SUPERANALOGS OF THE CALOGERO OPERATORS AND JACK POLYNOMIALS

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ABSTRACT. A depending on a complex parameter k superanalog $\mathcal{SL}$ of Calogero operator is constructed; it is related with the root system of the Lie superalgebra $\mathfrak{gl}(n|m)$. For $m = 0$ we obtain the usual Calogero operator; for $m = 1$ we obtain, up to a change of indeterminates and parameter k the operator constructed by Veselov Chalykh and M. Feigin. For $k = 1, \frac{1}{2}$ the operator $\mathcal{SL}$ is the radial part of the 2nd order Laplace operator for the symmetric superspaces corresponding to pairs $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V), \mathfrak{gl}(V))$ and $(\mathfrak{gl}(V), \mathfrak{osp}(V))$, respectively. We will show that for the generic m and n the superanalogs of the Jack polynomials constructed by Kerov, Okunkov and Olshanskii are eigenfunctions of $\mathcal{SL}$; for $k = 1, \frac{1}{2}$ they coincide with the spherical functions corresponding to the above mentioned symmetric superspaces.

We also study the inner product induced by Berezin's integral on these superspaces.

This paper is a detailed exposition of [S3] .I define superanalogs of Calogero operator and Jack polynomials for symmetric superspaces corresponding to pairs $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V), \mathfrak{gl}(V))$ and $(\mathfrak{gl}(V), \mathfrak{osp}(V))$.

Recently Desrosiers, Lapointe and P. Mathieu suggested a different approach to superization of Jack polynomials involving an odd indeterminates [DLM],[DLM1].

In [SchZ] Scheunert and Zhang proved the existence invariant integral for classical Lie superalgebras.In the section 7 an algebraic analog of Berezin integral for $\mathfrak{gl}(V)$ is constructed in more details.

1.1. The Hamiltonian of the quantum Calogero problem is of the form

$$\mathcal{L} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial t_i}\right)^2 - \frac{1}{2}k(k-1)\sum_{i<j} \frac{\omega^2}{\sinh^2 \frac{\omega}{2}(t_i - t_i)}. \tag{1.1.1}$$

In this form it is a particular case (corresponding to the root system R of $\mathfrak{gl}(n)$) of the operator constructed in the famous paper by Olshanetsky and Perelomov [OP]

$$\mathcal{L} = \Delta - \sum_{\alpha \in R^+} k_\alpha(k_\alpha - 1)\frac{(\alpha,\alpha)}{(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})^2}. \tag{1.1.2}$$

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Veselov, M. Feigin and Chalykh [CFV1] suggested the following generalization of operator (1.1.1)

$$\mathcal{L}' = \sum_{i=1}^{n}\left(\frac{\partial}{\partial t_i}\right)^2 + \left(\frac{\partial}{\partial t_{n+1}}\right)^2 + \tfrac{1}{2}k(k+1)\sum_{i<j}\frac{\omega^2}{\sinh^2\frac{\omega}{2}(t_i-t_i)} - \tfrac{1}{2}(k+1)\sum_{i=1}^{n}\frac{\omega^2}{\sinh^2\frac{\omega}{2}(t_i-\sqrt{k}t_{n+1})}. \tag{1.1.3}$$

It is known ([LV]) that eigenfunctions of operator (1.1.1) can be expressed in terms of Jack polynomials $P_\lambda(x_1,...,x_n;k)$, where λ is a partition of n. (For definition and properties of Jack polynomials see [M], [St].) It is known ([M]) that for $k=1,\frac{1}{2},2$ (our k is inverse of α, the parameter of Jack polynomials Macdonald uses in [M]) Jack polynomials are interpreted as spherical functions on symmetric spaces corresponding to pairs $(\mathfrak{gl}\oplus\mathfrak{gl},\mathfrak{gl})$, $(\mathfrak{gl},\mathfrak{sp})$ and $(\mathfrak{gl},\mathfrak{o})$, respectively. In these cases the corresponding operators are radial parts of the corresponding second order Laplace operators.

1.2. Superroots of $\mathfrak{gl}(n|m)$. Let $I = I_{\bar 0}\coprod I_{\bar 1}$ be the union of the "even" indices $I_{\bar 0} = \{1,\dots,n\}$ and "odd" indices $I_{\bar 1}=\{\bar 1,\dots,\bar m\}$. Let $\dim V=(n|m)$ and $e_1,\dots,e_n,e_{\bar 1},\dots,e_{\bar m}$ be a basis of V such that the parity of each vector is equal to that of its index. Let $\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{\bar 1},\dots,\varepsilon_{\bar m}$ be the left dual basis of V^*. Then the set of roots can be described as follows: $R=R_{11}\coprod R_{22}\coprod R_{12}\coprod R_{21}$, where

$$\begin{aligned} &R_{11}=\{\varepsilon_i-\varepsilon_j \mid i,j\in I_{\bar 0}\}, \quad R_{22}=\{\varepsilon_i-\varepsilon_j \mid i,j\in I_{\bar 1}\},\\ &R_{12}=\{\varepsilon_i-\varepsilon_j \mid i\in I_{\bar 0},\ j\in I_{\bar 1}\}, \quad R_{21}=\{\varepsilon_i-\varepsilon_j \mid i\in I_{\bar 1},\ j\in I_{\bar 0}\}. \end{aligned} \tag{1.2.1}$$

On V^*, define the depending on parameter k inner product by setting

$$(v_1^*,v_2^*)_k=\sum_{i=1}^{n}v_1^*(e_i)v_2^*(e_i)-k\sum_{j=1}^{m}v_1^*(e_{\bar j})v_2^*(e_{\bar j}) \tag{1.2.2}$$

and set $\rho_{(k)}=k\rho_1+\frac{1}{k}\rho_2-\rho_{12}$, where

$$\rho_1=\frac{1}{2}\sum_{\alpha\in R_{11}^+}\alpha;\quad \rho_2=\frac{1}{2}\sum_{\beta\in R_{22}^+}\beta;\quad \rho_{12}=\frac{1}{2}\sum_{\gamma\in R_{12}}\gamma.$$

For any $l\in V^*$, define e^l as a linear functional on $S(V)$ that extends l to a homomorphism of $S(V)$ (or as a formal series) and denote by $\mathcal{H}$ the subalgebra in the algebra of quotients of $S(V^*)$ generated by the elements e^l for $l\in V^*$ and $(1-e^\alpha)^{-1}$ for $\alpha\in R$. On $\mathcal{H}$, define operators $\partial_i,\partial_{\bar j}$ by setting

$$\partial_i(e^{v^*})=v^*(e_i)e^{v^*},\qquad \partial_{\bar j}(e^{v^*})=v^*(e_{\bar j})e^{v^*}.$$

Define the superanalog of the Calogero operator to be

$$\mathcal{SL}=\sum_{i=1}^{n}\partial_i^2-k\sum_{\bar j=1}^{m}\partial_{\bar j}^2-k(k-1)\sum_{\alpha\in R_{11}^+}\frac{(\alpha,\alpha)_k}{(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})^2}+\tfrac{1}{k}(\tfrac{1}{k}-1)\sum_{\beta\in R_{22}^+}\frac{(\beta,\beta)_k}{(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}})^2}-2\sum_{\gamma\in R_{12}}\frac{(\gamma,\gamma)_k}{(e^{\frac{\gamma}{2}}-e^{-\frac{\gamma}{2}})^2}. \tag{1.2.3}$$

It easy to verify, that the superanalog of the Calogero operator can be rewritten in following form

$$\mathcal{SL}=\sum_{i=1}^{n}\partial_i^2-k\sum_{\bar{j}=1}^{m}\partial_{\bar{j}}^2-k(k-1)\sum_{\alpha\in R_{11}^+}\frac{1}{(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})^2}+ \\ \frac{2(1-k)}{k}\sum_{\beta\in R_{22}^+}\frac{1}{(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}})^2}-2(1-k)\sum_{\gamma\in R_{12}}\frac{1}{(e^{\frac{\gamma}{2}}-e^{-\frac{\gamma}{2}})^2}. \tag{1.2.4}$$

Observe that the change of variables

$$k\mapsto -s,\quad \varepsilon_j\mapsto\sqrt{s}\varepsilon_j\quad \text{for } j\in I_{\bar{1}}$$

sends $\mathcal{SL}_2$ into $\bar{\mathcal{SL}}_2$

$$\bar{\mathcal{SL}}_2=\sum_{i=1}^{n}\partial_i^2+\sum_{\bar{j}=1}^{m}\partial_{\bar{j}}^2-s(s+1)\sum_{\alpha\in R_{11}^+}\frac{1}{(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})^2}+ \\ \frac{2(s+1)}{s}\sum_{\beta\in R_{22}^+}\frac{1}{(e^{\frac{\sqrt{s}\beta}{2}}-e^{-\frac{\sqrt{s}\beta}{2}})^2}-2(s+1)\sum_{\gamma\in R_{12}}\frac{1}{(e^{\frac{\gamma_s}{2}}-e^{-\frac{\gamma_s}{2}})^2}. \tag{1.2.5}$$

where $\gamma_s=\varepsilon_i-\sqrt{s}\varepsilon_j$, if $\gamma=\varepsilon_i-\varepsilon_j$.

It implies that if $\dim V_{\bar{1}}=1$, then (1.2.5) coincides with the generalization of the Calogero (1.1.3) operator suggested in [CFV1].

In order to describe the eigenfunctions of $\mathcal{SL}$, it is convenient to present $\mathcal{SL}$ in terms of operator $\mathcal{M}$ described below. Set

$$\delta^{(k)}=\prod_{\alpha\in R_{11}^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})^k\prod_{\beta\in R_{22}^+}(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}})^{1/k}\prod_{\gamma\in R_{12}}(e^{\frac{\gamma}{2}}-e^{-\frac{\gamma}{2}})^{-1}. \tag{1.2.6}$$

Set

$$\mathcal{M}=\left(\delta^{(k)}\right)^{-1}\left(\mathcal{L}-(\rho_{(k)},\rho_{(k)})_k\right)\delta^{(k)}.$$

1.2.1. Lemma . *The explicit form of $\mathcal{M}$ is*

$$\mathcal{M}=\sum_{i=1}^{n}\partial_i^2-k\sum_{\bar{j}=1}^{m}\partial_{\bar{j}}^2+k\sum_{\alpha\in R_{11}^+}\frac{e^\alpha+1}{e^\alpha-1}\partial_\alpha- \\ \sum_{\beta\in R_{22}^+}\frac{e^\beta+1}{e^\beta-1}\partial_\beta-\sum_{\gamma\in R_{12}}\frac{e^\gamma+1}{e^\gamma-1}\partial_{\gamma,k}, \tag{1.2.7}$$

where

$$\partial_\alpha=\partial_i-\partial_j\qquad \text{for } \alpha=\varepsilon_i-\varepsilon_j;$$
$$\partial_\beta=\partial_{\bar{i}}-\partial_{\bar{j}}\qquad \text{for } \beta=\varepsilon_{\bar{i}}-\varepsilon_{\bar{j}};$$
$$\partial_{\gamma,k}=\partial_i+k\partial_{\bar{j}}\quad \text{for } \gamma=\varepsilon_i-\varepsilon_{\bar{j}}.$$

In terms of new indeterminates $x_i=e^{\varepsilon_i}$ and $y_{\bar{j}}=e^{\varepsilon_{\bar{j}}}$ the operator $\mathcal{M}$ takes the form

$$\mathcal{M}=\sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2-k\sum_{\bar{j}=1}^{m}\left(y_j\frac{\partial}{\partial y_j}\right)^2+k\sum_{1\le i<j\le n}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right)- \\ \sum_{1\le i<j\le n}\frac{y_i+y_j}{y_i-y_j}\left(y_i\frac{\partial}{\partial y_i}-y_j\frac{\partial}{\partial y_j}\right)-\sum_{1\le i\le n,\,1\le j\le m}\frac{x_i+y_j}{x_i-y_j}\left(x_i\frac{\partial}{\partial x_i}+ky_j\frac{\partial}{\partial y_j}\right). \tag{1.2.8}$$

1.3. Following Kerov, Okunkov, and Olshanskii [KOO], determine superanalogs of Jack polynomials. Let us consider the polynomial algebra $\mathcal{A}$ in infinite number variables $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$. Let $p_r(x,y) = \sum x_i^r + \sum y_j^r$ be the power sum. Let us define the automorphism ω_k of $\mathcal{A}$ by the formula

$$\omega_k(p_r(x,y)) = \sum x_i^r - \frac{1}{k}\sum y_j^r$$

Let $P_\lambda(x,y,k)$ be the usual Jack polynomial. Then the superanalogs of Jack polynomials are of the form

$$SP_\lambda(x,y,k) = \omega_k(P_\lambda(x,y,k)) \tag{1.2.9}$$

If we set $x_{n+1} = \cdots = y_{m+1} = \cdots = 0$, then we can consider the superanalogs of Jack polynomials in finite number of variables $SP_\lambda(x_1,\ldots,x_n,y_1,\ldots,y_m,k)$.

Theorem . *The polynomials $SP_\lambda(x_1,\ldots,x_n,y_1,\ldots,y_m,k)$ are eigenfunctions of operator* (1.2.8).

1.4. Spherical functions. In this paper we adopt an algebraic approach to the theory of spherical functions.

Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra, $U(\mathfrak{g})$ its enveloping algebra, $\mathfrak{b} \subset \mathfrak{g}$ a subalgebra. Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be an irreducible representation and V^* the dual module. If $v \in V$ is a nonzero $\mathfrak{b}$-invariant vector, and there exists a nonzero vector $v^* \in V^*$ which is also $\mathfrak{b}$-invariant. The matrix coefficient $\theta^\pi(v^*,v) \in U(\mathfrak{g})^*$, where

$$\theta^\pi(v^*,v)(u) = (-1)^{p(u)p(v)}v^*(\pi(u)v) \quad \text{for any } u \in U(\mathfrak{g}),$$

will be called the *spherical function associated with the triple* (π, v^*, v).

Let L^* be the left co-regular representation of $\mathfrak{g}$; recall that it is given by the formula (in which t is the principal antiautomorphism of $U(\mathfrak{g})$)

$$L^*(u)l(v) = (-1)^{p(u)p(l)}l(u^tv) \quad \text{for any } u,v \in U(\mathfrak{g}).$$

Let $l \in U(\mathfrak{g})^*$ be a left and right $\mathfrak{b}$-invariant functional, i.e.,

$$l(xu) = l(uy) = 0 \quad \text{for any } x,y \in \mathfrak{b} \text{ and } u \in U(\mathfrak{g}).$$

Then $L^*(z)l$, where $z \in Z(\mathfrak{g})$, is also a left and right $\mathfrak{b}$-invariant functional.

