DUNKL OPERATORS FOR COMPLEX REFLECTION GROUPS

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ABSTRACT. Dunkl operators for complex reflection groups are defined in this paper. These commuting operators give rise to a parameter family of deformations of the polynomial De Rham complex. This leads to the study of the polynomial ring as a module over the "rational Cherednik algebra", and a natural contravariant form on this module. In the case of the imprimitive complex reflection groups $G(m,p,N)$, the set of singular parameters in the parameter family of these structures is described explicitly, using the theory of nonsymmetric Jack polynomials.

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1. Introduction

Let V be a finite dimensional Hilbert space. A finite complex reflection group in the unitary group $U(V)$ is a finite subgroup which is generated by pseudo-reflections (or "complex reflections"), transformations with all eigenvalues but one equal to unity. In this paper we introduce for each such group a commutative algebra of parameterized operators $T_i(k)$ generalizing the partial derivatives $\partial_i = T_i(0)$. Here k denotes an N-tuple of complex parameters where N equals the number of conjugacy classes of complex reflections in the group.

We prove the commutativity by showing that the perturbed De Rham complex, in which the role of the partial derivatives is played by the operators $T_i(k)$, is indeed a complex (i.e. the perturbed differential $d(k)$ satisfies $d(k)^2 = 0$). At the heart of this argument lies the computation of a "Laplacian" $E(k)$ for this complex. This operator gives rise to a parameter family of elements in the center of the group algebra, whose values on the irreducible representations of G are nonnegative integral linear combinations of the parameters k. These values seem to govern many of the properties of the operators $T_i(k)$. In particular, analyzing these values easily leads to a proof of the commutativity of the $T_i(k)$.

In the course of this argument various natural structures arise:

(i) The perturbed De Rham complex $P \otimes \bigwedge^\bullet(V^*)$ with differential $d(k)$, where P is the ring of polynomials on V.
(ii) Homogeneous, G-equivariant intertwining operators $\mathcal{V}(k)$ on P, such that $\partial_i \mathcal{V}(k) = \mathcal{V}(k) T_i(k)$.
(iii) An hermitian pairing $(\cdot,\cdot)_k$ on P such that $(x_i p, q)_k = (p, T_i(k) q)_k$.
(iv) The ring P as a module over the "rational Cherednik algebra" $\mathbb{A}(k)$, the algebra generated by $\mathbb{C}[G]$, $T_i(k)$ and P (acting on itself by multiplication).

A parameter value k is called *singular* if there exists a nonzero homogeneous polynomial p of positive degree such that for all i, $T_i(k)p = 0$. This turns out to be the only obstruction for the existence of a homogeneous equivariant intertwining isomorphism as in (ii). By this remark it is easy to see that k is singular if and only if there exists an $i > 0$ such that the cohomology group $H^i(k)$ of the De Rham complex with differential $d(k)$ is nonzero. Another equivalent formulation is the statement that $(\cdot,\cdot)_k$ is degenerate. From this one easily sees that k is singular if and only if P is not irreducible as an $\mathbb{A}(k)$-module.

By these equivalent descriptions it is clear that the set of singular parameter values is of fundamental importance, and one of the goals of this paper is to find this set explicitly. We are not able to solve this problem in general, but we will derive that the singular set is always a locally finite union of affine rational hyperplanes in the parameter space.

In the case of a Coxeter group one knows more about the above structures, and this was described in the paper [8]. The present paper grew out of an attempt to apply the methods discussed in [8] to the case of complex reflection groups.

The complex reflection groups were classified by Shephard and Todd [18]. There are 34 exceptional cases called $G_i, i = 4,..,37$ (containing the exceptional real reflection groups) and an infinite family of groups $G(m,p,N)$ with $m,p,N \in \mathbb{N}$ and $p|m$. The group $G(m,p,N)$ is a subgroup of $U(N)$ and consists of permutation matrices whose nonzero entries are m^{th} roots of unity and the product of the nonzero entries is an $(m/p)^{\text{th}}$ root of unity. If $N = 1$ we take $p = 1$ (cyclic groups of order

m acting on $\mathbb{C}$). The groups $G(1,1,N)$, $G(2,1,N)$, $G(2,2,N)$ are the Coxeter groups of types A_{N-1}, B_N, D_N respectively.

The infinite family $G(m,p,N)$ is studied in detail in the second half of this paper, by means of the theory of nonsymmetric Jack polynomials. A complete orthogonal decomposition for the pairing associated with $G(m,p,N)$ is obtained with explicit norm formulae. This leads to a precise description of the set of singular values and the construction of shift operators which transform between the structures for contiguous parameter values.

Various alternative interpretations are known for the singular parameter set in the case of Coxeter groups (see also [8]). It is closely related to the non-semisimple specializations of Hecke algebras (see [15]). Likewise it is closely related to the zeroes of the Bernstein-Sato polynomial of the discriminant (see [14]). In this regard it is also interesting to compare with the results of the paper [2]. It is an interesting question whether some of these interpretations survive in the general case of complex reflection groups. The Hecke algebras to be considered are the topological cyclotomic Hecke algebras studied in [1] (also [17]).

We note that the algebra $\mathbb{A}(k)$ naturally fits in the framework of the symplectic reflection algebras that were recently introduced by Etingof and Ginsburg [10]. As is mentioned in that paper, this provides an alternative proof of the commutativity of the operators $T_i(k)$. Yet another approach to the proof of the commutativity is to show the integrability of the related Knizhnik-Zamolodchikov connection directly, using geometric arguments along the lines of [13] (see [1] for the details of this argument).

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2. Dunkl operators for complex reflection groups

2.1. **Complex reflection groups.** Let V be a finite dimensional Hilbert space, and let $U(V)$ be the group of unitary linear transformations of V. An element $g \in U(V)$ is called a *complex reflection* if g has finite order and $H_g := \mathrm{Ker}(g - \mathrm{Id})$ is a hyperplane in V. A finite subgroup $G \subset U(V)$ is called a finite complex reflection group if G is generated by complex reflections.

Let $G \subset U(V)$ be a finite complex reflection group. If $W \subset V$ is a linear subspace, we denote by $G_W \subset G$ the subgroup of those elements of G which fix the elements of W. By a well known result of Steinberg [19], G_W is itself a finite complex reflection group. Clearly, G_W acts faithfully on $W^\perp$. In particular, if $H \subset V$ is a hyperplane, then G_H is a cyclic group. If $G_H \neq \{\mathrm{Id}\}$ we call H a reflection hyperplane. When H is a reflection hyperplane, we denote by e_H the order of the cyclic group G_H. The collection of all reflection hyperplanes is a central hyperplane arrangement $\mathcal{A}$ in V, on which the group G acts. Let us denote by $\mathcal{C}$ the set of G-orbits in $\mathcal{A}$. Obviously, e_H only depends on the orbit $C = G \cdot H \in \mathcal{C}$. We will write e_C instead of e_H whenever this is convenient.

Let det be the determinant character on $U(V)$. When $H \subset V$ is a reflection hyperplane, the characters of G_H also form a cyclic group, generated by the restriction

χ_H of det to G_H. We will thus label the character group of G_H by

$$\hat{G}_H = \{\chi_H^{-i} \mid i = 0, \dots, e_H - 1\}. \tag{1}$$

For each of the reflection hyperplanes we choose a functional $\alpha_H \in V^*$ such that $\mathrm{Ker}(\alpha_H) = H$.

The group G acts on the ring P of polynomials on V in the usual way, i.e. $p^g(x) := p(g^{-1}x)$. The functional α_H transforms under the action of G_H according to the character χ_H^{-1}. A polynomial p which transforms under the action of G_H according to a nontrivial character must vanish on H. Consequently, a polynomial which transforms under the action of G_H according to χ_H^{-i} with $i \in \{1, \dots, e_H - 1\}$ is divisible by α_H^i.

2.2. Dunkl operators. Given $H \in \mathcal{A}$ and $i \in \{0, 1, \dots, e_H - 1\}$, let

$$\epsilon_{H,i} := \frac{1}{e_H} \sum_{g \in G_H} \chi_H^i(g) g \in \mathbb{C}[G_H], \tag{2}$$

be the idempotent of $\mathbb{C}[G_H]$ of the character χ_H^{-i}. In addition, choose a list of complex numbers $k = (k_{C,i})$, where C runs over the set of orbits $\mathcal{C}$, and for each $C \in \mathcal{C}$, $i \in \{1, \dots, e_C - 1\}$. With these data we form, for each reflection hyperplane H, the element

$$a_H = a_H(k) = \sum_{i=1}^{e_H - 1} e_H k_{H,i} \epsilon_{H,i} \in \mathbb{C}[G_H]. \tag{3}$$

Notice that the $\epsilon_{H,i}$ with $i \in \{1, \dots, e_H - 1\}$ constitute a basis of the subalgebra in $\mathbb{C}[G_H]$ consisting of the elements $\sum_{g \in G_H} c_g g \in \mathbb{C}[G_H]$ such that $\sum c_g = 0$. Also notice that the elements a_H are equivariant with respect to conjugation in the group algebra: for all $g \in G$,

$$g a_H g^{-1} = a_{gH}. \tag{4}$$

Let $\xi \in V$, and denote by ∂_ξ the derivation of P associated to the constant vector field on V defined by ξ. By what was said above, we can define the following "Dunkl operator" on P:

$$T_\xi(k) = \partial_\xi + \sum_{H \in \mathcal{A}} \alpha_H(\xi) \alpha_H^{-1} a_H(k), \tag{5}$$

where $a_H(k)$ is considered as operator acting on P. The operator indeed maps polynomials to polynomials, since $\epsilon_{H,0} a_H = 0$, so that we can divide by α_H after applying $a_H(k)$. The next proposition is now clear:

Proposition 2.1. (i) *The operator $T_\xi(k)$ does not depend on the choice of the functionals α_H.*

(ii) *$T_\xi(k)$ is equivariant with respect to the action of G on P and V:*

$$g T_\xi(k) g^{-1} = T_{g\xi}(k). \tag{6}$$

(iii) *$T_\xi(k)$ is homogeneous of degree -1.*

Let $K^\bullet = P \otimes \bigwedge^\bullet V^*$ denote the algebra of polynomial differential forms on V. Let $\Omega(k) \in \mathrm{End}(P) \otimes K^1$ be given by

$$\Omega(k) = \sum_{H \in \mathcal{A}} a_H(k) \omega_H, \tag{7}$$

where $\omega_H := \alpha_H^{-1} d\alpha_H$ is the logarithmic differential of α_H (which is independent of the choice of α_H, and G_H-invariant).

Lemma 2.2. *The operator $\Omega(k) : P \to K^1$ is G-equivariant with respect to the usual action of G on P, and the diagonal action of G on K^1.*

Proof. This is clear since both $a_H(k)$ and ω_H are equivariant for the natural actions of G on P and on the space of (rational) 1-forms on V. □

We write $d(k) : P \to K^1$ to denote the map $d(k)(p) = dp + \Omega(k)(p)$. Thus we have:

$$T_\xi(k)(p) = c_\xi(d(k)(p)), \tag{8}$$

where c_ξ denotes the contraction with the constant vector field ξ. We extend the operator $d(k)$ to $K^\bullet$ in the usual way: for all $p \in P$ and $\omega \in \bigwedge^\bullet V^*$ we define $d(k)(p \otimes \omega) = d(k)(p) \wedge \omega$, and extend this linearly to $K^\bullet$. Note however that, unlike the case $k = 0$, this is not a derivation of the algebra $K^\bullet$.

Lemma 2.3. *We have the following equivalent definition of $d(k)$ on $K^\bullet$. Extend $\Omega(k)$ to the G-equivariant endomorphism $\Omega(k)$ on $K^\bullet$ which is defined by $\Omega(k)(\omega) = \sum_{H\in\mathcal{A}} a_H(k)(\omega_H \wedge \omega)$ for $\omega \in K^\bullet$, where $a_H(k)$ acts diagonally on $K^\bullet$. Then $d(k) = d + \Omega(k)$. The operator $d(k)$ is equivariant for the diagonal action of G on $K^\bullet$.*

Proof. If $g \in G_H$, then $x^g \in x + \mathbb{C}\alpha_H$, for all $x \in V^*$. In addition, ω_H is G_H-invariant. Hence if $\omega = p \otimes dx_1 \wedge \cdots \wedge dx_l$, then

$$(a_H(k)(p)) \otimes \omega_H \wedge dx_1 \wedge \cdots \wedge dx_l = a_H(k)(p \otimes \omega_H \wedge dx_1 \wedge \cdots \wedge dx_l), \tag{9}$$

where we used the diagonal action of G on the right hand side. The equivariance of $d(k)$ follows from the previous Lemma. □

Consider the Koszul differential ∂ on $K^\bullet$. This differential is defined by:

$$\partial : K^l \to K^{l-1}$$

$$p \otimes dx_1 \wedge \cdots \wedge dx_l \to \sum_{r=1}^{l} (-1)^{r+1} x_r p \otimes dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_l.$$

Observe that ∂ is $U(V)$-equivariant with respect to the diagonal action of $U(V)$ on $K^\bullet$.

Let $E(0)$ denote the Euler vector field on V. This vector field is the infinitesimal generator of the action of $\mathbb{C}^\times$ on V (by scalar multiplication). Differentiating the diagonal action of $\mathbb{C}^\times$ on $K^\bullet$ we obtain an action of $E(0)$ on $K^\bullet$, the "diagonal action". In other words, if we put $K^l_m = P_m \otimes \bigwedge^l V^*$, then $E(0)$ has eigenvalue $l + m$ on K^l_m. Notice that $d(k)(K^l_m) \subset K^{l+1}_{m-1}$ and that $\partial(K^l_m) \subset K^{l-1}_{m+1}$.

Proposition 2.4. *Let $E(k) = E(0) + \sum_{H\in\mathcal{A}} a_H(k)$, acting diagonally on $K^\bullet$. Then $\partial d(k) + d(k)\partial = E(k)$.*

Proof. It is well known that $\partial d(0) + d(0)\partial = E(0)$, since $d(0)$ is the ordinary De Rham differential on $K^\bullet$.

Extend the operators $d(k)$ and ∂ to the complex $\overline{K}^\bullet$ of rational differential forms on V in the natural way. For each $H \in \mathcal{A}$, define the operator $w_H : \overline{K}^\bullet \to \overline{K}^\bullet$ by

$w_H(\eta) := \omega_H \wedge \eta$. By the previous Lemma and the equivariance of ∂ we have

$$\partial a_H w_H + a_H w_H \partial = a_H(\partial w_H + w_H \partial) = a_H.$$

This finishes the proof. □

The next Lemma is of crucial importance in all that follows.

Lemma 2.5. *We put $z(k) = \sum_{H\in\mathcal{A}} a_H(k) \in \mathbb{C}[G]$. This element of the group algebra has the following properties:*

(i) *The element $z(k)$ is in the center of $\mathbb{C}[G]$.*

(ii) *For $(V_\tau, \tau) \in \hat{G}$, let $c_\tau(k)$ denote the scalar such that $z(k)$ acts on V_τ by multiplication with $c_\tau(k)$. Then $c_\tau(k)$ is a linear function of k, with nonnegative integer coefficients.*

(iii) *Let* triv *denote the trivial representation of G. Then $c_\tau(k) \equiv 0$ if and only if $\tau =$ triv.*

Proof. (i) This follows immediately from the equivariance of the elements a_H.

(ii) Let the restriction of τ to G_H be

$$\tau|_{G_H} = \sum_{i=0}^{e_H-1} n^\tau_{H,i} \chi_H^{-i} \tag{10}$$

for certain nonnegative integers $n^\tau_{H,i}$. Observe that these branching numbers only depend on the orbit $C = G\cdot H$ of H. Hence the trace of $z(k)$ on V_τ equals

$$\operatorname{trace}\tau(z(k)) = \sum_{C\in\mathcal{C}} \sum_{i=1}^{e_H-1} |C| e_C n^\tau_{C,i} k_{C,i}, \tag{11}$$

showing that $c_\tau(k)$ is linear with nonnegative rational numbers as coefficients. On the other hand, $z(k)$ is a linear expression in the $k_{C,i}$, with coefficients that are central elements of the algebra $A[G]$, where A denotes the ring of algebraic integers. By a well known result in the theory of representations of finite groups, the central elements of $A[G]$ assume algebraic integer values on the irreducible representations of G over $\mathbb{C}$. This proves the result.

(iii) If $c_\tau(k) \equiv 0$ then it follows from the proof of (ii) that for each $H \in \mathcal{A}$, $\tau|_{G_H}$ contains only the trivial character of G_H. Since G is generated by the subgroups G_H, this implies that $\tau =$ triv. □

Corollary 2.6. *Let X, Y, Z denote the generators of an associative algebra $\tilde{C}$, satisfying the relations $XY + YX = Z$, $X^2 = [Z, X] = [Z, Y] = 0$. The map $X \to \partial$, $Y \to d(k)$ and $Z \to E(k)$ extends to a representation of $\tilde{C}$ on $K^\bullet$.*

Proof. If $\tau \in \hat{G}$, we put $K^l_{m,\tau}$ for the τ-isotypic component of K^l_m. We have $\partial(K^l_{m,\tau}) \subset K^{l-1}_{m+1,\tau}$, $d(k)(K^l_{m,\tau}) \subset K^{l+1}_{m-1,\tau}$, and finally $E(k)(K^l_{m,\tau}) \subset K^l_{m,\tau}$. It also follows immediately that $E(k)$ commutes with ∂ and $d(k)$. □

We put $K(r,\tau) := \oplus_{l+m=r} K^l_{m,\tau}$, and $K(0) = K^0_0 = \mathbb{C}$.

Corollary 2.7. *$K(r,\tau)$ is a finite dimensional $\tilde{C}$-submodule, and $K^\bullet$ decomposes as direct sum $K^\bullet = \oplus_{r\geq 0,\tau} K(r,\tau)$.*

Furthermore we write $K^\bullet(+) := \oplus_{r>0,\tau} K(r,\tau)$, which is the $\tilde{C}$-submodule of $K^\bullet$ complementary to $K(0)$. We thus have the decomposition $K^\bullet = K^\bullet(+) \oplus K(0)$.

