PROJECTIVE SCHUR FUNCTIONS AS A BISPHERICAL FUNCTIONS ON CERTAIN HOMOGENEOUS SUPERSPACES

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ABSTRACT. I show that the projective Schur functions may be interpreted as bispherical functions of either the triple $(\mathfrak{q}(n), \mathfrak{q}(n)\oplus\mathfrak{q}(n), \mathfrak{q}(n))$, where $\mathfrak{q}(n)$ is the "odd" (queer) analog of the general linear Lie algebra, or the triple $(\mathfrak{pe}(n), \mathfrak{gl}(n|n), \mathfrak{pe}(n))$, where $\mathfrak{pe}(n)$ is the periplectic Lie superalgebra which preserves the nondegenerate odd bilinear form (either symmetric or skew-symmetric). Making use of this interpretation I characterize projective Schur functions as common eigenfunctions of an algebra of differential operators.

INTRODUCTION

1.1. In [Sch] I. Schur introduced projective Schur functions as characteristics of projective representations of symmetric groups. In [Se1] I showed that these characteristics are actually characters of tensor representations of Lie superalgebra $\mathfrak{q}(n)$.

In [St] Stembridge interpreted projective Schur functions as characteristics of of spherical functions of a certain twisted Gelfand pair. Here I will demonstrate that these Stembridge's characteristics are precisely bispherical functions of the triple $(\mathfrak{pe}(n), \mathfrak{gl}(n|n), \mathfrak{pe}(n))$, where $\mathfrak{pe}(n)$ can be embedded into $\mathfrak{gl}(n|n)$ in two different ways (corresponding to interpretation of $\mathfrak{pe}(n)$ as the algebra preserving a symmetric odd bilinear form or a skew-symmetric one).

One obtains one more realization of projective Schur functions if one considers bispherical functions of the triple $(\mathfrak{q}(n), \mathfrak{q}(n)\oplus\mathfrak{q}(n), \mathfrak{q}(n))$, where $\mathfrak{q}(n)$ can be embedded into $\mathfrak{q}(n)\oplus\mathfrak{q}(n)$ by one of the two ways: either as the diagonal or as a "twisted diagonal".

Both ways to realize projective Schur functions allow one to construct an algebra of differential operators for which the projective Schur functions are eigenfunctions. This algebra appears as the algebra of radial parts of Laplace operators for Lie superalgebras of series $\mathfrak{gl}$ and $\mathfrak{q}$.

1.2. Differential operators and projective Schur functions. Let $I = \{1, \dots, n\}$, V an n-dimensional vector space, $\{e_i\}_{i\in I}$ a basis of V, and $\{\varepsilon_i\}$ the dual basis of V^*. If $l \in V^*$, than e^l denotes a homomorphism $S(V) \longrightarrow \mathbb{C}$, where $\mathbb{C}$ is the ground field of complex numbers, and $S(V)$ is the symmetric algebra of V. Recall that $S(V)^*$ can be identified with the algebra of formal power series in n indeterminates.

On $S(V)^*$, define a family of differential operators. Set $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ for $i, j \in I$, $i \neq j$ and set

$$\partial_i^{(1)}(e^l) = l(e_i)e^l = \partial_i(e^l) \text{ for any } l \in V^*,$$

set further

$$\partial_i^{(k)} = \begin{cases} \partial_i\partial_i^{(k-1)} + \sum\limits_{j\neq i} \dfrac{2}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}}(\partial_i^{(k-1)} - \partial_j^{(k-1)}) & \text{for } k \text{ odd,} \\ (\partial_i - 1)\partial_i^{(k-1)} + \sum\limits_{j\neq i} \left(\dfrac{2}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}}\partial_i^{(k-1)} - \dfrac{2e^{\varepsilon_{ij}}}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}}\partial_i^{(k-1)}\right) & \text{for } k \text{ even.} \end{cases} \quad (1.2.1)$$

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Finally, for k odd, set

$$\Omega_k = \sum_{k=1}^{n} \partial_i^{(k)}.$$

It is not difficult to verify directly that

$$\begin{aligned}\Omega_3 = \sum_{1}^{n} \partial_i^3 + \sum_{i<j} \frac{6}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}}(\partial_i^2 - \partial_j^2) - \sum \frac{6}{(e^{\frac{1}{2}\varepsilon_{ij}} + e^{\frac{1}{2}\varepsilon_{ji}})^2}(\partial_i + \partial_j) \\ +24 \sum_{i \notin \{j,k\}} \frac{1}{(e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}})(e^{\varepsilon_{ik}} - e^{\varepsilon_{ki}})}\partial_i - \left(\sum \partial_i\right)^2,\end{aligned} \tag{1.2.2}$$

where $\{j,k\} \subset I$ is a two-element subset.

Introduce new indeterminates: $x_i = e^{\varepsilon_i}$. Then $\partial_i = x_i \frac{\partial}{\partial x_i}$ and

$$\partial_i^{(k)} = \begin{cases} \partial_i \partial_i^{(k-1)} + \sum_{j \neq i} \frac{2x_i x_j}{x_i^2 - x_j^2}\left(\partial_i^{(k-1)} - \partial_j^{(k-1)}\right) & \text{for } k \text{ odd,} \\ (\partial_i - 1)\partial_i^{(k-1)} + \sum_{j \neq i}\left(\frac{2x_i x_j}{x_i^2 - x_j^2}\partial_i^{(k-1)} - \frac{2x_i^2}{x_i^2 - x_j^2}\partial_j^{(k-1)}\right) & \text{for } k \text{ even.}\end{cases} \tag{1.2.3}$$

We have

$$\begin{aligned}\Omega_3 = \sum \partial_i^3 + 6\sum_{i<j} \frac{x_i x_j}{x_i^2 - x_j^2}(\partial_i^2 - \partial_j^2) - 6\sum_{i<j} \frac{x_i x_j}{(x_i + x_j)^2}(\partial_i + \partial_j) \\ +24 \sum_{i<j<k} \frac{x_i x_j x_k}{(x_i^2 - x_j^2)(x_i^2 - x_k^2)}\left(x_i(x_j^2 - x_k^2)\partial_i - x_j(x_i^2 - x_k^2)\partial_j + x_k(x_i^2 - x_j^2)\partial_k\right) \\ -\left(\sum \partial_i\right)^2.\end{aligned} \tag{1.2.4}$$

For $i = 1, \dots, n$ and any k, define differential operators $\tilde{\partial}_i^{(k)}$ and $\tilde{\partial}_{\bar{i}}^{(k)}$ by setting $\tilde{\partial}_i^{(1)} = \tilde{\partial}_i = \tilde{\partial}_{\bar{i}}^{(1)} = \tilde{\partial}_{\bar{i}} = \partial_i$ and

$$\begin{aligned}\tilde{\partial}_i^{(k)} &= \tilde{\partial}_i \tilde{\partial}_i^{(k-1)} + \sum_{j \neq i} \frac{e^{\frac{1}{2}\varepsilon_{ij}}}{e^{\frac{1}{2}\varepsilon_{ij}} - e^{\frac{1}{2}\varepsilon_{ji}}}(\tilde{\partial}_i^{(k-1)} - \tilde{\partial}_j^{(k-1)}) \\ &- \sum_{j \neq i} \frac{e^{\frac{1}{2}\varepsilon_{ij}}}{e^{\frac{1}{2}\varepsilon_{ij}} + e^{\frac{1}{2}\varepsilon_{ji}}}(\tilde{\partial}_i^{(k-1)} + \tilde{\partial}_{\bar{j}}^{(k-1)}),\end{aligned} \tag{1.2.5}$$

$$\begin{aligned}\tilde{\partial}_{\bar{i}}^{(k)} &= -\tilde{\partial}_i \tilde{\partial}_{\bar{i}}^{(k-1)} - \sum_{j \neq i} \frac{e^{\frac{1}{2}\varepsilon_{ij}}}{e^{\frac{1}{2}\varepsilon_{ij}} - e^{\frac{1}{2}\varepsilon_{ji}}}(\tilde{\partial}_{\bar{i}}^{(k-1)} - \tilde{\partial}_{\bar{j}}^{(k-1)}) \\ &+ \sum \frac{e^{\frac{1}{2}\varepsilon_{ij}}}{e^{\frac{1}{2}\varepsilon_{ij}} + e^{\frac{1}{2}\varepsilon_{ji}}}(\tilde{\partial}_{\bar{i}}^{(k-1)} + \tilde{\partial}_j^{(k-1)}).\end{aligned} \tag{1.2.6}$$

Set further

$$\widetilde{\Omega}_k = \sum_{i=1}^{n}(\tilde{\partial}_i^{(k)} + \partial_{\bar{i}}^{(k)}).$$

1.2.1. Lemma . *The algebra generated by operators Ω_k for $k = 1, 3, 5, \dots$ coincides with the algebra generated by operators $\widetilde{\Omega}_k$ for $k = 1, 2, 3, 4, \dots$.*

1.2.2. Let $x_1, \dots, x_n, t$ be indeterminates over $\mathbb{C}$. Introduce polynomials $q_k(x_1, \dots, x_n)$ from equation

$$\sum_{k=0}^{\infty} q_k(x_1, \dots, x_n)t^k = \frac{\prod(1 + x_i t)}{\prod(1 - x_i t)}. \tag{1.2.7}$$

Further, set

$$Q_{k,l} = q_k q_l + 2\sum_{p=1}^{l} q_{k+p} q_{l-p}.$$

We see that $Q_{(k,0)} = q_k$ and $Q_{k,l} = -Q_{l,k}$ for $k + l > 0$. If $\lambda = (\lambda_1, \dots, \lambda_n)$ is a strict partition, then

$$Q_\lambda(x_1, \dots, x_n) = \begin{cases} Pf(Q_{\lambda_i \lambda_j}) & \text{if } n \text{ is even,} \\ Q_{(\lambda,0)} & \text{if } n \text{ is odd.} \end{cases} \tag{1.2.8}$$

1.2.3. Lemma . *Set* $\delta = \prod_{i<j} \left(\frac{e^{\frac{1}{2}\varepsilon_{ij}} + e^{\frac{1}{2}\varepsilon_{ji}}}{e^{\frac{1}{2}\varepsilon_{ij}} - e^{\frac{1}{2}\varepsilon_{ji}}} \right)$, *then*

i) $\delta^{-1}\Omega_3\delta = \sum_1^n \partial_i^3 - (\sum \partial_i)^2$.

ii) *The polynomials* Q_λ *are common eigenfunctions of the operators* Ω_k *for* $k = 1, 3, 5, \dots$.

iii) *Let* $P(x_1, \dots, x_n)$ *be polynomial symmetric with respect to* $x_1, \dots, x_n$ *and such that after substitution* $x_i = t$, $x_j = -t$ *it becomes independent of* t. *If* P *is an eigenfunction of all the operators* Ω_k, $k = 1, 3, 5, \dots$, *then, up to a scalar multiple,* P *coinsides with one of the* Q_λ.