1.5. Let $\mathfrak{g} = \mathfrak{gl}(V) \oplus \mathfrak{gl}(V)$ and $\mathfrak{b} \simeq \mathfrak{g}(V)$ is the diagonal subalgebra, i.e., $\mathfrak{b} = \{(x,x) \mid x \in \mathfrak{g}(V)\}$, whereas $\mathfrak{b}_1 \simeq \mathfrak{g}(V)$ is the first summand of $\mathfrak{g}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{gl}(V)$, let λ be a partition of $l \in \mathbb{N}$ and V^λ an irreducible $\mathfrak{gl}(V)$-module in $V^{\otimes l}$, corresponding to λ, see [S1].

The $\mathfrak{g}$-module $W^\lambda = V^\lambda \otimes (V^\lambda)^*$ is irreducible and contains a unique, up to a constant factor, invariant vector v_λ. The dual module $(W^\lambda)^*$ contains a similar vector v_λ^*. Let $\varphi_\lambda = \theta^\pi(v_\lambda^*, v_\lambda)$ be the corresponding spherical function.

Let $\dim V = (n|m)$, and let $I_{\bar 0} = \{1,\ldots,n\}$ and $I_{\bar 1} = \{\bar 1,\ldots,\bar m\}$; let $\{e_{ij} \mid i,j \in I = I_{\bar 0} \coprod I_{\bar 1}\}$ be the basis of $\mathfrak{gl}(V)$ consisting of matrix units. Recall that $p(j) = \bar 0$ if $j \in I_{\bar 0}$, and $p(j) = \bar 1$ if $j \in I_{\bar 1}$. Set

$$C_2 = \sum_{i,j \in I}(-1)^{p(j)}e_{ij}e_{ji}.$$

As is easy to verify, C_2 is a central element in the enveloping algebra of $\mathfrak{gl}(V)$ and even in that of $\mathfrak{g}$, if $\mathfrak{gl}(V)$ is considered to be embedded in $\mathfrak{g}$ as the first summand.

1.5.1. Theorem . i) *Every left and right invariant functional $l \in U(\mathfrak{g})^*$ is uniquely determined by its restriction onto $S(\mathfrak{h}) \subset S(\mathfrak{b}_1)$.*

ii) *Let $(S(\mathfrak{h})^*)^{inv}$ be the set of restrictions of left and right invariant functional $l \in U(\mathfrak{g})^*$ onto $S(\mathfrak{h}) \subset S(\mathfrak{b}_1)$. Then for every $z \in Z(\mathfrak{g})$ there exists a uniquely determined operator Ω_z on $(S(\mathfrak{h})^*)^{inv}$ (the radial part of z). It is determined from the formula*

$$(\Omega_z l')(u) = (L^*(z)l)(u) \quad \text{for any } l' \in (S(\mathfrak{h})^*)^{inv} \text{ and its extension } l \in U(\mathfrak{g})^*.$$

iii) *The above defined operator Ω_{C_2} corresponding to C_2 coincides with the operator $\mathcal{M}$ determined by formula (1.2.5) for $k = 1$.*

iv) *The functions φ_λ, as functionals on $S(\mathfrak{h})$, coincide, up to a constant factor, with Jack polynomials $SP_\lambda(x, y; 1)$, where $x_i = e^{\varepsilon_i}$ for $i \in I_{\bar 0}$ and $y_j = e^{\varepsilon_j}$ for $j \in I_{\bar 1}$.*

1.6. Let $\mathfrak{g} = \mathfrak{gl}(V)$, $\dim V = (n|m)$ and $m = 2r$ is even. Let $\mathfrak{b} = \mathfrak{osp}(V)$ be the orthosymplectic Lie subsuperalgebra in $\mathfrak{gl}(V)$ which preserves the tensor

$$\sum_{i \in I_{\bar 0}} e_i^* \otimes e_i^* + \sum_{j \in I_{\bar 1}} (e_{\bar j}^* \otimes e_{\overline{j+r}}^* - e_{\overline{j+r}}^* \otimes e_{\bar j}^*). \tag{1.6.1}$$

Let ψ be an involutive automorphism of $\mathfrak{g}$ that singles out $\mathfrak{osp}(V)$:

$$\mathfrak{osp}(V) = \{x \in \mathfrak{gl}(V) \mid \psi(x) = -x\}.$$

Let V^λ be a $\mathfrak{g}$-module as in sec. 1.5. By [S2], V^λ contains a $\mathfrak{b}$-invariant vector $\tilde v_\lambda$ if and only if $\lambda = 2\mu$ and all its rows are of even length. The vector $\tilde v_\lambda^* \in (V^\lambda)^*$ is similarly defined. Let $\tilde\varphi_\lambda = \theta(v_\lambda^*, v_\lambda)$ be the corresponding matrix coefficient. Set $\mathfrak{h}^+ = \{x \in \mathfrak{h} \mid \psi(x) = x\}$, where $\mathfrak{h} \subset \mathfrak{g}$ is Cartan subalgebra.

1.6.1. Theorem . i) *Every left and right invariant functional on $U(\mathfrak{g})$ is uniquely determined by its restriction onto $S(\mathfrak{h}^+)$.*

ii) *Let $(S(\mathfrak{h}^+)^*)^{inv}$ be the set of restrictions of left and right invariant functionals. Then for every $z \in Z(\mathfrak{g})$ there exists a uniquely determined operator Ω_z on $(S(\mathfrak{h}^+)^*)^{inv}$ (the radial part of z). It is determined from the formula*

$$(\Omega_z l')(u) = (L^*(z)l)(u) \quad \text{for any } l' \in (S(\mathfrak{h}^+)^*)^{inv} \text{ and its extension } l \in U(\mathfrak{g})^*.$$

iii) *The operator Ω_{C_2} corresponding to C_2 coincides with the operator $\mathcal{M}$ determined by formula (1.2.5) for $m = r$ and $k = \frac{1}{2}$.*

iv) *The functions $\tilde\varphi_\lambda$, as functionals on $S(\mathfrak{h}^+)$, coincide, up to a constant factor, with Jack polynomials $SP_\mu(x, y; \frac{1}{2})$, where $\lambda = 2\mu$, $x_i = e^{2\varepsilon_i}$ for $1 \leq i \leq n$ and $y_j = e^{2\varepsilon_j}$ for $1 \leq j \leq r$.*

1.7. Invariant integral. For every $\mathfrak{g}$-module W, define in $U(\mathfrak{g})^*$ the subspace $C(W)$ consisting of the linear hull of the matrix coefficients of W. Denote by $\mathfrak{A}_{n,m}$ the subalgebra of $U(\mathfrak{g})^*$ generated by the matrix coefficients of the identity representation V of $\mathfrak{g} = \mathfrak{gl}(V)$ and its dual.

Theorem . i) *On $\mathfrak{A}_{n,m}$, there exists a unique up to a constant factor nontrivial left and right invariant (with respect to the left and right coregular representations) linear functional F.*

ii) *On $\mathfrak{A}_{n,m}$, define the inner product $\langle l_1, l_2 \rangle = F(l_1^t l_2)$, where $l \mapsto l^t$ is the principal automorphism of $U(\mathfrak{g})^*$ (the one corresponding to the principal antiautomorphism of $U(\mathfrak{g})$). Then $\langle l_1, l_2 \rangle = 0$ for any $l_1 \in C(V^\lambda)$, $l_2 \in C(V^\mu)$ and $\lambda \neq \mu$.*

iii) *If* $\dim V^\lambda_{\bar 0} \neq \dim V^\lambda_{\bar 1}$, *then* $\langle l_1, l_2\rangle = 0$ *for any* $l_1, l_2 \in C(V^\lambda)$.

§2. The dual of the enveloping algebra

In this section we rewrite some of the facts from Dixmier's book [Di] for Lie superalgebras.

2.1. Let $\mathfrak{g}$ be a Lie superalgebra, $U(\mathfrak{g})$ its enveloping algebra. On $U(\mathfrak{g})$, there is a canonical antiautomorphism $u \longrightarrow u^t$ given on $\mathfrak{g}$ by the formula $x^t = -x$ and extended on $U(\mathfrak{g})$ by the formula $(uv)^t = (-1)^{p(u)p(v)}v^t u^t$.

We endow $U(\mathfrak{g})^*$ with a coalgebra structure having defined the homomorphism $c : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by the formula

$$c(x) = x \otimes 1 + 1 \otimes x \quad \text{for any} \quad x \in \mathfrak{g}.$$

It is easy to check that $(c(x))^t = c(x^t)$, where the first t is the canonical antiautomorphism of $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \simeq U(\mathfrak{g} \oplus \mathfrak{g})$.

2.2. Lemma . *Let* $\dim \mathfrak{g} = n|m$. *Then* $U(\mathfrak{g})^*$ *is isomorphic to the algebra of formal power series in* n *even and* m *odd indeterminates.*

Recall that the Poincaré–Birkhoff–Witt theorem states that the graded algebra associated with the natural filtration of $U(\mathfrak{g})$ is the symmetric (polynomial) superalgebra in n even and m odd indeterminates. Let $\mathfrak{g}_{\bar 0} = \mathrm{Span}(e_1, \dots, e_n)$, $\mathfrak{g}_{\bar 1} = \mathrm{Span}(e_{\bar 1}, \dots, e_{\bar m})$. Let $I_{\bar 0} = \{1, \dots, n\}$, $I_{\bar 1} = \{\bar 1, \dots, \bar m\}$ and let

$$M = \{\nu = (\nu_1, \dots, \nu_n; \nu_{\bar 1}, \dots, \nu_{\bar m}) \mid \nu_i \in \mathbb{Z}_+ \ \text{for } i \in I_{\bar 0}; \nu_j \in \{0,1\} \ \text{for } j \in I_{\bar 1}\}.$$

Let $t_1, \dots, t_n; t_{\bar 1}, \dots, t_{\bar m}$ be the set of supercommuting indeterminates and

$$e_\nu = \frac{e_1^{\nu_1}}{\nu_1!} \cdots \frac{e_n^{\nu_n}}{\nu_n!} \frac{e_{\bar 1}^{\nu_{\bar 1}}}{\nu_{\bar 1}!} \cdots \frac{e_{\bar m}^{\nu_{\bar m}}}{\nu_{\bar m}!}.$$

The correspondence

$$U(\mathfrak{g})^* \ni F \longleftrightarrow \sum_{\nu \in M} F(e_\nu) t_{\bar m}^{\nu_{\bar m}} \dots t_{\bar 1}^{\nu_{\bar 1}} t_n^{\nu_n} \dots t_1^{\nu_1}$$

determines the isomorphism desired. □

2.3. Left and right coregular representations. Set

$$(L^*(u)F)(v) = (-1)^{p(u)p(F)} F(u^t v),$$
$$(R^*(u)F)(v) = (-1)^{p(u)(p(F)+p(v))} F(vu), \quad \text{for any } u, v \in U(\mathfrak{g}), F \in U(\mathfrak{g})^*.$$

The following statements are easy to check:

i) $u \mapsto L^*(u)$ is a representation of $U(\mathfrak{g})$ in $U(\mathfrak{g})^*$ (called the *left coregular* representation);

ii) $u \mapsto R^*(u)$ is a representation of $U(\mathfrak{g})$ in $U(\mathfrak{g})^*$ (called the *right coregular* representation);

iii) both $L^*(x)$ and $R^*(x)$ are superdifferentiations of superalgebra $U(\mathfrak{g})^*$.

Observe also that superalgebra $U(\mathfrak{g})^*$ possesses a canonical automorphism $F \mapsto F^t$, where

$$F^t(u) = F(u^t) \quad \text{for any } u \in U(\mathfrak{g}) \text{ and } F \in U(\mathfrak{g})^*.$$

On $\mathfrak{A}$, define the bilinear form

$$\langle L_1, L_2\rangle = F(L_1^t L_2). \tag{2.4.2}$$

2.4.5. Lemma . *Let W be an irreducible subrepresentation π of $\mathfrak{g}$ in $T(V)$. Then*

$$i)\quad \langle\theta^\pi(w_1^*, w_1), \theta^\pi(w_2^*, w_2))\rangle = (-1)d_W w_1^*(w_2)w_2^*(w_1),$$

for any $w_1, w_2 \in W$ and $w_1^*, w_2^* \in W^*$, where d_W only depends on W.
ii) *If* $\dim W_{\bar 0} \neq \dim W_{\bar 1}$, *then* $\langle L_1, L_2\rangle = 0$ *for any* $L_1, L_2 \in C(W)$.

Proof. i) Let $\varphi : W^* \otimes W \otimes W^* \otimes W^* \longrightarrow U(\mathfrak{g})^*$ be the map from Lemma 2.4.3. Then $F \circ \varphi$ is a $\mathfrak{g} \oplus \mathfrak{g}$-invariant map $W^* \otimes W \otimes W^* \otimes W^*$ to $\mathbb{C}$. But such a map is unique (up to a constant factor) and is of the form $w_1^* \otimes w_1 \otimes w_2^* \otimes w_2 = w_1^*(w_1)w_2^*(w_2)$. This proves i).

ii) Observe that, due to §7 $F(\varepsilon) = 0$ for the counit $\varepsilon \in U(\mathfrak{g})^*$. Now, apply L to both parts of equality from Lemma 2.4.4 we obtain

$$0 = \sum\langle\theta^\pi(w^*, w_i), \theta^\pi(w_i^*, w))\rangle = d_W w^*(w_1)(\dim W_{\bar 0} - \dim W_{\bar 1}).$$

Having selected w^* and w so that $w^*(w) \neq 0$ we deduce that $d_W = 0$. □

§3. Superanalogs of Calogero operator

3.1 Superroots of $\mathfrak{gl}(n|m)$. Let $I = I_{\bar 0} \coprod I_{\bar 1}$ be the union of the "even" indices $I_{\bar 0} = \{1, \dots, n\}$ and "odd" indices $I_{\bar 1} = \{\bar 1, \dots, \bar m\}$. Let $\dim V = (n|m)$ and $e_1, \dots, e_n, e_{\bar 1}, \dots, e_{\bar m}$ be a basis of V such that the parity of each vector is equal to that of its index. Let $\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\bar 1}, \dots, \varepsilon_{\bar m}$ be the left dual basis of V^*. Then the set of roots can be described as follows: $R = R_{11} \coprod R_{22} \coprod R_{12} \coprod R_{21}$, where

$$\begin{aligned} R_{11} &= \{\varepsilon_i - \varepsilon_j \mid i, j \in I_{\bar 0}\}, \quad R_{22} = \{\varepsilon_i - \varepsilon_j \mid i, j \in I_{\bar 1}\}, \\ R_{12} &= \{\varepsilon_i - \varepsilon_j \mid i \in I_{\bar 0},\ j \in I_{\bar 1}\}, \quad R_{21} = \{\varepsilon_i - \varepsilon_j \mid i \in I_{\bar 1},\ j \in I_{\bar 0}\}. \end{aligned} \tag{3.1.1}$$

