Topology of Lefschetz Fibrations in Complex and Symplectic Geometry

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1 Introduction

Let $X := Y \setminus V$ be a connected real space of dimension $2n$, $n \geq 2$, where Y and V are compact. We consider two classes of such spaces: complex analytic spaces and symplectic spaces. In both cases we allow singularities and we assume that Y is endowed with a Whitney stratification such that V is union of strata.

We say that X has the structure of a *Lefschetz fibration* if there exists a pencil on X with isolated singularities, in a general sense, to be precised bellow (2.1, 2.2).

In the complex analytic case it is well known that one has generic pencils on quasi-projective varieties, hence on spaces which can be embedded (e.g. by Kodaira embedding theorem) into some $\mathbb{P}^n(\mathbb{C})$. In Symplectic Geometry, it was shown by Donaldson [Do-1], [Do-2] that generic Lefschetz pencils exist for compact symplectic manifolds, such that the slices are symplectic submanifolds. See also the discution on symplectic fibrations in McDuff's survey [McD].

In this paper we consider mainly nongeneric pencils. We simultaneously prove, on the one hand, an estimation, under general conditions, of the level of connectivity, say q, of the pair space-section and on the other hand, how the q-th homotopy group of the space X can be described, by generators and relations, from the homotopy group of the section and the monodromies around atypical fibres.

This type of results have a long history in Complex Geometry, going back to Lefschetz [Lef], Zariski and van Kampen. The most recent results of Zariski–Van Kampen type, by Libgober [Li], respectively Chéniot [Ch-2], were proved for generic Lefschetz pencils of hyperplanes and in special cases, namely for the complement of a hypersurface in $\mathbb{P}^n$ or $\mathbb{C}^n$, respectively for a non-singular quasi-projective variety, at the level of homology.

We prove in this paper, under general conditions, that the q-th homotopy group of the space X is a quotient of the homotopy group of the section by the subgroup of images of

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variation maps (which we define below). Our results use only local complex coordinates at certain points of the space Y. Therefore this setting is suitable for Symplectic Geometry.

Our method is based on introspection within geometry of vanishing cycles, in the vein of original Lefshetz's ideas [Lef] (see Lamotke's survey [La]). We also prove a number of consequences, such as on topology of polynomial functions, viewed as nongeneric pencils, and on the topology of complements of hyperplane arrangements.

2   Preliminaries

Let $X := Y \setminus V$, where Y is a compact complex analytic space, respectively a compact symplectic space and V is some compact subspace (complex analytic, resp. symplectic).

In the complex analytic case, a *pencil* on Y is the ratio of two sections of a holomorphic line bundle over Y. This defines a meromorphic function $h := f/g : Y \dashrightarrow \mathbb{P}^1(\mathbb{C})$, which is holomorphic over the complement $Y \setminus A$ of the base locus $A := \{f = g = 0\}$.

In the symplectic case, Donaldson introduced "topological Lefschetz pencils" [Do-2]. We give below an extension of the definition in *loc.cit.*, for nongeneric pencils:

2.1 Definition A *topological (nongeneric) pencil* on Y is a smooth map $h : Y \setminus A \to S^2$ such that:

  (i) h is a submersion out of a finite set of points $\{\alpha_{ij}\} \subset Y \setminus A$.

  (ii) A is of real codimension 4. At any point $x \in A$, there are compatible local complex coordinates such that A is given by $\{f_x = g_x = 0\}$, for some holomorphic germs f_x, g_x and that, on the complement of A in some neighbourhood $\mathcal{N}_x$ of x, h is given by $f_x/g_x : \mathcal{N}_x \setminus A \to \mathbb{P}^1(\mathbb{C}) \simeq S^2$.

  (iii) at each point α_{ij} there are compatible local complex coordinates in which h is represented by a holomorphic function with isolated singularity.

Donaldson's Lefschetz pencils are those for which, at the points of A we have locally $h = z_1/z_2$, for two linear independent functions z_1, z_2, and moreover the isolated singularities α_{ij} of h are quadratic singularities (complex Morse singularities). Our definition extends Donaldson's along the main idea of having local holomorphic models. Donaldson proved a striking result: a compact symplectic manifold (Y, ω) with integral ω admits topological Lefschetz pencils, such that the fibres are symplectic submanifolds representing the Poincaré dual of $k[\omega]$, for given large enough positive integer k.

2.2 We may allow Y be singular and assume it is endowed with a Whitney (A)+(B) stratification $\mathcal{W}$, such that V is union of strata. Then, in Definition 2.1, we replace in (i) "submersion" by "stratified submersion" and in (iii) "isolated singularity" by "stratified isolated singularity", in the usual sense [GM], [Lê]. This also means that we impose to our stratification to be, locally at points of A and the points α_{ij}, a Whitney stratification with complex analytic strata.

Then we say that h defines a *pencil on* X if $X = Y \setminus V$ is as above and h is a pencil on Y as in Definition 2.1. Some of the points α_{ij} of h may be on V, hence outside X. We

consider them also as "singularities" of $h_{|X}$, since they produce jumps in the topology of the pencil.

Our condition (ii) being too general: one can see that the pencil may have singularities (in some sense) along the base locus A. Indeed, the local equations defining A may be "very" singular. On the other hand, even if A is non-singular, it might happen that A is not transversal to the strata of $\mathcal{W}$, which fact is known to produce jumps in the topology of the pencil. In order to have treatable pencils, we single out a class which we shall call pencils with *isolated singularities in the "axis"* A.

Let us first explain what we mean by singularities of a nongeneric pencil.

We define a blow-up along the base locus A and work on the new space. This originates in a suggestion of Thom, put to work by Andreotti and Frankel [AF], see also Lamotke's survey [La]. So, let $\mathbb{Y} := \text{closure}\{(y,t) \in (Y \setminus A) \times S^2 \mid h(y) - t = 0\} \subset Y \times S^2$. It is clear that the intersection $\mathbb{Y} \cap (Y \setminus A) \times S^2$ is just the graph of h, hence it is isomorphic to $Y \setminus A$.

Locally, at points of A, the total space $\mathbb{Y}$ has the following aspect: let $\mathcal{N}_x$ be a neighbourhood in Y of some point $x \in A$. Then $\mathbb{Y} \cap (\mathcal{N}_x \times S^2) = \text{closure}\{(y,t) \in (\mathcal{N}_x \setminus A) \times S^2 \mid \frac{f_x(y)}{g_x(y)} - t = 0\} = \{(y, [s:t]) \in \mathcal{N}_x \times \mathbb{P}^1 \mid sf_x(y) - tg_x(y) = 0\}$.

This is a hypersurface in $\mathcal{N}_x \times S^2$ obtained as a Nash blowing-up along A. Using Definition 2.1(ii), it follows that the space $\mathbb{Y}$ is well defined. Indeed, for two intersecting neighbourhoods of points, $\mathcal{N}_x$ and $\mathcal{N}_y$, we get that $f_y = uf_x$ and $g_y = ug_x$, for some holomorphic unit u, on $\mathcal{N}_x \cap \mathcal{N}_y$. It also follows that the subset $A \times \mathbb{P}^1$ is included into $\mathbb{Y}$.