1.3. Bispherical functions. Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra. Its enveloping algebra $U(\mathfrak{g})$ possesses a canonical antiautomorphism $t : u \mapsto {}^t u$ which extends the *principal antiautomorphism* of $\mathfrak{g}$ given by the formula $t(x) = -x$ for any $x \in \mathfrak{g}$ as follows (for brevity, I always just write $(-1)^u$ instead of $(-1)^{p(u)}$):

$${}^t(uv) = (-1)^{uv}\, {}^t(v){}^t(u)$$

The left and right coregular representations of $U(\mathfrak{g})$ are defined for any $l \in U(\mathfrak{g})^*$ and $v, u \in U(\mathfrak{g})$ by the formulas

$$(L^*(u)l)(v) = (-1)^{ul} l(u^t \cdot v) \text{ and } (R^*(u)l)(v) = (-1)^{u(l+v)} l(vu) \ .$$

Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be subalgebras of $\mathfrak{g}$. A functional $l \in U(\mathfrak{g})^*$ is called *two-side invariant* if

$$l(x_1 u) = l(u x_2) = 0 \text{ for any } x_1 \in \mathfrak{b}_1,\ x_2 \in \mathfrak{b}_2 \text{ and } u \in U(\mathfrak{g}).$$

Let V be a $\mathfrak{g}$-module containing a nonzero $\mathfrak{b}_2$-invariant vector $v \in V$; suppose also that there exists a $\mathfrak{b}_1$-invariant vector $v^* \in V^*$. Then we call the matrix coefficient $\Theta(v^*, v) \in U(\mathfrak{g})^*$ defined by the formula

$$\Theta(v^*, v)(u) = (-1)^{uv} v^*(uv) \quad \text{for} \quad u \in U(\mathfrak{g})$$

a *bispherical function associated with the triple* (V, v^*, v).

Observe that if $z \in Z(\mathfrak{g})$, then $L^*(z)l$ is two-side invariant if so is l. Therefore, on the space of invariant functionals, every $z \in Z(\mathfrak{g})$ determines a linear operator $\Omega_{(z)} : l \mapsto L^*(z)l$.

1.4. Let us endow $U(\mathfrak{g})$ with a $\mathfrak{g}$-module structure with respect to the action ([Se4], [G])

$$x * u = xu - (-1)^{x(u+\bar{1})} ux. \tag{1.4.1}$$

It is easy to verify that for a finite dimensional $\mathfrak{g}$-module V the functional $u \mapsto \mathrm{tr}_V(u)$ is invariant with respect to this action.

1.4.1. In $\mathfrak{g}\oplus\mathfrak{g}$, consider two subalgebras:

$$\mathfrak{g}_1=\{(x,(-1)^x x)\mid x\in\mathfrak{g}\} \text{ and } \mathfrak{g}_2=\{(x,x)\mid x\in\mathfrak{g}\}.$$

Lemma . *The algebra of functionals on $U(\mathfrak{g}\oplus\mathfrak{g})$ biinvariant with respect to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is isomorphic to the algebra of functionals on $U(\mathfrak{g})$ invariant with respect to the action* (1.4.1).

1.4.2. Let now $I=I_{\bar 0}\cup I_{\bar 1}=\{1,\dots,n\}\cup\{\bar 1,\dots,\bar n\}$ be the union of the "even" and "odd" indices. Let $\dim V=(n|n)$ and $\{e_i\}_i\in I$ a basis of V such that the parity of each vector of the basis is the same as that of its index. Assume that $\bar{\bar{i}}=i$ and define the odd operator $\Pi\in\mathfrak{gl}(V)$ by setting

$$\Pi(e_i)=(-1)^i e_{\bar i}, \text{ for any } i\in I.$$

Let

$$\mathfrak{q}(n)=\{X\in\mathfrak{gl}(V)\mid [X,\Pi]=0\}. \tag{1.4.2}$$

It is easy to verify that $\mathfrak{q}(n)=Span(e_{ij},f_{ij}\mid i,j\in I_{\bar 0})$, where

$$e_{ij}=e_i\otimes e_j+e_{\bar i}\otimes e_{\bar j},\quad f_{ij}=e_i\otimes e_{\bar j}^*+e_{\bar i}\otimes e_j^*, \text{ for any } i,j\in I_{\bar 0}$$

and where $\{e_i^*\}_{i\in I}$ is the left dual of the basis $\{e_i\}_{i\in I}$; we have also identified $\mathrm{End}(V)$ with $V\otimes V^*$.

Set $\mathfrak{g}=\mathfrak{q}(n)\oplus\mathfrak{q}(n)$ and select $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as at the beginning of sec. 1.4. Let $\sigma:\mathfrak{q}(n)\longrightarrow\mathfrak{g}$ be the embedding into the first summand, $\mathfrak{h}_{\bar 0}=Span(e_{ii}\mid i\in I_{\bar 0})$ be the even part of Cartan subalgebra of $\mathfrak{q}(n)$. Let us define inductively the following elements of $U(\mathfrak{q}(n))$:

$$e_{ij}^{(1)}=e_{ij},\ f_{ij}^{(1)}=f_{ij},$$

$$e_{ij}^{(p)}=\sum_{l=1}^{n}e_{il}e_{lj}^{(p-1)}+(-1)^{p-1}\sum_{l=1}^{n}f_{il}f_{lj}^{(p-1)}, \tag{1.4.3}$$

$$f_{ij}^{(p)}=\sum_{l=1}^{n}e_{il}f_{lj}^{(p-1)}+(-1)^{p-1}\sum_{l=1}^{n}f_{il}e_{lj}^{(p-1)}.$$

The following relations are subject to straightforward verification:

$$\begin{aligned}
[e_{ij},e_{kl}^{(p)}] &= \delta_{jk}e_{il}^{(p)}-\delta_{il}e_{kj}^{(p)},\\
[e_{ij},f_{kl}^{(p)}] &= \delta_{jk}f_{il}^{(p)}-\delta_{il}f_{kj}^{(p)},\\
[f_{ij},e_{kl}^{(p)}] &= (-1)^{p+1}\delta_{jk}f_{il}^{(p)}-\delta_{il}f_{kj}^{(p)},\\
[f_{ij},f_{kl}^{(p)}] &= (-1)^{p+1}\delta_{jk}e_{il}^{(p)}+\delta_{il}e_{kj}^{(p)}.
\end{aligned}\tag{1.4.4}$$

As is not difficult to verify, see [Se3], the elements $z_k=\sum_{i=1}^{n}e_{ii}^{(k)}$, $k=1,3,5,\dots$, are central ones if we embed $\mathfrak{q}(n)$ into $\mathfrak{g}=\mathfrak{q}(n)\oplus\mathfrak{q}(n)$ as the first summand, i.e., strictly speaking, I mean not z_k but $\sigma(z_k)\in U(\mathfrak{g})$.

Let $\{\varepsilon_i\}_{i\in I_{\bar 0}}$ be the basis of $\mathfrak{h}_{\bar 0}$ dual to $\{e_i\}_{i\in I_{\bar 0}}$.

Let $\lambda=(\lambda_1,\dots,\lambda_m)$ be a strong partition of k, i.e., $\lambda_1>\lambda_2>\dots>\lambda_m>0$. Let V^λ be an irreducible submodule of $V^{\otimes k}$ corresponding to λ, see [Se1]. Then the $\mathfrak{g}$-module $W^\lambda=V^\lambda\otimes(V^\lambda)$ is irreducible and contains a unique even $\mathfrak{g}_2$-invariant vector ω_λ corresponding to the identity operator under identification $V^\lambda\otimes(V^\lambda)^*=\mathrm{End}(V^\lambda)$. The dual module $(V^\lambda)^*\otimes V^\lambda$ is also irreducible and contains an even $\mathfrak{g}_1$-invariant vector ω_λ^* corresponding to the parity operator P (such that $Pv=(-1)^{p(v)}v$) under identification $(V^\lambda)^*\otimes V^\lambda=\mathrm{End}(V^\lambda)$.