Corollary 2.8. *Assume that for all $\tau \in \hat{G}$, $-c_\tau(k) \notin \mathbb{N}$. Then*

$$\mathrm{Ker}(d(k)) \cap K^\bullet(+) = \mathrm{Ker}(d(k)) \cap \mathrm{Im}(d(k)). \tag{12}$$

In particular, $\mathrm{Ker}(d(k)) \cap K^0(+) = \{0\}$.

Proof. $E(k)$ acts by scalar multiplication with $r + c_\tau(k)$ on the submodule $K(r,\tau)$. Therefore, by the above assumption, $E(k)$ is invertible on each $\tilde{C}$-submodule $K(r,\tau)$, with the exception of $K(0)$. Write $E(k)^{-1}$ for the inverse of $E(k)$ on $K^\bullet(+)$. Let $\omega \in \mathrm{Ker}(d(k)) \cap K^\bullet(+)$, then

$$\begin{aligned}\omega &= E(k)^{-1}E(k)(\omega)\\ &= E(k)^{-1}(\partial d(k) + d(k)\partial)(\omega)\\ &= d(k)(\partial(E(k)^{-1}\omega)).\end{aligned}$$

This shows that $\eta = \partial(E(k)^{-1}\omega) \in K^\bullet(+)$ is a solution of $d(k)(\eta) = \omega$, proving the desired result. □

Theorem 2.9. *Assume that for all $\tau \in \hat{G}$, $-c_\tau(k) \notin \mathbb{N}$. We call a linear operator $\mathcal{V} : K^\bullet \to K^\bullet$ completely homogeneous if $\mathcal{V}(K^l_m) \subset K^l_m$, for all l, m. There exists a unique completely homogeneous linear map $\mathcal{V}(k) : K^\bullet \to K^\bullet$ such that*

(i) *$\mathcal{V}(k)$ is the identity operator on $K(0) = \mathbb{C}$,*
(ii) *$\mathcal{V}(k)(p \otimes \omega) = (\mathcal{V}(k)p) \otimes \omega$, for all $\omega \in \bigwedge^\bullet V^* = K^\bullet_0$, and*
(iii) *$d(k)\mathcal{V}(k) = \mathcal{V}(k)d(0)$.*

Moreover, $\mathcal{V}(k)$ is a G-equivariant linear isomorphism.

Proof. First we show that if $\mathcal{V}(k)$ exists, it is necessarily a linear isomorphism. If not, let m be minimal such that $\mathcal{V}(k)$ has a nontrivial kernel in $K^\bullet_m$. By (ii) this implies that $P_m \cap \mathrm{Ker}(\mathcal{V}(k)) \neq 0$, and by (i) we see that $m > 0$. Let $0 \neq p \in P_m$ be such that $\mathcal{V}(k)p = 0$. By (iii) we see $\mathcal{V}(k)d(0)p = 0$. By the assumption on m this implies that $d(0)p = 0$, and since $m > 0$ this is a contradiction. Note that this argument is independent of the assumption on k, since it only uses that $d(0)p = 0$ implies that $p = 0$ if $p \in P_m$ with $m > 0$.

A similar argument shows that $\mathcal{V}(k)$ must be unique. If not, there exists a nonzero completely homogeneous operator $\mathcal{W}$ satisfying (ii) and (iii) but with $\mathcal{W}(K^\bullet_0) = 0$. Let $m > 0$ be minimal such that $\mathcal{W}(K^\bullet_m) \neq 0$. By (ii) this implies that $\mathcal{W}(P_m) \neq 0$. Let $p \in P_m$ such that $\mathcal{W}p \neq 0$. But then $d(k)\mathcal{W}p = \mathcal{W}d(0)p = 0$, which implies that $\mathcal{W}p \in \mathrm{Ker}(d(k)) \cap K^0(+)$. Given the assumption on k, this contradicts Corollary 2.8.

We now construct $\mathcal{V}(k)$ by induction on the degree m. Suppose that $m > 0$ and that we have already constructed $\mathcal{V}(k)$ on $K^\bullet_i$ for $i < m$, satisfying (i), (ii) and (iii). Let $p \in K^0_m = P_m$. Then $d(k)(\mathcal{V}(k)d(0)p) = 0$, and thus, by the previous corollary, there exists a unique $q \in P_m$ such that $d(k)q = \mathcal{V}(k)d(0)p$. Hence we define $\mathcal{V}(k)p = q$. For any $\omega \in K_0$ and $p \in P_m$ we now put $\mathcal{V}(k)(p \otimes \omega) = (\mathcal{V}(k)p) \otimes \omega$, and use this to define $\mathcal{V}(k)$ on $K^\bullet_m$. It is immediate that $\mathcal{V}(k)$ satisfies (i), (ii) and (iii).

The G-equivariance of $\mathcal{V}(k)$ now follows from the equivariance of $d(0)$ and $d(k)$. The equivariance implies that for any $g \in G$, $g \circ \mathcal{V}(k) \circ g^{-1}$ also meets the requirements (i), (ii) and (iii). By the uniqueness property we conclude that $\mathcal{V}(k) = g \circ \mathcal{V}(k) \circ g^{-1}$. □

Corollary 2.10. *The map $d(k)$ is a differential on $K^\bullet$, i.e. $d(k)^2 = 0$. In addition, the complex $(K^\bullet(+), d(k))$ (with $K^\bullet(+) := \oplus_{r>0,\tau} K(r,\tau)$) is acyclic (i.e. its cohomology is 0) if we assume that for all $\tau \in \hat{G}$, $-c_\tau(k) \notin \mathbb{N}$.*

Corollary 2.11. *The representation of $\tilde{C}$ on the submodule $K(r,\tau)$ (in which $Z = E(k)$ acts by scalar multiplication with $s := r + c_\tau(k)$) factors through the Clifford algebra $C(s)$, the associative algebra with generators X and Y, and relations $XY + YX = s$, $X^2 = Y^2 = 0$.*

The next reformulation is the main result of this section:

Theorem 2.12. *For all $\xi, \eta \in V$, $T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k)$.*

Proof. Choose a basis e_i of V, with dual basis x_j of V^*. Put $T_i = T_{e_i}(k)$. A simple direct computation shows that for all $p \in P$,

$$d(k)^2 p = \sum_{i<j} (T_i T_j - T_j T_i) p \otimes dx_i \wedge dx_j. \tag{13}$$

Hence the statement $d(k)^2 = 0$ is equivalent to the commutativity of the T_i. □

Corollary 2.13. *The restriction of $\mathcal{V}(k)$ to P satisfies $T_\xi(k)\mathcal{V}(k) = \mathcal{V}(k)\partial_\xi$.*

Proof. The point is that, because of defining property (ii) of $\mathcal{V}(k)$, $\mathcal{V}(k)$ commutes with the contraction c_ξ for each $\xi \in V$. Thus we obtain:

$$\begin{aligned} T_\xi(k)\mathcal{V}(k)p &= c_\xi(d(k)\mathcal{V}(k)p) \\ &= c_\xi(\mathcal{V}(k)d(0)p) = \mathcal{V}(k)\partial_\xi p. \end{aligned}$$

□

When $s := r + c_\tau(k) \neq 0$, the Clifford algebra $C(s)$ is semisimple and has only one irreducible module $M^s \simeq \mathbb{C}^2$, with basis a, b such that

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}. \tag{14}$$

Thus $K(r,\tau)$ is isomorphic to a direct sum of copies of the irreducible representation $\tau \otimes M^s$ of $\mathbb{C}[G] \otimes C(s)$. The "cohomology" $H(\tau \otimes M^s) := \mathrm{Ker}(Y)/\mathrm{Im}(Y)$ is equal to 0 in the module $\tau \otimes M^s$. However, the algebra $C(0)$ is the Grassmann algebra. This algebra is no longer semisimple. The space of equivalence classes of indecomposable modules of the Grassmann algebra is complicated, and the "cohomology" $\mathrm{Ker}(Y)/\mathrm{Im}(Y)$ is not necessarily 0 in the indecomposable modules. This happens for the only irreducible representation of $C(0)$, the trivial representation, but it may happen for nontrivial indecomposable modules as well. Note that for all parameters values k, the trivial representation of $C(0)$ is contained in $K^\bullet$ at least once, in the form of the submodule $K(0)$.

2.3. **Singular parameter values.** Let $H^i(k)$ denote the i-th cohomology group of the complex $(K^\bullet, d(k))$. The following are equivalent:

Corollary 2.14. (i) *$H^i(k) = 0$, for all $i > 0$.*
 (ii) *$H^0(k) = \mathbb{C}$.*
 (iii) *There exists a completely homogeneous intertwining map $\mathcal{V}(k)$ satisfying the conditions (i), (ii) and (iii) of Theorem 2.9.*

Proof. (i)⇒(ii): The complex $(K^\bullet, d(k))$ is a direct sum of the finite dimensional subcomplexes $K^\bullet(n) = \oplus_{l+m=n} K^l_m$. Let us denote by $H^i(n,k)$ the cohomology groups of $(K^\bullet(n), d(k))$. The Euler characteristic

$$\chi(n) := \sum_{i\geq 0}(-1)^i \dim(H^i(n,k)) = \sum_{i\geq 0}(-1)^i \dim(K^i_{n-i}) \tag{15}$$

is independent of k. Hence for all k, $\chi(n) = 0$ if $n > 0$. Thus for all $n > 0$ and k we have: if $H^i(n,k) = 0$ for all $i > 0$, then $H^0(n,k) = 0$.

(ii)⇒(iii): We carry out the proof of the existence of an intertwining map $\mathcal{W}(k)$ as in Theorem 2.9, but with the role of $d(0)$ and $d(k)$ interchanged. We can do this, because we now know that $d(k)^2 = 0$. Since we assume that $H^0(k) = \mathbb{C}$, we see that the proof of Theorem 2.9 showing that the intertwiner has to be a linear isomorphism also applies in this situation. Hence $\mathcal{V}(k) = \mathcal{W}(k)^{-1}$ is an intertwining operator as required.

(iii)⇒(i): The argument in the proof of Theorem 2.9 showing that $\mathcal{V}(k)$ has to be a linear isomorphism applies for all k. Hence $\mathcal{V}(k)$ defines an isomorphism between the cohomology spaces $H^i(0)$ and $H^i(k)$. □

Definition 2.15. *The parameter k is called regular if the equivalent statements of 2.14 hold. Otherwise we say that k is singular.*

We saw in Corollary 2.10 that if k is singular, then there exists a $\tau \in \hat{G}$ such that $-c_\tau(k) = m \in \mathbb{N}$. More precisely we have:

Corollary 2.16. *For $\tau \in \hat{G}$, let P_τ denote the τ-isotypical component of P, and $P_{m,\tau}$ the τ-isotypical component of P_m. If k is singular, there exists a $\tau \in \hat{G}$ and $m \in \mathbb{N}$ such that $c_\tau(k) + m = 0$ and $P_{m,\tau} \neq 0$.*

Proof. If k is singular then there exists a submodule $K(m,\tau)$ of $K^\bullet$ such that $H^0(K(m,\tau), d(k)) \neq 0$ and $m \in \mathbb{N}$. This implies that $(P_m : \tau) > 0$ and $c_\tau(k) + m = 0$. □

The converse statement is false. A counterexample occurs already in the case of the symmetric group S_4. Let τ be the irreducible 3 dimensional representation $(2,1,1)$. Then $P_{5,\tau} \neq 0$ and $c_\tau(k) = 8k$, but $k = -5/8$ is not a singular parameter.

In the case of a finite Coxeter group, the set of singular k can be determined exactly, cf. [8]. In the present generality we do not know how to prove a similar explicit description.

2.4. **An hermitian form.** For $\xi \in V$, denote by $\xi^* \in V^*$ the element such that for all $\eta \in V$, $\xi^*(\eta) = (\xi, \eta)$. The map $\xi \to \xi^*$ is an anti-linear isometric isomorphism. We extend this to an anti-linear isomorphism $* : S \to P$, where S denotes the symmetric algebra on V. The inverse map is denoted by $*$ as well. By the commutativity of the operators $T_\xi(k)$, we can uniquely extend the linear map $\xi \to T_\xi(k)$ to obtain a linear map from S to $\mathrm{End}(V)$. This map will be denoted by $s \to s(T)$ (or by $s \to s(T(k))$ if necessary). We define a sesquilinear pairing $(\cdot,\cdot)_k$ on P by

$$(p,q)_k := (p^*(T(k))q)(0). \tag{16}$$

Proposition 2.17. (i) *For all $g \in G$, $(p^g, q^g)_k = (p,q)_k$.*

(ii) *$(\xi^* p, q)_k = (p, T_\xi(k)q)_k$.*

(iii) *$(P_{m,\tau}, P_{l,\sigma})_k = 0$ if $m \neq l$ or if $\tau \neq \sigma$.*

Proof. (i) This follows from the remark that for all $p \in P$, $g \in G$, $(p^g)^* = g(p^*)$, since for we have, for all $x \in V^*$ and $v \in V$:

$$(gx^*, v) = (x^*, g^{-1}v) = x(g^{-1}v) = x^g(v) = ((x^g)^*, v). \tag{17}$$

Hence by the equivariance of $T_\xi(k)$ we have:

$$\begin{aligned}(p^g, q^g)_k &= ((g(p^*)(T(k)))q^g)(0) \\ &= ((g \circ p^*(T(k)) \circ g^{-1})q^g)(0) = (p^*(T(k))q)^g(0) = (p,q)_k.\end{aligned}$$

(ii) This follows immediately from the definition, using the commutativity of the $T_\xi(k)$.

(iii) From the definition we see that $(P_m, P_l)_k = 0$ if $l \neq m$. By (i) it follows that $(ep, q)_k = (ep, eq)_k = (p, eq)_k$ for every self-adjoint idempotent e of $\mathbb{C}[G]$. This implies (iii). □

The main theorem of this section is:

Theorem 2.18. *The pairing satisfies* $(p,q)_k = \overline{(q,p)_{\overline{k}}}$.

Proof. We prove this by induction on the degree m. By Proposition 2.17 it is enough to show that for every $\tau \in \hat{G}$ and $m \in \mathbb{Z}_+$, $(p,q)_k = \overline{(q,p)_{\overline{k}}}$ if $p, q \in P_{m,\tau}$. Assume by induction that this statement holds true for all $p, q \in P_{l,\sigma}$ with $l < m$ (the case $m = 0$ being trivial). Let e_i denote an orthonormal basis of V, and let $x_i = e_i^*$ denote the dual basis of coordinates on V. Then, since $\overline{c_\tau(k)} = c_\tau(\overline{k})$ by Lemma 2.5, we have for all $p, q \in P_{m;\tau}$:

$$\begin{aligned}(m + c_\tau(k))(p,q)_k &= (E(\overline{k})p, q)_k \\ &= (\sum_i x_i T_i(\overline{k})p, q)_k \\ &= \sum_i (T_i(\overline{k})p, T_i(k)q)_k \\ &= \sum_i \overline{(T_i(k)q, T_i(\overline{k})p)_{\overline{k}}} \\ &= \sum_i \overline{(x_i T_i(k)q, p)_{\overline{k}}} \\ &= (m + c_\tau(k))\overline{(q,p)_{\overline{k}}}.\end{aligned}$$

Since $(p,q)_k$ depends polynomially on k and since $(l + c_\tau(k)) \neq 0$ for generic values of k, this proves the necessary induction step. □

Corollary 2.19. *For all* $x \in V^*$ *and* $p, q \in P$ *we have* $(T_{x^*}(\overline{k})p, q)_k = (p, xq)_k$.

Proof. We have

$$\begin{aligned}(T_{x^*}(\overline{k})p, q)_k &= \overline{(q, T_{x^*}(\overline{k})p)_{\overline{k}}} \\ &= \overline{(xq, p)_{\overline{k}}} \\ &= (p, xq)_k.\end{aligned}$$

□

As was noticed in Proposition 2.17, the finite dimensional subspaces $P_{m,\tau} \subset P$ satisfy $(P_{m,\tau}, P_{l,\sigma})_k = 0$ unless $m = l$ and $\tau = \sigma$. It follows that there exists a polynomial $p \neq 0$ such that $(p, P)_k = 0$ if and only if there exists a polynomial $q \neq 0$ such that $(P, q)_k = 0$. In this case we call the sesquilinear pairing $(\cdot,\cdot)_k$ degenerate.

Proposition 2.20. *The following are equivalent:*

(i) *k is singular.*
(ii) *$(\cdot,\cdot)_k$ is degenerate.*
(iii) *There exists a proper graded ideal $I \subset P$ which is stable for the action of the operators $T_\xi(k)$.*
(iv) *$\overline{k}$ is singular.*

Proof. (i)$\Rightarrow$(ii) Choose $m > 0$ such that there exists a $0 \neq p \in P_m$ with $d(k)p = 0$. Then $(P, p)_k = 0$.

(ii)$\Rightarrow$(iii) Take $I := \{p \mid (P, p)_k = 0\}$.

(iii)$\Rightarrow$(i) Let $m > 0$ be minimal such that $I_m \neq 0$. Then $d(k)I_m = 0$.