Let us denote $\mathbb{X} := \mathbb{Y} \cap (X \times S^2)$. Consider the projection $p : \mathbb{Y} \to S^2$ to S^2, its restriction $p_{|\mathbb{X}} : \mathbb{X} \to S^2$ and the projection to the first factor $\sigma : \mathbb{Y} \to Y$. Notice that the restriction of p to $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$ can be identified with h.

The stratification $\mathcal{W}$ on Y, restricted to the open set $Y \setminus A$ induces a Whitney stratification on $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$, via the mentioned identification. We then denote by $\mathcal{S}$ the coarsest Whitney stratification on $\mathbb{Y}$ which coincides over $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$ with the one induced by $\mathcal{W}$ on $Y \setminus A$. This stratification exists within a neighbourhood of $A \times \mathbb{P}^1$, by usual arguments (see e.g. [GLPW]), hence such stratification is well defined on $\mathbb{Y}$. We call it the *canonical stratification* of $\mathbb{Y}$ generated by the stratification $\mathcal{W}$ of Y. The canonical stratification of $\mathbb{X}$ will be the restriction of $\mathcal{S}$ to $\mathbb{X}$.

2.3 Proposition *There exists a finite set $B \subset S^2$ such that the maps $p : \mathbb{Y} \setminus p^{-1}(B) \to S^2 \setminus B$ and $p_{|\mathbb{X}} : \mathbb{X} \setminus p^{-1}(B) \to S^2 \setminus B$ are stratified locally trivial fibrations. In particular, $h : Y \setminus (A \cup h^{-1}(B)) \to S^2 \setminus B$ is a locally trivial fibration.* $\square$

This result is a kind of Isotopy Theorem and can be traced back to Thom's paper [Th]. It is based on the fact that p is proper and complex analytic in a neighbourhood of $A \times \mathbb{P}^1$, and that $\mathcal{S}$ has finitely many strata. We tacitly assume that B denotes the minimal set B which satisfies Proposition 2.3.

2.4 Definition We call *singular locus* of p with respect to $\mathcal{S}$ the following closed subset of $\mathbb{Y}$ (and analytic in the neighbourhood of $A \times \mathbb{P}^1$):

$$\text{Sing}_{\mathcal{S}} p := \bigcup_{\mathcal{S}_\beta \in \mathcal{S}} \text{Sing} p_{|\mathcal{S}_\beta}.$$

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The set of *critical values* of p with respect to $\mathcal{S}$ is $\Lambda := p(\mathrm{Sing}_{\mathcal{S}} p)$.

It follows that the set Λ of critical values of p contains the minimal "bad set" B and, by the compactness of Y, that Λ is a finite set too.

2.5 Definition We say that the topological pencil defined by h (cf. Definitions 2.1, 2.2) is a *(nongeneric) pencil with isolated singularities in the axis* if the singularites of the function p at the blown-up base locus $A \times \mathbb{P}^1$ are at most isolated.

The definition of pencil with isolated singularities in the "axis" is equivalent to the condition $\dim \mathrm{Sing}_{\mathcal{S}} p \le 0$. Furthermore, in this case the singularities are finitely many, by the compactness of Y. They consist of the points $\{\alpha_{ij}\}$ together with the newly defined singularities in the blown-up axis $A \times \mathbb{P}^1$, of which some can be outside $\mathbb{X}$.

We shall use in our results the homotopical (or homological) depth.

2.6 Definition For a discrete subset $\Phi \subset \mathbb{X}$, we denote by $\mathrm{hd}_\Phi \mathbb{X}$ the homotopical depth of $\mathbb{X}$ at Φ. We say that $\mathrm{hd}_\Phi \mathbb{X} \ge q+1$ if, at any point $\alpha \in \Phi$, there is an arbitrarily small neighbourhood $\mathcal{N}$ of α such that the pair $(\mathcal{N}, \mathcal{N} \setminus \{\alpha\})$ is q-connected.

For a manifold M, at some point α, we have: $\mathrm{hd}_\alpha M \ge \dim_{\mathbb{R}} M$. In homology, one defines similarly the homological depth $\mathrm{Hd}_\Phi \mathbb{X}$. Complex V-manifolds are rational homology manifolds. So hd (resp. Hd) measures the defect of beeing a homotopy (resp. homology, for certain coefficients) manifold. In stratified complex spaces, Grothendieck [G] introduced the *rectified homotopical depth*; this was later investigated by Hamm and Lê [HL], who proved several of Grothendieck's conjectures regarding it. See also our Note 5.2.

3 Variation maps and the Main Theorem

We assume that our pencil h on Y has isolated singularities in the base locus. In local coordinates at singularities, the map $p : \mathbb{Y} \to S^2$ is holomorphic. We define global variation maps in homotopy, using the local properties of p.

3.1 Let us fix some notation. For any $M \subset S^2$, we denote $\mathbb{Y}_M := p^{-1}(M)$ and $\mathbb{X}_M := \mathbb{X} \cap \mathbb{Y}_M$ and $X_M := X \cap h^{-1}(M)$. We denote by a_i some point in the set of critical values Λ of p. We denote by $a_{ij} \in \mathbb{Y}$ some point of $\mathrm{Sing}_{\mathcal{S}} p \cap p^{-1}(a_i)$. This means the discrete set $\mathrm{Sing}_{\mathcal{S}} p$ is $\{a_{ij}\}_{i,j}$. For $c \in S^2 \setminus \Lambda$ we say that $\mathbb{Y}_c$ (resp. $\mathbb{X}_c$, resp. X_c) is a general fibre of $p : \mathbb{Y} \to S^2$ (resp. of $p_{|\mathbb{X}} : \mathbb{X} \to S^2$, resp. of $h_{|X} : X \to S^2$).

At some singularity a_{ij}, in local coordinates, we take a small ball B_{ij} centered at a_{ij}. For small enough radius of B_{ij}, this is a "Milnor ball" of the local holomorphic germ of function p at a_{ij}. Next we may take a small enough disc $D_i \subset S^2$ at $a_i \in S^2$, so that (B_{ij}, D_i) is Milnor data for p at a_{ij}. Moreover, we may do this for all (finitely many) singularities in the fibre $\mathbb{Y}_{a_i}$, keeping the same disc D_i, provided it is small enough.

Now the restriction of p to $\mathbb{Y}_{D_i} \setminus \cup_j B_{ij}$ is a trivial fibration if D_i is small enough. One may construct a stratified vector field which trivializes this fibration over D_i and such

that this vector field is tangent to the boundaries of the balls $\mathbb{Y}_{D_i} \cap \partial \bar{B}_{ij}$. Using this, we may also construct a geometric monodromy of the fibration $p_| : \mathbb{Y}_{\partial \bar{D}_i} \to \partial \bar{D}_i$, such that this monodromy is the identity on the complement of the balls, $\mathbb{Y}_{\partial \bar{D}_i} \setminus \cup_j B_{ij}$. The same is true when replacing $\mathbb{Y}_{\partial \bar{D}_i}$ by $\mathbb{X}_{\partial \bar{D}_i}$.