1.5. Theorem . *Let $\mathfrak{q}(n)$ be embedded into $\mathfrak{g} = \mathfrak{q}(n) \oplus \mathfrak{q}(n)$ as the first summand. Then*

i) *Each two-side invariant with respect to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ functional on $\mathfrak{g}$ is uniquely determined by its restriction onto $S(\mathfrak{h}_{\bar 0}) \subset \mathfrak{q}(n)$.*

ii) *Every $z \in Z(\mathfrak{q}(n))$ uniquely determines a differential operator $\Omega_{(z)}$ on the space of restrictions of left-invariant functionals $S(\mathfrak{h}_{\bar 0})^{*inv}$.*

iii) *$\Omega_{(z_k)} = \Omega_k$ for $z = z_k$.*

iv) *Let $\varphi_\lambda = \Theta(\omega_\lambda^*, \omega_\lambda)$. Then the restriction of φ_λ onto $S(\mathfrak{h}_{\bar 0})$ coincides up to a scalar multiple with $Q_\lambda(e^{\varepsilon_1}, \dots, e^{\varepsilon_n})$*

1.6. Let $\mathfrak{g} = \mathfrak{gl}(\dim V)$, where $\dim V = (n|n)$. Let $I = I_{\bar 0} \cup I_{\bar 1}$, where $I_{\bar 0} = \{1, \dots, n\}$ and $I_{\bar 1} = \{\bar 1, \dots, \bar n\}$. Let $\{e_i\}$ be a basis of V such that the parity of each vector coincides with that of its index. Denote by $\mathfrak{pe}_1(n)$ and $\mathfrak{pe}_2(n)$ the Lie subsuperalgebras in $\mathfrak{g}$ preserving the respective tensors:

$$\sum_{i \in I} e_i^* \otimes e_{\bar i}^*, \quad \sum_{i \in I} (-1)^i e_i^* \otimes e_{\bar i}^*$$

Let ψ_1 and ψ_2 be involutive antiautomorphisms of $\mathfrak{g}$ that single out $\mathfrak{pe}_1(n)$ and $\mathfrak{pe}_2(n)$, respectively, i.e.,

$$\mathfrak{pe}_i = \{X \in \mathfrak{gl}(v) \mid \psi_i(X) = -X\} \text{ for } i = 1, 2.$$

It is easy to see that the restrictions of ψ_1 and ψ_2 onto $\mathfrak{g}_{\bar 0}$ coincide. Set

$$\mathfrak{h}^+ = \{x \in \mathfrak{h} \mid \psi_1(x) = \psi_2(x) = X\},$$

where $\mathfrak{h}$ is Cartan subalgebra in $\mathfrak{g}$.

Let λ be a partition and V^λ the corresponding $\mathfrak{g}$-submodule in the tensor algebra of the identity representation, see [Se1].

It follows from [Se4] that, (in Frobenius's notations [Ma])

$$V^\lambda \text{ contains a } \mathfrak{pe}_1\text{-invariant vector if } \lambda = (\alpha_1, \dots, \alpha_p | \alpha_1 + 1, \dots, \alpha_p + 1)$$

and

$$V^\lambda \text{ contains a } \mathfrak{pe}_2\text{-invariant vactor if } \lambda = (\alpha_1 + 1, \dots, \alpha_p + 1 | \alpha_1, \dots, \alpha_p).$$

Similarly,

$$(V^\lambda)^* \text{ contains a } \mathfrak{pe}_1\text{-invariant vactor if } \lambda = (\alpha_1 + 1, \dots, \alpha_p + 1 | \alpha_1, \dots, \alpha_p)$$

and

$$(V^\lambda)^* \text{ contains a } \mathfrak{pe}_2\text{-invariant vactor if } \lambda = (\alpha_1, \dots, \alpha_p | \alpha_1 + 1, \dots, \alpha_p + 1).$$

Now, let $\lambda = (\alpha_1 + 1, \dots, \alpha_p + 1 | \alpha_1, \dots, \alpha_p)$, let $v_\lambda^* \in (V^\lambda)^*$ be a $\mathfrak{pe}_1$-invariant vector, $v_\lambda \in (V^\lambda)$ be a $\mathfrak{pe}_2$-invariant vector, and let $\varphi_\lambda = \Theta(v_\lambda^*, v_\lambda)$ be the corresponding bispherical function. In $\mathfrak{h}^+$, select the basis $\{e_{ii}^+ = \frac{1}{2}(e_{ii} + \psi_1(e_{ii})) \mid i \in I_{\bar 0}\}$ and let ε_i be the left dual vectors.

Let e_{ij} be the basis of matrix units in $\mathfrak{gl}(V)$. Set

$$e_{ij}^{(1)} = e_{ij}, \quad e_{ij}^{(k)} = \sum_{i \in I} (-1)^p e_{ip} e_{pj}^{(k-1)}.$$

It is easy to verify that

$$[e_{ij}, e_{pq}^{(k)}] = \delta_{jp} e_{ip}^{(k)} - (-1)^{(i+j)(p+q)} \delta_{iq} e_{pj}^{(k)}.$$

This easily implies that $z_k = \sum_{i \in I} e_{ii}^{(k)} \in Z(\mathfrak{g})$.

1.7. Theorem . i) *Each two-sided invariant with respect to* $\mathfrak{pe}_1$ *and* $\mathfrak{pe}_2$ *functional on* $U(\mathfrak{g})$ *is uniquely determined by its restriction onto* $S(\mathfrak{h}^+)$.

ii) *Every* $z \in Z(\mathfrak{g})$ *uniquely determines a differential operator* $\Omega_{(z)}$ *on the space* $S(\mathfrak{h}^+)^*$ *of restrictions of invariant functionals.*

iii) *If* $z = z_k$, *then* $\Omega_{(z_k)} = \tilde{\Omega}_k$.

iv) *The functional* φ_λ *coincides, up to a scalar multiple, with* $Q_\lambda(e^{\varepsilon_1}, \dots, e^{\varepsilon_k})$.

§2. The algebra dual to the enveloping algebra

In this section we follow Dixmier's book [Dix] applied *mutatis mutandis* to Lie superalgebras.

2.1. Let $\mathfrak{g}$ be a Lie superalgebra. We endow $U(\mathfrak{g})^*$ with a coalgebra structure by setting

$$c : \mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad c(x) = x \otimes 1 + 1 \otimes x \text{ for any } x \in \mathfrak{g},$$

so that ${}^t c(x) = c({}^t x)$ where the first t is the principal antiautomorphism of the superalgebra $U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \otimes \mathfrak{g})$, see sec. 1.3.

2.2. Lemma . *Let* $\dim \mathfrak{g} = (n|m)$. *Then* $U(\mathfrak{g})^*$ *is isomorphic to the supercommutative superalgebra of formal power series in* n *even and* m *odd indeterminates.*

Proof. Let $\mathfrak{g}_{\bar 0} = Span(e_1, \dots, e_n)$, $\mathfrak{g}_{\bar 1} = Span(e_{\bar 1}, \dots, e_{\bar m})$, $I_{\bar 0} = \{1, \dots, n\}$, $I_{\bar 1} = \{\bar 1, \dots, \bar m\}$. Denote:

$$M = \{\nu = (\nu_1, \dots, \nu_n, \nu_{\bar 1}, \dots, \nu_{\bar m}) \mid \nu_i \in \mathbb{Z}_{\geq 0} \text{ if } i \in I_{\bar 0} \text{ and } \nu_i \in \{0,1\} \text{ if } i \in I_{\bar 1}\}.$$

For $\nu \in M$, set

$$e_\nu = \frac{e_1^{\nu_1}}{\nu_1!} \cdots \frac{e_n^{\nu_n}}{\nu_n!} \frac{e_{\bar 1}^{\nu_{\bar 1}}}{(\nu_{\bar 1})!} \cdots \frac{e_{\bar m}^{\nu_{\bar m}}}{(\nu_{\bar m})!}$$

and let $t_1, \dots, t_n, t_{\bar 1}, \dots, t_{\bar m}$ be the set of even and odd supercommuting indeterminates. The correspondence

$$U(\mathfrak{g})^* \ni L \mapsto \sum_{\nu \in M} L(e_\nu) t_{\bar m}^{\nu_{\bar m}} \dots t_{\bar 1}^{\nu_{\bar 1}} t_n^{\nu_n} \dots t_1^{\nu_1}$$

determines the homomorphism desired. □

2.3. Left and right coregular representations. For any $u, v \in U(\mathfrak{g})$ and $L \in U(\mathfrak{g})^*$ set

$$(L^*(u)L)(v) = (-1)^{uL} L({}^t u v) \text{ and } (R^*(u)L)(v) = (-1)^{u(L+v)} L({}^t v u).$$

The following statements are easy to verify:

i) $u \longrightarrow L^*(u)$ is a representation of $U(\mathfrak{g})$ in $U(\mathfrak{g})^*$ (we call it the *left regular* one);

ii) $u \longrightarrow R^*(u)$ is a representation of $U(\mathfrak{g})$ in $U(\mathfrak{g})^*$ (we call it the *right regular* one);

iii) If $x \in \mathfrak{g}$, then $L^*(x)$ and $R^*(x)$ are superdifferentiations of the algebra $U(\mathfrak{g})^*$.

Observe also that algebra $U(\mathfrak{g})^*$ possesses an automorphism

$$L \mapsto L^T : L^T(u) = L({}^t u) \text{ for any } u \in U(\mathfrak{g}), \ L \in U(\mathfrak{g})^*.$$

2.4. Matrix coefficients. Let V be a $\mathfrak{g}$-module, V^* the dual module, let $v \in V$ and $v^* \in V^*$. Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be the corresponding representation. Denote by $\Theta^\pi(v^*, v)$ the linear form on $U(\mathfrak{g})^*$:

$$\Theta^\pi(v^*, v)(u) = (-1)^{uv} v^*(\pi(u)v). \tag{2.4.1}$$

Finally, denote by $C(\pi)$ (or $C(V)$) the subspace in $U(\mathfrak{g})^*$ generated by $\Theta^\pi(v^*, v)$ for all $v^* \in V^*$ and $v \in V$.

