THE q-DEFORMED ALGEBRA $U'_q(\mathrm{so}_n)$ RELATED TO MACDONALD SYMMETRIC POLYNOMIALS[1]

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Abstract

The aim of this paper is to give a review of our results on the nonstandard q-deformation $U'_q(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ of the Lie algebra $\mathrm{so}(n,\mathbb{C})$ which does not coincide with the Drinfeld–Jimbo quantum algebra $U_q(\mathrm{so}_n)$. It is exposed why this algebra appears. It is shown that finite dimensional irreducible representations of $U'_q(\mathrm{so}_n)$ separate elements of this algebra. Irreducible representations of the algebra $U'_q(\mathrm{so}_n)$ for q not a root of unity and for q a root of unity ($q^p = 1$) are given. The main class of representations in the last case acts on p^N-dimensional linear space (where N is a number of positive roots of the Lie algebra $\mathrm{so}(n,\mathbb{C})$) and are given by $r = \dim \mathrm{so}(n,\mathbb{C})$ complex parameters. The algebra $U'_q(\mathrm{so}_n)$ is related to Macdonald symmetric polynomials. Some consequenses of this relation for Macdonald polynomials are derived. The algebra $U'_q(\mathrm{so}_n)$ acts on the n-dimensional quantum vector space. This leads to the theory of harmonic polynomials on this space. Explicit expressions for q-analogue of the classical associated harmonic polynomials are given.

1. INTRODUCTION

Quantum orthogonal groups, quantum Lorentz groups and their corresponding quantized universal enveloping algebras are of special interest for modern mathematics and physics. M. Jimbo [1] and V. Drinfeld [2] defined q-deformations (quantized universal enveloping algebras) $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [3] and [4]). However, these approaches do not give a satisfactory presentation of the quantized algebra $U_q(\mathrm{so}(n,\mathbb{C}))$ from a viewpoint of some problems in mathematics and physics. When considering representations of the quantum groups $SO_q(n+1)$ and $SO_q(n,1)$ we are often interested in reducing them onto the quantum subgroup $SO_q(n)$. This reduction would give an analogue of the Gel'fand–Tsetlin basis for these representations. However, definitions of quantized universal enveloping algebras mentioned above do not allow the inclusions $U_q(\mathrm{so}(n+1,\mathbb{C})) \supset U_q(\mathrm{so}(n,\mathbb{C}))$ and $U_q(\mathrm{so}(n,1)) \supset U_q(\mathrm{so}(n))$. To be able to exploit such reductions we have to consider q-deformations of the algebra $U(\mathrm{so}(n,\mathbb{C}))$, when $\mathrm{so}(n,\mathbb{C})$ is defined in terms of the generators $I_{k,k-1} = E_{k,k-1} - E_{k-1,k}$ (where E_{is} is the matrix with elements $(E_{is})_{rt} = \delta_{ir}\delta_{st}$) rather than by means of the Cartan subalgebra and root elements. To construct such deformations we have to deform trilinear relations for elements $I_{k,k-1}$ instead of Serre's relations (used in the case of quantized universal enveloping algebras of Drinfeld and Jimbo). As a result, we obtain the associative algebra which will be denoted as $U'_q(\mathrm{so}_n)$.

This q-deformation was first constructed in [5]. It permits us to construct the reductions of $U'_q(\mathrm{so}_{n,1})$ and $U'_q(\mathrm{so}_{n+1})$ onto $U'_q(\mathrm{so}_n)$. The q-deformed algebra $U'_q(\mathrm{so}_n)$ leads for $n = 3$ to the

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q-deformed algebra $U_q'(\mathrm{so}_3)$ defined by D. Fairlie [6]. The cyclically symmetric algebra, similar to Fairlie's one, was also considered somewhat earlier by Odesskii [7]. The algebra $U_q'(\mathrm{so}_4)$ is a q-deformation of the algebra $U(\mathrm{so}(4,\mathbf{C}))$. In the case of the classical Lie algebra $\mathrm{so}(4,\mathbf{C})$ one has $\mathrm{so}(4,\mathbf{C}) = \mathrm{so}(3,\mathbf{C}) \oplus \mathrm{so}(3,\mathbf{C})$, while for our q-deformation $U_q'(\mathrm{so}_4)$ it is not a case (see, for example, [8]).

In the classical case, the imbedding $SO(n) \subset SU(n)$ (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian symmetric spaces. It is well known that in the framework of Drinfeld–Jimbo quantized universal enveloping algebras and the corresponding quantum groups one cannot construct such embedding. The algebra $U_q'(\mathrm{so}_n)$ allows to define such an embedding [9], that is, it is possible to define the embedding $U_q'(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$, where $U_q(\mathrm{sl}_n)$ is the Drinfeld–Jimbo quantized universal enveloping algebra.

As a disadvantage of the algebra $U_q'(\mathrm{so}_n)$ we have to mention the difficulties with Hopf algebra structure. Nevertheless, $U_q'(\mathrm{so}_n)$ turns out to be a coideal in $U_q(\mathrm{sl}_n)$ (see [9]) and this fact allows us to consider tensor products of finite dimensional irreducible representations of $U_q'(\mathrm{so}_n)$ for many interesting cases (see [10]).