Take some point $c_i \in \partial \bar{D}_i$. We have the geometric monodromy representation:

$$\rho_i : \pi_1(\partial \bar{D}_i, c_i) \to \mathrm{Iso}(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}),$$

where $\mathrm{Iso}(.,.)$ denotes the group of relative isotopy classes of stratified homeomorphisms (which are C^∞ along each stratum). As shown above, we may identify, in the trivial fibration over D_i, the fiber $X_{c_i} \setminus \cup_j B_{ij}$ with the fibre $X_{a_i} \setminus \cup_j B_{ij}$. Then the following morphism of groups is well defined, for any $k \geq 0$:

$$\pi_k(X_{c_i}, X_{a_i} \setminus \cup_j B_{ij}) \to \pi_k(X_{c_i}).$$

Furthermore, in local coordinates at a_{ij}, X_{a_i} is a germ of a complex analytic space; hence, for a small enough ball B_{ij}, the set $B_{ij} \cap X_{a_i} \setminus \cup_j a_{ij}$ retracts to $\partial \bar{B}_{ij} \cap X_{a_i}$, by the local conical structure of analytic sets [BV].

It follows that, if we denote $X_{a_i}^* := X_{a_i} \setminus \cup_j a_{ij}$, then $X_{a_i}^*$ is homotopy equivalent, by retraction, to $X_{a_i} \setminus \cup_j B_{ij}$.

3.2 Definition We call *variation map* the following well defined morphism of groups:

$$\mathrm{var}_i : \pi_k(X_{c_i}, X_{a_i}^*) \to \pi_k(X_{c_i}),$$

for any $k \geq 0$, which enters, as diagonal morphism, in the following diagram (in additive notation):

$$
\begin{array}{ccc}
\pi_k(X_{c_i}) & \overset{\nu_i - \mathrm{id}}{\longrightarrow} & \pi_k(X_{c_i}) \\
{\scriptstyle j_*} \downarrow & {\scriptstyle \mathrm{var}_i} \nearrow & \downarrow {\scriptstyle j_*} \\
\pi_k(X_{c_i}, X_{a_i}^*) & \overset{\nu_i - \mathrm{id}}{\longrightarrow} & \pi_k(X_{c_i}, X_{a_i}^*)
\end{array}
$$

where j_* is induced by inclusion and $\nu_i : \pi_k(X_{c_i}) \to \pi_k(X_{c_i})$ is the monodromy.

Variation morphisms can be defined in homology also and they enter traditionally in the description of global and local fibrations at singular fibres of holomorphic functions, see e.g. [Mi], [La], [Si], [Ch-2]. In dimension 2, already Zariski used $\nu_i - \mathrm{id}$ in his theorem.

For further use, we shall describe a decomposition of S^2. Let $K \subset S^2$ be a closed disc with $K \cap \Lambda = \emptyset$ and let $\mathcal{D}$ denote the closure of its complement in S^2. We denote by $S := K \cap \mathcal{D}$ the common boundary, which is a circle, and take a point $c \in S$. Then take standard paths γ_i (non self-intersecting, non mutually intersecting) from c to c_i, $\gamma_i \subset \mathcal{D} \setminus \cup_i D_i$. The configuration $\cup_i(\bar{D}_i \cup \gamma_i)$ is a deformation retract of $\mathcal{D}$ and we shall identify these two spaces in the following section. We shall also identify all fibres X_{c_i} to the fibre X_c, by parallel transport along paths γ_i.

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3.3 Definition We say that an inclusion of pairs of topological spaces $(N, N') \hookrightarrow (M, M')$ is a q-equivalence if the inclusion induces isomorphism of homotopy groups for $j < q$ and a surjection for $j = q$.

With these notations, we may now state our principal result.

3.4 Theorem *Let h define a Lefschetz fibration on $X = Y \backslash V$ with isolated singularities, including singularities in the axis. Let the axis A be not included in V and denote $\Sigma = \sigma(\text{Sing}_S p)$. For some $k \geq 0$, if the following conditions are fulfilled:*
(C1) $(X_c, X_c \cap A)$ *is k-connected,*
(C2) $(X_c, X_{a_i} \backslash \Sigma)$ *is k-connected, $\forall i$,*
(C3) $\text{hd}_{\overline{X} \cap \text{Sing}_S p} X \geq k + 3$
then

(i) *The inclusion $X_c \hookrightarrow X$ is a $k + 1$ equivalence.*

(ii) $\pi_{k+1}(X) \simeq \pi_{k+1}(X_c) / \langle \text{Im}(\text{var}_i) \rangle_{i=\overline{1,p}}$,
where $\langle \text{Im}(\text{var}_i) \rangle_{i=\overline{1,p}}$ denotes the subgroup normally generated by the images of the variation morphisms.

3.5 Note For the conclusion (i) we need a weaker condition (C3), namely:
(C3i) $\text{hd}_{\overline{X} \cap \text{Sing}_S p} X \geq k + 2.$
This will be clear from the proof, since (C3) is used (with $k + 3$) only in Corollary 4.6 and Proposition 4.7(ii). See also Note 5.2 for comparison with rectified homotopical depth condition.

We shall derive the form of this result in special cases, such as in case $\text{Sing}_S p \cap (A \times \mathbb{P}^1) \cap X = \emptyset$ and also in the complementary case $A \subset V$, see §5.

Let us discuss here one of the particular cases of the above theorem. Let Y be a compact manifold of real dimension $2n$ and let V be a compact subspace (complex analytic or symplectic). Assume that h defines a pencil on X without singularities in the axis. Then condition (C3) is satisfied for $k = 2n - 3$ and, if condition (C1) is satisfied, then condition (C2) is equivalent (see Observation 5.1(iii)) with:
(C2)' $(X_{a_i} \backslash \Sigma, A \cap X_{a_i})$ is $k - 1$ connected.