On V^*, define the depending on parameter k inner product by setting

$$(v_1^*, v_2^*)_k = \sum_{i=1}^{n} v_1^*(e_i)v_2^*(e_i) - k\sum_{j=1}^{m} v_1^*(e_{\bar j})v_2^*(e_{\bar j}) \tag{3.1.2}$$

and set $\rho_{(k)} = k\rho_1 + \frac{1}{k}\rho_2 - \rho_{12}$, where

$$\rho_1 = \frac{1}{2}\sum_{\alpha\in R_{11}^+} \alpha; \quad \rho_2 = \frac{1}{2}\sum_{\beta\in R_{22}^+} \beta; \quad \rho_{12} = \frac{1}{2}\sum_{\gamma\in R_{12}} \gamma.$$

For any $l \in V^*$, define e^l as a linear functional on $S(V)$ that extends l to a homomorphism of $S(V)$ (or as a formal series) and denote by $\mathcal{H}$ the subalgebra in the algebra of quotients of $S(V^*)$ generated by the elements e^l for $l \in V^*$ and $(1 - e^\alpha)^{-1}$ for $\alpha \in R$. On $\mathcal{H}$, define operators $\partial_i, \partial_{\bar j}$ by setting

$$\partial_i(e^{v^*}) = v^*(e_i)e^{v^*}, \qquad \partial_{\bar j}(e^{v^*}) = v^*(e_{\bar j})e^{v^*}.$$

Set for $\alpha \in R$

$$\Delta_\alpha^+ = e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}, \qquad \Delta_\alpha^- = e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}},$$

2.4. Matrix coefficients. Let V be a $\mathfrak{g}$-module, $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ the corresponding representation, V^* the dual module. For any $v \in V$ and $v^* \in V^*$, define a linear form $\theta^\pi(v^*, v)$ on $U(\mathfrak{g})^*$ by setting

$$\theta^\pi(v^*, v)(u) = (-1)^{p(u)p(v)} v^*(\pi(u)v). \tag{2.4.1}$$

Denote by $C(\pi)$ or by $C(V)$ the subspace of $U(\mathfrak{g})^*$ generated by the $\theta^\pi(v^*, v)$ for all $v \in V$ and $v^* \in V^*$.

2.4.1. Lemma . i) $\theta^{\pi_1 \otimes \pi_2}(v_1^* \otimes v_2^*, v_1 \otimes v_2) = (-1)^{p(v_1)p(v_2^*)} \theta^{\pi_1}(v_1^*, v_1)\theta^{\pi_2}(v_2^*, v_2)$.

ii) $C(\pi_1 \otimes \pi_2) = C(\pi_1)C(\pi_2)$.

iii) *If π is finite dimensional, then* $(\theta^\pi(v^*, v))^t = (-1)^{p(v_1)p(v_2^*)} \theta^{\pi^*}(v, v^*)$.

2.4.2. Lemma . *The map $V^* \otimes V \longrightarrow U(\mathfrak{g})^*$ given by the formula $(v^*, v) \mapsto \theta^\pi(v^*, v)$ is a $\mathfrak{g} \oplus \mathfrak{g}$-module homomorphism if we consider $U(\mathfrak{g})^*$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module with respect to the simultaneous action of the left and right coregular representations.*

If V is irreducible, the above map has no kernel.

2.4.3. Lemma . *Let V be a $\mathfrak{g}$-module. Consider the map $\varphi : V^* \otimes V \otimes V^* \otimes V \longrightarrow U(\mathfrak{g}^*)$:*

$$\varphi(v_1^* \otimes v_1 \otimes v_2^* \otimes v_2) = (-1)^{p(v_1^*)p(v_2^*) + p(v_1)p(v_2^*) + p(v_1^*)p(v_1^*)} \left(\theta^{\pi_1}(v_1^*, v_1)\right)^t \theta^{\pi_2}(v_2^*, v_2).$$

Consider $(V^* \otimes V) \otimes (V^* \otimes V)$ as $\mathfrak{g} \oplus \mathfrak{g}$-module. Then φ is a $\mathfrak{g} \oplus \mathfrak{g}$-module homomorpism, where we consider $U(\mathfrak{g})^*$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module with respect to the simultaneous action of the left and right coregular representations.

2.4.4. Lemma . *Let V be a finite dimensional $\mathfrak{g}$-module, π the corresponding representation. Let $\{v_i\}_{i \in I}$ be a basis of V, $\{v_i^*\}_{i \in I}$ the dual basis of V^*. Then*

$$\sum_i \left(\theta^\pi(v^*, v_i)\right)^t \theta^\pi(v_i^*, v) = (v^*, v)\varepsilon,$$

for any $v \in V$ and $v^* \in V^*$, and where ε is the counit of $U(\mathfrak{g})^*$.

Proof. The functional ε is uniquely, up to a constant factor, characterized by its invariance with respect to right coregular representation. Further, $w = \sum_i (-1)^{p(v_i)} v_i^* \otimes v_i$ is an invariant of the $\mathfrak{g}$-module $V^* \otimes V$. Hence, by the preceding Lemma the element $\varphi(w \otimes v^* \otimes v)$ is invariant with respect to the right coregular representation functional on $U(\mathfrak{g})$, so $\varphi(w \otimes v^* \otimes v) = \alpha\varepsilon$.

On the other hand,

$$\varphi(w \otimes v^* \otimes v) = \sum (-1)^{p(i)} \varphi(v_i^* \otimes v_i \otimes v^* \otimes v) = \sum \left(\theta^\pi(v^*, v_i)\right)^t \theta^\pi(v_i^*, v).$$

Hence, $\sum \left(\theta^\pi(v^*, v_i)\right)^t \theta^\pi(v_i^*, v) = \alpha\varepsilon$. To find α, substitute in both parts of this equality $u = 1$. We obtain

$$\alpha = \alpha\varepsilon(1) = \sum \left(\theta^\pi(v^*, v_i)\right)^t \theta^\pi(v_i^*, v)(1) = \sum (v^*(v_i) v_i^*(v)).$$

□

Let $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{A}$ the subalgebra in $U(\mathfrak{g})^*$ generated by $C(V)$ and $(C(V))^t$. It is not difficult to verify that $\mathfrak{A}$ is invariant with respect to the left and right coregular representations. In §7 we will prove that on $\mathfrak{A}$ there exists a nontrivial and invariant with respect to the left and right coregular representations functional F (the Berezin integral).

$$\Delta_1 = \prod_{\alpha\in R_{11}^+} \Delta_\alpha^- \quad \Delta_2 = \prod_{\beta\in R_{12}^+} \Delta_\beta^- \quad \Delta_{12} = \prod_{\gamma\in R_{12}^+} \Delta_\gamma^-$$

$$\delta^{(k)} = \Delta_1^k \Delta_2^{\frac{1}{k}} \Delta_{12}^{-1}$$

It is easily to verify that the operator $\mathcal{SL}$ can be rewritten in the following form

$$\mathcal{SL} = \sum_{i=1}^{n} \partial_i^2 - k\sum_{\bar{j}=1}^{m} \partial_{\bar{j}}^2 - k(k-1) \sum_{\alpha\in R_{11}^+} \frac{(\alpha,\alpha)_k}{(\Delta_\alpha^-)^2} + \frac{1}{k}(\frac{1}{k}-1) \sum_{\beta\in R_{22}^+} \frac{(\beta,\beta)_k}{(\Delta_\beta^-)^2} - 2\sum_{\gamma\in R_{12}} \frac{(\gamma,\gamma)_k}{(\Delta_\gamma^-)^2}. \tag{3.1.3}$$

Let us define the operator $\mathcal{M}^*$ by the formula

$$\mathcal{M}^* = \sum_{i=1}^{n} \partial_i^2 - k\sum_{\bar{j}=1}^{m} \partial_{\bar{j}}^2 + k \sum_{\alpha\in R_{11}^+} \frac{\Delta_\alpha^+}{\Delta_\alpha^-}\partial_\alpha + \sum_{\beta\in R_{22}^+} \frac{\Delta_\beta^+}{\Delta_\beta^-}\partial_\beta - \sum_{\gamma\in R_{12}} \frac{\Delta_\gamma^+}{\Delta_\gamma^-}\partial_{\gamma,k}. \tag{3.1.4}$$

where

$$\partial_\alpha = \partial_i - \partial_j \qquad \text{for } \alpha = \varepsilon_i - \varepsilon_j;$$
$$\partial_\beta = \partial_{\bar{i}} - \partial_{\bar{j}} \qquad \text{for } \beta = \varepsilon_{\bar{i}} - \varepsilon_{\bar{j}};$$
$$\partial_{\gamma,k} = \partial_i + k\partial_{\bar{j}} \quad \text{for } \gamma = \varepsilon_i - \varepsilon_{\bar{j}}.$$

3.2. Proof of Lemma 1.2.1. We will prove identity

$$\delta^{(k)}\mathcal{M}_2^*(\delta^{(k)})^{-1} = \mathcal{SL}_2 - (\rho_k,\rho_k)_k$$

equivalent to Lemma 1.2.1. The following identities are easy to verify:

$$\partial_i(\Delta_\alpha^+) = \frac{1}{2}\alpha(e_i)\Delta_\alpha^-, \qquad \partial_i(\Delta_\alpha^-) = \frac{1}{2}\alpha(e_i)\Delta_\alpha^+ \tag{3.2.1}$$

where $\alpha \in R$.

$$\delta^{(k)}\partial_i(\delta^{(k)})^{-1} = \partial_i - \frac{k}{2}\sum_{\alpha\in R_{11}^+} \alpha(e_i)\frac{\Delta_\alpha^+}{\Delta_\alpha^-} + \frac{1}{2}\sum_{\gamma\in R_{12}} \gamma(e_i)\frac{\Delta_\gamma^+}{\Delta_\gamma^-}. \tag{3.2.2}$$

where $i \in I_{\bar{0}}$.

$$\delta^{(k)}\partial_i(\delta^{(k)})^{-1} = \partial_i - \frac{1}{2k}\sum_{\beta\in R_{22}^+} \beta(e_i)\frac{\Delta_\beta^+}{\Delta_\beta^-} + \frac{1}{2}\sum_{\gamma\in R_{12}} \gamma(e_i)\frac{\Delta_\gamma^+}{\Delta_\gamma^-}. \tag{3.2.3}$$

where $i \in I_{\bar{1}}$. The operator $\mathcal{M}^*$ can be expressed in the form

$$\mathcal{M}^* = \sum_{i\in I_{\bar{0}}} \partial_i^2 - k\sum_{j\in I_{\bar{1}}} \partial_j^2 + k \sum_{\alpha\in R_{11}^+, i\in I_{\bar{0}}} \alpha(e_i)\frac{\Delta_\alpha^+}{\Delta_\alpha^-}\partial_i - \sum_{\beta\in R_{22}^+, j\in I_{\bar{1}}} \beta(e_j)\frac{\Delta_\beta^+}{\Delta_\beta^-}\partial_j - \sum_{\gamma\in R_{12}, i\in I_{\bar{0}}} \gamma(e_i)\frac{\Delta_\gamma^+}{\Delta_\gamma^-}\partial_i + k\sum_{\gamma\in R_{12}, j\in I_{\bar{1}}} \gamma(e_j)\frac{\Delta_\gamma^+}{\Delta_\gamma^-}\partial_j. \tag{3.2.4}$$

Further, set $X_\alpha = \frac{\Delta^+_\alpha}{\Delta^-_\alpha}$ for $\alpha \in R^+$ and let

$$\begin{aligned} \varphi_i &= \sum_{\alpha\in R^+_{11}} \alpha(e_i)X_\alpha \quad f_i = \sum_{\gamma\in R^+_{12}} \gamma(e_i)X_\gamma \quad i \in I_{\bar 0} \\ h_j &= \sum_{\beta\in R^+_{22}} \beta(e_j)X_\beta \quad g_j = \sum_{\gamma\in R^+_{12}} \gamma(e_j)X_\gamma \quad j \in I_{\bar 1}. \end{aligned} \tag{3.2.5}$$

The following identities are easy to verify:

$$\delta^{(k)}\partial_i^2(\delta^{(k)})^{-1} = \partial_i^2 + (f_i - k\varphi_i)\partial_i + \frac{1}{4}(f_i - k\varphi_i)^2 + \frac{1}{2}(\partial_i f_i - k\partial_i\varphi_i) \text{ where } \quad i \in I_{\bar 0} \tag{3.2.6}$$

$$\delta^{(k)}\partial_j^2(\delta^{(k)})^{-1} = \partial_j^2 + (g_j - \frac{1}{k}h_j)\partial_j + \frac{1}{4}(g_j - \frac{1}{k}h_j)^2 + \frac{1}{2}(\partial_j g_j - \frac{1}{k}\partial_j h_j) \text{ where } \quad j \in I_{\bar 1} \tag{3.2.7}$$

Therefore, after simple transformations we obtain

$$\begin{aligned} &\delta^{(k)}\mathcal{M}_2^*(\delta^{(k)})^{-1} = \delta^{(k)}(\sum_{i\in I_{\bar 0}} \partial_i^2 - k\sum_{j\in I_{\bar 1}} \partial_j^2 + \sum_{i\in I_{\bar 0}} (k\varphi_i - f_i)\partial_i + \\ &\sum_{j\in I_{\bar 1}} (kg_j - h_j)\partial_j)(\delta^{(k)})^{-1} = \sum_{i\in I_{\bar 0}} \partial_i^2 + \sum_{i\in I_{\bar 0}} (f_i - k\varphi_i)\partial_i + \tfrac{1}{4}\sum_{i\in I_{\bar 0}} (f_i - k\varphi_i)^2 + \\ &\tfrac{1}{2}\sum_{i\in I_{\bar 0}} (\partial_i f_i - k\partial_i\varphi_i) - \\ &k\left[\sum_{j\in I_{\bar 1}} \partial_j^2 + \sum_{j\in I_{\bar 1}} (g_j - \tfrac{1}{k}h_j)\partial_j + \tfrac{1}{4}\sum_{j\in I_{\bar 1}} (g_j - \tfrac{1}{k}h_j)^2 + \tfrac{1}{2}\sum_{j\in I_{\bar 1}} (\partial_j g_j - \tfrac{1}{k}\partial_j h_j)\right] + \\ &\sum_{i\in I_{\bar 0}} (k\varphi_i - f_i)(\partial_i + \tfrac{1}{2}f_i - \tfrac{1}{2}k\varphi_i) + \sum_{j\in I_{\bar 1}} (kg_j - h_j)(\partial_j + \tfrac{1}{2}g_j - \tfrac{1}{2k}h_j) = \\ &\sum_{i\in I_{\bar 0}} \partial_i^2 - k\sum_{j\in I_{\bar 1}} \partial_j^2 - \tfrac{1}{4}\sum_{i\in I_{\bar 0}} (f_i - k\varphi_i)^2 + \tfrac{1}{2}\sum_{i\in I_{\bar 0}} (\partial_i f_i - k\partial_i\varphi_i) + \\ &\tfrac{k}{4}\sum_{j\in I_{\bar 1}} (g_j - \tfrac{1}{k}h_j)^2 - \tfrac{k}{2}\sum_{j\in I_{\bar 1}} (\partial_j g_j - \tfrac{1}{k}\partial_j h_j) = \\ &\sum_{i\in I_{\bar 0}} \partial_i^2 - \tfrac{k^2}{4}\sum_{i\in I_{\bar 0}} \varphi_i^2 - \tfrac{k}{2}\sum_{i\in I_{\bar 0}} \partial_i\varphi_i - k\sum_{j\in I_{\bar 1}} \partial_j^2 + \tfrac{1}{4k}\sum_{j\in I_{\bar 1}} h_j^2 + \tfrac{1}{2}\sum_{j\in I_{\bar 1}} \partial_j h_j - \tfrac{1}{4}\sum_{i\in I_{\bar 0}} f_i^2 + \\ &+\tfrac{k}{2}\sum_{i\in I_{\bar 0}} f_i\varphi_i + \tfrac{1}{2}\sum_{i\in I_{\bar 0}} \partial_i f_i + \tfrac{k}{4}\sum_{j\in I_{\bar 1}} g_j^2 - \tfrac{1}{2}\sum_{j\in I_{\bar 1}} g_j h_j - \tfrac{k}{2}\sum_{j\in I_{\bar 1}} \partial_j g_j \end{aligned} \tag{3.2.8}$$