(i)$\Leftrightarrow$(iv) By the above text, there exists a $q \neq 0$ such that $(P, q)_k = 0$ if and only if there exists a $p \in P$, $p \neq 0$ such that $(p, P)_k = 0$. But by Theorem 2.18 this is also equivalent to $(P, p)_{\overline{k}} = 0$. Hence, using the equivalence of (i) and (ii), we see that (i) is indeed equivalent to (iv). □

Corollary 2.21. *The set K^{sing} of singular parameter values consists of an infinite union of hyperplanes $K_{m,\tau}$. Here $K_{m,\tau}$ denotes the hyperplane given by the equation $m + c_\tau(k) = 0$, and the union runs only over pairs (m, τ) with τ nontrivial and $P_{m,\tau} \neq 0$.*

Proof. By Proposition 2.16 we know that K^{sing} is contained in the above union of hyperplanes $K_{m,\tau}$. Notice that this is a locally finite set of hyperplanes. On the other hand, by the above Proposition, $K^{sing} = \cup_{m \in \mathbb{N}} K_m^{degen}$ where K_m^{degen} denotes the set of parameter values k such that $(\cdot,\cdot)_k$ is degenerate on P_m. Now K_m^{degen} is given by the condition that the determinant of $(\cdot,\cdot)_k$ with respect to a basis of P_m vanishes. Hence K_m^{degen} is an algebraic hypersurface in the parameter space. Since it has to be contained in $\cup K_{m,\tau}$, it follows that, as a subset of the parameter space, K_m^{degen} is a union of hyperplanes $K_{m,\tau}$. This proves the claim. □

Corollary 2.22. *Let K_r denote the $\mathbb{Q}$-vector space of rational parameters k (i.e. $k_{C,i} \in \mathbb{Q}$ for all C, i). The set $K^{sing} \cap K_r$ is a (locally finite) union of hyperplanes, and K^{sing} is the union of the complexifications of these rational hyperplanes.*

Proof. The hyperplanes $K_{m,\tau}$ are all rational, by Lemma 2.5. □

Corollary 2.23. *For every pair (m, τ), there exists a basis (b_i) of $P_{m,\tau}$ such that the determinant of $(b_i, b_j)_k$ is a product of linear factors of the form $l + c_\sigma(k)$ with $\sigma \in \hat{G}$ nontrivial, and $P_{l,\sigma} \neq 0$.*

Proof. The determinant is a polynomial in $\mathbb{C}[K]$ such that its zero set is contained in K_m^{degen}. Hence it contains only irreducible factors of the form $l + c_\sigma(k)$. The determinant is determined up to multiplication by an arbitrary nonzero positive real number by change of basis. When we specialize at $k = 0$, the pairing is clearly positive definite hermitian, and thus we can fix the normalization by choosing an appropriate basis. □

The above results show that, in order to describe the set K^{sing}, it suffices to describe $K^{sing} \cap K_\tau$. By Proposition 2.20 this is equal to the set $K^{degen} \cap K_\tau$. In particular we may restrict to real parameters, which has the advantage that, by Theorem 2.18, the form $(\cdot,\cdot)_k$ is hermitian:

Proposition 2.24. *Suppose that k is real. Then*

(i) *$(\cdot,\cdot)_k$ is hermitian.*

(ii) *Suppose moreover that, for all nontrivial irreducible representations $\tau \in \hat{G}$, $c_\tau(k) + m(\tau) > 0$, where $m(\tau)$ denotes the lowest homogeneous degree such that $(\tau : P_{m(\tau)}) \neq 0$ (this holds in particular when all the parameters satisfy $k_{C,j} \geq 0$). Then $(\cdot,\cdot)_k$ is positive definite.*

Proof. (i) This is immediate from Theorem 2.18.

(ii) This follows by induction on the homogeneous degree m, by taking $p = q$ in the computation in the proof of Theorem 2.18. □

2.5. **Lowest weight modules over the rational Cherednik algebra.** We assume that k is real throughout this subsection. Let us consider the structure of P as a module over the rational Cherednik algebra $\mathbb{A}(k)$, the algebra generated by $\mathbb{C}[G]$, $T_\xi(k)$ and P (acting on itself by multiplication) (see [10]).

Lemma 2.25. *For $p \in P$, denote by $m(p)$ the operator $m(p) : P \to P$, $m(p)(q) = pq$ (multiplication by p). For convenience, we put $k_{C,0} = 0$, and for $j \in \mathbb{Z}$ we define $k_{C,j} = k_{C,j'}$ if $j - j'$ is divisible by e_C. In $\mathbb{A}(k)$ we have*

(18)
$$[T_\xi(k), m(p)] = m(\partial_\xi p) + \sum_{H\in\mathcal{A}} \sum_{i,j=0}^{e_H-1} e_H(k_{H,i+j} - k_{H,j})\alpha_H(\xi)\alpha_H^{-1} m(\epsilon_{H,i}(p))\epsilon_{H,j}.$$

In particular, we have

$$[T_\xi(k), m(x)] = x(\xi) + \sum_{H\in\mathcal{A}} \sum_{g\in G_H, g\neq 1} c_g(k)\alpha_H(\xi)x(v_H)\alpha_H(v_H)^{-1}g, \tag{19}$$

where $v_H \in V$ is the vector such that $(v_H, v) = \alpha_H(v)$ for all $v \in V$, and where $c_g(k)$ is the constant

$$c_g(k) = \sum_{j=0}^{e_H-1} \chi_H^j(g)(k_{H,j+1} - k_{H,j}). \tag{20}$$

The function $g \to c_g(k)$ is invariant for conjugation of g by G.

Proof. This follows in a straightforward way from the equations

$$p = \sum_{i=0}^{e_H-1} \epsilon_{H,i}(p) \tag{21}$$

and

$$[\epsilon_{H,j}, m(\epsilon_{H,i}(p))] = m(\epsilon_{H,i}(p))(\epsilon_{H,j-i} - \epsilon_{H,j}). \tag{22}$$

□

Let us describe how $\mathbb{A}(k)$ fits into the framework of the paper [10]. Consider the abstract associative algebra $T(V + V^*) \otimes \mathbb{C}[G]$, the smash product where G

acts diagonally on the tensor algebra $T(V+V^*)$. We introduce in this algebra the relations $[\xi,\eta]=0$ (for all $\xi,\eta\in V$), $[x,y]=0$ (for all $x,y\in V^*$) and finally,

$$[\xi,x]=x(\xi)+\sum_{H\in\mathcal{A}}\sum_{g\in G_H,g\neq 1}c_g(k)\alpha_H(\xi)x(v_H)\alpha_H(v_H)^{-1}g, \tag{23}$$

where $c_g(k)$ is as in equation 20. The resulting algebra $\mathbb{A}'(k)$ is a symplectic reflection algebra in the sense of [10]. In particular, by Theorem 1.3 of [10], $\mathbb{A}'(k)$ has the PBW-property (this means that $\mathbb{A}'(k)$ is isomorphic as a vector space to $P\otimes S\otimes\mathbb{C}[G]$). By construction, $\mathbb{A}(k)$ is a quotient of $\mathbb{A}'(k)$ via $P\ni p\to m(p)$, $S\ni p^*\to p^*(T(k))$ and $g\to g$ (action in P). Using the PBW-property one easily identifies the $\mathbb{A}'(k)$-module P as

$$P=\mathrm{Ind}_{S\otimes\mathbb{C}[G]}^{\mathbb{A}'(k)}(triv), \tag{24}$$

where *triv* is the one dimensional representation such that G acts trivially, and $triv(V)=0$. In fact, it is not hard to see that P is a faithful module over $\mathbb{A}'(k)$ (see [10], Proposition 4.5). We will therefore identify $\mathbb{A}'(k)$ and $\mathbb{A}(k)$ from now on.

In the above construction we identified P with an induced module. We generalize this construction in the following way. Let (V,τ) be an irreducible module of G. We extend τ to the algebra $S\otimes\mathbb{C}[G]$ by demanding that $\tau(V)=0$. We define

$$M(\tau,k):=\mathrm{Ind}_{S\otimes\mathbb{C}[G]}^{\mathbb{A}'(k)}(\tau). \tag{25}$$

In the sequel we will usually suppress the parameter $k\in K$ in the notation if no confusion is possible. As a vector space, $M(\tau)\simeq P\otimes V$. The action of G is the diagonal action, P acts by multiplication in the left factor of the tensor product, and the action of $T_\xi(k)$ is given by

$$T_\xi(p\otimes v)=\partial_\xi p\otimes v+\sum_{H\in\mathcal{A}}\sum_{i,j=0}^{e_H-1}e_H(k_{H,i+j}-k_{H,j})\alpha_H(\xi)\alpha_H^{-1}\epsilon_{H,i}(p)\otimes\epsilon_{H,j}(v), \tag{26}$$

according to equation 18. So we have in particular that $P=M(triv)$.

Lemma 2.26. *Let $\sigma\in\hat{G}$ and $m\in\mathbb{Z}_+$. Let $M(\tau)_\sigma$ of denote the σ-isotypic component of $M(\tau)$, and let $M(\tau)_{m,\sigma}=M(\tau)_\sigma\cap(P_m\otimes V)$. The deformed Euler vector field $E(k)=\sum_i x_iT_i(k)$ acts on $M(\tau)_{m,\sigma}$ by multiplication with the scalar $m-c_\tau(k)+c_\sigma(k)$.*

Proof. From equation 26 we see that, for $p\in P_m$,

$$\begin{aligned}E(k)(p\otimes v)&=mp\otimes v-\sum_{H\in\mathcal{A}}\sum_{j=0}^{e_H-1}e_Hk_{H,j}p\otimes\epsilon_{H,j}(v)\\&\quad+\sum_{H\in\mathcal{A}}\sum_{l=0}^{e_H-1}e_Hk_{H,l}\sum_{i+j=l\,\mathrm{mod}\,e_H}\epsilon_{H,i}(p)\otimes\epsilon_{H,j}(v)\\&=(m-c_\tau(k))p\otimes v+\sum_{l=0}^{e_H-1}e_Hk_{H,l}\epsilon_{H,l}^{\Delta}(p\otimes v),\end{aligned}$$

where $\epsilon_{H,l}^{\Delta}$ denotes the idempotent $\epsilon_{H,l}$ acting diagonally on $P\otimes V$. Thus on the σ-isotypical part $M(\tau)_{m,\sigma}$ of $P_m\otimes V$, the action of $E(k)$ is scalar with eigenvalue $m-c_\tau(k)+c_\sigma(k)$, as claimed. ☐

Proposition 2.27. *All $\mathbb{A}(k)$-submodules of $M(\tau)$ are graded. In other words, if $M \subset M(\tau)$ is an $\mathbb{A}(k)$-submodule, then*

$$M = \oplus_{m \in \mathbb{Z}_+} M_m \tag{27}$$

with $M_m := M \cap (P_m \otimes V)$. With respect to this grading, $T_\xi(k)$ has degree -1, x has degree $+1$, and $g \in G$ has degree 0.

Proof. Let $M \subset M(\tau)$ be an $\mathbb{A}(k)$-submodule. Since $\mathbb{C}[G] \subset \mathbb{A}(k)$, we have $M = \oplus_{\sigma \in \hat{G}} M_\sigma$. By the preceding Lemma, the eigenvalues of the operator $E(k)$ separate elements of different homogeneous degree in each isotypic part M_σ. Therefore $M_\sigma = \oplus_n M_\sigma \cap (P_m \otimes V)$, and thus M itself is also the direct sum of its graded pieces. □

Corollary 2.28. *The module $M(\tau)$ has a unique proper maximal submodule. In particular, $M(\tau)$ is indecomposable.*

Proof. Since τ is irreducible for the action of G, a submodule $M \subset M(\tau)$ is proper if and only if $M \cap V = \{0\}$ (where $V = M(\tau)_0$ as before). All submodules $M \subset P$ are graded, and therefore the sum M' of all proper submodules of $M(\tau)$ also has the property that $M' \cap V = \{0\}$. Hence M' is the unique maximal proper submodule of $M(\tau)$. □

Definition 2.29. *We call a module M over $\mathbb{A}(k)$ a lowest weight module with lowest G-type τ if M is a nontrivial quotient of $M(\tau, k)$. We denote by $L(\tau, k)$ (or simply $L(\tau)$) the unique irreducible quotient of $M(\tau, k)$.*

The above Proposition 2.27 shows that all lowest weight modules M with lowest G-type τ have a unique "natural" grading

$$M = \oplus_{n \in \mathbb{Z}_+} M_n \tag{28}$$

such that M_0 is the irreducible $\mathbb{C}[G]$-module of type τ. Note that the submodules of lowest weight modules are also graded (as they are subquotients of $M(\tau)$). Note however that the grading induced by M on a lowest weight submodule M' of M is shifted with respect to the natural grading of M' (unless $M = M'$).

Notice the analogy with the theory of highest weight modules for a semisimple Lie algebra over $\mathbb{C}$. The role of Verma-modules is played by the modules $M(\tau)$. Let us call an element $m \in M$ of a lowest weight module M of $\mathbb{A}(k)$ *primitive* if $T_\xi(k)m = 0$ for all $\xi \in V$. Clearly the subspace M^p of primitive elements in M is a graded G-subspace of M.

Proposition 2.30. *Let M be an $\mathbb{A}(k)$-module generated by a subspace M_0 of primitive vectors such that M_0 is an irreducible $\mathbb{C}[G]$-module of type τ. Then there exists a surjective homomorphism $\phi : M(\tau) \to M$ such that $\phi(M(\tau)_0) = M_0$. The homomorphism ϕ is unique up to a scalar.*

Proof. This is clear by the universal property of induced modules. □

Proposition 2.31.
- (i) *Let M be a lowest weight module with lowest G-type τ, graded with its natural grading. Then the σ-isotypic component M^p_σ of M^p is contained in $M_{m(\tau,\sigma)}$ with $m(\tau, \sigma) := c_\tau(k) - c_\sigma(k)$. In particular, M^p is finite dimensional.*
- (ii) *Let $m \in \mathbb{Z}_+$ be such that $M^p_m \neq 0$, and let $H \subset M^p_m$ be an irreducible G-subspace of M^p_m of type σ. The subspace $J(H) := PH \subset M$ is a lowest weight submodule of M, of lowest G-type σ.*

(iii) *Let* $m \in \mathbb{Z}_+$ *be maximal such that* $M^p_m \neq 0$*, and let* $H \subset M^p_m$ *be an irreducible* G*-subspace of* M^p_m *of type* σ*. Then* $J(H) \simeq L(\sigma)$*.*

(iv) *If, for all* $\sigma \in \hat{G}$*,* $c_\tau(k) - c_\sigma(k) \notin \mathbb{N}$*, then* $M(\tau)$ *is irreducible.*

Proof. (i) Suppose $M^p_{m,\sigma} \neq 0$. By definition of primitivity, $E(k)(M^p_{m,\sigma}) = 0$. On the other hand, $E(k)$ acts by multiplication with the scalar $m - c_\tau(k) + c_\sigma(k)$ on $M(\tau)_{m,\sigma}$. Since $M_{m,\sigma}$ is a quotient of this space, $E(k)$ also acts on $M_{m,\sigma}$ via multiplication with this scalar. The equation $m = m(\tau, \sigma)$ for the degree m follows. In particular, the dimension of M^p is bounded by

$$d(M) := \sum_{\sigma \in \hat{G}} \dim(M_{m(\tau,\sigma)}). \tag{29}$$

(ii) $J(H)$ is an $\mathbb{A}(k)$-submodule of M, because $T_\xi(k)(ph) = [T_\xi(k), m(p)](h)$ and, by equation 18, we have $[T_\xi(k), m(p)] \in P \otimes \mathbb{C}[G]$. By the above Proposition, $J(H)$ is a quotient of $M(\sigma)$.

(iii) By the condition on m we see that $J(H)^p = H$, since clearly $J(H)^p \subset M^p$. Thus every nonzero submodule of $J(H)$ contains H, and is therefore equal to $J(H)$. Thus $J(H)$ is the unique irreducible quotient of $M(\sigma)$.

(iv) This is a special case of (iii), since the condition implies that the maximal value of m such that $M(\tau)^p_m \neq 0$ is equal to 0. Thus we can take $H = V = M(\tau)_0$ in (ii) to see that $M(\tau) = L(\tau)$ in this case. □

Corollary 2.32. *Each lowest weight module* M *has a finite Jordan-Hölder series whose irreducible quotients are isomorphic to modules of the form* $L(\sigma)$*.*

Proof. By proposition 2.31 it is clear that M contains an irreducible submodule J_1 isomorphic to $L(\sigma_1)$ for some σ_1. If $N_1 := M/J_1 = 0$ we are done. If not, we continue by choosing an irreducible submodule $J_2 \simeq L(\sigma_2)$ in N_1, and form the quotient $N_2 := N_1/J_2$. We thus construct a sequence of consecutive quotients $M \to N_1 \to N_2 \to \ldots$. Notice that by construction, $d(N_i) < d(N_{i-1})$ (see equation 29) in each step of the process. Hence the process has to stop in finitely many steps, say $N_n = 0$. Now put $M_i := \mathrm{Ker}(M \to N_{n-i})$. Then we get

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_n = 0, \tag{30}$$

with $M_i/M_{i+1} = \mathrm{Ker}(N_{n-i-1} \to N_{n-i}) \simeq L(\sigma_{n-i})$, as desired. □

We thus arrive at the following fundamental

Problem 2.33. *For a lowest weight module* M *over* $\mathbb{A}(k)$*, denote by* $[M : L(\sigma, k)]$ *the multiplicity of* $L(\sigma)$ *in a Jordan-Hölder series of* M*. Compute and interpret the multiplicities* $[M(\tau, k) : L(\sigma, k)]$*.*

There are many natural questions and problems related to the topics introduced in this subsection. One should study the structures that were introduced before in the case of $P = M(triv)$ (such as the contravariant pairing, the De Rham complex, the singular set) systematically for general lowest weight modules. In addition, one should consider the natural analogue of category $\mathcal{O}$ in the present situation. Except for the remarks below, we resist the temptation to address any of these questions here, since this would take us too far afield. The main goal of this paper is the study of the special case $\tau = triv$.

Some straightforward remarks are in order. By Proposition 2.31 it is clear that if M has lowest G-type τ, $[M : L(\sigma, k)]$ is bounded by the multiplicity $[M_{m(\tau,\sigma)} : \sigma]$ of σ in the degree $m(\tau, \sigma) = c_\tau(k) - c_\sigma(k)$ part of M. In particular we see that

$$[M(\tau, k) : L(\tau, k)] = 1 \tag{31}$$

and that for $\sigma \neq \tau$,

$$[M(\tau, k) : L(\sigma, k)] \neq 0 \text{ implies that } c_\tau(k) - c_\sigma(k) \in \mathbb{N}. \tag{32}$$

Hence if we introduce an ordering of $\hat{G}$ by defining $\sigma \geq \tau$ if and only if $c_\sigma(k) \leq c_\tau(k)$, then the matrix $[M(\tau) : L(\sigma)]$ is unipotent upper triangular. In particular, the matrix is invertible.