2.4.1. Lemma .

$$\Theta^{\pi_1 \otimes \pi_2}(v_1^* \otimes v_2^*, v_1 \otimes v_2) = (-1)^{v_1 v_2^*} \Theta^{\pi_1}(v_1^*, v_1) \otimes \Theta^{\pi_2}(v_2^*, v_2)$$

Proof. I remaind I will just write $(-1)^u$ insted of $(-1)^{p(u)}$. We have

$$\begin{aligned} &\Theta^{\pi_1 \otimes \pi_2}(v_1^* \otimes v_2^*, v_1 \otimes v_2)(u_1 \otimes u_2) \\ = \;& (-1)^{(u_1+u_2)(v_1+v_2)} v_1^* \otimes v_2^*(\pi_1(u_1) \otimes \pi_2(u_2)(v_1 \otimes v_2)) \\ = \;& (-1)^{(u_1+u_2)(v_1+v_2)+v_1 u_2 + v_2^*(u_1+v_1)} v_1^*(\pi_1(u_1)v_1) v_2^*(\pi_2(u_2)v_2) \\ = \;& \Theta^{\pi_1}(v_1^*, v_1) \otimes \Theta^{\pi_2}(v_2^*, v_2)(u_1 \otimes u_2) \\ = \;& (-1)^{u_1(v_2+v_2^*)} \Theta^{\pi_1}(v_1^*, v_1)(u_1) \Theta^{\pi_2}(v_2^*, v_2)(u_2) \\ = \;& (-1)^{u_1(v_2+v_2^*)+v_1 u_1 + v_2 u_2} v_1^*(\pi_1(u_1)v_1) v_2^*(\pi_2(u_2)v_2). \end{aligned}$$

□

2.4.2. Lemma . *The map $V^* \otimes V \longrightarrow U(\mathfrak{g})$ given by $v^* \otimes v \mapsto \Theta^\pi(v^*, v)$ is a $\mathfrak{g} \oplus \mathfrak{g}$-module homomorphism; here we consider $U(\mathfrak{g})^*$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module with respect to the left and right coregular representations. If V is irreducible, the above map is an isomorphism.*

Proof. Clearly, there exists a linear map: $\varphi : V^* \otimes V \longrightarrow U(\mathfrak{g})^*$ such that $\varphi(v^* \otimes v) = \Theta(v^*, v)$.

Let $x \in \mathfrak{g}$. Then

$$\varphi\left((x \otimes 1)(v^* \otimes v)\right)(u) = \varphi(xv^* \otimes v)(u) = \Theta(xv^*, v)(u) = (-1)^{vu}(xv^*)(u)$$

$= (-1)^{vu+\bar{1}+xv^*} v^*(xu)$. On the other hand,

$$\begin{aligned} L^*(x)\varphi(v^*, v)(u) &= (-1)^{x(v+v^*)+\bar{1}} \Theta(v^*, v)(xu) \\ &= (-1)^{x(v+v^*)+\bar{1}+(x+u)v} v^*(xuv) \\ &= (-1)^{uv+\bar{1}+xv^*} v^*(xu). \end{aligned}$$

Further,

$$\varphi((1 \otimes x)(v^* \otimes v))(u) = (-1)^{xv^*} \varphi(v^* \otimes xv)(u) = (-1)^{xv^*} \Theta(v^*, xv)(u) = (-1)^{xv^* + u(x+v)} v^*(uxv).$$

On the other hand,

$$R^*(x)\varphi(v^* \otimes v)(u) = (-1)^{x(v^*+v+u)} \varphi(v^* \otimes v)(ux) = (-1)^{x(v^*+v+u)} \Theta(v^* \otimes v)(ux) =$$
$$(-1)^{x(v^*+v+u)+(u+x)v} v^*(uxv) = (-1)^{xv^*+u(x+v)} v^*(uxv).$$

This proves the first statement. The second one is obvious. □

2.4.3. Lemma . *Let ρ be the tensor product of representations π_1 and π_2. Then*
i) $\Theta^{\rho}(v_1^*\otimes v_2^*, v_1\otimes v_2) = (-1)^{v_1v_2^*}\Theta^{\pi_1}(v_1^*, v_1)\Theta^{\pi_2}(v_2^*, v_2)$.
ii) $C(\rho) = C(\pi_1)C(\pi_2)$.
iii) *If π is finite dimensional, then* $(\Theta^{\pi}(v^*, v))^T = (-1)^{vv^*}\Theta^{\pi^*}(v, v^*)$.

Proof. i)

$$\begin{aligned}(-1)^{v_1v_2^*}(\Theta^{\pi_1}(v_1^*, v_1)\Theta^{\pi_2}(v_2^*, v_2))(u) &= (-1)^{v_1v_2^*}(\Theta^{\pi_1}(v_1^*, v_1)\otimes\Theta^{\pi_2}(v_2^*, v_2))(C(u)) \\ &= (-1)\Theta^{\pi_1\otimes\pi_2}(v_1^*\otimes v_2^*, v_1\otimes v_2)(C(u)) \\ &= (-1)^{(v_1+v_2)u}(v_1^*\otimes v_2^*)(\pi_1\otimes\pi_2\cdot C(u)(v_1\otimes v_2) \\ &= (-1)^{(v_1+v_2)u}(v_1^*\otimes v_2^*)(\rho(u)(v_1\otimes v_2)) \\ &= \Theta^{\rho}(v_1^*\otimes v_2^*, v_1\otimes v_2)(u).\end{aligned}$$

ii) follows from i)
iii)

$$\begin{aligned}(\Theta^{\pi}(v^*, v))^T(u) &= \Theta^{\pi}(v^*, v)({}^tu) \\ &= (-1)^{uv}v^*(\pi({}^tu)(v)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Theta^{\pi^*}(v, v^*)(u) &= (-1)^{v^*u}v(\pi^*(u)v^*) \\ &= (-1)^{v^*u+v(u+v^*)}(\pi^*(u)v^*)(v) \\ &= (-1)^{v^*u+v(u+v^*)+uv^*}v^*(\pi({}^tu)v).\end{aligned}$$

□

2.4.4. Lemma . *Let V be a $\mathfrak{g}$-module. Consider the map*

$$\varphi : V^*\otimes V\otimes V^*\otimes V \longrightarrow U(\mathfrak{g})^*$$
$$\varphi(v_1^*\otimes v_1\otimes v_2^*\otimes v_2) = (-1)^{v_1^*v_2^*+v_2^*v_1+v_1^*v_1}(\Theta(v_2^*, v_1))^T\cdot\Theta(v_1^*, v_2).$$

If we consider $V^*\otimes V\otimes V^*\otimes V = (V^*\otimes V)\otimes(V^*\otimes V)$ as a $\mathfrak{g}\oplus\mathfrak{g}$-module in such a way that the first two factors is one $\mathfrak{g}$-module and the last two ones is the other module, then φ is a $\mathfrak{g}\oplus\mathfrak{g}$-module homomorphism. (Recall that we consider $U(\mathfrak{g})^*$ as a $\mathfrak{g}\oplus\mathfrak{g}$-module with respect to the simultaneous left and right coregular representations.)

Proof. Let $x \in \mathfrak{g}$. Then

$$\varphi((x \otimes 1)(v_1^* \otimes v_1 \otimes v_2^* \otimes v_2)) = \varphi((xv_1^* \otimes v_1 + (-1)^{xv_1^*} v_1^* \otimes xv_1) \otimes v_2^* \otimes v_2))$$

$$= (-1)^{v_2^*(x+v_1^*)+v_2^* v_1 + (x+v_1^*)v_1} (\Theta(v_2^*, v_1))^T \Theta(xv_1^*, v_2) +$$

$$(-1)^{v_1^* v_2^* + v_2^*(x+v_1) + v_1^*(x+v_1) + xv_1^*} (\Theta(v_2^*, xv_1))^T \Theta(v_1^*, v_2)$$

$$= (-1)^{v_1^* v_2^* + v_2^* v_1 + v_1^* v_1} L^*(x) (\Theta(v_2^*, v_1))^T \Theta(v_1^*, v_2) +$$

$$(-1)^{v_1^* v_2^* + v_2^* v_1 + v_1^* v_1 + x(v_2^* + v_1)} (\Theta(v_2^*, v_1))^T L^*(x) \Theta(xv_1^*, v_2)$$

$$= (-1)^{v_1^* v_2^* + v_2^* v_1 + v_1^* v_1} L^*(x) [(\Theta(v_2^*, v_1))^T \Theta(v_1^*, v_2)]$$

$$= L^*(x) \varphi(v_1^* \otimes v_1 \otimes v_2^* \otimes v_2).$$

The identity

$$\varphi((1 \otimes x)(v_1^* \otimes v_1 \otimes v_2^* \otimes v_2)) = R^*(x) \varphi(v_1^* \otimes v_1 \otimes v_2^* \otimes v_2)$$

is similarly verified. □

2.4.5. Lemma . *Let V be a finite dimensional $\mathfrak{g}$-module, $\{v_i\}$ its basis, $\{v_i^*\}$ the left dual basis of V^*. Then*

$$\sum_i (\Theta(v^*, v_i))^T \Theta(v_i^*, v) = v^*(v) \cdot \varepsilon,$$

where $v \in V$, $v^* \in V^*$ and $\varepsilon \in U(\mathfrak{g})^*$ is the counit.

Proof. The functional ε is uniquely, up ot a scalar multiple, characterized by its invariance with respect to the right coregular representation. Further,

$$\omega = \sum (-1)^i v_i^* \otimes v_i$$

is an invariant of the $\mathfrak{g}$-module $V^* \otimes V$. Hence, by Lemma 2.4.4,

$$\varphi(\sum (-1)^i v_i^* \otimes v_i \otimes v^* \otimes v)$$

is an invariant with respect to the right coregular representation on $U(\mathfrak{g})$, i.e., φ is an invariant functional on $U(\mathfrak{g})$. Hence, $\varphi(\omega \otimes v^* \otimes v) = \alpha\varepsilon$.

On the other hand,

$$\varphi(\omega \otimes v^* \otimes v) = \sum (-1)^{i)} \varphi(v_i^* \otimes v_i \otimes v^* \otimes v) =$$
$$\sum (\Theta(v^*, v_i))^T \cdot \Theta(v_i^*, v).$$

Hence,

$$\sum_i (\Theta(v^*, v_i))^T \cdot \Theta(v_i^*, v) = \alpha\varepsilon.$$

To find α, let us substitute $u = 1$ into both parts of the identity. We obtain:

$$\alpha = \alpha\varepsilon(1) = \sum ((\Theta(v^*, v_i))^T \cdot \Theta(v_i^*, v)(1) = \sum v^*(v_i) v_i^*(v) = v^*(v).$$

□

Let $\mathfrak{g} = \mathfrak{gl}(V)$ and let $\mathfrak{A}$ be the subalgebra in $U(\mathfrak{g})^*$ generated by $C(V)$ and $C(V^*)$. It is not difficult to verify that $\mathfrak{A}$ is invariant with respect to the left and right coregular representations.