But due to the classical case we have

$$\sum_{i\in I_{\bar 0}} \partial_i^2 - \frac{k^2}{4}\sum_{i\in I_{\bar 0}} \varphi_i^2 - \frac{k}{2}\sum_{i\in I_{\bar 0}} \partial_i\varphi_i = \sum_{i\in I_{\bar 0}} \partial_i^2 - 2k(k-1)\sum_{\alpha\in R_{11}} \frac{1}{(\Delta^-_\alpha)^2} - (k\rho_1, k\rho_1)_1$$

$$\sum_{j\in I_{\bar 1}} \partial_j^2 - \frac{1}{4k^2}\sum_{j\in I_{\bar 1}} h_j^2 - \frac{1}{2k}\sum_{j\in I_{\bar 1}} \partial_j h_j = \sum_{j\in I_{\bar 1}} \partial_j^2 - \frac{2}{k}(\frac{1}{k} - 1)\sum_{\beta\in R_{22}} \frac{1}{(\Delta^-_\beta)^2} - (\frac{1}{k}\rho_2, \frac{1}{k}\rho_2)_2$$

where $(\cdot,\cdot)$ is the usual inner products in $V_{\bar 0}$ and $V_{\bar 1}$:

$$(l_1, l_2)_1 = \sum_{i\in I_{\bar 0}} l_1(e_i)l_2(e_i) \text{ for any } l_1, l_2 \in V^*_{\bar 0};$$

$$(l'_1, l'_2)_2 = \sum_{j\in I_{\bar 1}} l'_1(e_i)l'_2(e_i) \text{ for any } l'_1, l'_2 \in V^*_{\bar 1}$$

Hence,

$$\delta^{(k)}\mathcal{M}_2^*(\delta^{(k)})^{-1}=\sum_{i\in I_{\bar 0}}\partial_i^2-2k(k-1)\sum_{\alpha\in R_{11}}\frac{1}{(\Delta_\alpha^-)^2}-(k\rho_1,k\rho_1)_1-$$
$$k(\sum_{j\in I_{\bar 1}}\partial_j^2-\tfrac{2}{k}(\tfrac{1}{k}-1)\sum_{\beta\in R_{22}}\frac{1}{(\Delta_\beta^-)^2}-(\tfrac{1}{k}\rho_2,\tfrac{1}{k}\rho_2)_2)+(-\tfrac14\sum_{i\in I_{\bar 0}}f_i^2+$$
$$+\tfrac{k}{2}\sum_{i\in I_{\bar 0}}f_i\varphi_i+\tfrac12\sum_{i\in I_{\bar 0}}\partial_i f_i+\tfrac{k}{4}\sum_{j\in I_{\bar 1}}g_j^2-\tfrac12\sum_{j\in I_{\bar 1}}g_jh_j-\tfrac{k}{2}\sum_{j\in I_{\bar 1}}\partial_j g_j)$$

It is easy to verify that $\partial_i(X_\gamma)=\frac12\gamma(e_i)(1-X_\gamma^2)$. It remains to transform the summands in brackets. One can show that they can attain the form (we suppose that $\gamma,\gamma_1,\gamma_2\in R_{12},\quad \alpha\in R_{11}^+,\quad \beta\in R_{22}^+$)

$$-\tfrac14\sum_{i\in I_{\bar 0}}(\sum_\gamma\gamma(e_i)X_\gamma)^2+\tfrac{k}{2}\sum_{i\in I_{\bar 0}}(\sum_\gamma\gamma(e_i)X_\gamma)(\sum_\alpha\alpha(e_i)X_\alpha)+\tfrac14\sum_{i,\gamma}\gamma(e_i)^2(1-X_\gamma^2)+$$
$$\tfrac{k}{4}\sum_j(\sum_\gamma\gamma(e_j)X_\gamma)^2-\tfrac12\sum_{j\in I_{\bar 1}}(\sum_\gamma\gamma(e_j)X_\gamma)(\sum_\beta\beta(e_j)X_\beta)-\tfrac{k}{4}\sum_{j,\gamma}\gamma(e_j)^2(1-X_\gamma^2)=$$
$$-\tfrac12\sum_{i,\gamma}X_\gamma^2\gamma(e_i)^2+\tfrac14\sum_{i,\gamma}\gamma(e_i)^2+\tfrac{k}{2}\sum_{j,\gamma}X_\gamma^2\gamma(e_j)^2-\tfrac{k}{4}\sum_{j,\gamma}\gamma(e_j)^2-$$
$$\tfrac12\sum_{i,\gamma_1,\gamma_2}\gamma_1(e_i)\gamma_2(e_i)X_{\gamma_1}X_{\gamma_2}+\tfrac{k}{2}\sum_{i,\gamma,\alpha}\gamma(e_i)\alpha(e_i)X_\gamma X_\alpha+\tfrac{k}{2}\sum_{j,\gamma_1,\gamma_2}\gamma_1(e_j)\gamma_2(e_j)X_{\gamma_1}X_{\gamma_2}-$$
$$\tfrac12\sum_{i,\gamma,\beta}\gamma(e_j)\alpha(e_j)X_\gamma X_\beta=(\text{take into account that}\quad X_\gamma^2=1+\frac{4}{(\Delta_\gamma^-)^2})-$$
$$\tfrac14\sum_{i,\gamma}\gamma(e_i)^2+\tfrac{k}{4}\sum_{j,\gamma}\gamma(e_j)^2-2\sum_{i,\gamma}\frac{\gamma(e_i)^2}{(\Delta_\gamma^-)^2}+2k\sum_{j,\gamma}\frac{\gamma(e_j)^2}{(\Delta_\gamma^-)^2}-$$
$$\tfrac12\sum_{\gamma_1,\gamma_2}(\sum_i\gamma_1(e_i)\gamma_2(e_i)-k\sum_j\gamma_1(e_j)\gamma_2(e_j))X_{\gamma_1}X_{\gamma_2}+\tfrac{k}{2}\sum_{\gamma,\alpha}(\sum_i\gamma(e_i)\alpha(e_i))X_\gamma X_\alpha-$$
$$\tfrac12\sum_{\gamma,\beta}(\sum_j\gamma(e_j)\beta(e_j))X_\gamma X_\beta=-\sum_\gamma(\gamma,\gamma)_k-2\sum_\gamma\frac{(\gamma,\gamma)_k}{(\Delta_\gamma^-)^2}-\tfrac12\sum_{\gamma_1,\gamma_2}(\gamma,\gamma)_kX_{\gamma_1}X_{\gamma_2}+$$
$$\tfrac{k}{2}\sum_{\gamma,\alpha}(\gamma,\alpha)_kX_\gamma X_\alpha+\tfrac{1}{2k}\sum_{\gamma,\beta}(\gamma,\beta)_kX_\gamma X_\beta=-\sum_\gamma(\gamma,\gamma)_k-2\sum_\gamma\frac{(\gamma,\gamma)_k}{(\Delta_\gamma^-)^2}+$$
$$\tfrac{k}{2}[\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_2X_{\gamma_1}X_{\gamma_2}+\sum_{\gamma,\alpha}(\gamma,\alpha)_1X_\gamma X_\alpha]-\tfrac12[\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_2X_{\gamma_1}X_{\gamma_2}+\sum_{\gamma,\beta}(\gamma,\beta)_2X_\gamma X_\beta]$$

It is not difficult to verify that

$$\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_2X_{\gamma_1}X_{\gamma_2}+\sum_{\gamma,\alpha}(\gamma,\alpha)_1X_\gamma X_\alpha=\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_2+\sum_{\gamma,\alpha}(\gamma,\alpha)_1 \qquad (3.2.9)$$

Indeed, the set of ordered pairs (γ_1,γ_2) can be divided into equivalence classes for which $(\gamma_1,\gamma_2)\neq 0$. Each such class is of the form

$$\varepsilon_i-\varepsilon_j \text{ where } i\in I_{\bar 0}, j\in I_{\bar 1} \text{ for a fixed } j.$$

To prove (3.2.9) it suffices to verify that

$$(\gamma_1,\gamma_2)_2X_{\gamma_1}X_{\gamma_2}+(\gamma_1,\alpha)_1X_{\gamma_1}X_\alpha+(\gamma_2,\alpha)_1X_{\gamma_2}X_\alpha=(\gamma_1,\gamma_2)_2+(\gamma_1,\alpha)_1+(\gamma_2,\alpha)_1$$

for $\gamma_1=\varepsilon_1-\varepsilon_{\bar 1},\quad \gamma_2=\varepsilon_2-\varepsilon_{\bar 1}\quad \alpha=\varepsilon_1-\varepsilon_2$. This is not difficult. We similarly prove the identity

$$\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_1X_{\gamma_1}X_{\gamma_2}+\sum_{\gamma,\beta}(\gamma,\beta)_2X_\gamma X_\beta=\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_1+\sum_{\gamma,\alpha}(\gamma,\beta)_2$$

Therefore,

$$\delta^{(k)}\mathcal{M}_2^*(\delta^{(k)})^{-1} = \sum_{i\in I_{\bar 0}} \partial_i^2 - 2k(k-1)\sum_{\alpha\in R_{11}} \frac{1}{(\Delta_\alpha^-)^2} - (k\rho_1, k\rho_1)_1 -$$
$$k(\sum_{j\in I_{\bar 1}} \partial_j^2 - \tfrac{2}{k}(\tfrac{1}{k}-1)\sum_{\beta\in R_{22}} \tfrac{1}{(\Delta_\beta^-)^2} - (\frac{1}{k}\rho_2, \frac{1}{k}\rho_2)_2) - 2\sum_{\gamma_1,\gamma_2} \frac{(\gamma_1,\gamma_2)_k}{(\Delta_\gamma^-)^2} +$$
$$\tfrac{k}{2}(\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_2 + \sum_{\gamma,\alpha}(\gamma,\alpha)_1) - \tfrac{1}{2}(\sum_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)_1 + \sum_{\gamma,\alpha}(\gamma,\beta)_2)$$
$$\sum_{\gamma}(\gamma,\gamma)_k = \sum_{i\in I_{\bar 0}} \partial_i^2 - 2k(k-1)\sum_{\alpha\in R_{11}} \frac{1}{(\Delta_\alpha^-)^2} - (k\rho_1, k\rho_1)_1 -$$
$$k(\sum_{j\in I_{\bar 1}} \partial_j^2 - \tfrac{2}{k}(\tfrac{1}{k}-1)\sum_{\beta\in R_{22}} \tfrac{1}{(\Delta_\beta^-)^2} - (\tfrac{1}{k}\rho_2, \tfrac{1}{k}\rho_2)_2) - 2\sum_{\gamma_1,\gamma_2} \frac{(\gamma_1,\gamma_2)_k}{(\Delta_\gamma^-)^2} +$$
$$-(\rho_{12},\rho_{12})_k + (\rho_{12}, k\rho_1)_k - (\rho_{12}, \tfrac{1}{k}\rho_2)_k$$

□

§4. Superanalogs of Jack polynomials

4.1. The usual Jack polynomials. Let $t = (t_1, t_2, \dots)$ be sequences of independent indeterminates and Λ be the algebra of symmetric functions.The monomial symmetric functions m_λ is the sum of all distinct monomials that can be obtain from t^λ by permutations of the t's.We can define the power sum $p_r = \sum_i t^r$ and for any partition λ $p_\lambda = p_{\lambda_1}p_{\lambda_2}\dots$ There are symmetric functions P_λ, indexed by partitions and depending rationally on parameter k (I use parameter $k = \frac{1}{\alpha}$, the inverse of Macdonald's parameter). They are characterized by the following properties

$$P_\lambda = m_\lambda + \text{lower terms} \qquad (P_\lambda, P_\mu) = 0, \text{if} \quad \lambda \neq \mu$$

where the scalar product is defined by $(p_\lambda p_\mu) = \delta_{\lambda,\mu}k^{-l(\lambda)}z_\lambda$, where $z_\lambda = \prod i^{\mu_i}(\mu_i!)$ Let $t_1, \dots t_N$ be indeterminates and u an extra indeterminate. Consider the family of differential operators $D(u,k)$, called *Sekiguchi operators* defined by the formula

$$D(u,k) = \sum_{p=1}^N u^p D_p^{(k)} = V(t)^{-1} det\left[t_i{}^{N-j}\left(t_i\frac{\partial}{\partial t_i} + (N-j)k + u\right)\right]_{1\le i\le j\le N}$$

where $V(t) = \prod_{1\le i<j\le N}(t_i - t_j)$ .Then Jack polynomials $P_\lambda(t_1,\dots,t_N)$, where λ is a partition such that $\lambda_{N+1} = 0$ are uniquely determined by the following properties: i) $P_\lambda(t_1,\dots,t_N)$ is symmetric with respect to $t_1,\dots,t_N$;

ii) $P_\lambda(t_1,\dots,t_N) = t_1^{\lambda_1}\dots t_N^{\lambda_N}\dots$ (dots stand for monomials of lesser lexicographic order);

iii) $P_\lambda(t_1,\dots,t_N)$ are the eigenfunctions of the operators $D_p^{(k)}$ .More exactly

$$D(u,k)P_\lambda(t_1,\dots,t_N) = \left(\prod_1^N(\lambda_i + (N-i)k + u)\right) P_\lambda(t_1,\dots,t_N)$$

4.2. Following Kerov, Okounkov, and Olshanski [KOO], determine superanalogs of Jack polynomials. Let us consider the polynomial algebra $\mathcal{A}$ in infinite number variables $x_1, x_2, \dots$ and $y_1, y_2, \dots$. Let $p_r(x,y) = \sum x_i^r + \sum y_j^r$ be the power sum. Let us define the automorphism ω_k of $\mathcal{A}$ by the formula

$$\omega_k(p_r(x,y)) = \sum x_i^r - \frac{1}{k}\sum y_j^r$$

Let $P_\lambda(x,y,k)$ be the usual Jack polynomial. Then the superanalogs of Jack polynomials are of the form

$$SP_\lambda(x,y,k) = \omega_k(P_\lambda(x,y,k)) \tag{4.2.1}$$

If we set $x_{n+1} = \dots = y_{m+1} = \dots = 0$, then we can consider the superanalogs of Jack polynomials in finite number of variables $SP_\lambda(x_1,\dots,x_n,y_1,\dots,y_m,k)$. Observe, that our definition differs from the one in [KOO] by the change $y_j \to \left(-\frac{y_j}{\theta}\right)$, $k=\theta$.