Define the τ-singular set $K_\tau^{sing} \subset K$ as the set of $k \in K$ such that $M(\tau, k) \neq L(\tau, k)$. The description of K_τ^{sing} is just one part of Problem 2.33. By the above it is plain that $k \notin K_\tau^{sing}$ if for all σ, $c_\tau(k) - c_\sigma(k) \notin \mathbb{N}$. The converse is in general not true (as we saw in the special case $\tau = triv$). We see that K_τ^{sing} is contained in a locally finite collection of hyperplanes, such that the coefficients of the affine linear forms describing the hyperplanes are integral (but with signs). In the case where τ is the linear character of G whose restriction to G_H is $\chi_H^{-b_\tau(C)}$ (for some $b_\tau(C) \in \{0, \ldots, e_H - 1\}$) if $H \in C$, then it follows by formula 26 that $M(\tau, k) \simeq M(triv, \beta_\tau(k))$ where $\beta_\tau(k)_{C,j} := k_{C,j+b_\tau(C)} - k_{C,b_\tau(C)}$. In particular, $K_\tau^{sing} = \beta_\tau^{-1}(K^{sing})$.

2.6. **The module P over $\mathbb{A}(k)$.** Let us return to the module $P = M(triv)$. In [8] a polynomial $q \in P_+$ was called *singular* for the parameter k if $T_\xi(k)q = 0$ for all $\xi \in V$. In other words, q is singular if and only if $q \in H^0(k)$ (see subsection 2.3) and $q(0) = 0$. In the language of the previous subsection, a polynomial is singular if and only if it is a primitive element of positive degree for the module P. In addition, $H^0(k) = P^p$.

Recall that, by definition, nonzero singular polynomials for k exist if and only if k is a singular parameter. The space $H^0(k)$ of primitive polynomials for k is graded and is a G-space. For each $\tau \in \hat{G}$ and degree $m \in \mathbb{Z}_+$, we denote by $H^0_{m,\tau}(k)$ the space of primitive polynomials for k in degree m and of type τ. By Proposition 2.31, we have the relation

$$m = -c_\tau(k) \tag{33}$$

if $H^0_{m,\tau} \neq 0$, showing that $H^0(k)$ is finite dimensional. Suppose that $H \subset H^0_{m,\tau}(k)$ is an irreducible subspace. By Proposition 2.30, the ideal $J(H) := PH$ it generates in P is a lowest weight $\mathbb{A}(k)$-submodule of P with lowest G-type τ.

The form $(\cdot,\cdot)_k$ on P is an hermitian contravariant form on the module P over $\mathbb{A}(k)$, in the sense of Proposition 2.17 (i) and (ii). This construction can be extended to all lowest weight modules, and it plays a role quite similar to the Shapovalov form on Verma-modules for a semisimple Lie algebra. We will restrict ourselves to the special case at hand, the module P. The radical $\mathrm{Rad}(k) := \{p \in P \mid (p, P)_k = 0\}$ of this form is a graded ideal of P, which is stable for G and for the application of the operators $T_\xi(k)$ (see Proposition 2.20 and its proof). In other words, it is a (graded) $\mathbb{A}(k)$-submodule of P.

Proposition 2.34. *The radical* $\mathrm{Rad}(k)$ *of* $(\cdot,\cdot)_k$ *is the unique maximal proper* $\mathbb{A}(k)$-*submodule of P. In particular, P is not irreducible as an $\mathbb{A}(k)$-module if and only if k is singular, and $P/$*$\mathrm{Rad}(k)$ *is the unique simple quotient $L(triv, k)$ of P.*

Proof. Let $M \subset P$ be a proper $\mathbb{A}(k)$-submodule. Because $1 \notin M$ we have, since M is graded, $(1, M)_k = 0$. But this clearly implies $(P, M)_k = 0$, in other words: $M \subset \mathrm{Rad}(k)$. Thus $\mathrm{Rad}(k)$ is the unique maximal proper $\mathbb{A}(k)$-submodule of P. □

We are in the position to apply the technique of the Jantzen filtration (see [12], Chapter 5), so let us discuss this briefly. Given a real parameter $k_0 \in K$, consider the real line L in K through k_0 and 0. Note that $k = 0$ is a regular parameter, hence a generic point on L will be a regular parameter as well. We parameterize this line by $k(t) := (1+t)k_0$ and consider t as a real indeterminate. Let us denote $R = \mathbb{R}[t]$. For any complex vector space B we denote by $B_R := R \otimes_{\mathbb{R}} B$ the free $R_c = \mathbb{C} \otimes_{\mathbb{R}} R = \mathbb{C}[t]$-module that arises from B by extension of scalars. Then $\mathbb{A}_R$ is a free associative R_c-algebra. For $r \in R_c$ we define $r^*(t) := \overline{r(\bar{t})}$ (recall that $\bar{t} = t$). Thus $*$ is a $\mathbb{C}$-anti-linear involution on R_c. We have that P_R is a module for $\mathbb{A}_R$. The anti-linear isomorphism $* : P \to S$ extends naturally to an anti-linear isomorphism $* : P_R \to S_R$, where anti-linear means that $(rp)^* = r^* p^*$. We define a contravariant hermitian form $(\cdot,\cdot)_R$ with values in R_c on P_R by $(p,q)_R := (p^*(T(k(t))q)(0)$. The form is linear in the second factor and anti-linear in the first factor. It is hermitian in the sense that $(p,q)_R = (q,p)_R^*$, and contravariant in the sense that $(xp,q)_R = (p, T_{x^*}(k(t))q)_R$, $(T_{x^*}(k(t))p, q)_R = (p, xq)_R$ and $(p^g, q^g)_R = (p,q)_R$. Let m_0 denote the $*$-invariant maximal ideal $m_0 = tR_c$ of R_c. We introduce a sequence of R_c-linear subspaces $M_R^i \subset P_R$ defined by:

$$M_R^i := \{p \in P_R \mid (P_R, p)_R \subset m_0^i\}, \tag{34}$$

Lemma 2.35. *The M_R^i form a decreasing sequence of $\mathbb{A}_R$-submodules in P_R.*

Proof. The sequence M_R^i is clearly a decreasing sequence of R_c-linear subspaces. The fact that they are submodules follows from the contravariance of $(\cdot,\cdot)_R$. □

Let us denote by ψ the specialization functor $\psi(L) := L/tL$ (where L is an R_c-module) at $t = 0$. We thus obtain a $\mathbb{C}$-algebra homomorphism $\psi : \mathbb{A}_R \to \mathbb{A}(k_0)$. This is compatible with the module structures of P_R and P, in the sense that for $a \in \mathbb{A}_R$ and $p \in P_R$, we have $\psi(ap) = \psi(a)\psi(p)$. Also, $\psi((p,q)_R) = (\psi(p), \psi(q))_{k_0}$. We put

$$M^i := \psi(M_R^i). \tag{35}$$

For a graded subspace M of P we introduce the notation

$$\mathrm{Ch}(M) := \sum_{n \in \mathbb{Z}_+} \dim(M_n) X^n. \tag{36}$$

Proposition 2.36. (i) *$P_R = M^0 \supset M^1 \supset M^2 \dots$ is a sequence of $\mathbb{A}(k_0)$-submodules.*

(ii) *We have $\sum_{i>0} \mathrm{Ch}(M^i) = \sum_{n \geq 0} \nu(D(n)) X^n$, where $D(n)$ denotes the determinant of $(\cdot,\cdot)_R$ on $P_{R,n}$ (the R_c-module of polynomials of homogeneous degree n on V, with coefficients in R_c), and where ν denotes the m_0-adic valuation on the polynomial ring R_c.*

(iii) *For $i \gg 0$, $M^i = 0$.*

(iv) *For $p \in M^i$, let us denote by $\tilde{p} \in M_R^i$ an arbitrary element such that $\psi(\tilde{p}) = p$. For all $p, q \in M^i$, the expression $(p,q)^{(i)} := \psi(t^{-i}(\tilde{p}, \tilde{q})_R)$ depends only on p and q, not on the chosen lifts $\tilde{p}, \tilde{q}$. The form $(\cdot,\cdot)^{(i)}$ is hermitian* *and $\mathbb{A}(k_0)$-contravariant on M^i, and its radical is M^{i+1}. In particular, $M^1 = \mathrm{Rad}(k_0)$.*

Proof. (i) This follows directly from the above Lemma, by application of ψ to the sequence M^i_R.

(ii) Since the spaces $P_{R,n}$ are mutually orthogonal with respect to $(\cdot,\cdot)_R$, we see that $M^i_{R,n} = \{p \in P_{R,n} \mid (P_{R,n}, p)_R \in m_0^i\}$. We apply (an appropriately adapted version of) Lemma 5.1 of [12] to the R_c-module $P_{R,n}$, and obtain that for all n

$$\sum_{i>0} \dim M^i_n = \nu(D(n)), \tag{37}$$

where $M^i_n := \psi(M^i_{R,n}) = \psi(M^i_R)_n$. Note that $D(n)$ is determined up to a unit in R_c, so that $\nu(D(n))$ is well defined. The result follows.

(iii) By (ii) we see that given $n \in \mathbb{N}$, there exists a $b > 0$ such that $M^i_m = 0$ for all $m < n$ and $i > b$. Take $n > -\max_{\tau\in\hat{G}} c_\tau(k_0)$ and suppose that $M^i \neq 0$ for some $i > b$. Let $l \in \mathbb{N}$ be minimal such that $M^i_l \neq 0$, and let $\sigma \in \hat{G}$ be such that $M^i_{l,\sigma} \neq 0$. Then $T_\xi(k_0) M^i_{l,\sigma} = 0$ for each ξ, and thus $E(k_0) M^i_{l,\sigma} = 0$. On the other hand, the eigenvalue of $E(k_0)$ on $M^i_{l,\sigma}$ equals $l + c_\tau(k_0) > 0$, since $l \geq n > -c_\tau(k_0)$. This is a contradiction.

(iv) This follows by [12], Bemerkung after Lemma 5.1. Notice that the expression $(p,q)^{(i)}$ is independent of the lifts $\tilde{p}$ and $\tilde{q}$ since, if $a \in M^i_R \cap tP$, we have

$$(a, M^i_R)_R \subset t(P, M^i_R)_R \subset m_0^{i+1}. \tag{38}$$

When $i = 0$, we have $(\cdot,\cdot)^{(0)} = (\cdot,\cdot)_{k_0}$. Thus the result $M^1 = \mathrm{Rad}(k_0)$ is the special case $i = 0$. □

We end the section with a hint for the interpretation of the multiplicities $\delta_\tau(k) = [P = M(triv, k), L(\tau, k)]$. Suppose that G has a Coxeter-like presentation (in the sense of [1], Appendix 2) such that its diagram also provides a presentation of the fundamental group (the braid group) of the regular orbit space of G. Suppose that the cyclotomic Hecke algebra $H(G,u)$ corresponding to the diagram of G can be generated by $|G|$ elements over the ring $\mathbb{Z}[u, u^{-1}]$. The inhomogeneous relations for the simple generators of $H(G,q)$ are of the form

$$(T_s - u_{C,0})(T_s - u_{C,1}) \dots (T_s - u_{C,e_C-1}) = 0 \tag{39}$$

where s is a reflection in a certain cyclic group G_H with $H \in C \in \mathcal{C}$, with determinant $\det(s) = \exp(2\pi i/e_C)$. Let us now view the parameter value $u_{C,j}$ as a function of the parameters $k_{C,j}$ as follows

$$u_{C,j} \to q_{C,j} := \exp(2\pi i(j - e_C k_{C,j})/e_C). \tag{40}$$

We extend the ring of definition of $H(G,q)$ to the ring R of entire functions in the parameters $k_{C,j}$ via this substitution. The resulting algebra is denoted by $H(G)_R$. It is known that the Hecke algebra $H(G)_K$ over the quotient field K of R is split semi-simple (see [17], Corollary 6.6), so that we can uniquely parameterize the irreducible representations π_τ of $H(G)_K$ by the irreducible representations τ of G. Let U denote the principal indecomposable block of the trivial representation in the m-adic completion $H(G)_m$ of $H(G)_R$, and let K_m denote the quotient field of the ring R_m of formal power series in k centered at v. The results and method of [8] seem to suggest that the multiplicities $\delta'_\tau(v)$ defined by

$$K_m \otimes U = \oplus_{\tau\in\hat{G}} \delta'_\tau(v)(K_m \otimes (\pi_\tau)_m) \tag{41}$$

are related to the multiplicities $\delta_\tau(v)$ if $v_{C,j} < 0$ for all C, j.

In the next section we will turn to the study of the infinite family of imprimitive groups. The set K^{sing} will be described in detail for this class of complex reflection groups.

3. The groups of type $G(m,p,N)$

3.1. **Introduction.** In this section we study the particular case of the complex reflection group called $G(m,p,N)$, which is a finite subgroup of $U(N)$. Because of its close relation to the symmetric group it is possible to perform a detailed analysis of the Dunkl operators (constructed for real reflection groups in [3]), the pairing, and the analogues of the nonsymmetric Jack polynomials. In fact, the special case $G(2,1,N)$ is exactly the hyperoctahedral group (type B), and the results of one of the authors (Dunkl [6],[7]) on type-B polynomials motivate the methods used in this section. Some of the notation used in the first section is changed here to a mode better suited to deal with monomials and permutations. The fundamental objects are polynomials in $x = (x_1, x_2, \ldots, x_N) \in \mathbb{C}^N$ (considered as coordinate functions); the group is realized as a subgroup of the matrix group $U(N)$ acting on the row vector x. For a multi-index (composition) $\alpha \in \mathbb{N}_0^N$ let $|\alpha| = \sum_{i=1}^N \alpha_i$ and let x^α denote the monomial $\prod_{i=1}^N x_i^{\alpha_i}$. To a permutation $w \in S_N$ (the symmetric group on N letters) associate an $N \times N$ permutation matrix with 1's at the $(w(j), j)$ entries. The action on x is given by $(xw)_i = x_{w(i)}$; the action on polynomials is $wp(x) = p(xw)$. Thus the action on monomials is $w(x^\alpha) = x^{w\alpha}$ where $(w\alpha)_i = \alpha_{w^{-1}(i)}$ (consider α as a column vector). The symmetric group contains the transpositions (i,j), for $i \neq j$, defined by

$$x(i,j) = \left(x_1, \ldots, \overset{i}{x_j}, \ldots, \overset{j}{x_i}, \ldots\right).$$

For a fixed $m = 2, 3, \ldots$ let $\eta = e^{2\pi i/m}$, an m^{th} root of unity, then the complex reflection group W of type $G(m,1,N)$ consists of the $N \times N$ permutation matrices with the nonzero entries being powers of η. The group is generated by the transpositions $(i, i+1)$, $1 \leq i \leq N-1$, and by the complex reflection τ_N where τ_i is defined by

$$x\tau_i = \left(x_1, \ldots, \eta \overset{i}{x_i}, \ldots\right),$$

for $1 \leq i \leq N$. The powers τ_i^s are also reflections. Thus $\tau_i^s x^\alpha = \eta^{s\alpha_i} x^\alpha$ for $\alpha \in \mathbb{N}_0^N$. The symmetric group S_N is obviously a subgroup of W. The group W acts on polynomials by $wp(x) = p(xw)$ for $w \in W$. There are some obvious commutation relationships: (where x_i denotes the multiplication operator)

$$x_i \tau_i = \eta^{-1} \tau_i x_i,$$
$$(i,j)\,\tau_i = \tau_j\,(i,j),$$

and the elements $\tau_i^{-s}(i,j)\,\tau_i^s$ are ordinary (period 2) reflections in W. In terms of root vectors $v \neq 0$, such a reflection is given by $x\sigma_v = x - 2\left(\langle v, x\rangle / ||v||^2\right) v$ (where the hermitian inner product is $\langle x, y\rangle = \sum_{j=1}^N \overline{x}_j y_j$ and the norm $||x|| = \langle x, x\rangle^{1/2}$).

Definition 3.1. *For $1 \leq i \leq N$ let $e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots)$ denote the standard unit vector of $\mathbb{C}^N$. For $i \neq j$ and $0 \leq l \leq m-1$, let $v_{ij}^{(l)} = e_i - \eta^{-l} e_j$, and $R_0 = \left\{v_{ij}^{(l)} : 1 \leq i < j \leq N, 0 \leq l \leq m-1\right\}$.*

For $v = v_{ij}^{(l)}$ the reflection σ_v equals $\tau_i^{-l}(i,j)\tau_i^l$. Similarly $x\tau_i^l = x - (1-\eta^l)\langle e_i, x\rangle e_i$. To define the Dunkl operators introduce m parameters $\kappa_s, 0 \le s \le m-1$. In the notation of the Section 2, the class C_0 is $\{v^\perp : v \in R_0\}$ and $C_1 = \cup_{i=1}^N \{x : x_i = 0\}$, further $e_{C_0} = 2$, $k_{C_0} = \kappa_0$ and $e_{C_1} = m, k_{C_1,s} = \kappa_s$ for $1 \le s \le m-1$. The functional $\alpha_H = \langle v, \cdot\rangle$ where the hyperplane $H = v^\perp$.

Definition 3.2. *For* $1 \le i \le N$ *let*

$$T_i = \frac{\partial}{\partial x_i} + \kappa_0 \sum_{j\neq i}\sum_{s=0}^{m-1} \frac{1-\tau_i^{-s}(i,j)\tau_i^s}{x_i - \eta^s x_j} + \sum_{t=1}^{m-1}\kappa_t \sum_{s=0}^{m-1}\frac{\eta^{-st}\tau_i^s}{x_i},$$

where the divisions are understood to follow the numerator operations.

Note that T_i is the same object as $T_{e_i}(k)$; the nature of the group W makes it desirable to use coordinates. The action can be expressed in another useful way. Define associated projections on polynomials by the linear extension of

$$\pi_j(s)\,x^\alpha = \begin{cases} x^\alpha \text{ if } \alpha_j \equiv s \bmod m \\ 0 \text{ else} \end{cases},$$

for $1 \le j \le N$ and $s \in \mathbb{N}_0$. The projections can be expressed as $\pi_j(s) = \varepsilon_{H_j,s} = \frac{1}{m}\sum_{i=0}^{m-1}\eta^{-si}\tau_j^i$ (in the section 2 notation, where $H_i = \{x : x_i = 0\}$).