Now, if we assume in addition that Y is a complex projective manifold, then, for a general Lefschetz pencil, we get $\Sigma = \emptyset$. Under these circumstances, one may notice that conditions (C1) and (C2)' –cf. Obs. 5.1(iii)– are "inductive" conditions, that may be reproduced at each step of an inductive slicing procedure. This follows from the fact that, in quasi-projective varieties, we dispose of generic Lefschetz pencils and that the axis A can be regarded, in turn, as a generic slice of a slice (either X_c or X_{a_i}) of X. For instance, in cases $Y = \mathbb{P}^n$ or $Y = \mathbb{C}^n$, we get the following:

3.6 Corollary *Let $\mathcal{Y}$ denote either $\mathbb{P}^n$ or $\mathbb{C}^n$. Let $V \subset \mathcal{Y}$ be a complex algebraic subspace, not necessarily irreducible. For any hyperplane $H \subset \mathcal{Y}$ transversal to all strata of V, we have:*

(i) $H \cap (\mathcal{Y}^n \setminus V) \hookrightarrow \mathcal{Y}^n \setminus V$ *is a* $n + \operatorname{codim} V - 2$ *equivalence.*

(ii) *There exist generic pencils of hyperplanes such that H is a generic member of such a pencil. Then* $\pi_{n+\operatorname{codim}V-2}(\mathcal{Y} \setminus V) \simeq \pi_{n+\operatorname{codim}V-2}(H \cap (\mathcal{Y} \setminus V))/\langle \operatorname{Im}(\operatorname{var}_i)\rangle_{i=\overline{1,p}}.$

Proof We prove the statement (i) by induction, checking at each step the conditions in Theorem 3.4. We notice that $\Sigma = \emptyset$ at every step. Condition (C3) is empty at every step, since $\mathcal{Y} \setminus V$ is nonsingular and a generic pencil has no singularities within $\mathcal{Y} \setminus V$.

We give the proof in the case $\mathcal{Y} = \mathbb{C}^n$ since in the other case the proof is the same, except that in the affine case the generic pencils have the property that their axis is transversal not only to the strata of V but also to the strata of some Whitney stratification of the projective closure of V, which restricts on the affine part to the old stratification.

We may take a second generic pencil on $H \cap (\mathbb{C}^n \setminus V) = \mathbb{C}^{n-1} \setminus V$ having the first axis A as generic member. We may continue this procedure a number of $\dim V$ times, until we get as slice the complement of a finite set $V \cap \mathbb{C}^{n-\dim V}$ into an affine space $\mathbb{C}^{n-\dim V}$. At each step, conditions (C1) and (C2)'–see Observation 5.1(iii)– reproduce, increasing k by 1. In the first step, we get $k = 2n - 2\dim V - 3 = n + \operatorname{codim} V - 3 - \dim V$. Theorem 3.4 applies inductively to yield, on the one hand the conditions (C1) and (C2)' at each step, on the other hand the conclusion, which after the last step becomes our statement (i). Then statement (ii) follows as in the proof of Theorem 3.4. $\square$

In case of $\mathbb{P}^n \setminus V$, part (i) of Corollary 3.6 was proved at homology level, by Chéniot [Ch-1]; his article takes more than 100 pages. For generic pencils of hyperplanes on complex spaces, Chéniot [Ch-2] and Eyral [Ey] proved connectivity results which are particular cases of part (i) of our Theorem 3.4. Our proof, based on study of the geometry of vanishing cycles, is different. It gives strength and clarity to the Lefschetz Principle in a far reaching degree of generality, including the Second Lefschetz Theorem (Theorem 3.4(ii)).

We shall give more applications (§5) in case the Lefschetz structure of the space X is hereditary on slices. We especially draw consequences of the fact that the variation maps are injective or trivial, in two particular situations: X is $\mathbb{C}^n$ and the pencil is defined by a polynomial function and X is the complement of a (non central) arrangement of hyperplanes.

4 Proofs

When dealing with homotopy groups, we shall need to apply the homotopy excision theorem of Blackers and Massey [BM], see also [Gr]. The conditions for the homotopy excision have to be carefully checked. In homology, proofs would simplify quite a lot.

We shall use the notations introduced in the previous section. We let $A' := A \cap X_c$ and assume that $A' \neq \emptyset$.

4.1 Proposition *If (X_c, A') is k-connected, then the inclusion:*

$$(X_{\mathcal{D}}, X_c) \overset{\iota}{\hookrightarrow} (X, X_c)$$

is a $k + 2$ equivalence.

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Proof **Step 1.** If (X_c, A') is k-connected, then (X_S, X_c) is $k + 1$-connected. Note first that X_S is homotopy equivalent to the subset $\mathbb{X}_S \cup A' \times K$ of $\mathbb{X}_K$. Let I and J be two arcs which cover S, as in the proof of Proposition 4.7. We have the homotopy equivalence $(X_S, X_c) \overset{\text{ht}}{\simeq} (\mathbb{X}_I \cup (A' \times K) \cup \mathbb{X}_J \cup (A' \times K)), \mathbb{X}_J \cup (A' \times K))$.

Then, by homotopy excision, if we assume that the pairs $(\mathbb{X}_I \cup (A' \times K), \mathbb{X}_{\partial I} \cup (A' \times K))$ and $(\mathbb{X}_J \cup (A' \times K), \mathbb{X}_{\partial J} \cup (A' \times K))$ are $k + 1$ connected, then the following morphism:

$$\pi_j(\mathbb{X}_I \cup (A' \times K), \mathbb{X}_{\partial I} \cup (A' \times K)) \to \pi_j(\mathbb{X}_I \cup (A' \times K) \cup \mathbb{X}_J \cup (A' \times K), \mathbb{X}_J \cup (A' \times K))$$

is an isomorphism for $j \leq 2k + 1$. This implies that (X_S, X_c) is $k + 1$ connected.

To prove our assumption, notice the pair $(\mathbb{X}_I \cup (A' \times K), \mathbb{X}_{\partial I} \cup (A' \times K))$ is homotopy equivalent to $(\mathbb{X}_c \times I, \mathbb{X}_c \times \partial I \cup A' \times I)$ and this, in turn, is just the product of pairs $(\mathbb{X}_c, A') \times (I, \partial I)$. Since $(\mathbb{X}_c, A')$ is k-connected by hypothesis, our claim follows.

Step 2. $\pi_i(X_S, X_c) \simeq \pi_{i+1}(X_K, X_S)$ for all $i \geq 0$. This holds without restrictions since it follows directly from the exact sequence of the triple (X_K, X_S, X_c) and the fact that $X_K \overset{\text{ht}}{\simeq} X_c$.

Step 3. $(X_{\mathcal{D}}, X_S) \hookrightarrow (X, X_K)$ is a $k + 2$ equivalence, by homotopy excision, since (X_K, X_c) is $k + 2$-connected. This last fact follows directly from Steps 1 et 2.