§3. Bispherical functions and radial parts of Laplace operators for the triple $(\mathfrak{q}(n), \mathfrak{q}(n) \oplus \mathfrak{q}(n), \mathfrak{q}(n))$

3.1. Let $\mathfrak{g} = \mathfrak{q}(n)$, let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$ and $\mathfrak{h} = Span(e_{ii}, f_{jj} \mid \text{ for } i, j \in I_{\bar{0}})$ the Cartan subalgebra (for definition of e_{ij} and f_{ij} see sec. 1.4.2). Let $\mu \in \mathfrak{h}_{\bar{0}}^*$ and $M(\mu)$ the coresponding Verma module with the highest weight with respect to the decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_- = Span(e_{ij}, f_{ij} \mid \text{ for } i > j)$ and $\mathfrak{n}_+ = Span(e_{ij}, f_{ij}) \mid \text{for } i < j)$, see [Se4], [Pe]. Such module $M(\mu)$ has a central character, i.e., there exists a homomorphism

$$\chi_\mu : Z(\mathfrak{g}) \longrightarrow \mathbb{C}, \quad zv = \chi_\mu(z)v \text{ for any } v \in M(\mu) \text{ and } z \in Z(\mathfrak{g}).$$

Let ε_i be the left dual vector to e_{ii}. Set $R^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ and

$$B(\mu) = \{\nu \mid \nu = \lambda - \sum n_\alpha \alpha \text{ for any } n_\alpha \in \mathbb{Z}_{\geq 0}, \text{ where } \alpha \in R^+, \text{ and } \chi_\nu = \chi_\mu\}.$$

3.2. Lemma . *Let $L(\mu)$ be an irreducible module with highest weight μ. Then*

$$\mathrm{ch}L(\mu) = \sum_{\nu \in B(\mu)} C_{\mu\nu}\, \mathrm{ch}M(\nu) \tag{3.2.1}$$

where for any fixed $\tau \in \mathfrak{h}_{\bar{0}}^*$ only a finite number of summands in the right hand side contains τ as a weight.

Proof. Let $\alpha_1, \dots, \alpha_{n-1}$ be a base (system of simple roots) in the root system R. Set

$$ht(\nu) = \sum_{1}^{n-1} k_i \quad \text{for any } \nu = \mu - \sum_{i=1}^{n-1} k_i \alpha_i.$$

By induction on t we prove that

$$\mathrm{ch}L(\mu) = \sum_{\nu \in B(\mu),\, ht(\nu) \leq t} C_{\mu\nu}\, \mathrm{ch}M(\nu) + \sum_{i=1}^{m(t)} \mathrm{ch}V_i \tag{3.2.2}$$

for any $\tau \in \mathrm{Supp\,ch}V_i$, $ht(\tau) > t$, $\chi_{V_i} = \chi_\mu$ and $\mathrm{Supp\,ch}V_i \subset B(\mu)$.

Let $t = 0$. Then we obtain an exact sequence

$$0 \longrightarrow N \longrightarrow M(\mu) \longrightarrow L(\mu) \longrightarrow 0$$

with $N \subset \bigoplus_{\nu \neq \mu} M(\mu)_\nu$. Therefore, $ht(\nu) > 0$ and $\mathrm{ch}L(\mu) = \mathrm{ch}M(\mu) - \mathrm{ch}(N)$; moreover, $\mathrm{Supp\,ch}N \subset B(\mu)$.

Let (3.2.2) hold. Consider the set $ht(\nu)$, where $\nu \in \mathrm{Supp\,ch}V_i$ and let k be the least element of this set. Clearly, $k > t$. There exists then a finite number of vectors $v_1^i, \dots, v_l^i \in V_i$ with weights $\mu_{1i}, \dots, \mu_{li}$ such that $ht(\mu_{1i}) = ht(\mu_{2i}) \dots = k$. For all the other weights ν of V_i we have $ht(\nu) > k \geq t+1$. The vectors $v_1^i, \dots, v_l^i$ are, obviously, the highest weight ones, so we have an exact sequence

$$0 \longrightarrow N \longrightarrow \bigoplus_{j=1}^{l} M(\mu_{ji}) \longrightarrow V_i \longrightarrow K \longrightarrow 0. \tag{3.2.3}$$

This implies:

$$\mathrm{ch}V_i = \sum_{j=1}^{l} \mathrm{ch}M(\mu_{ji}) + \mathrm{ch}K - \mathrm{ch}N$$

and if $\nu \in \operatorname{Supp} \operatorname{ch}K \cup \operatorname{Supp} \operatorname{ch}N$, then $ht(\nu) > t+1$. This proves (3.2.2).

Let $\nu \in \mathfrak{h}_{\bar{0}}^*$ be any weight. Set $t = ht(\nu)$; apply (3.2.2) to see that $\nu \notin \cup \operatorname{Supp} \operatorname{ch}V_i$. This proves (3.2.1). □

3.3. Lemma . *Let* $z_3 = \sum e_{ii}^{(3)}$ *(see* 1.4.2*) and let* v_μ *be the highest weight vector of the* $\mathfrak{g}$*-module* $M(\mu)$. *Then*

$$z_3 v_\mu = \left(\sum \mu_i^3 - (\sum \mu_i)^2\right) v_\mu.$$

Proof. First, observe that

$$e_{ii}^{(3)} = \sum_{j=1}^{n} e_{ij} e_{ji}^{(2)} + \sum_{j=1}^{n} f_{ij} f_{ji}^{(2)} = \sum_{i<j} e_{ij} e_{ji}^{(2)} + \sum_{i<j} f_{ij} f_{ji}^{(2)}$$
$$+ \sum_{i>j} e_{ij} e_{ji}^{(2)} + \sum_{i>j} f_{ij} f_{ji}^{(2)} + \sum e_{ii} e_{ii}^{(2)} + \sum f_{ii} f_{ii}^{(2)}$$

But, as is easy to verify, $e_{ij}^{(2)} v_\mu = f_{ij}^{(2)} v_\mu = 0$. Therefore,

$$z_3 v_\mu = \left(\sum e_{ii} e_{ii}^{(2)} + \sum f_{ii} f_{ii}^{(2)}\right) v_\mu + \left(\sum_{i<j} e_{ij} e_{ji}^{(2)} + \sum f_{ij} f_{ji}^{(2)}\right) v_\mu$$
$$= \left(\sum e_{ii} e_{ii}^{(2)} + \sum f_{ii} f_{ii}^{(2)} + \sum_{i<j} e_{ii}^{(2)} - e_{jj}^{(2)} + \sum_{i<j} e_{jj}^{(2)} - e_{ii}^{(2)}\right) v_\mu$$
$$= \left(\sum e_{ii} e_{ii}^{(2)} + \sum f_{ii} f_{ii}^{(2)}\right) v_\mu.$$

Further, it is easy to verify that

$$e_{ii}^{(2)} v_\mu = \left(\mu_i^2 - \mu_i - 2 \sum_{k>i} \mu_k\right) v_\mu \text{ and } f_{ii}^{(2)} v_\mu = 0.$$

Hence, $z_3 v_\mu = (\sum \mu_i^3 - (\sum \mu_i)^2) v_\mu$. □

3.4. Corollary . *Let* $\delta = \prod_{\alpha \in R^+} \frac{e^{\alpha/2} + e^{-\alpha/2}}{e^{\alpha/2} - e^{-\alpha/2}}$ *(see* 1.4.2*) and let* $\Omega_3^* = \sum \partial_i^3 - (\sum \partial_i)^2$, *where* $\partial_i e^l = l(e_i) e^l$. *Then* $\Omega_3^* \delta^{-1} = 0$ *and* $(\sum \partial_i^3)\delta^{-1} = 0$.

Proof. Let us prove a more general statement; namely, let φ_μ be the character of an irreducible $\mathfrak{g}$-module with highest weight μ. Then

$$\delta^{-1}\varphi_\mu \textit{ is an eigenfunction of } \Omega_3^* \textit{ with eigenvalue } \sum \mu_i^3 - (\sum \mu_i)^3.$$

Indeed, by Lemma 3.2

$$\varphi_\mu = \sum_{\nu \in B(\mu)} C_{\mu\nu} \operatorname{ch} M(\nu).$$

By multiplying both parts of the inequality by δ^{-1} we obtain

$$\delta^{-1}\varphi_\mu = \sum_{\nu \in B(\mu)} \tilde{C}_{\mu\nu} e^\nu. \tag{3.4.1}$$

If $z \in Z(\mathfrak{g})$, then $z v_\mu = \chi_\mu(z) v_\mu$. Since $\nu \in B(\mu)$, it follows that $\chi_\mu(z) = \chi_\nu(z)$. But by Lemma 3.3

$$\chi_\mu(z_3) = \sum \mu_i^3 - \left(\sum \mu_i\right)^2 = \chi_\nu(z_3) = \sum \nu_i^3 - \left(\sum \nu_i\right)^2.$$

So for $\Omega_3^* = \sum \partial_i^3 - (\sum \partial_i)^2$, we have

$$\Omega_3^*\delta^{-1}\varphi_\mu = \sum_{\nu\in B(\mu)} \Omega_3^*(\tilde{C}_{\mu\nu}e^\nu) = \sum_{\nu\in B(\mu)} \tilde{C}_{\mu\nu}\left(\sum v_i^3 - (\sum v_i)^2\right)e^\nu$$
$$= \left(\sum \mu_i^3 - (\sum \mu_i)^2\right)\delta^{-1}\varphi_\mu.$$

In particular, applying this statement to the trivial module we obtain$\Omega_3^*\delta^{-1} = 0$, implying $(\sum \partial_i^3)\delta^{-1} = 0$. □

3.5. Proof of heading i) of Lemma 1.2.3. Let $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ and $R^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. Set $\Delta_\alpha^+ = e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha}$ and $\Delta_\alpha^- = e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}$, for any $\alpha \in R$. Then

$$\partial_i(\Delta_\alpha^+) = \frac{1}{2}\alpha(e_i)\Delta_\alpha^-, \quad \partial_i(\Delta_\alpha^-) = \frac{1}{2}\alpha(e_i)\Delta_\alpha^+.$$