Proposition 3.3. *For* $\alpha \in \mathbb{N}_0^N$ *and for* $1 \le i \le N$*:*

$$T_i x^\alpha = \frac{\partial}{\partial x_i}x^\alpha + m\kappa_0 \sum_{j\neq i}\pi_j(\alpha_j)\frac{x^\alpha - (i,j)\,x^\alpha}{x_i - x_j}$$
$$+\begin{cases} m\kappa_s x^\alpha / x_i \textit{ if } \alpha_i \equiv s \bmod m \textit{ and } 1 \le s \le m-1 \\ 0 \textit{ if } \alpha_i \equiv 0 \bmod m. \end{cases}$$

Proof. The part involving (period 2) reflections (κ_0) is proven using the following formulae: (stated for $i = 1, j = 2$, which suffices)

$$\frac{x_1^{\alpha_1}x_2^{\alpha_2} - (1,2)\,x_1^{\alpha_1}x_2^{\alpha_2}}{x_1 - x_2} = \operatorname{sign}(\alpha_1 - \alpha_2)\sum_{t=\min(\alpha_1,\alpha_2)}^{\max(\alpha_1,\alpha_2)-1} x_1^{\alpha_1+\alpha_2-t-1}x_2^t, \tag{42}$$

and for any s with $0 \le s \le m-1$, we have

$$\frac{x_1^{\alpha_1}x_2^{\alpha_2} - \tau_1^{-s}(1,2)\,\tau_1^s x_1^{\alpha_1}x_2^{\alpha_2}}{x_1 - \eta^s x_2} = \eta^{-\alpha_2 s}\frac{x_1^{\alpha_1}(\eta^s x_2)^{\alpha_2} - x_1^{\alpha_1}(\eta^s x_2)^{\alpha_1}}{x_1 - \eta^s x_2}$$
$$= \operatorname{sign}(\alpha_1 - \alpha_2)\sum_{t=\min(\alpha_1,\alpha_2)}^{\max(\alpha_1,\alpha_2)-1} x_1^{\alpha_1+\alpha_2-t-1}x_2^t\eta^{s(t-\alpha_2)};$$

now the sum over $0 \le s \le m-1$ in effect applies $m\pi_2(\alpha_2)$ to the sum in formula 42. □

Remark 3.4. *The complex reflection group* $G(m,p,N)$ *(defined when* p *divides* m*) contains the reflections* $\tau_j^{-s}(i,j)\tau_j^s$ *(for* $i < j$ *and* $0 \le s \le m-1$*) and* τ_i^{sp} *for* $1 \le i \le N, 1 \le s \le \frac{m}{p} - 1$*. The appropriate modification of* $\{T_i\}_{i=1}^N$ *is to require* $\kappa_{sm/p} = 0$ *and* $\kappa_{t+sm/p} = \kappa_t$ *for* $1 \le s \le p-1$ *and* $1 \le t \le \frac{m}{p} - 1$.

Proof. Let $c_s = \sum_{j=1}^{m-1} \kappa_j \eta^{-sj}$ for $0 \leq s \leq m-1$, then c_s is the coefficient of τ_i^s in the formula for T_i (for any i), and $1 \leq s \leq m-1$. The inversion formula is $\kappa_j = \frac{1}{m} \sum_{s=0}^{m-1} c_s \eta^{sj}$ for $j \geq 1$, and $\sum_{s=0}^{m-1} c_s = 0$. The condition $c_s = 0$ unless $s \equiv 0 \bmod p$ is equivalent to the periodicity condition on the values of κ_j stated above. □

We refer to the list of residues mod m of the index α as the parity type and say that x^α and x^β have the same parity type if $\alpha_i \equiv \beta_i \bmod m$ for $1 \leq i \leq N$. If each monomial in a polynomial has the same parity type then we say the polynomial has that type. Proposition 3.3 implies that if a polynomial $p(x)$ has the same type as x^α for some $\alpha \in \mathbb{N}_0^N$ then $T_i p(x)$ has the same type as $x^\alpha x_i^{-1}$ (or $x^\alpha x_i^{m-1}$ if $\alpha_i = 0$). Thus the operators $\{T_i x_i\}_{i=1}^N$ preserve parity type. We will define an inner-product structure on polynomials in which these are self-adjoint. Just as in the symmetric group case they do not commute but can be modified to form a commutative set. First we need to calculate the commutant $[x_j, T_i] = x_j T_i - T_i x_j$.

Proposition 3.5. *For $i \neq j$, $[x_j, T_i] = \kappa_0 \sum_{s=0}^{m-1} \eta^s \tau_j^{-s} (i,j) \tau_j^s$.*

Proof. Use the definition of T_i and consider $x_j T_i - T_i x_j$. The terms involving differentiation and transpositions (i,t) with $t \neq i,j$ cancel, leaving only

$$x_j T_i - T_i x_j = \kappa_0 \sum_{s=0}^{m-1} \left(x_j \frac{1 - \tau_i^{-s}(i,j)\tau_i^s}{x_i - \eta^s x_j} - \frac{x_j - \tau_i^{-s}(i,j)\tau_i^s x_j}{x_i - \eta^s x_j} \right)$$

$$= \kappa_0 \sum_{s=0}^{m-1} \frac{x_i \eta^{-s} - x_j}{x_i - \eta^s x_j} \tau_i^{-s}(i,j)\tau_i^s = \kappa_0 \sum_{s=0}^{m-1} \eta^{-s} \tau_i^{-s}(i,j)\tau_i^s.$$

This completes the proof. □

Corollary 3.6. *For $i \neq j$ the following hold:*

$$[T_i x_i, T_j x_j] = \kappa_0 (T_i x_i - T_j x_j) \sum_{s=0}^{m-1} \tau_j^{-s}(i,j)\tau_j^s,$$

$$\left[T_i x_i, T_j x_j - \kappa_0 \sum_{s=0}^{m-1} \tau_j^{-s}(i,j)\tau_j^s \right] = 0.$$

Proof. Indeed

$$T_i x_i T_j x_j - T_j x_j T_i x_i = T_i (x_i T_j - T_j x_i) x_j - T_j (x_j T_i - T_i x_j) x_i,$$

and by the Proposition

$$T_i (x_i T_j - T_j x_i) x_j = \kappa_0 T_i \sum_{s=0}^{m-1} \eta^s \tau_i^{-s}(i,j)\tau_i^s x_j$$

$$= \kappa_0 T_i \sum_{s=0}^{m-1} \eta^s \tau_i^{-s} x_i (i,j) \tau_i^s$$

$$= \kappa_0 T_i x_i \sum_{s=0}^{m-1} \tau_i^{-s}(i,j)\tau_i^s$$

and similarly $T_j (x_j T_i - T_i x_j) x_i = \kappa_0 T_j x_j \sum_{s=0}^{m-1} \tau_j^{-s}(i,j)\tau_j^s$. Finally observe that $\sum_{s=0}^{m-1} \tau_i^{-s}(i,j)\tau_i^s = \sum_{s=0}^{m-1} \tau_i^{-s}\tau_j^s (i,j) = \sum_{s=0}^{m-1} \tau_i^s \tau_j^{-s} (i,j) = \sum_{s=0}^{m-1} \tau_j^{-s}(i,j)\tau_j^s$

(by changing the index of summation from s to $-s$ and using the relation $\tau_i\tau_j = \tau_j\tau_i$).

Furthermore the commutant satisfies $\left[T_ix_i, \tau_j^{-s}(i,j)\tau_j^s\right] = T_ix_i\tau_j^{-s}(i,j)\tau_j^s - \tau_j^{-s}(i,j)\tau_j^sT_ix_i = T_ix_i\tau_j^{-s}(i,j)\tau_j^s - T_jx_j\tau_j^{-s}(i,j)\tau_j^s$ for each s, since τ_j commutes with T_jx_j. □

Definition 3.7. *For $1 \le i \le N$ define the operators*

$$\mathcal{U}_i = T_ix_i - \kappa_0 \sum_{j<i}\sum_{s=0}^{m-1} \tau_i^{-s}(i,j)\tau_i^s.$$

Theorem 3.8. $\mathcal{U}_i\mathcal{U}_j = \mathcal{U}_j\mathcal{U}_i$.

Proof. Let $\lambda_{rt} = \sum_{s=0}^{m-1}\tau_r^{-s}(r,t)\tau_r^s = \sum_{s=0}^{m-1}\tau_t^{-s}(r,t)\tau_t^s$. Suppose that $i<j$, then we have $[\mathcal{U}_i, \mathcal{U}_j] = [T_ix_i, T_jx_j - \kappa_0\lambda_{ij}] - \kappa_0\sum_{r<j, r\neq i}[T_ix_i, \lambda_{rj}] - \kappa_0\sum_{r<i}[\lambda_{ri}, T_jx_j] + \kappa_0^2\sum_{r<i}([\lambda_{ri},\lambda_{rj}] + [\lambda_{ri},\lambda_{ij}])$. The commutators $[\lambda_{ri},\lambda_{tj}]$ with $t\neq r,i$ vanish. In the previous expansion all terms but the last are zero (by the Corollary). For $r<i<j$ (this computation is in the group algebra $\mathbb{C}W$)

$$\begin{aligned}\lambda_{ri}\lambda_{rj} &= \sum_{s=0}^{m-1}\tau_r^{-s}(r,i)\tau_r^s\sum_{t=0}^{m-1}\tau_j^{-t}(r,j)\tau_j^t = \sum_{s,t=0}^{m-1}\tau_r^{-s}\tau_i^{s+t}\tau_j^{-t}(r,i)(r,j)\\ &= \sum_{s,t=0}^{m-1}\tau_r^{-s}\tau_i^{s+t}\tau_j^{-t}(i,j)(r,i) = \sum_{t=0}^{m-1}\tau_j^{-t}(i,j)\tau_j^t\sum_{s=0}^{m-1}\tau_r^{-s}(r,i)\tau_r^s\\ &= \lambda_{ij}\lambda_{ri},\end{aligned}$$

and similarly $\lambda_{ri}\lambda_{ij} = \lambda_{rj}\lambda_{ri}$. Thus $[\lambda_{ri},\lambda_{rj}] + [\lambda_{ri},\lambda_{ij}] = 0$. □

The group algebra elements $\sum_{r<i}\lambda_{ri}$ (for $1<i\le N$) are the analogues of the Jucys-Murphy elements for the symmetric group. The inner product we will use is the pairing $(p,q)_k = p^*(T)\,q(x)\,|_{x=0}$, which means that the operator $p^*(T)$, obtained from $p^*(x) = \overline{p(\overline{x})}$ (each coefficient of p is conjugated) by replacing each x_i by T_i, is applied to the polynomial $q(x)$ and the resulting polynomial is evaluated at $x=0$. Obviously if p,q are homogeneous of the same degree then $p^*(T)\,q(x)$ is a constant (degree 0); and if p,q are homogeneous of different degrees then $(p,q)_k = 0$. Of course this is the coordinatized form of the general definition given in 16. The known results for S_N are used to analyze the action of T_i by factoring polynomials into a "parity part" and a part invariant under the complex reflections τ_i. Introduce a new variable

$$y = (y_1,\ldots,y_N) = (x_1^m,\ldots,x_N^m).$$

Say that a composition $\alpha\in\mathbb{N}_0^N$ is a *parity type* if $0\le\alpha_i<m$ for each i. If all the terms of a polynomial $p(x)$ have the same parity type then p can be expressed in the form $p(x) = x^\alpha g(y)$ for some parity type α. The type-A Dunkl operators (see Section 8.3 in the monograph [9] by Dunkl and Xu) are defined by

$$\mathcal{D}_ig(y) = \frac{\partial}{\partial y_i}g(y) + \kappa_0\sum_{j\neq i}\frac{g(y)-(i,j)\,g(y)}{y_i-y_j},$$

for $1\le i\le N$ (using the same notation (i,j) for the transpositions acting on y as on x). Often we write $\frac{1-(i,j)}{y_i-y_j}g(y)$ for the inner term.

Proposition 3.9. *Suppose $g(y)$ is a polynomial and α is a parity type then*

(1) *if* $\alpha_i > 0$ *then*

$$T_i x^\alpha g(y) = m x^\alpha x_i^{-1}\left(\frac{\alpha_i}{m} + \kappa_{\alpha_i} + \mathcal{D}_i y_i - 1 - \kappa_0 \sum_{j\neq i, \alpha_j \geq \alpha_i} (i,j)\right) g(y);$$

(2) *if* $\alpha_i = 0$ *then* $T_i x^\alpha g(y) = m x^\alpha x_i^{m-1} \mathcal{D}_i g(y)$.

Proof. Suppose $\alpha_i > 0$ then $T_i x^\alpha g(y) = (\alpha_i + m\kappa_{\alpha_i}) x^\alpha x_i^{-1} + m x^\alpha x_i^{m-1} \frac{\partial}{\partial y_i} g(y) + m\kappa_0 \sum_{j\neq i} E_j$, where (for $j \neq i$)

$$\begin{aligned} E_j &= \pi_j(\alpha_j) \frac{x^\alpha g(y) - (i,j) x^\alpha g(y)}{x_i - x_j} \\ &= \pi_j(\alpha_j)\left(x^\alpha \frac{y_i - y_j}{x_i - x_j}\frac{1-(i,j)}{y_i - y_j} g(y) + \frac{x^\alpha - (i,j)x^\alpha}{x_i - x_j}(i,j) g(y)\right). \end{aligned}$$

For the first term note $\pi_j(\alpha_j) x^\alpha \dfrac{y_i - y_j}{x_i - x_j} = \pi_j(\alpha_j) x^\alpha \sum_{s=0}^{m-1} x_i^{m-1-s} x_j^s = x_i^{m-1} x^\alpha$. By formula 42 (with $1, 2$ replaced by i, j)

$$\frac{x^\alpha - (i,j) x^\alpha}{x_i - x_j} = \operatorname{sign}(\alpha_i - \alpha_j) \sum_{s=\min(\alpha_i,\alpha_j)}^{\max(\alpha_i,\alpha_j)-1} x^\alpha x_i^{\alpha_j - s - 1} x_j^{s-\alpha_j},$$

(note that the power of x_j is s) and it follows that $\pi_j(\alpha_j) \dfrac{x^\alpha - (i,j)x^\alpha}{x_i - x_j} = x^\alpha x_i^{-1}$ if $\alpha_i > \alpha_j$ and $= 0$ if $\alpha_i \leq \alpha_j$; if $\alpha_i > \alpha_j$ then the summation extends over $0 \leq \alpha_j \leq s \leq \alpha_i - 1 \leq m - 2$ and the projection $\pi_j(\alpha_j)$ picks out the term with $s = \alpha_j$, and when $\alpha_i \leq \alpha_j$ the case $s = \alpha_j$ cannot occur. Also $\mathcal{D}_i y_i g(y) = g(y) + y_i \frac{\partial}{\partial y_i} g(y) + \kappa_0 \sum_{j\neq i} \dfrac{y_i g(y) - y_j (i,j) g(y)}{y_i - y_j}$ and the inner term equals $y_i \dfrac{1-(i,j)}{y_i - y_j} g(y) + (i,j) g(y)$. Consequently, for each $j \neq i$ we obtain the equality $x^\alpha x_i^{m-1} \dfrac{1-(i,j)}{y_i - y_j} g(y) = x^\alpha x_i^{-1} y_i \dfrac{1-(i,j)}{y_i - y_j} g(y) = x^\alpha x_i^{-1}\left(\dfrac{1-(i,j)}{y_i - y_j} y_i g(y) - (i,j) g(y)\right)$. The transpositions (i,j) for $\alpha_i > \alpha_j$ are cancelled out.

If $\alpha_i = 0$ then $T_i x^\alpha g(y) = m x^\alpha x_i^{m-1} \frac{\partial}{\partial y_i} g(y) + m\kappa_0 \sum_{j\neq i} E_j$ with the same E_j as before. The case $\alpha_i > \alpha_j$ can not occur so that $E_j = x_i^{m-1} x^\alpha \dfrac{1-(i,j)}{y_i - y_j} g(y)$. This completes the proof. □

3.2. **Simultaneous Eigenfunctions.** We use the nonsymmetric Jack polynomials from the type-A machinery to produce a complete set of simultaneous eigenfunction for the commuting operators $\{\mathcal{U}_i : 1 \leq i \leq N\}$. The nicest case is for polynomials of "*standard parity type*", that is, of the form $x^\alpha g(y)$ such that $i < j$ implies $\alpha_i \geq \alpha_j$. This means that in the list $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ the values $\alpha_i = m - 1$ appear first, then the values $\alpha_i = m - 2$ and so on until the end of the list composed of $\alpha_i = 0$ values (roughly $(m-1, \ldots, m-1, m-2, \ldots, m-2, \ldots, 0)$). Not every value need appear, of course. First we restate Proposition 3.9 for the commuting operators.

Proposition 3.10. *Suppose* $g(y)$ *is a polynomial and* α *is a parity type then*

(1) *if* $0 \leq \alpha_i < m-1$ *then* $\mathcal{U}_i x^\alpha g(y) =$

$$= m x^\alpha \left(\frac{\alpha_i + 1}{m} + \kappa_{\alpha_i+1} + \mathcal{D}_i y_i - 1 - \kappa_0 \left(\sum_{\alpha_j > \alpha_i} + \sum_{j<i, \alpha_j = \alpha_i} \right) (i,j) \right) g(y);$$

(2) *if* $\alpha_i = m-1$ *then* $\mathcal{U}_i x^\alpha g(y) = m x^\alpha \left(\mathcal{D}_i y_i - \kappa_0 \sum_{j<i, \alpha_j = m-1} (i,j) \right) g(y)$.

Proof. For any $\beta \in \mathbb{N}_0^N$ we have $\sum_{s=0}^{m-1} \tau_i^{-s} (i,j) \tau_i^s x^\beta = \sum_{s=0}^{m-1} \eta^{s(\beta_i - \beta_j)} (i,j) x^\beta = m (i,j) x^\beta$ if $\beta_j \equiv \beta_i \bmod m$ and else equals 0. Thus the second part of $\mathcal{U}_i$ contributes $-\kappa_0 \sum_{j<i, \alpha_j = \alpha_i} (i,j) g(y)$ (and $(i,j) x^\alpha = x^\alpha$ for such j). When $\alpha_i = m-1$ then write $x_i x^\alpha$ as $x^\alpha x_i^{1-m} y_i$ and use part (2) of Proposition 3.9. □

Following Definition 8.3.4 in [9], the type-A (commuting) operators are defined by

$$\mathcal{U}_i^A = \mathcal{D}_i y_i + \kappa_0 - \kappa_0 \sum_{j<i} (i,j).$$

The nonsymmetric Jack polynomials are defined by means of a partial order on compositions.

