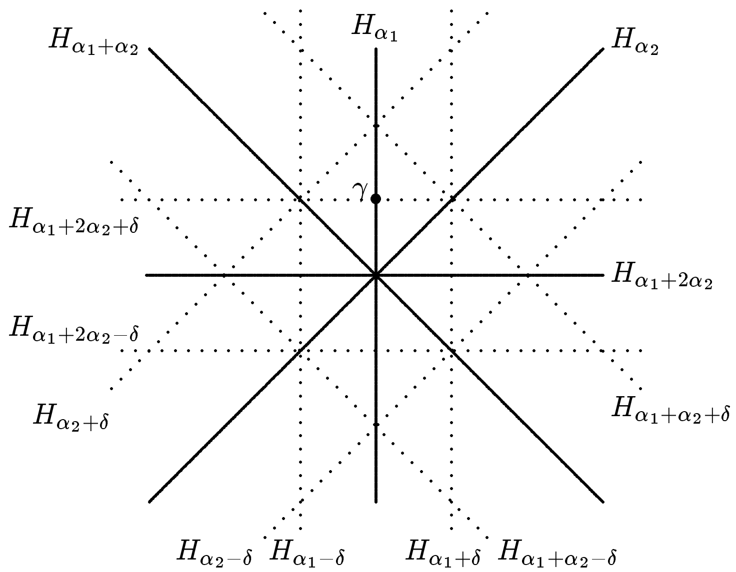


FIGURE 3. Hyperplanes and a nonintegral weight for C_2

Example 5.4. The integrality of γ is necessary in Theorem 5.2. Let $W = I_2(4) = WC_2$ be the dihedral group of order 8 (the Weyl group of type C_2). The root system for type C_2 is determined by simple roots

$$\alpha_1 = 2\varepsilon_1 \qquad \text{and} \qquad \alpha_2 = \varepsilon_2 - \varepsilon_1$$

where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis of $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^2$. Let $c_1 = c_2 = 1$ be the parameters for $\mathbb{H}$. If $\gamma = (1/2)\varepsilon_2$ (see Figure 3) then $Z(\gamma) = \{\alpha_1\}$, $P(\gamma) = \{\alpha_1 + 2\alpha_2\}$, and γ is dominant but $\gamma(\alpha_2)$ is not integral. The set $J = P(\gamma)$ satisfies the condition in Theorem 5.2, but $\bar{J} = J$ is not an inversion set for any $w \in W$ since $\bar{J}^c$ is not closed.

The following method of reducing to the integral root subsystem of a weight is standard in the theory of highest weight modules for finite dimensional complex semisimple Lie algebras, see [Ja]. This method turns out to be an efficient tool for reducing the nonemptiness conjecture of [Ra3] to the statement in Theorem 5.2.

Let $R_{[\gamma]} = \{\alpha \in R \mid \langle \gamma, \alpha^\vee \rangle \in \mathbb{Z}\}$. For any $\alpha, \beta \in R_{[\gamma]}$,

$$\langle \gamma, (s_\alpha \beta)^\vee \rangle = \langle s_\alpha \gamma, \beta^\vee \rangle = \langle \gamma, \beta^\vee \rangle - \langle \gamma, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle \in \mathbb{Z},$$

and so $R_{[\gamma]}$ is a root system with Weyl group $W_{[\gamma]} = \langle s_\alpha \mid \alpha \in R_{[\gamma]} \rangle \subseteq W$. If $\tau \in W_{[\gamma]}$ then the $R_{[\gamma]}$-inversion set of τ is

$$R_{[\gamma]}(\tau) = \{\alpha > 0 \mid \tau\alpha < 0,\ \alpha \in R_{[\gamma]}\} = R(\tau) \cap R_{[\gamma]}.$$

Theorem 5.5. *Let W be a crystallographic reflection group and let R be the crystallographic root system of W. Let $\gamma \in \mathfrak{h}_{\mathbb{C}}$ such that* $\mathrm{Re}(\gamma)$ *is dominant and set*

$$Z(\gamma) = \{\alpha > 0 \mid \langle \gamma, \alpha \rangle = 0\} \qquad \textit{and} \qquad P(\gamma) = \{\alpha > 0 \mid \langle \gamma, \alpha \rangle = 1\}.$$

Let $J \subseteq P(\gamma)$ be such that

$$\textit{if } \beta \in J,\ \alpha \in Z(\gamma) \textit{ and } \beta - \alpha \in R^+ \quad \textit{then} \quad \beta - \alpha \in J.$$

Then $\mathcal{F}^{(\gamma, J)} = \{w \in W \mid R(w) \cap Z(\gamma),\ R(w) \cap P(\gamma) = J\}$ is nonempty.