Finally, by examining the exact sequence of the triple $(X_{\mathcal{D}}, X_S, X_c)$, we see that $(X_{\mathcal{D}}, X_c) \hookrightarrow (X_{\mathcal{D}}, X_c)$ is a $k + 2$ equivalence, since (X_S, X_c) is $k + 1$-connected (by Step 1). Comparing this to Step 3., we conclude that $(X_{\mathcal{D}}, X_c) \hookrightarrow (X, X_K)$ is a $k + 2$ equivalence. $\square$

Consider the commutative diagram:

(1)
$$
\begin{array}{ccc}
\pi_{k+2}(X_{\mathcal{D}}, X_c) & \overset{\iota_*}{\longrightarrow} & \pi_{k+2}(X, X_c) \\
_{\partial_1} \searrow & & \swarrow _{\partial} \\
& \pi_{k+1}(X_c) &
\end{array}
$$

where ∂ and ∂_1 are boundary morphisms. Since Proposition 4.1 shows that ι_* is an epimorphism, we get:

4.2 Corollary *If (X_c, A') is k-connected then, in the diagram (1), we have $\operatorname{Im} \partial = \operatorname{Im} \partial_1$.* $\square$

4.3 Lemma (i) $(X_{\mathcal{D}}, X_c)$ *is homotopy equivalent to* $(\mathbb{X}_{\mathcal{D}}, X_c)$.

(ii) *If $(\mathbb{X}_{D_i}, X_{c_i})$ is $k + 1$-connected for any i, then the inclusion $(\cup_i \mathbb{X}_{D_i}, \cup_i X_{c_i}) \to (\mathbb{X}_{\mathcal{D}}, X_c)$ is a $k + 1$ equivalence.*

Proof (i) Notice that, for any $M \subset S^2$, X_M is homotopy equivalent to $\mathbb{X}_M$ to which one attaches along $A' \times M$ the product $A' \times \operatorname{Cone}(M)$. Since $\mathcal{D}$ is contractible, it follows from this that $X_{\mathcal{D}} \overset{\text{ht}}{\simeq} \mathbb{X}_{\mathcal{D}}$.

(ii) follows by homotopy excision. $\square$

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Consider next the following commutative diagram:

$$\oplus_i \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \simeq \pi_{k+2}(\cup_i \mathbb{X}_{D_i}, \cup_i X_{c_i})$$

(2)
$$\iota_* \downarrow \qquad\qquad\qquad \searrow \partial_2$$
$$\pi_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c) \xrightarrow{\partial_1} \pi_{k+1}(X_c),$$

where X_{c_i} is identified with X_c by parallel transport along the paths and the boundary operator ∂_2 is identified to the sum of the boundary operators $\partial_i : \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \to \pi_{k+1}(X_{c_i})$. With these notations we have the following:

4.4 Proposition *If $(\mathbb{X}_{D_i}, X_{c_i})$ is $k+1$-connected for all i, then $\mathrm{Im}\,\partial_1 = \mathrm{Im}\,\partial_2$ in diagram (2)*.

Proof We use Hurewicz maps between homotopy and homology groups (denoted h_i, h_1 and h_2 below). We have the following commutative diagram induced by inclusions:

$$\oplus_i \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \simeq \pi_{k+2}(\cup_i \mathbb{X}_{D_i}, \cup_i X_{c_i}) \xrightarrow{\iota_*} \pi_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c)$$
$$\oplus_i h_i \downarrow \qquad\qquad h_1 \downarrow \qquad\qquad \downarrow h_2$$
$$\oplus_i H_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \simeq H_{k+2}(\cup_i \mathbb{X}_{D_i}, \cup_i X_{c_i}) \xrightarrow{j_*} H_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c).$$

By our hypothesis and by Hurewicz theorem (see e.g. [Sp]), we have that the Hurewicz map $h_i : \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \to H_{k+2}(\mathbb{X}_{D_i}, X_{c_i})$ is an isomorphism, for all i, hence h_1 is an isomorphism. The same is h_2, since $(\mathbb{X}_{\mathcal{D}}, X_c)$ is $(k+1)$-connected (by using Lemma 4.3(ii)). Next, j_* is an excision in homology, hence isomorphism. It follows that ι_* is an isomorphism. This shows that, in the diagram (2), ∂_1 is "equivalent" to ∂_2. $\square$

At this point, our problem reduces to the following:
(a). to prove that $(\mathbb{X}_{D_i}, X_{c_i})$ is $k+1$-connected, for all i, and
(b). to find the image of the map $\partial_i : \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \to \pi_{k+1}(X_{c_i})$, for all i.

We shall reduce these problems again, by replacing $\mathbb{X}_{D_i}$ by $\mathbb{X}_{D_i}^* := \mathbb{X}_{D_i} \setminus \mathrm{Sing}_{\mathcal{S}} p$. For this, we use condition (C3) for (b), respectively condition (C3i) for (a).

4.5 Lemma *If $\mathrm{hd}_{\mathbb{X} \cap \mathrm{Sing}_{\mathcal{S}} p} \mathbb{X} \geq q+1$ then the inclusion of pairs $(\mathbb{X}_{D_i}^*, X_{c_i}) \xhookrightarrow{j} (\mathbb{X}_{D_i}, X_{c_i})$ is a q-equivalence, for all i.*

Proof Due to the exact sequence of the triple $(\mathbb{X}_{D_i}, \mathbb{X}_{D_i}^*, X_{c_i})$, it will be sufficient to prove that $(\mathbb{X}_{D_i}, \mathbb{X}_{D_i}^*)$ is q-connexe, for all i. This is true since, by homotopy excision, the inclusion

$$(\cup_j B_{ij} \cap \mathbb{X}_{D_i}, \cup_j \partial B_{ij} \cap X_{c_i}) \hookrightarrow (\mathbb{X}_{D_i}, \mathbb{X}_{D_i}^*)$$

is a q equivalence. We have denoted here by $B_{ij} \subset \mathbb{X}$ a small enough ball centered at the singular point $a_{ij} \in \mathrm{Sing}_{\mathcal{S}} p$. We have used the hypothesis $\mathrm{hd}_{\mathbb{X} \cap \mathrm{Sing}_{\mathcal{S}} p} \mathbb{X} \geq q+1$ and that $\mathbb{X}_{D_i} \cap B_{ij} \setminus \{a_{ij}\}$ retracts to $\mathbb{X}_{D_i} \cap \partial B_{ij}$, by the local conical structure, which respects the stratification. $\square$

4.6 Corollary *If $\mathrm{hd}_{\mathbb{X} \cap \mathrm{Sing}_{\mathcal{S}} p} \mathbb{X} \geq k+3$, then, for all i:*

$$\mathrm{Im}(\partial_i : \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) \to \pi_{k+1}(X_{c_i})) = \mathrm{Im}(\partial_i' : \pi_{k+2}(\mathbb{X}_{D_i}^*, X_{c_i}) \to \pi_{k+1}(X_{c_i})).$$

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Proof From the proof of Lemma 4.5, it follows that $(\mathbb{X}^*_{D_i}, X_{c_i})$ is $(k+2)$-connected. We have that $\partial'_i = \partial_i \circ j_*$, where $j_* : \pi_{k+2}(\mathbb{X}^*_{D_i}, X_{c_i}) \to \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i})$ is induced by the inclusion. By Lemma 4.5, j_* is surjective, hence $\mathrm{Im}\,\partial'_i = \mathrm{Im}\,\partial_i$. $\qquad\square$

We shall use the notation $X^*_{a_i} := X_{a_i} \setminus \Sigma$. The last step in the proof of Theorem 3.4 is the following result, where the variation maps come in:

4.7 Proposition *If* $(X_{c_i}, X^*_{a_i})$ *is k-connected for all i, then:*

(i) $(\mathbb{X}^*_{D_i}, X_{c_i})$ *is $k + 1$-connected.*

(ii) $\mathrm{Im}\,\partial'_i = \mathrm{Im}(\mathrm{var}_i : \pi_{k+1}(X_{c_i}, X^*_{a_i}) \to \pi_{k+1}(X_{c_i}))$.