4.3. Set

$$\varphi(t) = \prod_{j=1}^{m}(1-y_jt)\prod_{i=1}^{n}(1-x_it)^{-k}$$

and $\varphi(t_1,\dots,t_N) = \varphi(t_1)\dots\varphi(t_N)$.

Set further

$$\mathcal{H} = \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2 - k\sum_{j=1}^{m}\left(y_j\frac{\partial}{\partial y_j}\right)^2 + k\sum_{1\le i<j\le n}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right) - \sum_{1\le i<j\le n}\frac{y_i+y_j}{y_i-y_j}\left(y_i\frac{\partial}{\partial y_i} - y_j\frac{\partial}{\partial y_j}\right) - \sum_{1\le i\le n,\, 1\le j\le m}\frac{x_i+y_j}{x_i-y_j}\left(x_i\frac{\partial}{\partial x_i} + ky_j\frac{\partial}{\partial y_j}\right). \tag{4.3.1}$$

and

$$\mathcal{L}_N = \sum_{i=1}^{n}\left(t_i\frac{\partial}{\partial t_i}\right)^2 + k\sum_{1\le i<j\le n}\frac{t_i+t_j}{t_i-t_j}\left(t_i\frac{\partial}{\partial t_i} - t_j\frac{\partial}{\partial t_j}\right) \tag{4.3.2}$$

4.4. Lemma .

$$\mathcal{H}\varphi_N - \mathcal{L}_N\varphi_N = (k(n-N)-m)\left(\sum_{i=1}^{n} t_i\frac{\partial}{\partial t_i}\right)\varphi_N \tag{4.4.1}$$

Proof: induction in N.

For $N=1$ formula (4.4.1) takes the form:

$$\mathcal{H}\varphi(t) = \left(t\frac{\partial}{\partial t}\right)^2\varphi(t) + (k(n-1)-m)\left(t\frac{\partial}{\partial t}\right)\varphi(t)$$

The following identities are easy to verify:

$$\left(x_i\frac{\partial}{\partial x_i}\right)\varphi(t) = \frac{kx_it}{1-x_it}\varphi(t)$$

$$\left(x_i\frac{\partial}{\partial x_it}\right)^2\varphi(t) = \left(\frac{kx_it}{1-x_it} + \frac{k(k+1)x_i^2t^2}{(1-x_it)^2}\right)\varphi(t)$$

$$\left(y_j\frac{\partial}{\partial y_j}\right)\varphi(t) = -\frac{y_jt}{1-y_jt}\varphi(t)$$

$$\left(y_j\frac{\partial}{\partial y_j}\right)^2\varphi(t) = -\frac{y_jt}{1-y_jt}\varphi(t)$$

Now, present $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_x - k\mathcal{H}_y + k\mathcal{H}_{xx} - \mathcal{H}_{yy} - \mathcal{H}_{xy}$$

then

$$\mathcal{H}_x\varphi(t) = \left(\sum_{i=1}^{n}\left(\frac{kx_it}{1-x_it} + \frac{k(k+1)x_i^2t^2}{(1-x_it)^2}\right)\right)\varphi(t)$$

$$\mathcal{H}_y\varphi(t) = -\left(\sum_{j=1}^{m}\frac{ky_jt}{1-y_jt}\right)\varphi(t)$$

$$\mathcal{H}_{xx}\varphi(t) = \left(\sum_{i=1}^{n}\left(\sum_{j\neq i}\frac{x_i+x_j}{x_i-x_j}\right)\frac{kx_it}{1-x_it}\right)\varphi(t)$$

$$\mathcal{H}_{yy}\varphi(t) = -\left(\sum_{j=1}^{m}\left(\sum_{l\neq j}\frac{y_j+y_l}{y_j-y_l}\right)\frac{y_jt}{1-y_jt}\right)\varphi(t)$$

$$\mathcal{H}_{xy}\varphi(t) = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\frac{x_i+y_j}{x_i-y_j}\right)\frac{kx_it}{1-x_it} - \sum_{j=1}^{m}\left(\sum_{i=1}^{m}\frac{x_i+y_j}{x_i-y_j}\right)\frac{ky_jt}{1-y_jt}\right)$$

Therefore

$$\frac{\mathcal{H}\varphi(t)}{\varphi(t)} = \sum_{i=1}^{n}\left(\frac{kx_it}{1-x_it} + \frac{k(k+1)x_i^2t^2}{(1-x_it)^2}\right) + k\sum_{j=1}^{m}\frac{y_jt}{1-y_jt} + k\sum_{i=1}^{n}\left(\sum_{j\neq i}\frac{x_i+x_j}{x_i-x_j}\right)\frac{kx_it}{1-x_it}$$
$$+\sum_{j=1}^{m}\left(\sum_{l\neq j}\frac{y_j+y_l}{y_j-y_l}\right)\frac{y_jt}{1-y_jt} - \sum_{i=1}^{n}\left(\sum_{j=1}^{m}\frac{x_i+y_j}{x_i-y_j}\right)\frac{kx_it}{1-x_it} + \sum_{j=1}^{m}\left(\sum_{i=1}^{m}\frac{x_i+y_j}{x_i-y_j}\right)\frac{y_jt}{1-y_jt}$$

Further

$$\frac{\left(t\frac{\partial}{\partial t}\varphi(t)\right)}{\varphi(t)} = \left(\sum_{i=1}^{n}\frac{kx_it}{1-x_it} - \sum_{j=1}^{m}\frac{y_jt}{1-y_jt}\right)$$

$$\frac{\left(\left(t\frac{\partial}{\partial t}\right)^2\varphi(t)\right)}{\varphi(t)} = \left(\sum_{i=1}^{n}\frac{kx_it}{1-x_it} - \sum_{j=1}^{m}\frac{y_jt}{1-y_jt}\right)^2 +$$
$$\left(\sum_{i=1}^{n}\frac{kx_it}{1-x_it} + \frac{kx_i^2t^2}{(1-x_it)^2} - \sum_{j=1}^{m}\frac{y_jt}{1-y_jt} + \frac{y_j^2t^2}{(1-y_jt)^2}\right)$$

Hence,

$$\frac{(\mathcal{H}-\mathcal{L}_1)\varphi}{\varphi} = (k+1)\sum_{j=1}^{m}\frac{y_jt}{1-y_jt} + k^2(n-1)\sum_{i=1}^{n}\frac{x_it}{1-x_it} + (k-1)\sum_{j=1}^{m}\frac{y_jt}{1-y_jt} -$$
$$m\sum_{i=1}^{n}\frac{kx_it}{1-x_it} - n\sum_{j=1}^{m}\frac{ky_jt}{1-y_jt} = (k(n-1)-m)\left(\sum_{i=1}^{n}\frac{kx_it}{1-x_it} - \sum_{j=1}^{m}\frac{y_jt}{1-y_jt}\right)$$

Let now $N > 1$. Then by inductive hypothesis we have

$$\mathcal{H}\varphi_{N-1} - \mathcal{L}_{N-1}\varphi_{N-1} = (k(n-N+1)-m)\left(\sum_{i=1}^{N-1} t_i \frac{\partial}{\partial t_i}\right)\varphi_{N-1}$$

Set $C_p = (k(n-p)-m)$. We have

$$\mathcal{H}(\varphi_N) = \mathcal{H}(\varphi_{N-1}\varphi(t_N)) = (\mathcal{H}(\varphi_{N-1}))\,\varphi(t_N) + \varphi_{N-1}\,(\mathcal{H}(\varphi(t_N))) +$$
$$2\left(\sum_{i=1}^{n}\left(x_i\tfrac{\partial}{\partial x_i}\varphi_{N-1}\right)\left(x_i\tfrac{\partial}{\partial x_i}\varphi(t_{N-1})\right) - k\sum_{j=1}^{m}\left(y_j\tfrac{\partial}{\partial y_j}\varphi_{N-1}\right)\left(y_j\tfrac{\partial}{\partial y_j}\varphi(t_{N-1})\right)\right)$$

$$\mathcal{L}_N\varphi_N = (\mathcal{L}_{N-1}\varphi_{N-1})\,\varphi(t_N) + \varphi_{N-1}\left(t_N\tfrac{\partial}{\partial t_N}\right)^2\varphi(t_N) +$$
$$k\left(\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}t_l\tfrac{\partial}{\partial t_l}\varphi_{N-1}\right)\varphi(t_N) - k\left(\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}t_N\tfrac{\partial}{\partial t_N}\varphi(t_N)\right)\varphi_{N-1}$$

Terefore,

$$\frac{(\mathcal{H}-\mathcal{L}_N)\varphi_N}{\varphi_N} = \frac{\left(C_{N-1}\sum_{i=1}^{N-1} t_l\frac{\partial}{\partial t_l} + C_1 t_N\frac{\partial}{\partial t_N}\right)\varphi_N}{\varphi_N} +$$
$$2\sum_{i=1}^{n}\sum_{l=1}^{N-1}\frac{k^2x_i^2t_lt_N}{(1-x_it_l)(1-x_it_N)} - 2\sum_{j=1}^{m}\sum_{l=1}^{N-1}\frac{ky_j^2t_lt_N}{(1-y_jt_l)(1-y_jt_N)} -$$
$$\sum_{l=1}^{N-1}k\frac{t_l+t_N}{t_l-t_N}\left(\sum_{i=1}^{n}\frac{kx_it_l}{1-x_it_l} - \sum_{j=1}^{m}\frac{y_jt_l}{1-y_jt_l}\right) + k\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}\left(\sum_{i=1}^{n}\frac{kx_it_N}{1-x_it_N} - \sum_{j=1}^{m}\frac{y_jt_l}{1-y_jt_l}\right)$$

Hence,

$$\frac{(\mathcal{H}-\mathcal{L}_N)\varphi_N}{\varphi_N} = \frac{\left(C_{N-1}\sum_{i=1}^{N-1} t_l\frac{\partial}{\partial t_l} + C_1 t_N\frac{\partial}{\partial t_N}\right)\varphi_N}{\varphi_N} + 2\sum_{i=1}^{n}\sum_{l=1}^{N-1}\frac{k^2x_i^2t_lt_N}{(1-x_it_l)(1-x_it_N)} -$$
$$2\sum_{j=1}^{m}\sum_{l=1}^{N-1}\frac{ky_j^2t_lt_N}{(1-y_jt_l)(1-y_jt_N)} - \sum_{i=1}^{n}\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}\frac{k^2x_it_l}{1-x_it_l} + \sum_{j=1}^{m}\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}\frac{ky_jt_l}{1-y_jt_l} +$$
$$\sum_{i=1}^{n}\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}\frac{k^2x_it_N}{1-x_it_N} - \sum_{j=1}^{m}\sum_{l=1}^{N-1}\frac{t_l+t_N}{t_l-t_N}\frac{ky_jt_N}{1-y_jt_N} = \frac{\left(C_{N-1}\sum_{i=1}^{N-1} t_l\frac{\partial}{\partial t_l} + C_1 t_N\frac{\partial}{\partial t_N}\right)\varphi_N}{\varphi_N} +$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{N-1}\frac{t_l}{t_N-t_l}\frac{2k^2t_Nx_i}{1-x_it_N} + \frac{t_N}{t_l-t_N}\frac{2k^2t_lx_i}{1-x_it_l} - \frac{t_l+t_N}{t_l-t_N}\frac{k^2t_lx_i}{1-x_it_l} + \frac{t_l+t_N}{t_l-t_N}\frac{k^2t_Nx_i}{1-x_it_N} -$$
$$\frac{t_l}{t_N-t_l}\frac{2kt_Ny_j}{1-y_jt_N} - \frac{t_N}{t_l-t_N}\frac{2kt_ly_j}{1-y_jt_l} + \frac{t_l+t_N}{t_l-t_N}\frac{kt_ly_j}{1-y_jt_l} - \frac{t_l+t_N}{t_l-t_N}\frac{kt_Ny_j}{1-y_jt_N} =$$

$$\frac{\left(C_{N-1}\sum\limits_{i=1}^{N-1} t_i\frac{\partial}{\partial t_i}+C_1t_N\frac{\partial}{\partial t_N}\right)\varphi_N}{\varphi_N}+$$

$$\sum_{l=1}^{N-1}\left(-\sum_{i=1}^{n}\frac{k^2x_it_N}{1-x_it_N}-\sum_{i=1}^{n}\frac{k^2x_it_l}{1-x_it_l}+\sum_{j=1}^{m}\frac{ky_jt_N}{1-y_it_N}+\sum_{j=1}^{m}\frac{ky_it_l}{1-y_it_l}\right)=$$

$$\sum_{l=1}^{N-1}\left((N-1)k\sum_{j=1}^{m}\frac{y_it_N}{1-y_it_N}+\sum_{j=1}^{m}\frac{ky_it_l}{1-y_it_l}\right)-$$

$$\sum_{l=1}^{N-1}\left((N-1)k^2\sum_{i=1}^{n}\frac{x_it_N}{1-x_it_N}\right)-\sum_{i=1}^{n}\frac{k^2x_it_l}{1-x_it_l}+\frac{\left(C_{N-1}\sum\limits_{l=1}^{N-1} t_l\frac{\partial}{\partial t_l}+C_1t_N\frac{\partial}{\partial t_N}\right)\varphi_N}{\varphi_N}=$$

$$\frac{\left(C_{N-1}\sum\limits_{i=1}^{N-1} t_i\frac{\partial}{\partial t_i}\right)\varphi_N+C_1\left(t_N\frac{\partial}{\partial t_N}\right)\varphi_N-((N-1)k\left(t_N\frac{\partial}{\partial t_N}\right)\varphi_N-k\left(\sum\limits_{l=1}^{N-1} t_l\frac{\partial}{\partial t_l}\right)\varphi_N}{\varphi_N}=$$

$$\frac{C_N\left(\sum\limits_{l=1}^{N-1} t_l\frac{\partial}{\partial t_l}+t_N\frac{\partial}{\partial t_N}\right)\varphi_N}{\varphi_N}$$