We also set

$$\varphi_i = \sum_{\alpha\in R^+} \frac{\alpha(e_i)}{\Delta_\alpha^+\Delta_\alpha^-}, \qquad \psi_i = \sum_{\alpha\in R^+} \frac{\alpha^2(e_i)}{(\Delta_\alpha^+)^2}, \qquad \Theta_i = \sum_{\{\alpha,\beta\}\subset R^+} \frac{\alpha(e_i)\beta(e_i)}{\Delta_\alpha^+\Delta_\alpha^-\Delta_\beta^+\Delta_\beta^-} \tag{3.5.1}$$

where the last sum runs over the two-element subsets of R^+. It is not difficult to verify that one can express the operator $\hat{\Omega}_3 = \Omega_3 + (\sum \partial_i)^2$ in the form

$$\hat{\Omega}_3 = \sum \partial_i^3 + 6\sum \varphi_i\partial_i^2 - 6\sum \psi_i\partial_i + 24\sum \Theta_i\partial_i. \tag{3.5.2}$$

It is easy to verify that $\delta^{-1}\partial_i\delta = \partial_i - 2\varphi_i$; hence,

$$\delta^{-1}\partial_i^2\delta = \partial_i^2 - 4\varphi_i\partial_i + 4\varphi_i^2 - 2\partial_i(\varphi_i) \tag{3.5.3}$$

$$\delta^{-1}\partial_i^3\delta = \partial_i^3 - 6\varphi_i\partial_i^2 + 3(4\varphi_i^2 - 2\partial_i(\varphi_i))\partial_i - 8\varphi_i^3 - 2\partial_i^2(\varphi_i) + 12\varphi_i\partial_i(\varphi_i) \tag{3.5.4}$$

Therefore,

$$\delta^{-1}\hat{\Omega}\delta = \textstyle\sum_i[\partial_i^3 + (24\Theta_i - 6\psi_i - 12\varphi_i^2 - 6\partial_i(\varphi_i))\partial_i$$
$$+(16\varphi_i^3 - 2\partial_i^2(\varphi_i) - 48\Theta_i\varphi_i + 12\psi_i\varphi_i)].$$

Direct calculations show that

$$24\Theta_i - 6\psi_i - 12\varphi_i^2 - 6\partial_i(\varphi_i) = 0.$$

Hence,

$$\delta^{-1}\hat{\Omega}\delta = \sum \partial_i^3 + f.$$

But, $\hat{\Omega}(1) = 0$, so

$$(\delta^{-1}\hat{\Omega}\delta)\cdot(\delta^{-1}) = \left(\sum \partial_i^3\right)(\delta^{-1}) + f\delta^{-1} = 0.$$

But due to Corollary 3.4, $(\sum \partial_i^3)(\delta^{-1}) = 0$; hence, $f\delta^{-1} = 0$ and $f = 0$. □

3.6. Lemma . *Let $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ be selected as in sec. 1.4, let I be the left ideal in $U(\mathfrak{g}\oplus\mathfrak{g})$ generated by $\mathfrak{g}_2$ and $M = U(\mathfrak{g}\oplus\mathfrak{g})/I$. Let $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}\oplus\mathfrak{g}$ be the embedding into the first summand, i.e., $\sigma(x) = (x, 0)$. Let $\tilde{\sigma} : U(\mathfrak{g}) \longrightarrow M$ be the map induced by the homomorphism $U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}\oplus\mathfrak{g})$ that extends σ and $\rho(x) = (x, (-1)^x x)$ an isomorphism of $\mathfrak{g}$ with $\mathfrak{g}_1$. Then $\tilde{\sigma}(x * u) = \rho(x)\tilde{\sigma}(u)$.*

Proof.

$$\begin{aligned}
\tilde{\sigma}(x * u) &= \tilde{\sigma}(xu - (-1)^{x(u+\bar{1})}ux) = xu \otimes 1 - (-1)^{x(u+\bar{1})}ux \otimes 1 \\
&= xu \otimes 1 - (-1)^{x(u+\bar{1})}ux \otimes 1 - \rho(x)\tilde{\sigma}(u) + \rho(x)\tilde{\sigma}(u) \\
&= \rho(x)\tilde{\sigma}(u) + xu \otimes 1 - (-1)^{x(u+\bar{1})}ux \otimes 1 \\
&\quad -(x \otimes 1 + (-1)^{x} 1 \otimes x)(u \otimes 1) \\
&= \rho(x)\tilde{\sigma}(u) - (-1)^{x(u+\bar{1})}(ux \otimes 1 + u \otimes x) \\
&= \rho(x)\tilde{\sigma}(u) - (-1)^{x(u+\bar{1})}(u \otimes 1)(x \otimes 1 + 1 \otimes x) \\
&\equiv_{(\text{mod } I)} \rho(x)\tilde{\sigma}(u).
\end{aligned}$$

□

3.6.1. Corollary . *Statement of Lemma 1.4.1 is true.*

3.7. Proof of heading i) of Theorem 1.5. Due to [Se4] and [G] we have an isomorphism of $\mathfrak{g}$-modules with respect to the action (1.4.1):

$$U(\mathfrak{g}) \simeq \mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}(U(\mathfrak{g}_{\bar{0}})).$$

Therefore, there exists a bijection between the space of $\mathfrak{g}$-invariant with respect to the action (1.4.1) functionals on $U(\mathfrak{g})$ and the space of $\mathfrak{g}_{\bar{0}}$-invariant functionals on $U(\mathfrak{g}_{\bar{0}})$. Moreover, any $\mathfrak{g}$-invariant functional is uniquely determined by its restriction onto $U(\mathfrak{g}_{\bar{0}})$. On the other hand, every $\mathfrak{g}_{\bar{0}}$-invariant functional is uniquely determined by its restriction onto $U(\mathfrak{h}_{\bar{0}}) = S(\mathfrak{h}_{\bar{0}})$. □

3.8. Let $l \in (U(\mathfrak{g})^*)^{\mathfrak{g}}$ and φ_l the generating function of its restriction onto $S(\mathfrak{h}_{\bar{0}})$, i.e.,

$$\varphi_l(t_1, \dots, t_n) = \sum \frac{l(e_{11}^{\nu_1} \dots e_{nn}^{\nu_n})}{(\nu_1)! \dots (\nu_n)!} t_1^{\nu_1} \dots t_n^{\nu_n}. \tag{3.8.1}$$

On $S(\mathfrak{h}_{\bar{0}})^*$, define the following operators by setting for any $f \in S(\mathfrak{h}_{\bar{0}})$:

$$(\partial_i^{(k)} l)(f) = l(e_{ii}^{(k)} f), \quad (\delta_i^{(k)} l)(f) = L(f_{ii}^{(k)} f), \tag{3.8.2}$$

$$(D_{ij}^{(k)} l)(f) = l(e_{ij} e_{ji}^{(k)} f), \quad (\Delta_{ij}^{(k)} l)(f) = L(f_{ij} f_{ji}^{(k)} f). \tag{3.8.3}$$

3.8.1. Lemma . *Let* $\alpha = \varepsilon_i - \varepsilon_j$. *Then*

i) $D_{ij}^{(k)} = \dfrac{e^{\alpha}}{e^{\alpha} - 1}(\partial_i^{(k)} - \partial_j^{(k)})$.

ii) $\Delta_{ij}^{(k)} = \dfrac{e^{\alpha}}{e^{\alpha} + 1}(\partial_j^{(k)} + (-1)^{k+1}\partial_i^{(k)})$.

iii) $l(f \cdot f_{ii} f_{ii}^{(k)}) = l(f e_{ii}^{(k)})$ *for* k *odd.*

iv) $l(f \cdot f_{ii} f_{ii}^{(k)}) = 0$ *for* k *even.*

Proof. i) and ii) are similarly proved. Consider i):

$$(D_{ij}^{(k)}l)(f) = l(fe_{ij}e_{ji}^{(k)}) = L(e_{ij}f(h+\alpha(h))e_{ij}^{(k)} = l(f(h+\alpha(h))e_{ij}^{(k)}e_{ij})$$

$$= l(f(h+\alpha(h))e_{ij}e_{ji}^{(k)}) - l(f(h+\alpha(h))[e_{ij}, e_{ji}^{(k)}])$$

$$= (e^{\alpha}D_{ij}^{(k)}l)(f) - l(f(h+\alpha(h))(e_{ii}^{(k)} - e_{jj}^{(k)}))$$

$$= (e^{\alpha}D_{ij} - e^{\alpha}(\partial_i^{(k)} - \partial_j^{(k)}))(l)(f).$$

This proves i).

iii) We have

$$f_{ii} * (f \cdot f_{ii}^{(k)}) = f_{ii}f \cdot f_{ii}^{(k)} - f \cdot f_{ii}^{(k)} f_{ii}$$

$$= f(f_{ii}f_{ii}^{(k)} - f_{ii}^{(k)}f_{ii}) = f(2f_{ii}f_{ii}^{(k)} - [f_{ii}, f_{ii}^{(k)}]) = 2ff_{ii} \cdot f_{ii}^{(k)} - 2e_{ii}^{(k)}f$$

or $f \cdot f_{ii} \cdot f_{ii}^{(k)} = \frac{1}{2}f_{ii} * (f \cdot f_{ii}^{(k)}) + f \cdot e_{ii}^{(k)}$. This proves iii). Heading iv) is similar. □

3.8.2. Corollary . *Heading* iii) *of Theorem* 1.5 *is true.*

3.9. Proof of heading iv) of Theorem 1.5. Let $V^{\lambda} = Span(v_p)$, where the v_p form a basis of V^{λ}, the v_p^* is the left dual basis, and $(V^{\lambda})^* = Span(v_p^*)$; let $\omega_{\lambda}^* = \sum v_p^* \otimes v_p$, $\omega_{\lambda} = \sum v_p \otimes v_p^*$.