Definition 3.11. *For* $\mu, \nu \in \mathbb{N}_0^N$ *the relation* $\mu \succ \nu$ *means* $\sum_{i=1}^j \mu_i \geq \sum_{i=1}^j \nu_i$ *for each* j *and* $\mu \neq \nu$ *(dominance order);* μ^+ *is defined to be the (unique) partition* $w\mu$ *(for some* $w \in S_N$*; that is,* $1 \leq i < j \leq N$ *implies* $(\mu^+)_i \geq (\mu^+)_j$ *); and* $\mu \rhd \nu$ *means* $|\mu| = |v|$ *and* $\mu^+ \succ \nu^+$ *or* $\mu^+ = \nu^+$ *and* $\mu \succ \nu$.

Define a convenient basis $\{p_\mu : \mu \in \mathbb{N}_0^N\}$ for homogeneous polynomials (see [5]) by the generating function:

$$\sum_{\mu \in \mathbb{N}_0^N} p_\mu(y) z^\mu = \prod_{i=1}^N \left\{ (1 - y_i z_i)^{-1} \prod_{j=1}^N (1 - y_j z_i)^{-\kappa_0} \right\};$$

with the useful property that $\mu_i = 0$ implies $\mathcal{D}_i p_\mu = 0$. For each $\mu \in \mathbb{N}_0^N$ there is a unique simultaneous eigenfunction of $\{\mathcal{U}_i^A\}_{i=1}^N$ of the form $\zeta_\mu = p_\mu + \sum_{\nu \rhd \mu} B(v, \mu) p_\nu$ (where $B(v,\mu) \in \mathbb{Q}(\kappa_0)$ and does not depend on N provided that $N \geq M$ and $\mu_i = 0$ for all $i > M$). See Theorem 8.4.13 in [9].

Then $\zeta_\mu(y)$ satisfies $\mathcal{U}_i^A \zeta_\mu(y) = \xi_i(\mu) \zeta_\mu(y)$ for $\mu \in \mathbb{N}_0^N$ and $1 \leq i \leq N$; where the eigenvalues are given by

$$\xi_i(\mu) = \kappa_0 (N - \#\{j : \mu_j > \mu_i\} - \#\{j : j < i \,\&\, \mu_j = \mu_i\}) + \mu_i + 1.$$

Suppose $\beta \in \mathbb{N}_0^N$ then there is a unique parity type α and a composition $\gamma \in \mathbb{N}_0^N$ so that $\beta = \alpha + m\gamma$ (as vectors; that is, $\beta_i = \alpha_i + m\gamma_i$ and $\gamma_i = \lfloor \beta_i / m \rfloor$ for each i). We will construct a simultaneous eigenfunction for each β. When α is a standard parity type the nonsymmetric Jack polynomials work directly.

Proposition 3.12. *Suppose* α *is a standard parity type,* g *is any polynomial in* y*, and* $1 \leq i \leq N$*, then*

(1) *if* $0 \leq \alpha_i < m-1$ *then* $\mathcal{U}_i x^\alpha g(y) = m x^\alpha \left(\frac{\alpha_i+1}{m} + \kappa_{\alpha_i+1} - \kappa_0 - 1 + \mathcal{U}_i^A \right) g(y)$;

(2) *if* $\alpha_i = m-1$ *then* $\mathcal{U}_i x^\alpha g(y) = m x^\alpha \left(\mathcal{U}_i^A - \kappa_0 \right) g(y)$.

Proof. By definition of standard parity type, for any i the set $\{j : \alpha_j > \alpha_i\} \cup \{j : j < i \,\&\, \alpha_j = \alpha_i\}$ is exactly the set $\{j : 1 \leq j < i\}$. □

Corollary 3.13. *Suppose α is a standard parity type, and $\gamma \in \mathbb{N}_0^N$ then*

(1) $\mathcal{U}_i x^\alpha \zeta_\gamma(y) = m\left(\frac{\alpha_i+1}{m} - 1 + \kappa_{\alpha_i+1} + \xi_i(\gamma) - \kappa_0\right) x^\alpha \zeta_\gamma(y)$ *when* $0 \leq \alpha_i < m-1$;

(2) $\mathcal{U}_i x^\alpha \zeta_\gamma(y) = m(\xi_i(\gamma) - \kappa_0) x^\alpha \zeta_\gamma(y)$ *when* $\alpha_i = m-1$.

Suppose α' is any parity type, but not standard. The idea is to use a permutation which maps α' to a standard parity type in such a way that the original order of coordinates with the same value of α'_i is preserved. For technical reasons it is easier to do this backwards. Suppose that α is a standard parity type and suppose $w \in S_N$ (a permutation) and has the property that $1 \leq i < j \leq N$ and $\alpha_i = \alpha_j$ implies $w(i) < w(j)$. We will show that $wx^\alpha \zeta_\beta(y)$ is an eigenvector of each $\mathcal{U}_i$. We use the transformation properties (holding for any $w \in S_N$) : $w\mathcal{D}_i y_i = \mathcal{D}_{w(i)} y_{w(i)} w$ and $(r,s)w = w\left(w^{-1}(r), w^{-1}(s)\right)$ for $r \neq s$ (the action of permutations is as follows: w is an $N \times N$ permutation matrix with 1's at the $(w(i), i)$ entries, x is a row vector, compositions are column vectors; so that $w(x^\alpha) = x^{w\alpha}$ where $(w\alpha)_{w(i)} = \alpha_i$ for any i). In the following $wx^\alpha \zeta_\gamma(y)$ is the polynomial $x^{w\alpha}\zeta_\gamma(yw)$ (with parity type $w\alpha$).

Proposition 3.14. *Suppose α is a standard parity type, $\gamma \in \mathbb{N}_0^N$, and $w \in S_N$ has the property that $w(i) < w(j)$ whenever $1 \leq i < j \leq N$ and $\alpha_i = \alpha_j$, then*

(1) $\mathcal{U}_{w(i)} w x^\alpha \zeta_\gamma(y) = m\left(\frac{\alpha_i+1}{m} - 1 + \kappa_{\alpha_i+1} + \xi_i(\gamma) - \kappa_0\right) w x^\alpha \zeta_\gamma(y)$ *when* $0 \leq \alpha_i < m-1$;

(2) $\mathcal{U}_{w(i)} w x^\alpha \zeta_\gamma(y) = m(\xi_i(\gamma) - \kappa_0) w x^\alpha \zeta_\gamma(y)$ *when* $\alpha_i = m-1$.

Proof. Suppose $\alpha_i < m-1$ then part (1) of Proposition 3.10 applies (and note that $(w\alpha)_{w(i)} = \alpha_i$) so that

$$\begin{aligned}
&\mathcal{U}_{w(i)} w x^\alpha \zeta_\gamma(y) \\
&= m x^{w\alpha}\left(\frac{\alpha_i+1}{m} + \kappa_{\alpha_i+1} + \mathcal{D}_{w(i)} y_{w(i)} - 1 - \kappa_0 \sum_{j \in E} (w(i), j)\right) w\zeta_\gamma \\
&= m x^{w\alpha} w\left(\frac{\alpha_i+1}{m} + \kappa_{\alpha_i+1} + \mathcal{D}_i y_i - 1 - \kappa_0 \sum_{j\in E}\left(i, w^{-1}(j)\right)\right)\zeta_\gamma;
\end{aligned}$$

where the set $E = \left\{j : (w\alpha)_j > (w\alpha)_{w(i)}\right\} \cup \left\{j : j < w(i) \;\&\; (w\alpha)_j = (w\alpha)_{w(i)}\right\}$. But $\sum_{j\in E}\left(i, w^{-1}(j)\right) = \sum_{w(s)\in E}(i,s) = \sum_{s \in w^{-1}E}(i,s)$, and $w^{-1}E = \{s : \alpha_s > \alpha_i\} \cup \{s : w(s) < w(i) \;\&\; \alpha_s = \alpha_i\}$. By the condition on w, the set $w^{-1}E$ is equal to $\{s : 1 \leq s < i\}$. Indeed, consider any $j < i$; $\alpha_j < \alpha_i$ is impossible by definition of standard parity type so $\alpha_j \geq \alpha_i$; furthermore if $\alpha_j = \alpha_i$ for some j then $j < i$ if and only if $w(j) < w(i)$. The proof of part (1) is now completed similarly to the proof of Proposition 3.12.

The proof of part (2) is an obvious modification of that for part (1). □

3.3. Evaluation of the Pairing.

We recall some facts from Proposition 2.17 and Theorem 2.18.

Theorem 3.15. *The pairing $(\cdot,\cdot)_k$ has the following properties (with $p, q \in \mathcal{P}_n$):*

(1) $(q,p)_k = \overline{(p,q)_k}$;

(2) $(T_i x_i p, q)_k = (p, T_i x_i q)_k$ *for* $1 \leq i \leq N$;

(3) $(wp, wq)_k = (p, q)_k$ *for any* $w \in W$.

Proof. From the definition it is clear that $(p, T_i x_i q)_k = (x_i p, x_i q)_k$, and by part (1) we have $(T_i x_i p, q)_k = \overline{(q, T_i x_i p)_k} = \overline{(x_i q, x_i p)_k} = (x_i p, x_i q)_k$.

For the real case ($m = 2$) there is an associated inner product with respect to the measure $\prod_{i=1}^{N} |x_i|^{2\kappa_1} \prod_{i<j} |x_i^2 - x_j^2|^{2\kappa_0} \exp\left(-\frac{|x|^2}{2}\right) dx$ on $\mathbb{R}^N$ (see [4] or Theorem 5.2.7 and Section 9.6.3 in [9]). It is an interesting question whether there is a similar situation for the complex group. □

By the properties of the pairing in Theorem 3.15 there are obvious orthogonality relations.

Lemma 3.16. *The operators* $\mathcal{U}_i$ *are self-adjoint for* $(\cdot, \cdot)_k$.

Proof. The reflections $\sigma = \tau_i^{-s}(i, j)\tau_i^s$ are self-adjoint (by part (2) of Theorem 3.15 because $\sigma^2 = 1$. By part (3) of this theorem $T_i x_i$ is self-adjoint, thus, $\mathcal{U}_i$ is also. □

Proposition 3.17. *Suppose* α, β *are parity types and* $\mu, \nu \in \mathbb{N}_0^N$ *then:*

(1) *if* $g_1(y), g_2(y)$ *are polynomials and* $\alpha \neq \beta$ *then* $\left(x^\alpha g_1(y), x^\beta g_2(y)\right)_k = 0$;
(2) *if* $\mu \neq \nu$ *and* α *is a standard parity type then* $(x^\alpha \zeta_\mu(y), x^\alpha \zeta_\nu(y))_k = 0$.

Proof. For part (1) suppose $\alpha_i \neq \beta_i$ for some i then $\left(x^\alpha g_1(y), x^\beta g_2(y)\right)_k = \left(\tau_i x^\alpha g_1(y), \tau_i x^\beta g_2(y)\right)_k = \eta^{\beta_i - \alpha_i}\left(x^\alpha g_1(y), x^\beta g_2(y)\right)_k$ and $\eta^{\beta_i - \alpha_i} \neq 1$.

For part (2), suppose that $\mu_i \neq \nu_i$ for some i. By Lemma 3.16, we have the equality $(\mathcal{U}_i x^\alpha \zeta_\mu(y), x^\alpha \zeta_\nu(y))_k = (x^\alpha \zeta_\mu(y), \mathcal{U}_i x^\alpha \zeta_\nu(y))_k$. This proves the claim, because the two polynomials are eigenfunctions with different eigenvalues. □

For other parity types see Proposition 3.14. To do the nontrivial pairings we collect some facts and notation from the S_N case. An element $\lambda \in \mathbb{N}_0^N$ is a *partition* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. For any $\gamma \in \mathbb{N}_0^N$ let γ^+ be the unique partition such that $\gamma^+ = w\gamma$ for some $w \in S_N$. For $\gamma \in \mathbb{N}_0^N$ and $\varepsilon = \pm$ let

$$\mathcal{E}_\varepsilon(\gamma) = \prod\left\{1 + \frac{\varepsilon\kappa_0}{\xi_j(\gamma) - \xi_i(\gamma)} : i < j \,\&\, \gamma_i < \gamma_j\right\}.$$

For a partition λ and indeterminate t (and implicit parameter κ_0) the *generalized Pochhammer symbol* is defined by

$$(t)_\lambda = \prod_{i=1}^{N} (t - (i-1)\kappa_0)_{\lambda_i},$$

where $(a)_0 = 1$ and $(a)_{n+1} = (a)_n (a + n)$; further suppose $\lambda_M > \lambda_{M+1} = 0$ (or $M = N$) then the *hook length product* is

$$h(\lambda, t) = \prod_{i=1}^{M} \prod_{j=1}^{\lambda_i} (\lambda_i - j + t + \kappa_0 \# \{s : s > i \,\&\, j \leq \lambda_s\}).$$

For $0 \leq l \leq m-1$ and $1 \leq i \leq m-1$ define the indicator functions $\chi_i(l) = \lfloor \frac{l}{i} \rfloor$ and extend coordinate-wise to parity types, that is, $\chi_i(\alpha) \in \mathbb{N}_0^N$ and $(\chi_i(\alpha))_j = \chi_i(\alpha_j)$ for each j. The rest of the section is mostly concerned with proving the following.

Theorem 3.18. *Let α be a standard parity type and let $\gamma \in \mathbb{N}_0^N$, then*

$$\left(x^{\alpha}\zeta_{\gamma}(y), x^{\alpha}\zeta_{\gamma}(y)\right)_k = m^{|\alpha|+m|\gamma|}\prod_{i=1}^{m-1}\left((N-1)\kappa_0 + \frac{i}{m} + \kappa_i\right)_{(\gamma+\chi_i(\alpha))^+}$$
$$\times (N\kappa_0+1)_{\gamma^+}\frac{h(\gamma^+,\kappa_0+1)}{h(\gamma^+,1)}\mathcal{E}_+(\gamma)\mathcal{E}_-(\gamma).$$

The argument has three main steps:

(1) find $(x^{\alpha}\zeta_{\gamma}(y), x^{\alpha}\zeta_{\gamma}(y))_k$ in terms of $(\zeta_{\gamma}(y), \zeta_{\gamma}(y))_k$;
(2) find $(\zeta_{\gamma}(y), \zeta_{\gamma}(y))_k$ in terms of $(\zeta_{\lambda}(y), \zeta_{\lambda}(y))_k$, where $\lambda = \gamma^+$;
(3) find $(\zeta_{\lambda}(y), \zeta_{\lambda}(y))_k$ in terms of $(\zeta_{\lambda-\varepsilon_M}(y), \zeta_{\lambda-\varepsilon_M}(y))_k$ where $\lambda_M > 0$ and $\lambda_{M+1} = 0$ or $M = N$.

The formula is valid for the trivial case $\alpha = 0 = \gamma$. The method of proof is to assume the formula for lower degrees and show the above mentioned ratios are consistent with the formula.

Proposition 3.19. *Let α be a standard parity type and let $\gamma \in \mathbb{N}_0^N$ then*

$$T^{\alpha}x^{\alpha}\zeta_{\gamma}(y) = m^{|\alpha|}\frac{\prod_{i=1}^{m-1}\left((N-1)\kappa_0 + \frac{i}{m} + \kappa_i\right)_{(\gamma+\chi_i(\alpha))^+}}{\prod_{i=1}^{m-1}\left((N-1)\kappa_0 + \frac{i}{m} + \kappa_i\right)_{\gamma^+}}\zeta_{\gamma}(y).$$

Proof. We reduce the degree of x^{α} by one at each step so that each intermediate stage is a standard parity type. Suppose that $\alpha_s = l > \alpha_{s+1}$ for some s, or $s = N$ and $l > 0$, then $\{j : j \neq s, \alpha_j \geq l\} = \{j : j < s\}$. By Proposition 3.9

$$T_s x^{\alpha}\zeta_{\gamma}(y) = m x^{\alpha}x_s^{-1}\left(\frac{l}{m} + \kappa_l + \mathcal{D}_s y_s - 1 - \kappa_0\sum_{j<s}(j,s)\right)\zeta_{\gamma}(y)$$
$$= m x^{\alpha}x_s^{-1}\left(\frac{l}{m} + \kappa_l + \xi_s(\gamma) - 1 - \kappa_0\right)\zeta_{\gamma}(y),$$

(because $\mathcal{D}_s y_s - \kappa_0\sum_{j<s}(j,s) = \mathcal{U}_s^A - \kappa_0$); recall $\xi_s(\gamma) = \kappa_0(N - \#\{j : \gamma_j > \gamma_s\} - \#\{j : \gamma_j = \gamma_s \,\&\, j < s\} + \gamma_s + 1)$. There is a unique $w \in S_N$ such that $w\gamma = \gamma^+$ and $i < j$ and $\gamma_i = \gamma_j$ implies $w(i) < w(j)$. We claim that $w(\gamma + \chi_t(\alpha)) = (\gamma + \chi_t(\alpha))^+$ for $1 \leq t \leq m-1$. We must show that $\gamma_i + \chi_t(\alpha_i) > \gamma_j + \chi_t(\alpha_j)$ implies $w(i) < w(j)$; the condition is equivalent to $\gamma_i - \gamma_j > \chi_t(\alpha_j) - \chi_t(\alpha_i)$. If $\chi_t(\alpha_j) \geq \chi_t(\alpha_i)$ then $\gamma_i > \gamma_j$ and $w(i) < w(j)$. By definition of standard type the finite sequence $(\chi_t(\alpha_i))_{i=1}^N$ is nonincreasing (a partition), thus $\chi_t(\alpha_j) < \chi_t(\alpha_i)$ implies $\chi_t(\alpha_i) = 1, \chi_t(\alpha_j) = 0$ and $i < j$. So $\gamma_i + 1 > \gamma_j$; if $\gamma_i > \gamma_j$ or if $\gamma_i = \gamma_j$ and $i < j$ then $w(i) < w(j)$.