4.5. Proof of the Theorem 1.3 It is easy to verify that

$$\omega_k\left(\frac{1}{\prod\limits_i(1-x_it)^k\prod\limits_j(1-y_jt)^k}\right)=\frac{\prod\limits_j(1-y_jt)}{\prod\limits_i(1-x_it)^k}$$

Therefore

$$\omega_k\left(\frac{1}{\prod\limits_{i,l}(1-x_it_l)^k\prod\limits_{j,l}(1-y_jt_l)^k}\right)=\frac{\prod\limits_{j,l}(1-y_jt_l)}{\prod\limits_{i,l}(1-x_it_l)^k}$$

Then by Couchy identity

$$\frac{1}{\prod\limits_{i,l}(1-x_it_l)^k\prod\limits_{j,l}(1-y_jt_l)^k}=\sum_\lambda\frac{1}{J_\lambda}P_\lambda(x,y,k)P_\lambda(t,k)$$

Let us apply the automorphism ω_k, then

$$\omega_k\left(\frac{1}{\prod\limits_{i,l}(1-x_it_l)^k\prod\limits_{j,l}(1-y_jt_l)^k}\right)=\sum_\lambda\frac{1}{J_\lambda}\omega_k\left(P_\lambda(x,y,k)\right)P_\lambda(t,k)$$

Therefore

$$\frac{\prod\limits_{j,l}(1-y_jt_l)}{\prod\limits_{i,l}(1-x_it_l)^k}=\sum_\lambda\frac{1}{J_\lambda}SP_\lambda(x,y,k)P_\lambda(t,k)$$

Hence

$$\varphi_N = \sum_{\lambda_{N+1}=0} \frac{1}{j_\lambda} SP_\lambda(x_1,\dots,x_n,y_1\dots,y_m,k)P_\lambda(t_1,\dots,t_N,k)$$

Now, by Lemma 4.4

$$\mathcal{H}\varphi_N = \mathcal{L}_N\varphi_N + (k(n-N)-m)\left(\sum_{i=1}^{n} t_i\frac{\partial}{\partial t_i}\right)\varphi_N$$

Let us denote by $\mathcal{L}_N^*$ the operator

$$\mathcal{L}_N\varphi_N + (k(n-N)-m) + \left(\sum_{i=1}^{n} t_i\frac{\partial}{\partial t_i}\right)$$

Then,

$$\sum_{\lambda_{N+1}=0} \frac{1}{j_\lambda}\mathcal{H}\left(SP_\lambda(x_1,\dots,x_n,y_1\dots,y_m,k)\right)P_\lambda(t_1,\dots,t_N,k) =$$

$$\sum_{\lambda_{N+1}=0} \frac{1}{j_\lambda} SP_\lambda(x_1,\dots,x_n,y_1\dots,y_m,k)\mathcal{L}^*\left(P_\lambda(t_1,\dots,t_N,k)\right)$$

It is well known that $P_\lambda(t_1,\dots,t_N,k)$ are the eigenfunctions of the operator $\mathcal{L}^*$. Therefore $SP_\lambda(x_1,\dots,x_n,y_1\dots,y_m,k)$ are the eigenfunctions of the operator $\mathcal{H}$. □

§5. Spherical functions and radial parts of Laplace operators for the pair $(\mathfrak{gl}(V)\oplus\mathfrak{gl}(V),\mathfrak{gl}(V))$

5.1. Let $\mathfrak{g}=\mathfrak{gl}(V)$ be the Lie superalgebra of linear transformations of $n|m$-dimensional superspace V, let $\mathfrak{h}$ be the Cartan Subalgebra, R the root system, $U(\mathfrak{g})$ the enveloping algebra, and $U(\mathfrak{g})^*$ the dual space endowed with the *superalgebra* structure . For any ad-invariant functional on $U(\mathfrak{g})$ (i.e., for any l such that $l(u,v)=(-1)^{p(u)p(v)}l(v,u)$) denote by φ_l the generating function of the restriction of l onto $S(\mathfrak{h})$, namely

$$\varphi_l(t_1,\dots,t_n) = \sum \frac{l(e_{11}^{\nu_1}\dots e_{nn}^{\nu_n})}{(\nu_1)!\dots(\nu_n)!} t_1^{\nu_1}\dots t_n^{\nu_n}. \tag{5.1.1}$$

On $S(\mathfrak{h})^*$, define the following operators by setting for any $f\in S(\mathfrak{h})$:

$$(\partial_i l)(f) = l(e_{ii}f), \quad (D_{ij}l)(f) = l(e_{ij}e_{ji}f), \tag{5.1.2}$$

5.2. Lemma . *Let $\alpha=\varepsilon_i-\varepsilon_j$. Then*

i) $D_{ij} = \dfrac{e^\alpha}{e^\alpha-1}(\partial_i - (-1)^{p(i)+p(j)}\partial_j)$.

Proof.

$$(D^l_{ij})(f) = l(fe_{ij}e_{ji}) = L(e_{ij}f(h+\alpha(h)e_{ji}) =$$
$$(-1)^{p(i)+p(j)}l(f(h+\alpha(h))e_{ji}e_{ij}) =$$
$$= l(f(h+\alpha(h))e_{ij}e_{ji}) - l(f(h+\alpha(h))[e_{ij}, e_{ji}])$$
$$= (e^\alpha D_{ij}l)(f) - l(f(h+\alpha(h))(e_{ii} - (-1)^{p(i)+p(j)}e_{jj})$$
$$= (e^\alpha D_{ij} - e^\alpha(\partial_i - (-1)^{p(i)+p(j)}\partial_j))(l)(f).$$

□

5.3. Lemma . *Let* $\mathfrak{g}$ *be a Lie superalgebra* $\mathfrak{g}_1 = \{(x,x) \mid x \in \mathfrak{g}\}$, *be the diagonal subalgebra, let* I *be the left ideal in* $U(\mathfrak{g}\oplus\mathfrak{g})$ *generated by* $\mathfrak{g}_1$ *and* $M = U(\mathfrak{g}\oplus\mathfrak{g})/I$. *Let* $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}\oplus\mathfrak{g}$ *be the embedding into the first summand, i.e.,* $\sigma(x) = (x, 0)$. *Let* $\tilde\sigma : U(\mathfrak{g}) \longrightarrow M$ *be the map induced by the homomorphism* $U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}\oplus\mathfrak{g})$ *that extends* σ *and* $\varrho(x) = (x,x)$ *an isomorphism of* $\mathfrak{g}$ *with* $\mathfrak{g}_1$. *Then* $\tilde\sigma([x,u]) = \rho(x)\tilde\sigma(u)$.

Proof.

$$\begin{aligned}
\tilde\sigma([x,u]) &= \tilde\sigma(xu - (-1)^{p(x)p(u)} = xu\otimes 1 - (-1)^{p(x)p(u)}ux\otimes 1\\
&= xu\otimes 1 - (-1)^{p(x)p(u)}ux\otimes 1 - \rho(x)\tilde\sigma(u) + \varrho(x)\tilde\sigma(u)\\
&= \rho(x)\tilde\sigma(u) + xu\otimes 1 - (-1)^{p(x)p(u)}ux\otimes 1\\
&\quad -(x\otimes 1 + 1\otimes x)(u\otimes 1)\\
&= \rho(x)\tilde\sigma(u) - (-1)^{p(x)p(u)}(ux\otimes 1 + u\otimes x)\\
&= \rho(x)\tilde\sigma(u) - (-1)^{p(x)p(u)}(u\otimes 1)(x\otimes 1 + 1\otimes x)\\
&\equiv_{(\mathrm{mod}\ I)} \rho(x)\tilde\sigma(u).
\end{aligned}$$

□

Corollary . *The algebra of functionals on* $U(\mathfrak{g}\oplus\mathfrak{g})$ *biinvariant with respect to* $\mathfrak{g}_1$ *is isomorphic to the algebra of functionals on* $U(\mathfrak{g})$ *invariant with respect to the adjoint action.*

5.4. Lemma . *Let* $\mathfrak{g} = \mathfrak{gl}(V)$, $\mathfrak{h}$ *be a Cartan subalgebra in* $\mathfrak{g}$, *then*

$$U(\mathfrak{g}) = S(\mathfrak{h}) + [U(\mathfrak{g}), U(\mathfrak{g})]$$

Proof. Any element from $U(\mathfrak{g})$ can be represented as a sum of elements of the form

$$fX_{\alpha_1}\dots X_{\alpha_r}$$

where $f \in S(\mathfrak{h})$, $\alpha_i \in R$, R is a roots system of $\mathfrak{g}$ and X_{α_i} is an element of weight α_i. Therefore, to prove Lemma, it suffices to demonstrate that

$$fX_{\alpha_1}\dots X_{\alpha_r} \in [U(\mathfrak{g}), U(\mathfrak{g})]$$

for $r > 0$. Let us induct on r.

If $r = 1$, then

$$[X_\alpha, f] = (f(h-\alpha(h)) - f(h))\,X_\alpha = R_\alpha(f)X_\alpha$$

But, as is easy to see, any element of $S(\mathfrak{h})$ can be represented as $R_\alpha(f)$. Let $r > 1$. Then

$$[X_{\alpha_1}, fX_{\alpha_2}\dots X_{\alpha_r}] = [X_{\alpha_1}, f]X_{\alpha_2}\dots X_{\alpha_r} + f[X_{\alpha_1}, X_{\alpha_2}]X_{\alpha_3}\dots X_{\alpha_r} +$$
$$(-1)^{p(X_{\alpha_1})p(X_{\alpha_2})} fX_{\alpha_2}[X_{\alpha_1}, X_{\alpha_3}]X_{\alpha_4}\dots X_{\alpha_r} +$$
$$(-1)^{p(X_{\alpha_1})(p(X_{\alpha_2})+p(X_{\alpha_3}))} fX_{\alpha_2}X_{\alpha_3}[X_{\alpha_1}, X_{\alpha_4}]X_{\alpha_5}\dots X_{\alpha_r} + \dots$$

But $[X_\alpha, f]X_\alpha = R_\alpha(f)X_\alpha$ and by the above and inductive hypothesis

$$fX_{\alpha_1}\dots X_{\alpha_r} \in [U(\mathfrak{g}), U(\mathfrak{g})]$$

□

Lemma 5.4 immediately implies statement of heading i) of theorem 1.5.1. Statement of heading ii) is obvious.

Proof of iii). Let l be an invariant functional on $U(\mathfrak{b}_1)$ and φ_l the generating function of its restriction onto $S(\mathfrak{b})$. Then

$$\Omega(\varphi_l) = \left(\sum_{i,j\in I}(-1)^{p(j)}e_{ij}e_{ji}\right)\varphi_l = (\text{ Lemma 5.2}) = \left(\sum_{i\in I_{\bar 0}}\partial_i^2 - \sum_{j\in I_{\bar 1}}\partial_j^2\right)\varphi_l +$$

$$\sum_{\alpha\in R^+}\left((-1)^{p(j)}\frac{e^\alpha}{e^\alpha - 1}(\partial_i - (-1)^{p(i)+p(j)}\partial_j) + (-1)^{p(i)}\frac{e^{-\alpha}}{e^{-\alpha}-1}(\partial_j - (-1)^{p(i)+p(j)}\partial_i)\right)\varphi_l =$$

$$\left(\sum_{i\in I_{\bar 0}}\partial_i^2 - \sum_{j\in I_{\bar 1}}\partial_j^2 - \sum_{\alpha\in R^+}\frac{1+e^\alpha}{1-e^\alpha}((-1)^{p(j)}\partial_i - (-1)^{p(i)}\partial_j)\right)\varphi_l$$

which proves heading iii) of Theorem 1.5.1.

5.5. Proof of iv) of Theorem 1.5.1. As is easy to verify, $\theta = \sum\limits_{i\in I} e_i \otimes e_i^*$ is a $\mathfrak{b}$-invariant, provided $\{e_i\}_{i\in I}$ is a basis in V and $\{e_i^*\}_{i\in I}$ is its left dual. Similarly, $\theta^* = \sum\limits_{i\in I} e_i\otimes e_i^*$ is also a $\mathfrak{b}$-invariant.

By [S2], the invariants in $W = V^{\otimes p}(V^*)^{\otimes p}$ lie in the linear span of $(\mathfrak{S}_p \otimes \mathfrak{S}_p)(\theta^{\otimes p})$ under the natural $\mathfrak{S}_p\otimes\mathfrak{S}_p$ -action on W. Moreover, the stabilizer of $\theta^{\otimes p}$ is $\mathfrak{S}_p$ embedded diagonally. Hence, the space of $\mathfrak{b}$-invariant vectors is, as $\mathfrak{S}_p\times\mathfrak{S}_p$-module, isomorphic to

$$Ind_{\mathfrak{S}_p}^{\mathfrak{S}_p\times\mathfrak{S}_p}(id) = \oplus_\lambda S^\lambda \otimes S^\lambda$$

where λ runs over partitions of p such that $\lambda_{n+1} \leq m$. This implies that, up to a constant multiple,

$$\varphi_\lambda(u) = \theta^\pi((\theta^*)^{\otimes p}, \theta^{\otimes p})(u) = (\theta^*)^{\otimes p}\left(e_\lambda \times e_\lambda u\theta^{\otimes p}\right) \quad \text{for any } u \in U(\mathfrak{g}),$$

where e_λ the idempotent corresponding to the partition λ, and $\varphi_\lambda(u)$ is proportional to

$$(\theta^*)^{\otimes p}\left(e_\lambda \times 1u\theta^{\otimes p}\right)$$

By expanding $e_\lambda = \frac{1}{p!}\sum \chi^\lambda(\sigma)\sigma$ where χ^λ is a character of the corresponding representation of $\mathfrak{S}_p$, we see that thanks to the identity

$$\theta^{\pi_1\otimes\pi_2}(v_1^*\otimes v_2^*, v_1\otimes v_2) = (-1)^{p(v_1)p(v_2^*)}\theta^{\pi_1}(v_1^*, v_1)\theta^{\pi_2}(v_2^*, v_2)$$

it suffices to consider the case when σ is the cycle $(12\dots p)$. Let $\{e_{ii}^*\}_{i\in I}$ be the basis of $\mathfrak{h}$ and $\{\varepsilon_i\}_{i\in I}$ is the left dual basis; let e^l denotes the homomorphism $S(\mathfrak{h}) \longrightarrow k$ extending the linear form l. It suffices to take into account only summands with $i_1 = i_2 = \dots = i_p$ in

$$\theta^{\otimes p} = \sum_{i_1,\dots,i_p\in I} e_{i_1}\otimes e_{i_1}^*\otimes\dots\otimes e_{i_p}\otimes e_{i_p}^*$$

Therefore, if $\sigma = (12\dots p)$ then

$$(\theta^*)^{\otimes p}\left(u\sigma\theta^{\otimes p}\right) = (\theta^*)^{\otimes p}\left(\sum_{i\in I}(e_i\otimes e_i^*)^{\otimes p}\right) = \sum_{i\in I}(\theta^*)^{\otimes p}\left(u\sigma\theta^{\otimes p}\right) =$$

$$\sum_{i\in I}(\theta^*)^{\otimes p}\left((-1)^{(p-1)i}u(e_i\otimes e_i^*)^{\otimes p}\right) = \sum_{i\in I}(-1)^{(p-1)i}e^{p\varepsilon_i}(u)(\theta^*)^{\otimes p}(e_i\otimes e_i^*) =$$

$$\sum_{i\in I}(-1)^i e^{p\varepsilon_i}(u) = \sum_{i\in I_{\bar 0}} x_i^p - \sum_{j\in I_{\bar 1}} y_j^p$$

this implies that, up to a constant scalar, we have $\varphi_\lambda = \sum\frac{\chi_\mu^\lambda}{Z_\mu}sp_\mu$

where χ_μ^λ is the value of the character of the symmetric group on the element of cycle type μ and $Z_\mu = \prod i^{\mu_i}(\mu_i)!$.This coincides with $SP_\lambda(x, y, 1)$.