Proof Let us take here Milnor data (B_{ij}, D_i) at the (stratified) singularities a_{ij}, namely small enough balls B_{ij} and "very small" discs D_i, such that radius $D_i \ll$ radius B_{ij}. We shall give the proof for a fixed index i and therefore we suppres the lower indices i in the following.

(i). Let D^* denote $D \setminus \{a\}$. By retraction, we identify D^* to a circle and cover this circle with the union of two arcs $I \cup J$, like follows: for the standard circle S^1, we take $I := \{\exp i\pi t \mid t \in [-\frac{1}{2}, 1]\}$, $J := \{\exp i\pi t \mid t \in [\frac{1}{2}, 2]\}$. Then $\mathbb{X}_{D^*} \overset{\mathrm{ht}}{\simeq} \mathbb{X}_I \cup \mathbb{X}_J$ and $X_c \overset{\mathrm{ht}}{\simeq} \mathbb{X}_J = X_c \times J$. With these, we have the following homotopy equivalences:

$$(\mathbb{X}^*_D, X_c) \overset{\mathrm{ht}}{\simeq} (\mathbb{X}_{D^*} \cup X^*_a \times D, X_c \cup X^*_a \times D) \overset{\mathrm{ht}}{\simeq} (\mathbb{X}_I \cup \mathbb{X}_J \cup X^*_a \times D, \mathbb{X}_J \cup X^*_a \times D).$$

We then excise $\mathbb{X}_J \cup X^*_a \times D$ from the last pair and get $(\mathbb{X}_I, X_c \times \partial I \cup X^*_a \times I)$, which in turn is homotopy equivalent to $(X_c, X^*_a) \times (I, \partial I)$. Now, since (X_c, X^*_a) is k-connected, this last product is $k + 1$ connected.

It remains to examine the condition for the homotopy excision. This is the level of connectivity of the pair $(\mathbb{X}_J \cup X^*_a \times D, X_c \times \partial I \cup X^*_a \times I)$. We transform this pair by the following sequence of homotopy equivalences (we replace I by D and this by J):

$$(\mathbb{X}_J \cup X^*_a \times D, X_c \times \partial I \cup X^*_a \times I) \overset{\mathrm{ht}}{\simeq}$$
$$(\mathbb{X}_J \cup X^*_a \times D, X_c \times \partial I \cup X^*_a \times D) \overset{\mathrm{ht}}{\simeq}$$
$$(\mathbb{X}_J \cup X^*_a \times J, X_c \times \partial J \cup X^*_a \times J) \overset{\mathrm{ht}}{\simeq}$$
$$(\mathbb{X}_J, X_c \times \partial J \cup X^*_a \times J) \overset{\mathrm{ht}}{\simeq} (X_c, X^*_a) \times (J, \partial J).$$

The result implies that the pair on top is $k + 1$-connected. It then follows that the excision in cause is a $2k + 2$ equivalence. This proves that $(\mathbb{X}^*_D, X_c)$ is $k + 1$-connected.

(ii). The variation map is now identifiable, on the excision diagram, by the arrow which makes the following diagram commutative:

$$\begin{array}{ccc}
\pi_{k+2}(\mathbb{X}^*_{D_i}, X_{c_i}) & \xrightarrow{\ \partial'_i\ } & \pi_{k+1}(X_{c_i}) \\
{\scriptstyle\mathrm{excision}}\uparrow{\scriptstyle\iota_*} & & \uparrow{\scriptstyle\mathrm{var}_i} \\
\pi_{k+2}(\mathbb{X}_I, X_{c_i} \times \partial I \cup X^*_{a_i} \times I) & \simeq & \pi_{k+1}(X_{c_i}, X^*_{a_i}) \times \pi_1(I, \partial I).
\end{array}$$

$\qquad\square$

This ends the proof of Theorem 3.4.

5 Comments and further results

Several other statements can be derived from Theorem 3.4 and its proof, by taking into account the following (still under the condition $A \cap X \neq \emptyset$):

5.1 Observations   (i) In case $\mathbb{X} \cap \mathrm{Sing}_S p = \emptyset$, the condition (C3) is void.

(ii) In case $(A \times S^2) \cap \mathbb{X} \cap \mathrm{Sing}_S p = \emptyset$, we may replace condition (C3) by the more global condition, which is also more general:

(C3)' $(X, X \setminus \Sigma)$ is $k + 2$-connected.

(iii) In case $(A \times S^2) \cap \mathrm{Sing}_S p = \emptyset$, the condition (C2) is equivalent, under the assumption (C1), with the following:

(C2)' $(X_{a_i}^*, A \cap X_{a_i}^*)$ is $(k-1)$-connected.

Proof (ii). By examining the Proof of Theorem 3.4, we see that we have used the homotopy depth condition only to compare $\mathbb{X}_{D_i}$ to $\mathbb{X}_{D_i}^*$. We may cut off the proof this comparison (Lemma 4.5 and Corollary 4.6) and start from the beginning with the space $X \setminus \Sigma$ instead of the space X. Taking into account that $\mathbb{X}_{D_i}^* = \mathbb{X}_{D_i}^* \setminus \Sigma$, for all i (since of (C3)'), the effect of this change is that the proof yields the conclusion "$X_c \hookrightarrow X \setminus \Sigma$ is a $k + 1$ equivalence" and the corresponding second part (ii). But at this final stage, we may substitute $X \setminus \Sigma$ by X since they have isomorphic homotopy groups up to π_{k+1}, by condition (C3)'.

(iii) When there are no singularities in the axis, we have $A \cap X_{a_i}^* = A \cap X_c$, for any i. Then the exact sequence of the triple $(X_c, X_{a_i}^*, A \cap X_{a_i}^*)$ shows that the boundary morphism

$$\pi_q(X_c, X_{a_i}^*) \to \pi_{q-1}(X_c, A \cap X_{a_i}^*)$$

is an isomorphism, for $q \leq k$, by condition (C1). This implies our claimed equivalence. $\square$

5.2 Note For a pencil on a complex space, with isolated singularities in the axis, conditions (C2) and (C3) can be replaced by:
(C4) $\mathrm{rhd}\, X \geq k + 3$,
    respectively by (see Note 3.5):
(C4i) $\mathrm{rhd}\, X \geq k + 2$.

Indeed, $\mathrm{rhd}\, X \geq q$ implies $\mathrm{rhd}\, \mathbb{X} \geq q$ (since $\mathbb{X}$ is a hypersurface in $X \times \mathbb{P}^1$ and then apply the result of Hamm and Lê [HL, Theorem 3.2.1]) and this in turn implies $\mathrm{hd}_\alpha \mathbb{X} \geq q$, for any point $\alpha \in \mathbb{X}$, by definition.