Let $\beta = \alpha - \varepsilon_s$ (the parity type of $T_s x^{\alpha}\zeta_{\gamma}(y)$), then $\chi_t(\beta) = \chi_t(\alpha)$ for $t \neq l$. By the claim the only difference between $(\gamma + \chi_t(\alpha))^+$ and $(\gamma + \chi_t(\beta))^+$ is $(\gamma + \chi_t(\alpha))_{w(s)}^+ = \gamma_s + 1 = (\gamma + \chi_t(\beta))_{w(s)}^+ + 1$ (also note that $(\gamma + \chi_t(\alpha))_{w(j)}^+ = (\gamma + \chi_t(\alpha))_j$ and similarly for β). Up to a factor depending only on γ, and where C_l denotes the factors involving $\gamma + \chi_t(\alpha)$ for $t \neq l$ which do not change from β to

α, $T^{\beta}x^{\beta}\zeta_{\gamma}(y)$ equals

$$C_l m^{|\beta|}\prod_{i=1}^{N}\left((N-i)\kappa_0+\frac{l}{m}+\kappa_l\right)_{(\gamma+\chi_l(\beta))_i^+}$$
$$=C_l m^{|\beta|}\prod_{j=1}^{N}\left((N-w(j))\kappa_0+\frac{l}{m}+\kappa_l\right)_{(\gamma+\chi_l(\beta))_j}.$$

By construction $\gamma^+_{w(i)}=\gamma_i$ and thus $\xi_s(\gamma)=\xi_{w(s)}(\gamma^+)=(N-w(s)+1)\kappa_0+\gamma_s+1$. We showed that $T_s x^{\alpha}\zeta_{\gamma}(y)=m\left(\frac{l}{m}+\kappa_l+\xi_s(\gamma)-1-\kappa_0\right)x^{\beta}\zeta_{\gamma}(y)=m(A+\gamma_s)x^{\beta}\zeta_{\gamma}(y)$, where $A=\frac{l}{m}+\kappa_l+(N-w(s))\kappa_0$. This is the desired factor since $(\gamma+\chi_l(\beta))_s=\gamma_s$ and $(A)_{\gamma_s}(A+\gamma_s)=(A)_{\gamma_s+1}$ and $(\gamma+\chi_l(\alpha))^+_{w(s)}=(\gamma+\chi_l(\alpha))_s=\gamma_s+1$. This process is used repeatedly to find $T^{\alpha}x^{\alpha}\zeta_{\gamma}(y)$ as a multiple of $\zeta_{\gamma}(y)$. □

We state the definition of an admissible inner product $\langle\cdot,\cdot\rangle$ on polynomials associated with S_N. For polynomials $g_1(y), g_2(y)$ we require that (1) $\langle g_1(y), g_2(y)\rangle=0$ if g_1, g_2 are homogeneous of different degrees, (2) for any $w\in S_N$, $\langle wg_1(y), wg_2(y)\rangle=\langle g_1(y), g_2(y)\rangle$ and (3) $\langle \mathcal{D}_i y_i g_1(y), g_2(y)\rangle=\langle g_1(y), \mathcal{D}_i y_i g_2(y)\rangle$ for $1\le i\le N$. The pairing $\langle\cdot,\cdot\rangle_T$ is permissible; the first two properties have been established already, and by Proposition 3.9 we have $T_i x_i g(y)=m\left(\frac{1}{m}+\kappa_1-1+\mathcal{D}_i y_i\right)g(y)$. This proves property (3). The following is a consequence of Theorem 8.5.8 in [9].

Proposition 3.20. *Suppose $\gamma\in\mathbb{N}_0^N$. Then we have*

$$(\zeta_{\gamma}(y),\zeta_{\gamma}(y))_k=\mathcal{E}_+(\gamma)\,\mathcal{E}_-(\gamma)\left(\zeta_{\gamma^+}(y),\zeta_{\gamma^+}(y)\right)_k.$$

Next we consider the problem of lowering the degree of ζ_λ for a partition λ. Suppose that $\lambda_M>0=\lambda_{M+1}$ (where $\lambda_{N+1}=0$; by using several results from Section 8.6 in [9] we will compute $\dfrac{(\zeta_\lambda(y),\zeta_\lambda(y))_k}{(\zeta_{\lambda-\varepsilon_M}(y),\zeta_{\lambda-\varepsilon_M}(y))_k}$. The starting point is the equation $(T_M^m\zeta_\lambda(y), T_M^m\zeta_\lambda(y))_k=(x_M^m T_M^m\zeta_\lambda(y),\zeta_\lambda(y))_k=(y_M T_M^m\zeta_\lambda(y),\zeta_\lambda(y))_k$.

We claim that $T_M^m\zeta_\lambda(y)=m^m K_\lambda\mathcal{D}_M\zeta_\lambda(y)$, where

$$K_\lambda=\prod_{t=1}^{m-1}\left(\frac{t}{m}+\kappa_t+(N-M)\kappa_0+\lambda_M-1\right).$$

Indeed, $T_M\zeta_\lambda(y)=mx_M^{m-1}\mathcal{D}_M\zeta_\lambda(y)$. By Lemma 8.6.3(ii) of [9] we know in addition that $\mathcal{D}_M y_M\mathcal{D}_M\zeta_\lambda(y)=((N-M)\kappa_0+\lambda_M)\mathcal{D}_M\zeta_\lambda(y)$. By induction suppose that

$$T_M^n\zeta_\lambda(y)=m^n\prod_{t=m-n+1}^{m-1}\left(\frac{t}{m}+\kappa_t+(N-M)\kappa_0+\lambda_M-1\right)x_M^{m-n}\mathcal{D}_M\zeta_\lambda(y).$$

Then

$$T_M x_M^{m-n}\mathcal{D}_M\zeta_\lambda(y)=mx_M^{m-n-1}\left(\frac{m-n}{m}+\kappa_{m-n}+\mathcal{D}_M y_M-1\right)\mathcal{D}_M\zeta_\lambda(y)$$
$$=mx_M^{m-n-1}\left(\frac{m-n}{m}+\kappa_{m-n}+(N-M)\kappa_0+\lambda_M-1\right)\mathcal{D}_M\zeta_\lambda(y),$$

where we let n take the values $1,2,\ldots,m-1$ in turn.

Suppose that $\theta_M = (1,2)(2,3)\ldots(M-1,M) \in S_N$ is a cycle. Denote by $\widetilde{\lambda} = \theta_M(\lambda - \varepsilon_M) = (\lambda_M - 1, \lambda_1, \lambda_2, \ldots, \lambda_{M-1}, 0, \ldots)$. By Theorem 8.6.5 of [9] $\mathcal{D}_M \zeta_\lambda = b_\lambda \theta_M^{-1} \zeta_{\widetilde{\lambda}}$, where $b_\lambda = (N - M + 1)\kappa_0 + \lambda_M$, and by Lemma 8.6.2(iii) $(y_M \mathcal{D}_M \zeta_\lambda, \zeta_\lambda)_k = \dfrac{\lambda_M}{\kappa_0 + \lambda_M} b_\lambda (\zeta_\lambda, \zeta_\lambda)_k$. Combining these equations we obtain

$$\begin{aligned}(T_M^m \zeta_\lambda(y), T_M^m \zeta_\lambda(y))_k &= m^{2m} K_\lambda^2 (\mathcal{D}_M \zeta_\lambda, \mathcal{D}_M \zeta_\lambda)_k = (y_M T_M^m \zeta_\lambda(y), \zeta_\lambda(y))_k \\ &= m^m K_\lambda (y_M \mathcal{D}_M \zeta_\lambda, \zeta_\lambda)_k = m^m K_\lambda b_\lambda \frac{\lambda_M}{\kappa_0 + \lambda_M} (\zeta_\lambda, \zeta_\lambda)_k,\end{aligned}$$

and thus

$$\begin{aligned}(\zeta_\lambda, \zeta_\lambda)_k &= m^m K_\lambda b_\lambda \frac{\kappa_0 + \lambda_M}{\lambda_M} \left(\theta_M^{-1} \zeta_{\widetilde{\lambda}}, \theta_M^{-1} \zeta_{\widetilde{\lambda}}\right)_k \\ &= m^m K_\lambda b_\lambda \frac{\kappa_0 + \lambda_M}{\lambda_M} (\zeta_{\lambda - \varepsilon_M}, \zeta_{\lambda - \varepsilon_M})_k \, \mathcal{E}_+\left(\widetilde{\lambda}\right) \mathcal{E}_-\left(\widetilde{\lambda}\right),\end{aligned}$$

because $\left(\theta_M^{-1} \zeta_{\widetilde{\lambda}}, \theta_M^{-1} \zeta_{\widetilde{\lambda}}\right)_k = \left(\zeta_{\widetilde{\lambda}}, \zeta_{\widetilde{\lambda}}\right)_k$ and $\left(\widetilde{\lambda}\right)^+ = \lambda - \varepsilon_M$. Now notice that $\mathcal{E}_\varepsilon\left(\widetilde{\lambda}\right) = \prod_{i=2}^M \left(1 + \dfrac{\varepsilon \kappa_0}{(M-i)\kappa_0 + \lambda_i + 1 - \lambda_M}\right)$ for $\varepsilon = \pm$, and it can be shown (see Section 8.7 of [9]) that

$$\frac{\kappa_0 + \lambda_M}{\lambda_M} \mathcal{E}_+\left(\widetilde{\lambda}\right) \mathcal{E}_-\left(\widetilde{\lambda}\right) = \frac{h(\lambda, \kappa_0 + 1)\, h(\lambda - \varepsilon_M, 1)}{h(\lambda - \varepsilon_M, \kappa_0 + 1)\, h(\lambda, 1)}.$$

Further $K_\lambda = \prod_{t=1}^{m-1} \dfrac{\left((N-1)\kappa_0 + \frac{t}{m} + \kappa_t\right)_\lambda}{\left((N-1)\kappa_0 + \frac{t}{m} + \kappa_t\right)_{\lambda - \varepsilon_M}}$ and $b_\lambda = \dfrac{(N\kappa_0 + 1)_\lambda}{(N\kappa_0 + 1)_{\lambda - \varepsilon_M}}$. This completes the proof of Theorem 3.18.

We turn to skew-symmetric polynomials in y. Suppose λ is a partition with all distinct parts, that is, $\lambda_1 > \lambda_2 > \ldots > \lambda_N \geq 0$ then the polynomial $a_\lambda = \sum_{w \in S_N} \operatorname{sign}(w)\, w \zeta_\lambda = \sum_{w \in S_N} \operatorname{sign}(w) \dfrac{\mathcal{E}_-\left(\lambda^R\right)}{\mathcal{E}_-(w\lambda)} \zeta_{w\lambda}$ is skew-symmetric (that is, $(i,j)\, a_\lambda = -a_\lambda$ for any transposition), where $\lambda^R = (\lambda_N, \lambda_{N-1}, \ldots, \lambda_1)$; further $(a_\lambda, a_\lambda)_k = N! \mathcal{E}_-\left(\lambda^R\right) (\zeta_\lambda, \zeta_\lambda)_k$ (see Theorem 8.5.11 in [9]). For the minimal example $\lambda = \delta = (N-1, N-2, \ldots, 2, 1, 0)$ the polynomial a_δ has the simple form $\prod_{1 \leq i < j \leq N} (y_i - y_j)$ and $\mathcal{E}_-\left(\delta^R\right) = \dfrac{h(\delta, 1)}{h(\delta, \kappa_0 + 1)}$. Let $v = (1, 1, \ldots, 1) \in \mathbb{N}_0^N$ and for any $0 \leq t \leq m-1$ let $f(x) = x^{tv} \prod_{1 \leq i < j \leq N} \left(x_i^m - x_j^m\right)$ (so that the parity type is $(t, t, \ldots, t)$. Specializing Theorem 3.18 to $\gamma = \delta$ and $\alpha = tv$, note that $a_\delta(y) = \sum_{w \in S_N} \operatorname{sign}(w)\, w \zeta_\delta(y)$ and thus

$$\begin{aligned}T^{tv} x^{tv} a_\delta(y) &= \sum_{w \in S_N} \operatorname{sign}(w)\, w \left(T^{tv} x^{tv} \zeta_\delta(y)\right) \\ &= m^{mt} \prod_{i=1}^{t} \frac{\left((N-1)\kappa_0 + \frac{i}{m} + \kappa_i\right)_{\delta + v}}{\left((N-1)\kappa_0 + \frac{i}{m} + \kappa_i\right)_\delta} a_\delta(y)\end{aligned}$$

which implies

$$(f,f)_k = N!\,m^{m(t+N(N-1)/2)} \prod_{i=1}^{t} \left((N-1)\,\kappa_0 + \frac{i}{m} + \kappa_i\right)_{\delta+v}$$
$$\times \prod_{i=t+1}^{m-1} \left((N-1)\,\kappa_0 + \frac{i}{m} + \kappa_i\right)_{\delta} (N\kappa_0+1)_{\delta}\,.$$

We used the fact that $\langle f,g\rangle / \langle \zeta_\lambda,\zeta_\lambda\rangle$ is the same in any permissible inner product for each $f,g \in \operatorname{span}\{\zeta_{w\lambda} : w \in S_N\}$ and the value of $\langle a_\delta,a_\delta\rangle / \langle \zeta_\delta,\zeta_\delta\rangle$ from Theorem 8.7.15 in [9]. The formula was conjectured by P. Hanlon in 1995.

3.4. **The Radical.** The radical for the hermitian form $(\cdot,\cdot)_k$ is the linear space $\operatorname{Rad}(\kappa)$ of polynomials p such that $(q,p)_k = 0$ for any polynomial q; the space depends on the parameter values $\kappa_0,\kappa_1,\ldots,\kappa_{m-1}$. For the group $G(m,p,N)$ the set of singular values (when $\operatorname{Rad}(\kappa) \neq \{0\}$) can be explicitly stated with reference to the discriminant. Briefly, the radical is nontrivial exactly when any of the linear functions occurring in $(f,f)_k$ take on values in $0,-1,-2,-3\ldots$ where $f = (x_1x_2\ldots x_N)^{m-1}\prod_{1\le i<j\le N}(x_i^m - x_j^m)$. The proof of this requires extra care when $\kappa_0 \in -\mathbb{N}$ because some of the nonsymmetric Jack polynomials fail to exist for such values. By careful analysis of their construction one sees that the poles can occur only at values of the form $n\kappa_0 + l = 0$ for $1 \le n \le N$ and $l \in \mathbb{N}$. In the following κ refers to the parameters $(\kappa_i)_{i=0}^{m-1}$

Definition 3.21. *Let* $\mathcal{K}_0 = \left\{\kappa : \kappa_0 = -\frac{j}{n} - l : l \in \mathbb{N}_0, 2 \le j-1 \le n \le N\right\}$ *and let* $\mathcal{K}_1 = \cup_{i=1}^{m-1}\cup_{n=0}^{N-1}\left\{\kappa : n\kappa_0 + \frac{i}{m} + \kappa_i \in -\mathbb{N}_0\right\}$.

We claim the set of singular values is $\mathcal{K}_0 \cup \mathcal{K}_1$.

Firstly, suppose $\kappa \notin \mathcal{K}_0$ and $\kappa_0 \notin -\mathbb{N}$; in this case all the functions ζ_α exist, the simultaneous eigenfunctions of $\{\mathcal{U}_i\}_{i=1}^N$ span the polynomials and the pairing formulae are valid. If $\kappa \in \mathcal{K}_1$ then there is a (nonzero) polynomial $x^\alpha\zeta_\lambda(y)$ (some standard parity type α, a partition λ) which is orthogonal to each polynomial, thus in $\operatorname{Rad}(\kappa)$. On the other hand, suppose there is a nonzero $p \in \operatorname{Rad}(\kappa)$ then expand p in terms of the eigenvector basis and let $wx^\alpha\zeta_\gamma(y)$ appear in the expansion with a nonzero coefficient (see Proposition 3.14); by the orthogonality relations this implies $(x^\alpha\zeta_\gamma(y), x^\alpha\zeta_\gamma(y))_k = 0$ and thus $\kappa \in \mathcal{K}_1$.

Secondly, suppose $\kappa \in \mathcal{K}_1$ and $\kappa_0 = l \in -\mathbb{N}$; but $\mathcal{K}_1$ is closed so take a sufficiently close $\kappa' \in \mathcal{K}_1$ with $\kappa_0' \notin -\mathbb{N}$ (in the same component as κ), for this value there is a nonzero polynomial p (a simultaneous eigenfunction) in $\operatorname{Rad}(\kappa)$; for some $n = 1,2,3,\ldots$ $(\kappa_0 - l)^n p$ has no poles and a nonzero limit at $\kappa_0 = l$. For any polynomial q we have $(q,(\kappa_0-l)^n p)_k = 0$ for all κ' close to κ, thus $\operatorname{Rad}(\kappa)$ is nontrivial.

Thirdly, let $\kappa \in \mathcal{K}_0$; by the results in [8] there is a nonzero polynomial $g(y)$ of least degree in the type-A radical, which implies $\mathcal{D}_i g(y) = 0$ for $1 \le i \le N$. By Proposition 3.9 $T_i g(y) = 0$ for all i and so $g(y) \in \operatorname{Rad}(\kappa)$.

It remains to show that if $\kappa \notin \mathcal{K}_0 \cup \mathcal{K}_1$ and $\kappa_0 \in -\mathbb{N}$ then $\operatorname{Rad}(\kappa) = \{0\}$. The problem is that the simultaneous eigenfunctions of $\{\mathcal{U}_i\}_{i=1}^N$ no longer span the polynomials; nevertheless it is still possible to give an argument based on triangularity properties. We will show that $\kappa \notin \mathcal{K}_1$ and $\operatorname{Rad}(\kappa) \neq \{0\}$ implies $\kappa \in \mathcal{K}_0$. Suppose that $p \in \operatorname{Rad}(\kappa)$ and $p \neq 0$; because $\operatorname{Rad}(\kappa)$ is an ideal the polynomial $g(y) = x^\alpha p(x) \in \operatorname{Rad}(\kappa)$ (if the parity type of p is β let $\alpha_i = m - \beta_i$ for each i).