Finite dimensional irreducible representations of the algebra $U_q'(\mathrm{so}_n)$ were constructed in [5]. The formulas of action of the generators of $U_q'(\mathrm{so}_n)$ upon the basis (which is a q-analogue of the Gel'fand–Tsetlin basis) are given there. A proof of these formulas and some their corrections were given in [11]. However, finite dimensional irreducible representations described in [5] and [11] are representations of the classical type. They are q-deformations of the corresponding irreducible representations of the Lie algebra $\mathrm{so}(n,\mathbb{C})$, that is, at $q \to 1$ they turn into representations of $\mathrm{so}(n,\mathbb{C})$.

The algebra $U_q'(\mathrm{so}_n)$ has other classes of finite dimensional irreducible representations which have no classical analogue. These representations are singular at the point $q = 1$. They are described in [12]. Note that the description of these representations for the algebra $U_q'(\mathrm{so}_3)$ is given in [13]. A classification of irreducible $*$-representations of real forms of the algebra $U_q'(\mathrm{so}_3)$ is given in [14].

Irreducible representations of the algebra $U_q'(\mathrm{so}_n)$ in the case when q is a root of unity were considered in the paper [15]. It is proved that in this case all irreducible representations of $U_q'(\mathrm{so}_n)$ are finite dimensional. In order to prove the corresponding theorem an analogue of the Poincaré–Birkhoff–Witt theorem for $U_q'(\mathrm{so}_n)$ and the description of central elements of this algebra for q a root of unity (given in [16]) were used. For construction of irreducible representations of $U_q'(\mathrm{so}_n)$ for q a root of unity, the method of D. Arnaudon and A. Chakrabarti [17] for construction of irreducible representations of the quantum algebra $U_q(\mathrm{sl}_n)$ when q is a root of unity was applied. As in the case of irreducible representations of the quantum algebra $U_q(\mathrm{sl}_n)$, it is difficult to enumerate all irreducible representations of $U_q'(\mathrm{so}_n)$ for q not a root of unity. Only the main classes of such representations were constructed.

It was shown in [9] that the q-analogue of the Riemannian symmetric space $SU(n)/SO(n)$, constructed by means of the quantum group $SU_q(n)$ and the algebra $U_q'(\mathrm{so}_n)$ (instead of the group $SO(n)$) leads to the quantized space of functions $\mathcal{F}_q(SU(n)/SO(n))$ whose zonal spherical functions are multiple to the Macdonald symmetric polynomials $P_m(x;q,t)$ with $t = q^{1/2}$.

It was shown (see [18] and [19]) that by using the algebra $U_q'(\mathrm{so}_n)$ and its representations we can construct a q-deformation of the theory of harmonic polynomials on the Euclidean space and as a result we obtain a theory of harmonic polynomials on the n-dimensional quantum vector space.

The aim of this preprint is to give a review of the results on the algebra $U_q'(\mathrm{so}_n)$, on its representations and on applications to Macdonald symmetric polynomials and harmonic polynomials on the quantum vector space.

2. The q-deformed algebra $U'_q(\mathrm{so}_n)$

The origin of existing a q-deformation of the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$, different from the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$, consists in the following. The Lia algebra $\mathrm{so}(n,\mathbb{C})$ has two structures:

(a) The structure related to existing in $\mathrm{so}(n,\mathbb{C})$ a Cartan subalgebra and root elements. A quantization of this structure leads to the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$.

(b) The structure related to realization of $\mathrm{so}(n,\mathbb{C})$ by skew-symmetric matrices. In $\mathrm{so}(n,\mathbb{C})$ there exists a basis consisting of the matrices I_{ij}, $i>j$, defined as $I_{ij}=E_{ij}-E_{ji}$, where E_{ij} is the matrix with entries $(E_{ij})_{rs}=\delta_{ir}\delta_{js}$. These matrices are not root elements.

Using the structure (b), we may say that the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C})0$ is generated by the elements I_{ij}, $i>j$. But in order to generate the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$, it is enough to take only the elements $I_{21}, I_{32},\cdots,I_{n,n-1}$. It is a minimal set of elements necessary for generating $U(\mathrm{so}(n,\mathbb{C}))$. These elements satisfy the relations

$$I_{i,i-1}^2 I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i},$$

$$I_{i,i-1}I_{i+1,i}^2 - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1},$$

$$I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j|>1.$$

The following theorem is true [20] for the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$.

Theorem 1. *The universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ is isomorphic to the complex associative algebra (with a unit element) generated by the elements $I_{21}, I_{32},\cdots,I_{n,n-1}$ satisfying the above relations.*

We make the q-deformation of these relations by fulfilling the deformation of the integer 2 in these relations as

$$2 \to [2] := (q^2-q^{-2})/(q-q^{-1}) = q+q^{-1}.$$

As a result, we obtain the complex unital (that is, with a unit element) associative algebra generated by elements $I_{21}, I_{32},\cdots,I_{n,n-1}$ satisfying the relations

$$I_{i,i-1}^2 I_{i+1,i} - (q+q^{-1})I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i}, \tag{1}$$

$$I_{i,i-1}I_{i+1,i}^2 - (q+q^{-1})I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \tag{2}$$

$$I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j|>1. \tag{3}$$

This algebra was introduced by us in [5] and is denoted by $U'_q(\mathrm{so}_n)$.