If $\mathfrak{g}$ is embedded into $\mathfrak{g} \oplus \mathfrak{g}$ as the first summand, and $u \in U(\mathfrak{g})$, then

$$\varphi_{\lambda}(u \otimes 1) = \Theta(\omega_{\lambda}^*, \omega_{\lambda})(u \otimes 1) = \omega_{\lambda}^*(u \otimes 1(\omega_{\lambda}))$$

$$= \omega_{\lambda}^*(\sum uv_p \otimes v_p^*) = \sum v_p^*(uv_p) = tr(u).$$

In other words, the matrix coefficient $\Theta^{\pi}(\omega_{\lambda}^*, \omega_{\lambda})$ coincides with the functional $\mathrm{tr}_{V^{\lambda}}(u)$ after restriction onto $U(\mathfrak{g})$. But due to [Se1] we have, up to a scalar multiple,

$$\mathrm{tr}_{V^{\lambda}}|_{\mathfrak{h}_{\bar{0}}} = Q_{\lambda}(e^{\varepsilon_1}, \dots, e^{\varepsilon_n}). \quad \square$$

3.10. Proof of headings ii) and iii) of Lemma 1.2.3. Heading ii) immediately follows from our proof of Corollary 3.4. Let us prove iii).

Let Ω be the algebra generated by all the Ω_k, $k = 1, 3, 5, \dots$ It is not difficult to see that Ω is the image of $Z(\mathfrak{q}(n))$ under the homomorphism

$$r : Z(\mathfrak{q}(n)) \longrightarrow \Omega, \quad r(z)Q_{\lambda} = \varphi(z)(\lambda)Q_{\lambda},$$

where φ is the Harish–Chandra homomorphism, see [Se3]. Therefore, statement of heading iii) can be reformulated as follows:

Let R_n be the algebra of polynomials $r(t_1, \dots, t_n)$ symmetric with respect to $(t_1, \dots, t_n)$ and which do not depend on s after substitution $t_i = s$, $t_j = -s$. Define an R_n-action on the algebra generated by Q_{λ} by setting

$$r * Q_{\lambda} = r(\lambda)Q_{\lambda}.$$

If P is an eigenvector for R_n and belongs to the subalgebra generated by Q_{λ}, then P coincides, up to a scalar multiple, with one of the Q_{λ}.

Indeed, let $P = \sum_{\lambda \in \Lambda} C_\lambda Q_\lambda$. Since $r * P = C(r)P$ it follows that $r(\lambda) = const$ for any $\lambda \in \Lambda$. Let $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$; then $r(\lambda) = r(\mu)$ for any $r \in R_n$. Hence,

$$\prod \frac{t - \lambda_i}{t + \lambda_i} = \prod \frac{t - \mu_j}{t + \mu_j}.$$

We may assume that $\lambda_i > 0$ and $\mu_i > 0$ for all i and j. Then the identity

$$\prod(t - \lambda_i) \prod(t + \mu_j) = \prod(t + \lambda_i) \prod(t + \mu_j)$$

implies $\prod(t - \lambda_i) = \prod(t - \mu_j)$, hence, $\lambda = \mu$. Contradiction. □

§4. Bispherical functions and the radial parts of Laplace operators for the triple $(\mathfrak{pe}(n), \mathfrak{gl}(n|n), \mathfrak{pe}(n))$

4.1. Let V be an $(n|n)$-dimensional superspace, $I = I_{\bar 0} \cup I_{\bar 1} = \{1, \dots, n\} \cup \{\bar 1, \dots, \bar n\}$, the union of the even and odd indices. On I, there is defined a map $i \mapsto \bar i$ such that $\bar{\bar i} = i$. Let $\{e_i\}_{i \in I}$ be the basis of V consisting of vectors whose parity is equal to that of their indices and $\{e_{ij}\}_{i,j \in I}$ be the basis fo $\mathrm{Mat}(V)$ consisting of matrix units, cf. sec. 1.4.2.

The supertransposition antiautomorphism in $\mathfrak{gl}(V)$ is in these terms of the form

$$e_{ij}^t = (-1)^{i)(j+1)} e_{ji}. \tag{4.1.1}$$

Define also an operator $S:\ Se_i = e_{\bar i}$ and the parity operator $P:\ Pe_i = (-1)^i e_i$. Define two antiautomorphisms ψ_1 and ψ_2 by setting

$$\psi_1(x) = (-1)^x S(x^t)S \text{ and } \psi_2 = P\psi_1 P. \tag{4.1.2}$$

Direct calculations show that

$$\psi_1(e_{ij}) = (-1)^{j(i+1)} e_{\bar j \bar i}, \quad \psi_2(e_{ij}) = (-1)^{i(j+1)} e_{\bar j \bar i}. \tag{4.1.3}$$

Define two Lie subsuperalgebras of $\mathfrak{gl}(V)$:

$$\mathfrak{pe}_1(V) = \{x \in \mathfrak{gl}(V) \mid \psi_1(x) = -x\} \tag{4.1.4}$$

and

$$\mathfrak{pe}_2(V) = \{x \in \mathfrak{gl}(V) \mid \psi_2(x) = -x\}. \tag{4.1.5}$$

Observe that $\psi_1(x) = \psi_2(x)$ if $p(x) = 0$ and $\psi_1(x) = -\psi_2(x)$ if $p(x) = 1$. Therefore,

$$\mathfrak{pe}_1(V) = \mathfrak{gl}(V)_{\bar 0}^- \oplus \mathfrak{gl}(V)_{\bar 1}^-, \quad \mathfrak{pe}_2(V) = \mathfrak{gl}(V)_{\bar 0}^- \oplus \mathfrak{gl}(V)_{\bar 1}^+,$$

where

$$\begin{aligned} \mathfrak{gl}(V)_{\bar 0}^- &= \{x \in \mathfrak{gl}(V)_{\bar 0} \mid \psi_2(x) = -x\}, \\ \mathfrak{gl}(V)_{\bar 1}^- &= \{x \in \mathfrak{gl}(V)_{\bar 1} \mid \psi_1(x) = -x\}, \\ \mathfrak{gl}(V)_{\bar 1}^+ &= \{x \in \mathfrak{gl}(V)_{\bar 1} \mid \psi_1(x) = x\}. \end{aligned}$$

For every $x \in \mathfrak{gl}(V)$, set

$$x^+ = \frac{1}{2}(x + \psi_1(x)), \quad x^- = \frac{1}{2}(x - \psi_1(x)).$$

Also set

$$\mathfrak{h}^+ = Span(e_{ii}^+ \mid i \in I_{\bar 0}).$$

4.2. Lemma . *For $f \in S(\mathfrak{h}^+)$ and $\alpha = \varepsilon_i - \varepsilon_j$ set*

$$R^-_{ij} f = \tfrac{1}{2}[f(h - \alpha(h)) - f(h + \alpha(h))],$$
$$R^+_{ij} f = \tfrac{1}{2}[f(h - \alpha(h)) + f(h + \alpha(h))].$$

Then the following identities hold:

i) $e^-_{ij} f = R^+_{ij} f e^-_{ij} + R^-_{ij} f \cdot e^+_{ij}$.

ii) $R^-_{ij} f e_{ij} e^{(k)}_{ji} - (R^-_{ij} - R^+_{ij}) f \cdot [e^-_{ij}, e^{(k)}_{ji}] \in \mathfrak{pe}_1 U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{pe}_2$ *if* $p(i) + p(j) = 0$.

iii) $R^+_{ij} f e_{ij} e^{(k)}_{ji} - (R^+_{ij} - R^-_{ij}) f \cdot [e^+_{ij}, e^{(k)}_{ji}] \in \mathfrak{pe}_1 U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{pe}_2$ *if* $p(i) + p(j) = 1$.

iv) $[e^-_{ij}, e^{(k)}_{ji}] = \dfrac{1}{2}[e^{(k)}_{ii} - e^{(k)}_{jj}]$ *if* $p(i) + p(j) = 0$.

v) $[e^+_{ij}, e^{(k)}_{ji}] = \dfrac{1}{2}[e^{(k)}_{ii} + e^{(k)}_{jj}]$ *if* $\bar{\imath} \neq j$ *and* $p(i) + p(j) = 1$.

vi) $[e^+_{i\bar{i}}, e^{(k)}_{\bar{i}i}] = e^{(k)}_{ii} + e^{(k)}_{\bar{i}\bar{i}}$ *if* $p(i) = 1$.

vii) $[e^+_{i\bar{i}}, e^{(k)}_{\bar{i}i}] = 0$ *if* $p(i) = 0$.

Proof is reduced to a direct verification. □

4.3. Proof of heading i) of Theorem 1.7. It suffices to prove that $U(\mathfrak{g}) = S(\mathfrak{h}^+) + \mathfrak{pe}_1 U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{pe}_2$.

Indeed, any element of $U(\mathfrak{g})$ can be represented in the form

$$u = u_0 + u_1 u'_0 u_2, \text{ where } u_1 \in U(\mathfrak{g}^-_{\bar{1}}),\ u_2 \in U(\mathfrak{g}^+_{\bar{1}}),\ u_0, u'_0 \in U(\mathfrak{g}_{\bar{0}}).$$

Therefore, we may assume that

$$u = f e^+_{\alpha_1} \dots e^+_{\alpha_k},$$

where $f \in S(\mathfrak{h}^+)$ and $e^+_{\alpha_1}, \dots, e^+_{\alpha_k} \in \mathfrak{g}_{\bar{0}}$ are the weight vectors.

Now, induction on k. If $k = 1$, then $R^-_{ij} f e^+_{ij} = e^-_{ij} f - R^+_{ij} f e^-_{ij}$ by Lemma 4.2.i). Hence, $f e^+_{ij} \in \mathfrak{g}^-_{\bar{0}} U(\mathfrak{g}_{\bar{0}}) \mathfrak{g}^-_{\bar{0}}$.