Suppose that $g_0(y) \in \mathrm{Rad}(\kappa)$ and has minimal degree (among polynomials in y). Consider the polynomials $T_i g_0(y) = m x_i^{m-1} \mathcal{D}_i g_0(y)$; if each $\mathcal{D}_i g_0 = 0$ then g_0 is in the type-A radical and $\kappa \in \mathcal{K}_0$. Suppose not, we may assume $\mathcal{D}_1 g_0(y) \neq 0$. By Proposition 3.9 $T_1^m g_0(y) = m^m \prod_{i=1}^{m-1} \left(\frac{i}{m} + \kappa_i + \mathcal{D}_1 y_1 - 1\right) \mathcal{D}_1 g_0(y)$. We now use the fact that $\mathcal{D}_1 y_1$ is triangular (see p.454 in [7]) with respect to the partial order $\rhd$ (see Definition 3.11). Let $c y^\gamma$ be a nonzero term in $\mathcal{D}_1 g_0$ with maximal (for $\rhd$) γ;this implies that the coefficient of y^γ in $T_1^m g_0(y)$ is $m^m \prod_{i=1}^{m-1} \left(\frac{i}{m} + \kappa_i + v\right) c$ where $\nu = \kappa_0 (N - 1 - \#\{j : \gamma_j > \gamma_1\}) + \gamma_1$; the hypothesis $\kappa \notin \mathcal{K}_1$ implies $\frac{i}{m} + \kappa_i + v \neq 0$ for each i but then $T_1^m g_0(y)$ is a nonzero polynomial in y in the radical and of lower degree than g_0, a contradiction.

Thus the detailed knowledge of type-A polynomials makes it possible to describe the set of singular values for $G(m,1,N)$ and indeed for any $G(m,p,N)$. For the latter impose the periodicity conditions in Remark 3.4 on κ.

3.5. **Shift Operators.** Suppose G is a complex reflection group and recall the definitions of Section 2. Let p be a G-invariant polynomial, and C a G-orbit of reflection hyperplanes. Given a rational function f on V, we denote by $m(f)$ the operator "multiplication by f". Observe that, for $s \in \mathbb{N}$ with $s < e_C$, the operator

$$m\left(\prod_{H \in C} \alpha_H^{-s}\right) \circ p^*(T(k)) \circ m\left(\prod_{H \in C} \alpha_H^{s}\right)$$

has the property that it maps P^G to P^G.

Question 3.22. *Is the above operator on P^G equal to the restriction to P^G of an operator of the form $p^*(T(k'))$, where k' is obtained by incrementing some of the values of $k = (k_{C,i})$?*

This is well known in the case of real reflection groups [11], and in that case the relation plays an important role in many applications [14]. The proof of this "shift relation" is based in this case on the presence of the invariant $p = \sum x_i^2$ of order two. For this invariant the relation can be checked by simple direct computation. Then one remarks that this forces the relation also to be true for the higher order invariants, using $\mathfrak{sl}_2$ representation theory (see [11]).

We do not know of any general argument that works in the present case of complex reflection groups. Nevertheless, in this section we shall show that the answer to this question is affirmative for the groups $G(m,p,N)$. The argument is again based on a reduction to the case of S_N.

First we deal with shifting the parameters $\kappa_i, 1 \leq i \leq m-1$, for the group $G(m,1,N)$. We recall the type-A commutation $\mathcal{D}_i y_i - y_i \mathcal{D}_i = 1 + \kappa_0 \sum_{j \neq i} (i,j)$, where $y = (x_1^m, \ldots, x_N^m)$; thus $y_i \mathcal{D}_i = Y_i$ with $Y_i = \mathcal{D}_i y_i - 1 - \kappa_0 \sum_{j \neq i} (i,j)$ and $(\mathcal{D}_i y_i) \mathcal{D}_i = \mathcal{D}_i Y_i$, for each i. Let $v = (1, 1, \ldots, 1) \in \mathbb{N}^N$.

Proposition 3.23. *Let $1 \leq t \leq m-1$ and let g be any polynomial in y then*

$$T_i(\kappa)^m x^{tv} g(y) = x^{tv} T_i(\kappa')^m g(y),$$

for $1 \leq i \leq N$, where $\kappa' = (\kappa_0, \kappa_1 + 1, \ldots, \kappa_t + 1, \kappa_{t+1}, \ldots, \kappa_{m-1})$.

Proof. By Proposition 3.9

$$\begin{aligned}
&T_i(\kappa)^m x^{tv} g(y) \\
&= m^m x^{tv} \prod_{s=m-t}^{m-1} \left(\frac{s}{m} + \kappa_s + \mathcal{D}_i y_i - 1\right) \mathcal{D}_i \prod_{s=1}^{t} \left(\frac{s}{m} + \kappa_s + Y_i\right) \\
&= m^m x^{tv} \mathcal{D}_i \prod_{s=m-t}^{m-1} \left(\frac{s}{m} + \kappa_s + Y_i - 1\right) \prod_{s=1}^{t} \left(\frac{s}{m} + \kappa_s + Y_i\right) \\
&= m^m x^{tv} \mathcal{D}_i \prod_{s=1}^{m-1} \left(\frac{s}{m} + \kappa'_s + Y_i - 1\right) \\
&= x^{tv} T_i(\kappa')^m g(y).
\end{aligned}$$

In each of the products the terms commute pairwise so the order does not matter. The relation $(\mathcal{D}_i y_i)\mathcal{D}_i = \mathcal{D}_i Y_i$ is used to move $\mathcal{D}_i$ to the front (last in the order of operation). The formula for $T_i(\kappa')^m g(y)$ is obtained by setting $t = 0$ in the starting calculation. □

The argument must be modified for $G(m,p,N)$ for $1 < p < m$. Recall from Remark 3.4 that $\kappa_{sm/p} = 0$ and $\kappa_{t+sm/p} = \kappa_t$ for $1 \le s \le p-1$ and $1 \le t \le \frac{m}{p} - 1$. We will use κ to denote $(\kappa_0, \kappa_1, \ldots, \kappa_{m/p-1})$ in this discussion. The invariants for $G(m,p,N)$ are generated by the elementary symmetric functions of degrees $1, 2, \ldots, N-1$ in y and $x^{(m/p)v}$. Here is the modification of the previous proposition.

Proposition 3.24. *For the group $G(m,p,N)$ let $1 \le t \le \frac{m}{p} - 1$, $0 \le s \le p-1$ and let g be any polynomial in y then*

$$T_i(\kappa)^{m/p} x^{(t+sm/p)v} g(y) = x^{tv} T_i(\kappa')^{m/p} x^{(sm/p)v} g(y),$$

where $\kappa' = \left(\kappa_0, \kappa_1 + \frac{1}{p}, \ldots, \kappa_t + \frac{1}{p}, \kappa_{t+1}, \ldots, \kappa_{m/p-1}\right)$.

Proof. Suppose first that $1 \le s \le p-1$ then

$$\begin{aligned}
&T_i(\kappa)^{m/p} x^{(t+sm/p)v} g(y) \\
&= m^{m/p} x^{(t+(s-1)m/p)v} \prod_{j=t+1+(s-1)m/p}^{t+sm/p} \left(\frac{j}{m} + \kappa_j + Y_i\right) g(y) \\
&= m^{m/p} x^{(t+(s-1)m/p)v} \left(\frac{s}{p} + Y_i\right) \prod_{j=t+1}^{m/p-1} \left(\frac{s-1}{p} + \frac{j}{m} + \kappa_j + Y_i\right) \\
&\times \prod_{j=1}^{t} \left(\frac{s}{p} + \frac{j}{m} + \kappa_j + Y_i\right) g(y) \\
&= m^{m/p} x^{(t+(s-1)m/p)v} \left(\frac{s}{p} + Y_i\right) \prod_{j=1}^{m/p-1} \left(\frac{s-1}{p} + \frac{j}{m} + \kappa'_j + Y_i\right) g(y) \\
&= x^{tv} T_i(\kappa')^{m/p} x^{(sm/p)v} g(y).
\end{aligned}$$

Again the order of the product does not matter because each term is a linear function of the operator Y_i. The periodicity conditions were applied to yield the third line.

Further (the case $s = 0$)

$$\begin{aligned}
&T_i(\kappa)^{m/p} x^{tv} g(y) \\
&= m^{m/p} x^{tv} \prod_{j=m-m/p+t+1}^{m-1} \left(\frac{j}{m} + \kappa_j + \mathcal{D}_i y_i - 1 \right) \mathcal{D}_i \prod_{j=1}^{t} \left(\frac{j}{m} + \kappa_j + Y_i \right) g(y) \\
&= m^{m/p} x^{tv} \mathcal{D}_i \prod_{j=t+1}^{m/p-1} \left(\frac{p-1}{p} + \frac{j}{m} + \kappa_j + Y_i - 1 \right) \\
&\times \prod_{j=1}^{t} \left(\frac{j}{m} + \kappa_j + Y_i \right) g(y) \\
&= m^{m/p} x^{tv} \mathcal{D}_i \prod_{j=1}^{m/p-1} \left(\frac{j}{m} - \frac{1}{p} + \kappa'_j + Y_i \right) g(y) \\
&= x^{tv} T_i(\kappa')^{m/p} g(y).
\end{aligned}$$

In the second line the product over $m - \frac{m}{p} + t + 1 \leq j \leq m - 1$ is changed to $t + 1 \leq j \leq \frac{m}{p} - 1$ by replacing j by $j + \frac{m(p-1)}{p}$ and the periodicity of κ_j is also used. This completes the proof. □

Remark 3.25. *It may appear that the shift of $\frac{1}{p}$ is significantly different from the shift of 1 for the group $G(m,1,N)$, but in fact, the parameters κ_i should be replaced by $p\kappa_i$ (for $1 \leq i \leq \frac{m}{p} - 1$) to conform to the general setup of Section 2 and so the shift is effectively* 1.

We turn to the parameter κ_0 associated with the action of the symmetric group. Using the known results for S_N, we are going to prove that $f(T(\kappa))\, a_\delta(y)\, g(y) = a_\delta(y)\, f(T(\kappa'))\, g(y)$, where $\delta = (N-1, N-2, \ldots, 1, 0) \in \mathbb{N}_0^N$, and $a_\delta(y) = \prod_{1 \leq i < j \leq N} (y_i - y_j)$, $\kappa' = (\kappa_0 + 1, \kappa_1, \ldots, \kappa_{m-1})$ and f, g are (real) symmetric polynomials in y.

Recall some facts from Section 3.3: for any partition $\lambda \in \mathbb{N}_0^N$ the space $X_\lambda(\kappa_0) = \operatorname{span}\{w\zeta_\lambda : w \in S_N\} = \operatorname{span}\{\zeta_{w\lambda} : w \in S_N\}$ (the $\mathbb{R}$-span) is equipped with two permissible inner products $(\cdot,\cdot)_k$ and $\langle \cdot,\cdot \rangle_{\kappa_0}$, where the inner product $\langle f, g \rangle_{\kappa_0} = f(\mathcal{D}(\kappa_0))\, g(y)$ for $f, g \in X_\lambda(\kappa_0)$. We have shown that

$$(f,g)_k = m^{m|\lambda|} \prod_{i=1}^{m-1} \left((N-1)\kappa_0 + \frac{i}{m} + \kappa_i \right)_\lambda \langle f, g \rangle_{\kappa_0}.$$

First we establish the formula $(a_\delta f, a_\delta g)_k = (a_\delta, a_\delta)_k (f,g)_{k'}$ where $(f,g)_{k'}$ denotes the pairing for κ' and f, g are symmetric polynomials in y. The result of Heckman [11] shows that $\langle a_\delta f, a_\delta g \rangle_{\kappa_0} = \langle a_\delta, a_\delta \rangle_{\kappa_0} \langle f, g \rangle_{\kappa_0+1}$. Further, there is a unique S_N-invariant j_λ (up to scalar multiple) in X_λ, and if $\lambda_1 > \lambda_2 > \ldots > \lambda_N$ then there is a unique skew-invariant $a_\lambda \in X_\lambda$ (see Section 3.3). Further the invariant polynomial $a_{\lambda+\delta}(\kappa_0)/a_\delta = j_\lambda(\kappa_0 + 1)$ (note the dependence on the parameter), see Opdam [16]; we take the constant as 1 for convenience. If λ, μ are partitions in $\mathbb{N}_0^N$ then $\lambda \neq \mu$ implies $X_\lambda(\kappa_0) \perp X_\mu(\kappa_0)$ in any permissible inner product, hence $(a_{\lambda+\delta}(\kappa_0), a_{\mu+\delta}(\kappa_0))_k = 0$. The polynomials $a_{\lambda+\delta}(\kappa_0)/a_\delta$ form a basis for the symmetric polynomials so it suffices to prove the formula for the cases

$f = g = a_{\lambda+\delta}(\kappa_0)/a_\delta$. Thus

$$\begin{aligned}
&(a_{\lambda+\delta}(\kappa_0), a_{\lambda+\delta}(\kappa_0))_k \\
&= m^{m|\lambda+\delta|}\prod_{i=1}^{m-1}\left((N-1)\kappa_0+\frac{i}{m}+\kappa_i\right)_{\lambda+\delta}\langle a_{\lambda+\delta}(\kappa_0), a_{\lambda+\delta}(\kappa_0)\rangle_{\kappa_0} \\
&= m^{m|\lambda+\delta|}\prod_{i=1}^{m-1}\prod_{j=1}^{N}\left((N-j)\kappa_0+\frac{i}{m}+\kappa_i\right)_{\lambda_j+N-j} \\
&\times\langle a_\delta, a_\delta\rangle_{\kappa_0}\langle j_\lambda(\kappa_0+1), j_\lambda(\kappa_0+1)\rangle_{\kappa_0+1} \\
&= m^{m|\lambda+\delta|}\prod_{i=1}^{m-1}\prod_{j=1}^{N}\left((N-j)\kappa_0+\frac{i}{m}+\kappa_i\right)_{N-j}\left((N-j)(\kappa_0+1)+\frac{i}{m}+\kappa_i\right)_{\lambda_j} \\
&\times\langle a_\delta, a_\delta\rangle_{\kappa_0}\langle j_\lambda(\kappa_0+1), j_\lambda(\kappa_0+1)\rangle_{\kappa_0+1} \\
&= m^{m|\delta|}\prod_{i=1}^{m-1}\left((N-1)\kappa_0+\frac{i}{m}+\kappa_i\right)_\delta\langle a_\delta, a_\delta\rangle_{\kappa_0}(j_\lambda(\kappa_0+1), j_\lambda(\kappa_0+1))_{k'}
\end{aligned}$$

which used the relation between κ_0+1 and k' inner products on $X_\lambda(\kappa_0+1)$, and the known S_N results. Setting $\lambda = 0$ shows again that $(a_\delta, a_\delta)_k = m^{m|\delta|}\prod_{i=1}^{m-1}\left((N-1)\kappa_0+\frac{i}{m}+\kappa_i\right)_\delta\langle a_\delta, a_\delta\rangle_{\kappa_0}$.

Proposition 3.26. *Suppose f, g are symmetric polynomials in y then*

$$f(T(\kappa))\,a_\delta(y)\,g(y) = a_\delta(y)\,f(T(\kappa'))\,g(y),$$

where $\kappa' = (\kappa_0+1, \kappa_1, \ldots, \kappa_{m-1})$.

Proof. Without loss of generality assume that f, g are homogeneous. By the group-invariance properties of the pairing $f(T(\kappa))\,a_\delta(y)\,g(y)$ is a skew- symmetric polynomial and hence divisible by $a_\delta(y)$ with a symmetric quotient. If $\deg f > \deg g$ the quotient is zero. If $\deg f = \deg g$ then $f(T(\kappa))\,a_\delta(y)\,g(y) = c a_\delta(y)$ for some constant c; thus by the preceding formula we obtain $a_\delta(T(\kappa))\,f(T(\kappa))\,a_\delta(y)\,g(y) = (a_\delta(T(\kappa))\,a_\delta(y))\,(f(T(\kappa'))\,g(y))$ (recall $(a_\delta, a_\delta)_k \neq 0$ for generic κ, which suffices to prove this $\mathbb{Q}[\kappa]$ polynomial identity).

Suppose $\deg f = l < \deg g = n$ and let f_1 be an arbitrary homogeneous symmetric polynomial of degree $n-l$. Then we have $f_1(T(\kappa))\,f(T(\kappa))\,a_\delta(y)\,g(y) = a_\delta(y)\,f_1(T(\kappa'))\,f(T(\kappa'))\,g(y)$, but $f(T(\kappa'))\,g(y)$ is symmetric of degree $n-l$. Hence $a_\delta(y)\,f_1(T(\kappa'))\,f(T(\kappa'))\,g(y) = f_1(T(\kappa))\,a_\delta(y)\,f(T(\kappa'))\,g(y)$. This identity holds for all κ and implies the claimed formula when κ is not singular. The formula is again a polynomial in κ, hence is valid for all κ. □

Now we can state the analogue of Corollary 4.5 of [8]:

Corollary 3.27. *For $G = G(m,p,N)$, let $\pi := x^{(m/p-1)v}a_\delta$. Let 1 denote the parameter tuple such that $1_{C,i} = 1$ for both the orbits of reflection hyperplanes C, and $i = 1, \ldots, e_C - 1$ (in the notation $\mathbf{k} = (k_{C,i})$ of Section 2, not the tuple (κ_i) of the present section; cf. Remark 3.25; if $p = m$ or $N = 1$ we have only one orbit). For all $p, q \in P^G$ we have*

$$(\pi p, \pi q)_k = (\pi, \pi)_k (p, q)_{k+1}. \qquad (43)$$

Proof. As in [8] we have, using Theorem 2.18, Proposition 3.24 and Proposition 3.26

$$\begin{aligned}(\pi p, \pi q)_k &= (\pi, p^*(T(k))\pi q)_k \\ &= (\pi, \pi p^*(T(k+1))q)_k \\ &= (\pi, \pi)_k (p, q)_{k+1},\end{aligned}$$

since we may assume that the degrees of p and q are equal. □

This result in fact gives an alternative proof of the explicit description of the radical which was derived in the previous subsection, analogous to the proof given in [8].

These results may help in formulating more general statements for arbitrary complex reflection groups.

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