§6. Spherical functions and radial parts of the Laplace operators for the pair $(\mathfrak{gl}, \mathfrak{osp})$

6.1. Let $\mathfrak{g} = \mathfrak{gl}(V)$, where $\dim V = n|2r$; let $I_{\bar 0} = \{1,\dots,n\}$, $I_{\bar 1} = \{\bar 1,\dots,\overline{2r}\}$, and $\{e_{ij}\}_{i,j\in I}$ the basis of matrix units in $\mathfrak{gl}(V)$ It is easy to verify that the antiautomorphism of supertransposition in $\mathfrak{gl}(V)$ is of the form

$$e_{ij}^t = (-1)^{i)(j+1)}e_{ji}. \tag{6.1.1}$$

Further, set$\varepsilon(i) = 1$, if $i \in I_{\bar 0}\cup\{\overline{r+1},\dots,\overline{2r}\}$,and $\varepsilon(i) = -1$, if $i\in\{\bar 1,\dots,\bar r\}$. Set also $\delta(i) = i + \bar r(\mod \overline{2}r)$, if $i\in I_{\bar 1}$, and $\delta(i) = i$, if $i\in I_{\bar 0}$.

Now, define operator S by setting

$$Se_i = \varepsilon(i)e_{\delta(i)} \tag{6.1.2}$$

Clearly, $S^2 = J$, where J is the parity operator in V i.e., $Je_i = (-1)^{p(i)}$ and, therefore, $SJ = JS$ as well as $S^t = SJ = JS = S^3 = S^{-1}$.For any $x\in\mathfrak{gl}(V)$ set $\psi(x) = Sx^tS^{-1}$. It is not difficult to verify that ψ is an involutive antiautomorphism of the associative superalgebra $\mathrm{Mat}(V)$, i.e., $\psi^2 = 1$ and $\psi(xy) = (-1)^{p(x)p(y)}$. Therefore,$\mathfrak{g} = \mathfrak{g}^-\oplus\mathfrak{g}^+$ where $\mathfrak{g}^- = \{x\in\mathfrak{g}\mid\psi(x) = -x\}$ and $\mathfrak{g}^+ = \{x\in\mathfrak{g}\mid\psi(x) = x\}$.

Observe that $\mathfrak{g}^-$ is a Lie subsuperalgebra of $\mathfrak{gl}(V)$ isomorphic to $\mathfrak{osp}(V)$ and $\mathfrak{g}^+$ is a $\mathfrak{g}^-$-module. For any $x \in \mathfrak{g}$ set

$$x^+ = \frac{1}{2}(x + \psi(x)) \quad \text{and} \quad x^- = \frac{1}{2}(x - \psi(x))$$

corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^+$.

Let $\mathfrak{h}$ be Cartan subalgebra in $\mathfrak{g}$ and also set

$$\mathfrak{h}^+ = \mathrm{Span}(e^+_{ii} \mid i \in I).$$

It is easy to verify, that$\psi(e_{ij}) = (-1)^{p(i)p(j)}\varepsilon(j)\varepsilon(\delta(i))e_{\delta(j)\delta(i)}$.

6.2. Lemma . *For $f \in S(\mathfrak{h}^+)$ and $\alpha = \varepsilon_i - \varepsilon_j$ set*

$$R^-_{ij}f = \tfrac{1}{2}[f(h - \alpha(h)) - f(h + \alpha(h))],$$
$$R^+_{ij}f = \tfrac{1}{2}[f(h - \alpha(h)) + f(h + \alpha(h))].$$

Then the following identities hold:

i) $e^-_{ij}f = R^+_{ij}fe^-_{ij} + R^-_{ij}f \cdot e^+_{ij}$

ii) $R^-_{ij}fe_{ij}e_{ji} - (R^-_{ij} - R^+_{ij})f \cdot [e^-_{ij}, e^+_{ji}] \in \mathfrak{g}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^-$

iii) $[e^-_{ij}, e^+_{ij}] = \frac{1}{2}(e^+_{ii} - (-1)^{p(i)+p(j)}e^+_{jj})$

iv) *If $h \in \mathfrak{h}$, and $\alpha = \varepsilon_i - \varepsilon_j$, then*

$$[h, e^+_{ij}] = \alpha(h)e^-_{ij}, \quad [h, e^-_{ij}] = \alpha(h)e^+_{ij}$$

Proof is reduced to a direct verification. □

6.3. Lemma . *Let $\mathcal{I} = \mathfrak{g}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^-$.Then*

$$U(\mathfrak{g}) = S(\mathfrak{h}^+) + \mathcal{I}.$$

Proof. It suffices to show that for $q > 0$ we have

$$u = fe^+_{\alpha_1} \dots e^+_{\alpha_q} \in \mathcal{I}$$

where $f \in S(\mathfrak{h}^+)$ and $e^+_{\alpha_1}, \dots, e^+_{\alpha_q} \in \mathfrak{g}^+$ are the weight vectors. Let us induct on q. If $q = 1$ and $f \in S(\mathfrak{h}^+)$ we have $R^-_{ij}fe^+_{ij} = e^-_{ij}f - R^+_{ij}fe^-_{ij}$ by Lemma 6.2.i). Hence, $R^-_{ij}fe^+_{ij} \in \mathfrak{g}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^-$.Hence $fe^+_{ij} \in \mathcal{I}$ Let $q > 1$. Then

$$\begin{aligned} R^-_{\alpha_1}fe^+_{\alpha_1}\dots e^+_{\alpha_q} &= e^-_{\alpha_1}fe^+_{\alpha_2}\dots e^+_{\alpha_n} - R^+_{\alpha_1}fe^-_{\alpha_1}e^+_{\alpha_2}\dots \\ &\equiv_{(\mathrm{mod}\ \mathcal{I})} -R^+_{\alpha_1}fe^-_{\alpha_1}e^+_{\alpha_2}\dots e^+_{\alpha_q} \\ &\equiv R^+_{\alpha_1}f \cdot [e^-_{\alpha_1}, e^+_{\alpha_2}\dots e^+_{\alpha_q}] \\ &= -R^+_{\alpha_1}f \cdot [e^-_{\alpha_1}, e^+_{\alpha_2}]e^+_{\alpha_3}\dots e^+_{\alpha_q} \\ &\quad -R^+_{\alpha_1}fe^+_{\alpha_2}[e^-_{\alpha_1}, e^+_{\alpha_3}]\dots e^+_{\alpha_q} + \dots \in \mathcal{I} \end{aligned}$$

□

6.4. Lemma . *Let $\alpha = \varepsilon_i - \varepsilon_j$ and $j \neq \delta(i)$, let l be a two-sided $\mathfrak{b}$-invariant functional on $U(\mathfrak{g})$ and φ_l the generating function of its restriction onto $S(\mathfrak{h}^+)$. Let $D_{ij}(l)(f) = l(fe_{ij}e_{ji})$, $\partial_i^+(l)(f) = l(fe_{ii}^+)$ where $f \in S(\mathfrak{h}^+)$. Then*

$$D_{ij} = \frac{e^\alpha}{e^\alpha - e^{-\alpha}}(\partial_i^+ - (-1)^{p(i)+p(j)}\partial_j^+)$$

The proof is the consequence of statement ii) Lemma 6.2.

6.5. Proof of Theorem 1.6.1. i) follows immediately from Lemma 6.3.

ii) Let $\sum\limits_{i,j\in I}(-1)^{p(j)}e_{ij}e_{ji}$ be the Laplace operator for $\mathfrak{gl}(V)$. By Lemma 6.4 the radial part of its restriction onto $S(\mathfrak{h})$ is of the form (we have excluded the roots $\beta = \varepsilon_i - \varepsilon_{\delta(i)}$, because $e_{i\delta(i)}^+ = 0$):

$$\begin{gathered}\mathcal{M} = \sum_{i=1}^{n}\partial_i^2 - \sum_{j=1}^{2r}\partial_{\bar{j}}^2 + \sum_{\alpha\in R_{11}^+}\frac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{\alpha}}\partial_\alpha^+ - \\ \sum_{\beta\in R_{22}^+}\frac{e^\beta + e^{-\beta}}{e^\beta - e^{-\beta}}\partial_\beta^+ - \sum_{\gamma\in R_{12}}\frac{e^\gamma + e^{-\gamma}}{e^\gamma - e^{-\gamma}}\partial_{\gamma,1}^+\end{gathered} \tag{6.5.1}$$

where

$$\begin{aligned}\partial_\alpha &= \partial_i^+ - \partial_j^+ && \text{for } \alpha = \varepsilon_i - \varepsilon_j;\\ \partial_\beta &= \partial_{\bar{i}}^+ - \partial_{\bar{j}}^+ && \text{for } \beta = \varepsilon_{\bar{i}} - \varepsilon_{\bar{j}};\\ \partial_{\gamma,1} &= \partial_i^+ + \partial_{\bar{j}}^+ && \text{for } \gamma = \varepsilon_i - \varepsilon_{\bar{j}}.\end{aligned}$$

Now, observe that $\varepsilon_i|_{\mathfrak{h}^+} = \varepsilon_{\delta(i)}|_{\mathfrak{h}^+}$ so (6.5.1) takes the form

$$\begin{gathered}\sum_{i=1}^{n}\partial_i^2 - 2\sum_{j=1}^{r}\partial_{\bar{j}}^2 + \sum_{\alpha\in R_{11}^+}\frac{e^{2\alpha}+1}{e^{2\alpha}-1}\partial_\alpha^+ - \\ 4\sum_{\beta\in\frac{1}{4}R_{22}^+}\frac{e^{2\beta}+1}{e^{2\beta}-1}\partial_\beta^+ - 2\sum_{\gamma\in\frac{1}{2}R_{12}}\frac{e^{2\gamma}+1}{e^{2\gamma}-1}\partial_{\gamma,1}^+\end{gathered} \tag{6.5.2}$$

where

$$\tfrac{1}{4}R_{22} = \{\varepsilon_i - \varepsilon_j \mid i,j \in \tfrac{1}{2}I_{\bar{1}}\},\ \tfrac{1}{2}R_{12} = \{\varepsilon_i - \varepsilon_j \mid i \in I_{\bar{0}},\ j \in \tfrac{1}{2}I_{\bar{1}}\},\quad \tfrac{1}{2}I_{\bar{1}} = \{\bar{1},\dots,\bar{r}\}$$

Set $T(e^l) = e^{l+l_{\bar{0}}}$ where $l_{\bar{0}}$ is the even part of l. Then, as is easy to verify,

$$T^{-1}\partial_i^+T = \partial_i^+,\quad \text{if}\quad p(i) = \bar{1},\quad T^{-1}\partial_i^+T = 2\partial_i^+,\quad \text{if}\quad p(i) = \bar{0},\quad T^{-1}e^lT = e^{l_{\bar{1}}+\frac{1}{2}l_{\bar{0}}}$$

Therefore, this transformation sends (6.5.2) into

$$\begin{gathered}4\sum_{i=1}^{n}\partial_i^2 - 2\sum_{j=1}^{r}\partial_{\bar{j}}^2 + \sum_{\alpha\in R_{11}^+}\frac{e^{\alpha}+1}{e^{\alpha}-1}2\partial_\alpha^+ - \\ 4\sum_{\beta\in\frac{1}{4}R_{22}^+}\frac{e^{\beta}+1}{e^{\beta}-1}\partial_\beta^+ - 4\sum_{\gamma\in\frac{1}{2}R_{12}}\frac{e^{2\gamma}+1}{e^{2\gamma}-1}\partial_{\gamma,\frac{1}{2}}^+\end{gathered} \tag{6.5.3}$$

The latter expression is equal to $4\mathcal{M}_2$.

iii) First, let us describe the invariant vector $v_\lambda \in V_\lambda$, where all the rows of the diagram λ are of even length. It is easy to verify that

$$\theta = \sum_{i\in I} \varepsilon(i) e_i \otimes e_i \quad \text{and} \theta = \sum_{i\in I} \varepsilon(i) e_i^* \otimes e_i^*$$

are $\mathfrak{b}$-invariants. Hence, $\theta^{\otimes p}$ is also a $\mathfrak{b}$-invariant. By [S2] the linear span of all invariants is $k[\mathfrak{S}_{2p}]\,\theta^{\otimes p}$ and, as is easy to show, as $\mathfrak{S}_{2p}$-module it is isomorphic to

$$Ind_{H_p}^{\mathfrak{S}_{2p}}(id) = \oplus_\lambda S^\lambda$$

where λ runs over partitions of $2p$ such that $\lambda_{n+1} \leq 2r$ and all the rows of λ are of even length and H_p is the stabilizer of $\theta^{\otimes p}$. This implies that, up to a constant multiple,

$$\varphi_\lambda(u) = \theta^\pi((\theta^*)^{\otimes p}, \theta^{\otimes p})(u) = (\theta^*)^{\otimes p}\left(e_\lambda u \theta^{\otimes p}\right) \quad \text{for any } u \in U(\mathfrak{g}),$$

where e_λ is a primitive idempotent in the in the Hecke algebra $(\mathfrak{S}_{2p}, H_p)$ The explicit form of e_λ is known ([M]):

$$e_\lambda = \sum \frac{\omega_\mu^\lambda}{Z_{2\mu}} \sigma$$

where σ runs over the set of representatives of double cosets .