Next, $\mathrm{rhd}\, \mathbb{X} \geq q$ implies that the pair $(\mathbb{X}_{D_i}, X_{c_i})$ is $q-1$ connected, by [Ti-2, Proposition 4.1]. This shows that conditions (C1) + (C4i) imply the connectivity statement (i) of Theorem 3.4, by only using, as a shortcut, Proposition 4.1 and Lemma 4.3.

Furthermore, if we assume (C4) instead of (C4i), then, besides that $(\mathbb{X}_{D_i}, X_{c_i})$ is $k+2$ connected (shown just above), it follows that the pair $(\mathbb{X}_{D_i}^*, X_{c_i})$ is $k + 1$ connected, by Lemma 4.5. By the proof of Proposition 4.7, the $k + 1$ connectivity of $(\mathbb{X}_{D_i}^*, X_{c_i})$ is equivalent to the k-connectivity of the pair $(X_{c_i}, X_{a_i}^*)$, which is condition (C2). $\square$

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The case $A \subset V$.

We discuss in the following the case $A' = \emptyset$, or equivalently $A \subset V$, which is complementary to the one we have considered until now. Then, for a pencil $h_{|X} : X \to S^2$, the condition (C1) would be replaced by the condition "X_c is k-connected", which is too restricting.

Nevertheless, in case $h_{|X}$ is not onto S^2, the situation becomes interesting. So let us assume that $A \subset V$ and that V contains a fibre of the pencil $h : Y \setminus A \to S^2$. Even if the axis A is outside the space X, the "singularities in the axis" influence on the topology of the pencil.

We have the following result on a class of nongeneric pencils, disjoint from the class considered in Theorem 3.4.

5.3 Theorem *Let $X = Y \setminus V$ have a structure of Lefschetz fibration with isolated singularities in the axis, defined by the pencil $h : Y \setminus A \to S^2$, such that V contains a member of the pencil. For some fixed $k \geq 0$, assume that $(X_c, X^*_{a_i})$ is k connected, for a general member X_c and any atypical one $X^*_{a_i}$. Then:*

  (i) *If $(X, X \setminus \Sigma)$ is $k + 1$ connected, then the inclusion $X_c \hookrightarrow X$ is a $k + 1$-equivalence.*

  (ii) *If $(X, X \setminus \Sigma)$ is $k + 2$ connected, then:*

$$\pi_{k+1}(X) \simeq \pi_{k+1}(X_c)/\langle \mathrm{Im}(\mathrm{var}_i)\rangle_{i=\overline{1,p}}.$$

Proof The proof follows the lines of the proof of Theorem 3.4 and we shall only point out the differences, using the same notations. In our case, the target of $h_{|X}$ is $S^2 \setminus \{\alpha\}$ for some $\alpha \in S^2$. We have $\mathcal{D} \stackrel{\mathrm{ht}}{\simeq} S^2 \setminus \{\alpha\}$ and therefore $X_{\mathcal{D}} \stackrel{\mathrm{ht}}{\simeq} X$. Examining the proofs of Proposition 4.1 and Corollary 4.2, we see that, under our assumptions, their conclusions hold without any restrictions on k. Hence (C1) does not enter as condition in our proof. On the other hand, from Observation 5.1(ii) and (iii), we can use (C2)' and (C3)' instead of (C2), resp. (C3). $\qquad\square$

Polynomials on $\mathbb{C}^n$ as nongeneric pencils.

Let $X = \mathbb{C}^n$ and $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. This extends as a meromorphic function on $\mathbb{P}^n$: if $\deg f = d$, then $h = \tilde{f}/z^d : \mathbb{P}^n \setminus A \to \mathbb{P}^1$, where $\tilde{f}$ is the homogeneized of f with respect to the new variable z and $A = \{f_d = 0\} \subset H_\infty$ is the indeterminacy locus (axis of the pencil). Here we have $Y = \mathbb{P}^n$, $V = H_\infty = \{z = 0\} \subset \mathbb{P}^n$. Since $H_\infty = h^{-1}(\infty)$, we are in the situation described above, namely we have a pencil $h_{|\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}$, where $h_{|\mathbb{C}^n} = f$. In particular, $\Sigma \cap X = \mathrm{Sing} f$.

For such a pencil, we may work under more general hypotheses, namely we assume that f has isolated singularities, but no condition on singularities in the axis, which may be non-isolated. Indeed, take the complement of a big ball $B \subset \mathbb{C}^n$, centered at the origin of a fixed system of coordinates on $\mathbb{C}^n$. The complement $C_B := \mathbb{C}^n \setminus B$ is a kind

of "uniform" neighbourhood of the whole H_∞ and of all singularities in the axis together. For big enough radius of B, we have

$$X_{a_i} \cap C_B \stackrel{\text{ht}}{\simeq} X_{a_i},$$

for any i, since the distance function has a finite set of critical values on the algebraic sets X_{a_i}. This implies, as in 3.1, that there is a well defined geometric monodromy representation at each $a_i \in \Lambda \subset \mathbb{C}$, $\rho_i : \pi_i(\partial \bar{D}_i, c_i) \to \text{Iso}(X_{c_i}, X_{c_i} \setminus (B \cup \cup_j B_{ij}))$, where the B_{ij}'s are small enough balls around the critical points of f on X_{a_i}, if any. This induces, exactly as before (see Definition 3.2), a variation map:

$$\text{var}_i : \pi_k(X_{c_i}, X_{a_i}^*) \to \pi_k(X_{c_i}),$$

where $X_{a_i}^* := X_{a_i} \setminus \text{Sing} f$ embeds into X_{c_i}.

Therefore, under these notations, Theorem 5.3 holds for a pencil defined by a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$. When working in homology, we have in addition a more precise grip on variation maps. Firstly, the boundary map $H_{*+1}(\mathbb{C}^n, X_c) \stackrel{\partial}{\to} \tilde{H}_*(X_c)$ is an isomorphism and secondly, we have by excision: $H_{*+1}(\mathbb{C}^n, X_c) \simeq \oplus_i H_{*+1}(X_{D_i}, X_c)$. These show that $H_*(X_c)$ decomposes into the direct sum of vanishing cycles at each atypical fibre X_{a_i}.

In case of a holomorphic function germ with isolated singularity on a germ $(\mathbb{C}^n, 0)$, the variation map of the local monodromy is an isomorphism on the level of homology. In our global case of a polynomial function with isolated singularities, the variation maps cannot be isomorphisms since the homology of the fibre $H_*(X_c)$ captures information on vanishing cycles at all fibres X_{a_i} together. Nevertheless, we can prove the most we can ask.