Let $k > 1$. Then

$$\begin{aligned}
R^-_{\alpha_1} f e^+_{\alpha_1} \dots e^+_{\alpha_k} &= e^-_{\alpha_1} f e^+_{\alpha_2} \dots e^+_{\alpha_n} - R^+_{\alpha_1} f e^-_{\alpha_1} e^+_{\alpha_2} \dots \\
&\equiv_{(\mathrm{mod}\ \mathfrak{g}^-_{\bar{0}} U(\mathfrak{g}) \mathfrak{g}^-_{\bar{0}})} -R^+_{\alpha_1} f e^-_{\alpha_1} e^+_{\alpha_2} \dots e^+_{\alpha_n} \\
&\equiv R^+_{\alpha_1} f \cdot [e^-_{\alpha_1}, e^+_{\alpha_2} \dots e^+_{\alpha_k}] \\
&= -R^+_{\alpha_1} f \cdot [e^-_{\alpha_1}, e^+_{\alpha_2}] e^+_{\alpha_3} \dots e^+_{\alpha_k} \\
&\quad -R^+_{\alpha_1} f e^+_{\alpha_2} [e^-_{\alpha_1}, e^+_{\alpha_3}] \dots e^+_{\alpha_n} + \dots \in \mathfrak{g}^-_{\bar{0}} U(\mathfrak{g}_{\bar{0}}) \mathfrak{g}^-_{\bar{0}}.
\end{aligned}$$

4.4. Proof of Lemma 1.2.1. Set

$$\Delta^{(k)+}_i = \tilde{\partial}^{(k)}_i + \tilde{\partial}^{(k)}_{\bar{i}}, \qquad \Delta^{(k)-}_i = \tilde{\partial}^{(k)}_i - \tilde{\partial}^{(k)}_{\bar{i}}.$$

Now, it is not difficult to verify the following identities:

$$\Delta^{(k)+}_i - \Delta^{(k-1)+}_i = \partial_i \Delta^{(k-1)-}_i + \sum_{j \neq i} \frac{2}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}} (\Delta^{(k-1)-}_i - \Delta^{(k-1)-}_j), \tag{4.4.1}$$

$$\Delta^{(k)-}_i = (\partial_i - 1)\Delta^{(k-1)+}_i + \sum \left(\frac{2}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}} \Delta^{(k-1)+}_i - \frac{2e^{\varepsilon_{ij}}}{e^{\varepsilon_{ij}} - e^{\varepsilon_{ji}}} \Delta^{(k-1)+}_j \right). \tag{4.4.2}$$

Further, if $\Delta_i^{2k+} = f_{2k}(\partial_i^{(1)}, \dots, \partial_i^{(2k-1)})$ and $\Delta_i^{(2k+1)} = f_{2k+1}(\partial_i^{(1)}, \dots, \partial_i^{(2k+1)})$, where f_{2k} and f_{2k+1} are any linear functions, then

$$\begin{aligned} f_{2k+1} &= f_{2k} + f_{2k-1}(\partial_i^{(3)}, \dots, \partial_i^{(2k+1)}), \\ f_{2k} &= f_{2k-1} + f_{2k-2}(\partial_i^{(3)}, \dots, \partial_i^{(2k-1)}). \end{aligned} \tag{4.4.3}$$

It is easy to verify that

$$\tilde{\Omega}_1 = 2\Omega_1 = \tilde{\Omega}_2, \quad \tilde{\Omega}_3 = 2\Omega_3 + 2\Omega_1, \quad \tilde{\Omega}_4 = 4\Omega_3 + 2\Omega_1$$

and, by induction, (4.4.3) implies that $\tilde{\Omega}_k$ is a linear combination of the Ω_{2l+1}. One can show that, the other way round, Ω_{2l+1} can be expressed via $\tilde{\Omega}_k$. □

4.5. Proof of headings ii), iii) of Theorem 1.5. Statement of heading ii) is obvious. Heading iii) follows from Lemma 4.2 and the fact that $z_k = \sum_{i\in I} e_{ii}^{(k)}$.

4.6. Proof of heading iv) of Theorem 1.5. It is easy to verify that $\Theta^* = \sum e_i^* \otimes e_{\bar{i}}^*$ is a $\mathfrak{pe}_1$-invariant whereas $\Theta = \sum\limits_{i\in I} e_i \otimes e_{\bar{i}}$ is a $\mathfrak{pe}_2$-invariant. According to [Se2] the linear hull of all the $\mathfrak{pe}_1$-invariants in $V^{\otimes 2k}$ is isomorphic to

$$\mathrm{Ind}_{H_k}^{\mathfrak{S}_{2k}}(\varepsilon) = \bigoplus_{\lambda=(\alpha_1+1,\dots,\alpha_p+1|\alpha_1,\dots,\alpha_p)} S^\lambda, \tag{4.6.1}$$

where $H_k = \mathfrak{S}_k \circ \mathbb{Z}_2^k$ is the semidirect product, and
$h((\Theta^*)^{\otimes k}) = \varepsilon(h)(\Theta^*)^{\otimes k}$ for any $h \in H_k$.

Similarly, the module of all the $\mathfrak{pe}_2$-invariants in $V^{\otimes 2k}$ is of the form (4.6.1). This implies that

$$\varphi_\lambda(u) = \Theta(v_\lambda^*,\ v_\lambda)(u) = \Theta(e_\lambda\Theta^{*2k},\ e_\lambda\Theta^{2k})(u),$$

where e_λ is the minimal idempotent in the Hecke algebra $H(\mathfrak{S}_{2k}, H_{2k}, \varepsilon)$ corresponding to partition λ, see [St].

If $\sigma \in \mathfrak{S}_{2k}$, then the map

$$\sigma \mapsto \varphi_\sigma, \quad \text{where} \quad \varphi_\sigma(u) = (-1)^{\frac{1}{2}k(k-1)}\Theta^{*\otimes k}(\sigma u\Theta^{\otimes k}),$$

satisfies

$$\varphi_{h_1\sigma h_2} = \varepsilon(h_1h_2)\varphi_\sigma \quad \text{for any } h_1, h_2 \in H_k.$$

Therefore, we obtain a map $H(\mathfrak{S}_{2k}, H_k, \varepsilon) \longrightarrow U(\mathfrak{g})^{*inv}$. By restricting this map onto $S(\mathfrak{h}_{\bar{0}})$ we obtain a map

$$ch : H(\mathfrak{S}_{2k}, H_k, \varepsilon) \longrightarrow S(\mathfrak{h}_{\bar{0}})^*. \tag{4.6.2}$$

Let $K = K_1 \cup K_2$, where $K_1 = \{1, \dots, 2k_1\}$, $K_2 = \{2k_1 + 1, \dots, 2k_1 + 2k_2\}$ and let σ_1 permute the elements of K_1 whereas $\sigma_2$permutes the elements ofK_2; let $\sigma = \sigma_1\sigma_2$. Now, one can verify that

$$\mathrm{ch}(\sigma_1\sigma_2) = \mathrm{ch}(\sigma_1)\mathrm{ch}(\sigma_2).$$

Let now k be odd and $\sigma = (1, \dots, 2k)$ a cycle. Let us calculate $\mathrm{ch}(\sigma)$. We have

$$\Theta^{\otimes k} = \sum e_\psi, \quad \Theta^{*\otimes k} = \sum e_\psi^*, \tag{4.6.3}$$

where $e_\psi = e_{\psi(1)} \otimes \dots \otimes e_{\psi(2k)}$ and the sum runs over all the maps $\psi : \{1, \dots, 2k\} \mapsto \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ such that $\psi(2i) = \overline{\psi(2i-1)}$ for $i = 1, \dots, k$. The element e_ψ^* is similarly defined. Therefore,

$$\mathrm{ch}(\sigma)(u) = (-1)^{\frac{1}{2}k(k-1)}\Theta^{*\otimes k}(u\sum \sigma e_\psi),$$

where in the sum one has to take into account only the summands e_ψ for which $\sigma\psi$ possesses the same property as ψ does. Therefore, we may assume that

$$\psi(1) = \overline{\psi(2)} = \psi(3) = \overline{\psi(4)} = \ldots = \psi(2k-1) = \overline{\psi(2k)}.$$

i.e., $e_\psi = (e_i \otimes e_{\bar{i}})^{\otimes k}$ for $i \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. But for such ψ and $u \in S(\mathfrak{h}_{\bar{0}})$ we have

$$\begin{aligned} ch(\sigma)(u) &= (-1)^{\frac{1}{2}k(k-1)}\Theta^{*\otimes k}(u \textstyle\sum_{i\in I} \sigma(e_i \otimes e_{\bar{i}})^{\otimes k}) \\ &= (-1)^{\frac{1}{2}k(k-1)}\Theta^{*\otimes k}(u \textstyle\sum_{i\in I} (e_i \otimes e_{\bar{i}})^{\otimes k}) \\ &= (-1)^{\frac{1}{2}k(k-1)}\Theta^{*\otimes k}(\textstyle\sum_{i\in I_{\bar{0}}} 2e^{\varepsilon_i}(u)(e_i \otimes e_{\bar{i}})^{\otimes k}) \\ &= 2\textstyle\sum_{i\in I_{\bar{0}}} 2e^{k\varepsilon_i}(u). \end{aligned}$$

Hence, $\mathrm{ch}(\sigma) = 2\sum_{i=1}^n 2e^{k\varepsilon_i}$. Therefore, if $\sigma = \sigma_{\nu_1} \ldots \sigma_{\nu_p}$ is the product of independent cycles of odd lengths, then

$$\mathrm{ch}(\sigma) = 2^{l(\nu)} P_\nu, \text{ where } P_\nu = P_{\nu_1} \ldots P_{\nu_p} \text{ and } P_l = \sum_{i=1}^n e^{l\varepsilon_i}.$$

We see that the map (4.6.2) coincides with the characteristic map Stembridge constructed in [St], p. 85. Therefore, by Theorem 5.2 from [St] we have

$$\mathrm{ch}(e_\lambda) = \Theta_\lambda(e^{\varepsilon_1}, \ldots, e^{\varepsilon_n}) \cdot 2^{n-e(\lambda)} \cdot g_\lambda,$$

where g_λ is the number of shifted standard tableaux of shape λ. □

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