Proof. Since γ is dominant and integral for the root system $R_{[\gamma]}$, it follows from Theorem 5.2 that there is an element w in $W_{[\gamma]}$ such that

$$R_{[\gamma]}(w) \cap Z(\gamma) = \emptyset \qquad \text{and} \qquad R_{[\gamma]}(w) \cap P(\gamma) = J,$$

where $R_{[\gamma]}(w) = \{\alpha \in R_{[\gamma]} \mid \alpha > 0,\ w\alpha < 0\}$. Usually $R(w)$ is strictly larger than $R_{[\gamma]}(w)$ but it is still true that

$$R(w) \cap Z(\gamma) = \emptyset \qquad \text{and} \qquad R(w) \cap P(\gamma) = J,$$

since all roots of $P(\gamma)$ and $Z(\gamma)$ are in $R_{[\gamma]}$. So $w \in \mathcal{F}^{(\gamma,J)}$. □

When W is crystallographic we can use the method of the proof of Theorem 5.5 in combination with the result of Theorem 5.2 to give a precise description of the set $\mathcal{F}^{(\gamma,J)}$ for all central characters $\gamma \in \mathfrak{h}_{\mathbb{C}}$. By choosing γ appropriately in its W-orbit we may assume that $\mathrm{Re}(\gamma)$ is dominant.

Define

$$W^{[\gamma]} = \{\sigma \in W \mid R(\sigma) \cap R_{[\gamma]} = \emptyset\}.$$

Each $w \in W$ has a unique expression

$$w = \sigma\tau \quad \text{with} \quad \sigma \in W^{[\gamma]}, \tau \in W_{[\gamma]}, \text{ and } R(w) \cap R_{[\gamma]} = R(\tau) \cap R_{[\gamma]} = R_{[\gamma]}(\tau).$$

In this way the elements of $W^{[\gamma]}$ are coset representatives of the cosets in $W/W_{[\gamma]}$.

Since $P(\gamma) \subseteq R_{[\gamma]}$ and $Z(\gamma) \subseteq R_{[\gamma]}$ it follows that

$$\mathcal{F}^{(\gamma,J)} = \{\sigma\tau \in W \mid \sigma \in W^{[\gamma]}, \tau \in \mathcal{F}_{[\gamma]}^{(\gamma,J)}\}, \qquad \text{where} \tag{5.2}$$

$$\mathcal{F}_{[\gamma]}^{(\gamma,J)} = \{\tau \in W_{[\gamma]} \mid R_{\gamma}(\tau) \cap P(\gamma) = J, \quad R(w) \cap Z(\gamma) = \emptyset\}. \tag{5.3}$$

Since $\mathcal{F}^{(\gamma,J)} = \mathcal{F}^{(\mathrm{Re}(\gamma),J)}$ and γ is dominant and integral for the root system $R_{[\gamma]}$, Theorem 5.2 has the following corollary.

Corollary 5.6. *With notations and assumptions as in* Theorem 5.5

$$\mathcal{F}^{(\gamma,J)} = \mathcal{F}_{[\gamma]}^{(\gamma,J)} = W^{[\gamma]} \cdot [\tau_{\max}, \tau_{\min}],$$

where, $\mathcal{F}_{[\gamma]}^{(\gamma,J)}$ *is as in* (5.3) *and* $\tau_{\max}$ *and* $\tau_{\min}$ *in* $W_{[\gamma]}$ *are determined by* $R_{[\gamma]}(\tau_{\max}) = \overline{J}$ *and* $R_{[\gamma]}(\tau_{\min}) = \overline{(P(\gamma)\backslash J) \cup Z(\gamma)}^{c}$, *where the complement is taken in the set of positive roots of* $R_{[\gamma]}$.

This refined version of Theorem 5.2 is reminiscent of the reduction to real central character given in [BM2].

The following example shows that Theorem 5.5 does not naturally extend to noncrystallographic reflection groups. Note that such a generalization necessarily involves modifying the closure condition on J to be

$$\text{if } \beta \in J,\ \alpha \in Z(\gamma),\ a \in \mathbb{R}_{>0}, \text{ and } \beta - a\alpha \in R^+ \quad \text{then} \quad \beta - a\alpha \in J.$$

Example 5.7. Let $W = I_2(n)$ be the dihedral group of order $2n$, n even, with root system chosen as in Section 3 (so all roots are the same length). Let γ be such that $Z(\gamma) = \{\beta_0\}$ and $P(\gamma) = \{\beta_{n/4}, \beta_{n/2}, \beta_{3n/4}\}$ (this γ is an example of γ_q in Table 1). Then the subset $J = \{\beta_{n/4}, \beta_{3n/4}\} \subseteq P(\gamma)$ satisfies the generalized closure condition above since $\beta_{n/2}$ cannot be written as $\beta_{n/4} - a\beta_0$ for any $a \in \mathbb{R}_{>0}$. However, $\mathcal{F}^{(\gamma,J)} = \emptyset$ since there are no chambers which are on the positive side of both H_{β_0} and $H_{\beta_{n/2}}$ and on the negative side of both $H_{\beta_{n/4}}$ and $H_{\beta_{3n/4}}$.

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Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085
E-mail address: krilcath@isu.edu

Department of Mathematics, University of Wisconsin, Madison, WI 53706
E-mail address: ram@math.wisc.edu