5.4 Proposition *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function with isolated singularities. Then:*

(i) *If $(X_c, X_{a_i}^*)$ is k-connected for any i, then $(\mathbb{C}^n, X_c)$ is $k+1$-connected.*

(ii) *In homology, the variation map $\text{var}_i : H_*(X_{c_i}, X_{a_i}^*) \to H_*(X_{c_i})$ is injective, for any i.*

Proof Since the fibres of f are Stein spaces of dimension $n-1$, their homology groups are trivial in dimensions $\geq n$. The condition (C3)' is largely satisfied, since $(\mathbb{C}^n, \mathbb{C}^n \setminus \text{Sing} f)$ is $(2n-1)$-connected. Hence part (i) follows from Theorem 5.3. For part (ii), remark first that, by the above arguments, the boundary map $\partial_i : H_{*+1}(X_{D_i}, X_{c_i}) \to \tilde{H}_*(X_{c_i})$ is injective, for any i. Next one may replace X_{D_i} by $X_{D_i}^*$ since $(X_{D_i}, X_{D_i}^*)$ is $(2n-1)$-connected. It follows that the boundary morphism $\partial_i' : H_{*+1}(X_{D_i}^*, X_{c_i}) \to \tilde{H}_*(X_{c_i})$ is injective. As in Proposition 4.7, one may identify $H_{*+1}(X_{D_i}^*, X_{c_i})$ to $H_*(X_{c_i}, X_{a_i}^*)$, by excision and ∂_i' can be identified with $\text{var}_i : H_*(X_{c_i}, X_{a_i}^*) \to H_*(X_{c_i})$. $\qquad\square$

The subset $X_{a_i}^* \stackrel{\iota}{\hookrightarrow} X_{c_i}$ plays here the role of the boundary of the Milnor fibre in the local case. We say that $\text{Im}\,\iota_*$ is the group of "boundary cycles" at a_i. Then it also follows that:

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5.5 Corollary *Boundary cycles are exactly those which are invariant under the monodromy at a_i, i.e.:*

$$\operatorname{Im} \iota_* = \operatorname{Ker}(\nu_i - \operatorname{id}).$$

Proof We have the following commutative diagram, where the first row is the exact sequence of the pair $(X_{c_i}, X_{a_i}^*)$:

$$
\begin{array}{ccccc}
H_*(X_{a_i}^*) & \xrightarrow{\iota_*} & H_*(X_{c_i}) & \xrightarrow{j_*} & H_*(X_{c_i}, X_{a_i}^*) \\
& {\scriptstyle \nu_i - \operatorname{id}} \searrow & & \swarrow {\scriptstyle \operatorname{var}_i} & \\
& & H_*(X_{c_i}) & &
\end{array}
$$

We have that $\operatorname{Im} \iota_* = \operatorname{Ker} j_*$. Since $\nu_i - \operatorname{id} = \operatorname{var}_i \circ j_*$, and since var_i is injective, by Proposition 5.4, our claim follows. $\qquad\square$

Complements of arrangements and minimal models.

Consider the complex space $X = \mathbb{C}^n \setminus V$, where V is a hypersurface. Since $\mathbb{C}^n \setminus V$ is a Stein space, it has the homotopy type of a CW-complex of dimension $\leq n$. For a generic hyperplane $H \in \mathbb{C}^n$ we have that $(\mathbb{C}^n \setminus V, H \setminus V)$ is $(n-1)$-connected, by Corollary 3.6. We may deduce:

5.6 Corollary *The space $\mathbb{C}^n \setminus V$ is obtained, up to homotopy type, from the slice $H \setminus V$ to which one attaches n-cells.* $\qquad\square$

Slicing again $H \setminus V$ by a generic hyperplane and repeating this, we get a CW-complex model of the space $\mathbb{C}^n \setminus V$, which starts with a bouquet of d circles, the 1-dimensional skeleton. One may ask if this model is minimal, in the sense that the number of q-cells equals the betti number $b_q(\mathbb{C}^n \setminus V)$, for any q. This question was raised to us by S. Papadima in connection to our paper [Ti-1] (in which we construct models of hypersurfaces in families) and in connection to his paper with A. Suciu [PS], in which they use minimal models to get information on homotopy groups of complements of central arrangements.

We show here that the model of $\mathbb{C}^n \setminus V$ is indeed minimal in case of a (not necessarily central) arrangement of hyperplanes. This will be done by induction, using at each step the particular behaviour of the variation maps within a generic pencil of hyperplanes in $\mathbb{C}^n$. Namely, let H be defined by $l = 0$ and consider the pencil $\{l = \alpha\}_{\alpha \in \mathbb{C}}$. The genericity of the pencil amounts to the condition that the direction of the pencil is chosen such that all members of the pencil are transversal to all strata of positive dimension of the canonical stratification of the arrangement V (which is generically fulfilled). With these conventions, we have the following result:

5.7 Corollary *For a (noncentral) arrangement of hyperplanes $\mathcal{A} \subset \mathbb{C}^n$ and a generic hyperplane H, we have: $H_j(\mathbb{C}^n \setminus \mathcal{A}) \simeq H_j(H \setminus \mathcal{A})$, for $j \leq n-1$ and $H_n(\mathbb{C}^n \setminus \mathcal{A}) \simeq H_n(\mathbb{C}^n \setminus \mathcal{A}, H \setminus \mathcal{A})$.*
In particular $\mathbb{C}^n \setminus \mathcal{A}$ has a minimal model.

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Proof We work in homology. Having the result in Corollary 5.6, we still have to look at the exact sequence:

$$0 \to H_n(\mathbb{C}^n \setminus \mathcal{A}) \to H_n(\mathbb{C}^n \setminus \mathcal{A}, H \setminus \mathcal{A}) \to H_{n-1}(H \setminus \mathcal{A}) \overset{\iota_*}{\to} H_{n-1}(\mathbb{C}^n \setminus \mathcal{A}) \to 0.$$

We claim that ι_* is injective. Indeed, by our results, in particular Corollary 3.6(ii), $\mathrm{Ker}\,\iota_* = \langle \mathrm{Im}(\mathrm{var}_i)\rangle_i$. We observe now that var_i is trivial, for any i. The reason for this is that the atypical fibres X_{a_i} of our pencil are exactly those which intersect the point-strata of the canonical stratification of $\mathcal{A}$. There are no singularities in the axis and the only singularities of the pencil are in fact these points. We localize at such a point and we notice that we may define a local geometric monodromy of the pencil, respecting the hyperplanes of $\mathcal{A}$, by $x \mapsto x\exp(i\pi t)$, for any local coordinate x. This shows that the local monodromy is trivial and we can extend the local monodromies to a global one around X_{a_i}, which is the identity on X_{a_i} as subset of X_{c_i}.

We have proved that ι_* is injective, which also means that the above exact sequence splits in the middle. This proves our first statement. In particular, we get that the number of the n-cells attached to $H \setminus \mathcal{A}$ in order to obtain $\mathbb{C}^n \setminus \mathcal{A}$ is equal to $b_n(\mathbb{C}^n \setminus \mathcal{A})$. Our second statement follows then by induction. $\qquad\square$

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