If $\mu = \mu_1 \dots \mu_q$ we may assume that $\sigma = 2\mu_1 \dots 2\mu_q$, as far as the cycle structure is concerned.

Now, let us calculate the functional

$$\varphi_{\lambda,\mu}(u) = (\theta^*)^{\otimes p}\left(\sigma u \theta^{\otimes p}\right) \quad \text{where} u \in S(\mathfrak{h})$$

Thanks to the identity

$$\theta^{\pi_1\otimes\pi_2}(v_1^* \otimes v_2^*, v_1 \otimes v_2) = (-1)^{p(v_1)p(v_2^*)} \theta^{\pi_1}(v_1^*, v_1) \theta^{\pi_2}(v_2^*, v_2)$$

it suffices to assume that σ is a cycle of length $2p$. We have

$$\theta^{\otimes p} = \sum \varepsilon(\psi) e_\psi, \quad (\theta^*)^{\otimes p} = \sum \varepsilon(\psi) e_\psi^*,$$

where the sum runs over all the maps

$$\psi : [1, \dots, 2p] \longrightarrow [1, \dots, n, \bar{1}, \dots, \overline{2r}] \quad \text{and} \quad \delta(\psi(2i)) = \psi(2i-1) \quad \text{for} \quad i = 1, \dots, l.$$

and

$$e_\psi = e_{\psi(1)} \otimes \dots \otimes e_{\psi(2p)}, \quad \varepsilon(\psi) = \varepsilon(\psi(1))\varepsilon(\psi(3)) \dots \varepsilon(\psi(2l-1))$$

If σ is a cycle, we only have to take into account the summands of the sum $\varphi_{\lambda,\mu}(u)$ for which $\sigma\psi$ possesses the same property as ψ. But then

$$\psi(1) = \psi(3) = \dots \psi(2l-1) = \delta(\psi(2)) = \delta(\psi(4)) \dots = \delta(\psi(2l))$$

and $e_\psi = (e_i \otimes e_{\delta(i)})^{\otimes p}$ where $i \in I$

Direct calculations show that

$$(\theta^*)^{\otimes p}\left(u\sigma(e_i \otimes e_{\delta(i)})^{\otimes p}\right) = (-1)^{p(i)} e^{p\varepsilon_i}(u) e^{p\varepsilon_{\delta(i)}}(u) = (-1)^{p(i)} e^{2p\varepsilon_i}(u)$$

Therefore,

$$(\theta^*)^{\otimes p}\left(u\sigma\theta^{\otimes p}\right) = \left(\sum_{i\in I_{\bar{0}}} (e^{2\varepsilon_i})^p - 2\sum_{i\in\frac{1}{2}I_{\bar{1}}} (e^{2\varepsilon_j})^p\right)(u)$$

Hence, setting $x_i = e^{2\varepsilon_i}, i \in I_{\bar 0}$ $y_j = e^{2\varepsilon_j}, j \in \frac{1}{2}J_{\bar 1}$ we obtain

$$\phi_\lambda = \sum \frac{\omega_\mu^\lambda}{Z_{2\mu}} SP_{mu}(x, y, \frac{1}{2})$$

where$SP_\mu(x, y, \frac{1}{2}) = SP_{\mu_1}(x, y, \frac{1}{2}) \dots SP_{mu_q}(x, y, frac12)$ and

$$SP_{\mu_p} = \sum_{i \in I_{\bar 0}} x_i^p - 2 \sum_{j \in \frac{1}{2} I_{\bar 1}} y_j^p$$

§7. An algebraic analog of Berezin integral

7.1. For the usual Jack polynomials corresponding to Lie algebra $\mathfrak{gl}(n)$ there exists an inner product induced by the invariant integral on $U(n)$. In [B] Berezin constructed an invariant integral on the unitary supergroup $U(n|m)$ and established a number of its properties.

In particular, matrix coefficients of any finite dimensional irreducible representation V such that $\dim V_{\bar 0} \neq \dim V_{\bar 1}$ are isotropic with respect to the natural inner product related with Berezin integral.

In this section I construct an algebraic analog of Berezin integral and established a number of its properties.

For every $\mathfrak{g}$-module W, define in $U(\mathfrak{g})^*$ the subspace $C(W)$ consisting of the linear hull of the matrix coefficients of W. Denote by $\mathfrak{A}_{n,m}$ the subalgebra of $U(\mathfrak{g})^*$ generated by the matrix coefficients of the identity representation V of $\mathfrak{g} = \mathfrak{gl}(V)$ and its dual, V^*. Let$\{e_1, \dots, e_n, e_{\bar 1} \dots, e_{\bar m}\}$ be a basis of V and $\{e_1^*, \dots, e_n^*, e_{\bar 1}^* \dots, e_{\bar m}^*\}$ the left dual basis of V^*; let $x_{ij} = \theta(e_i^*, e_j)$ be the corresponding matrix coefficient. Let $\Delta_{\bar 0} = det(x_{ij})$, where $i, j \in I_{\bar 0}$ and $\Delta_{\bar 0} = det(x_{ij})$ where $i, j \in I_{\bar 1}$

7.2. Lemma . *The algebra $\mathfrak{A}_{n,m}$ is isomorphic to $S(V^* \otimes V)\left[\Delta_{\bar 0}^{-1}, \Delta_{\bar 1}^{-1}\right]$ as algebra and as a $\mathfrak{g} \oplus \mathfrak{g}$-module (provided we have established the natural $\mathfrak{g} \oplus \mathfrak{g}$-module structure on $V^* \otimes V$).*

Proof. Consider the natural map

$$V^* \otimes V \longrightarrow U(\mathfrak{g})^*, \quad v^* \otimes v \longmapsto \theta(v^*, v)$$

as in Lemma 2.4.2 and extend it to a homomorphism $\varphi : S(V^* \otimes V)) \longrightarrow U(\mathfrak{g})^*$ Select in $\mathfrak{g}$ a basis of matrix units e_{ij} and identify $U(\mathfrak{g})^*$ with the algebra of formal power series in t_{ij}. Then $\varphi(x_{ij}) = t_{ij} + \alpha_{ij}$ where α_{ij} is a formal series that begins with terms of degree ≥ 2. This implies that $\varphi(x_{ij})$ are algebraically independent and φ is an embedding. Clearly,

$$\varphi(S(V^* \otimes V)) \subset \mathfrak{A}_{n,m} \quad \text{and} \quad \varphi\left(S\left(V^* \otimes V\right)\left[\Delta_{\bar 0}^{-1}, \Delta_{\bar 1}^{-1}\right]\right) = \mathfrak{A}_{n,m}.$$

Moreover, by Lemma 2.4.2 this is a $\mathfrak{g} \oplus \mathfrak{g}$-module isomorphism. □

7.3. Lemma . *Consider $\mathfrak{A}_{n,m}$ as a left $\mathfrak{g}$-module and a right $\mathfrak{g}_{\bar 0}$-module. Then*

$$\mathfrak{A}_{n,m} = \bigoplus_{\chi} \left(Ind_{\mathfrak{g}_{\bar 0}}^{\mathfrak{g}} (V_{\bar 0}^*)^\chi\right) \otimes V_{\bar 0}^\chi \tag{7.3.1}$$

where χ runs over the set of collections (χ_1, χ_2) such that χ_1 is an integer highest weight of $\mathfrak{gl}(n)$,χ_2 is an integer highest weight of $\mathfrak{gl}(m)$ and $V_{\bar 0}^\chi$ is an irreducible $\mathfrak{gl}(n) \oplus \mathfrak{gl}(m)$-module with highest weight χ and $(V_{\bar 0}^*)^\chi$ is the dual module.

Proof. Set

$$(\mathfrak{A}_{n,m})_1 = S\left(V^* \otimes V_{\bar 0}\right)\left[\Delta_{\bar 0}^{-1}\right]$$

and

$$(\mathfrak{A}_{n,m})_2 = S\left(V^* \otimes V_{\bar 1}\right)\left[\Delta_{\bar 1}^{-1}\right]$$

We have an isomorphism of algebras and $\mathfrak{g} \oplus \mathfrak{g}_{\bar 0}$ -modules

$$\mathfrak{A}_{n,m} = (\mathfrak{A}_{n,m})_1 \otimes (\mathfrak{A}_{n,m})_2 \tag{7.3.2}$$

where $\mathfrak{gl}(n)$ acts trivially on the right on the second factor and $\mathfrak{gl}(m)$ trivially on the right on the first factor.

Theorem 1.3 from [S2] implies that

$$S\left(V^* \otimes V_{\bar 0}\right) = \bigoplus_{\lambda_{n+1}=0} (V^*)^\lambda \otimes V_{\bar 0}^\lambda$$

where λ is a partition. Let us expand $(\mathfrak{A}_{n,m})_1$ into the sum of isotypical $\mathfrak{gl}(n)$-modules:

$$(\mathfrak{A}_{n,m})_1 = \sum_{i=0}^{\infty} \Delta_{\bar 0}^{-i} S\left(V^* \otimes V_{\bar 0}\right) = \sum_{i=0}^{\infty} \Delta_{\bar 0}^{-i}\left(\oplus_\lambda (V^*)^\lambda \otimes V_{\bar 0}^\lambda\right) \tag{7.3.3}$$

Let

$$\chi_1 = (\chi_1^{(1)}, \ldots, \chi_n^{(1)}) \in \mathbb{Z}^n \quad : \chi_1^{(1)} \geq \chi_2^{(1)} \geq \cdots \geq \chi_n^{(1)}$$

be an integer highest weight for $\mathfrak{gl}(n)$. Then the isotypical component of type χ in $(\mathfrak{A}_{n,m})_1$is, due to (7.3.3),

$$W^\chi = \sum_\lambda \Delta_{\bar 0}^{-i}\left((V^*)^\lambda \otimes V_{\bar 0}^\lambda\right) \tag{7.3.4}$$

where $\lambda - i\delta_n = \chi$, and where λ is a partition, δ_n is the highest weight of $\bigwedge^n(V_{\bar 0}) \otimes \mathbb{C}$ $\mathfrak{gl}(n) \oplus \mathfrak{gl}(m)$ module, where $\mathbb{C}$ is a trivial $\mathfrak{gl}(m)$ module. In the set of all such diagrams, select a one, $\lambda(\chi)$, such that $\lambda(\chi) - j\delta_n = \chi$ and which is the least with respect to the lexicographic ordering and containing an $n \times m$ rectangle, or, equivalently: the module $(V^*)^\lambda$ is a typical one.

Then for any summand in (7.3.4) we have

$$\Delta_{\bar 0}^{-i}\left((V^*)^\lambda \otimes V_{\bar 0}^{\lambda(\chi)}\right) \subset \Delta_{\bar 0}^{-j}\left((V^*)^{\lambda(\chi)} \otimes V_{\bar 0}^{\lambda(\chi)}\right).$$

Indeed, if $i < j$ then $\lambda(\chi) - j\delta_n = \lambda - i\delta_n$ hence,$\lambda = \lambda(\chi) - (j-i)\delta_n$. Therefore,

$$\Delta_{\bar 0}^{j-i}\left((V^*)^\lambda \otimes V_{\bar 0}^\lambda\right) \subset (V^*)^{\lambda(\chi)} \otimes V_{\bar 0}^{\lambda(\chi)}$$

Multiplying both parts of (7.3.4) by $\Delta_{\bar 0}^{-j}$ we get the statement desired.

If $i > j$ then

$$\Delta_{\bar 0}^{i-j}\left((V^*)^{\lambda(\chi)} \otimes V_{\bar 0}^{\lambda(\chi)}\right) = (V^*)^\lambda \otimes V_{\bar 0}^\lambda$$

from identity of dimensions and the fact that $\Delta_{\bar 0}$ is not a zero divisor. Thus, in either case we have

$$W^\chi = \Delta_{\bar 0}^{-j}\left((V^*)^{\lambda(\chi)} \otimes V_{\bar 0}^{\lambda(\chi)}\right)$$

Since $(V^*)^{\lambda(\chi)}$ is typical, we have

$$W^\chi = Ind_{\mathfrak{g}_1 \oplus \mathfrak{g}_{\bar 0}}^{\mathfrak{g}}\left((V_{\bar 0}^*)^\chi \otimes V_{\bar 0}^\chi\right).$$

Therefore,

$$(\mathfrak{A}_{n.m})_1 = \bigoplus_{\chi_1} Ind^{\mathfrak{g}}_{\mathfrak{g}_1 \oplus \mathfrak{g}_{\bar 0}} \left((V_{\bar 0}^*)^{\chi_1} \otimes V_{\bar 0}^{\chi_1} \right).$$

We similarly prove that

$$(\mathfrak{A}_{n.m})_2 = \bigoplus_{\chi_2} Ind^{\mathfrak{g}}_{\mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar 0}} \left((V_{\bar 1}^*)^{\chi_2} \otimes V_{\bar 1}^{\chi_2} \right).$$

Lemma from [S2] implies that

$$Ind^{\mathfrak{g}}_{\mathfrak{g}_1 \oplus \mathfrak{g}_{\bar 0}} \left((V_{\bar 0}^*)^{\chi_1} \otimes V_{\bar 0}^{\chi_1} \right) \otimes Ind^{\mathfrak{g}}_{\mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar 0}} \left((V_{\bar 1}^*)^{\chi_1} \otimes V_{\bar 1}^{\chi_2} \right) =$$
$$Ind^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \left((V_{\bar 0}^*)^{\chi_1} \otimes (V_{\bar 1}^*)^{\chi_2} \otimes V_{\bar 0}^{\chi_1} \otimes V_{\bar 1}^{\chi_2} \right)$$

□

7.4. Proof of tehorem 1.7. i) Let F be a left-invariant functional on $\mathfrak{A}_{n.m}$ Lemma 7.3 implies that F determines a unique $\mathfrak{g}_{\bar 0}$-invariant functional on $\mathfrak{A}_{n,m}$ and the other way round. This proves the existence and uniqueness and, moreover, shows that any left-invariant functional is right-invariant with respect to $\mathfrak{g}_{\bar 0}$. This implies two-sided invariance.

Set

$$\omega_1 = \Delta_{\bar 0}^{-m} \prod_{i \in I_{\bar 0}, j \in I_{\bar 1}} x_{ij} \quad \omega_2 = \Delta_{\bar 1}^{-n} \prod_{i \in I_{\bar 1}, j \in I_{\bar 0}} x_{ij}$$

Then

$$1 = \left(\prod_{i \in I_{\bar 0}, j \in I_{\bar 1}} e_{ij} \prod_{i \in I_{\bar 0}, j \in I_{\bar 1}} e_{ij} \right)$$

Hence, $F(1) = 0$.

ii) and iii) are immediate corollaries of Lemma 2.4.6 □

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