The analogue of the elements $I_{ij}, i>j$, can be introduced into $U'_q(\mathrm{so}_n)$ (see [18] and [21]). In order to give them we use the notation $I_{k,k-1}\equiv I^+_{k,k-1}\equiv I^-_{k,k-1}$. Then for $k>l+1$ we define recursively

$$I^+_{kl} := [I_{l+1,l}, I_{k,l+1}]_q \equiv q^{1/2}I_{l+1,l}I_{k,l+1} - q^{-1/2}I_{k,l+1}I_{l+1,l}, \tag{4}$$

$$I^-_{kl} := [I_{l+1,l}, I_{k,l+1}]_{q^{-1}} \equiv q^{-1/2}I_{l+1,l}I_{k,l+1} - q^{1/2}I_{k,l+1}I_{l+1,l}.$$

The elements I^+_{kl}, $k>l$, satisfy the commutation relations

$$[I^+_{ln}, I^+_{kl}]_q = I^+_{kn}, \quad [I^+_{kl}, I^+_{kn}]_q = I^+_{ln}, \quad [I^+_{kn}, I^+_{ln}]_q = I^+_{kl} \quad \text{for} \quad k>l>n, \tag{5}$$

$$[I^+_{kl}, I^+_{nr}] = 0 \quad \text{for} \quad k>l>n>r \ \text{ and } \ k>n>r>l, \tag{6}$$

$$[I^+_{kl}, I^+_{nr}]_q = (q-q^{-1})(I^+_{lr}I^+_{kn} - I^+_{kr}I^+_{nl}) \quad \text{for} \quad k>n>l>r. \tag{7}$$

For I_{kl}^-, $k > l$, the commutation relations are obtained from these relations by replacing I_{kl}^+ by I_{kl}^- and q by q^{-1}.

The algebra $U'_q(\mathrm{so}_n)$ can be defined as a unital associative algebra generated by I_{kl}^+, $1 \le l < k \le n$, satisfying the relations (5)–(7). In fact, using the relations (4) we can reduce the relations (5)–(7) to the relations (1)–(3) for $I_{21}, I_{32}, \cdots, I_{n,n-1}$.

The Poincaré–Birkhoff–Witt theorem for the algebra $U'_q(\mathrm{so}_n)$ can be formulated as follows (a proof of this theorem is given in [15]).

Theorem 2. *The elements*

$$I_{21}^{+\,m_{21}} I_{31}^{+\,m_{31}} \cdots I_{n1}^{+\,m_{n1}} I_{32}^{+\,m_{32}} I_{42}^{+\,m_{42}} \cdots I_{n2}^{+\,m_{n2}} \cdots I_{n,n-1}^{+\quad m_{n,n-1}}, \qquad m_{ij} = 0, 1, 2, \cdots, \tag{8}$$

form a basis of the algebra $U'_q(\mathrm{so}_n)$. This assertion is true if I_{ij}^+ are replaced by the corresponding elements I_{ij}^-.

Example 1. Let us consider the case of the algebra $U'_q(\mathrm{so}_3)$. It is generated by two elements I_{21} and I_{32}, satisfying the relations

$$I_{21}^2 I_{32} - (q - q^{-1}) I_{21} I_{32} I_{21} + I_{32} I_{21}^2 = -I_{32}, \tag{9}$$

$$I_{21} I_{32}^2 - (q + q^{-1}) I_{32} I_{21} I_{32} + I_{32}^2 I_{21} = -I_{21}. \tag{10}$$

Introducing the element $I_{31} = q^{1/2} I_{21} I_{32} - q^{-1/2} I_{32} I_{21}$ we have for the elements I_{21}, I_{32}, I_{31} the relations

$$[I_{21}, I_{32}]_q = I_{31}, \quad [I_{32}, I_{31}]_q = I_{21}, \quad [I_{31}, I_{21}]_q = I_{32}, \tag{11}$$

where the q-commutator $[\cdot,\cdot]_q$ is defined as $[A,B]_q = q^{1/2}AB - q^{-1/2}BA$. The algebra $U'_q(\mathrm{so}_3)$ can be defined as the associative algebra generated by the elements I_{21}, I_{32}, I_{31} satisfying relations (11).

Note that the algebra $U'_q(\mathrm{so}_3)$ has a big automorphism group. In fact, it is seen from (9) and (10) that these relations do not change if we permute I_{21} and I_{32}. From relations (11) we see that the set of these relations do not change under cyclical permutation of the elements I_{21}, I_{32}, I_{31}. The change of a sign at I_{21} or at I_{32} also does not change the relations (9) and (10). Generating by these automorphisms a group, we may find that they generate the group isomorphic to the modular group $SL(2,\mathbb{Z})$. It is why the algebra $U'_q(\mathrm{so}_3)$ is interesting for algebraic topology and algebraic geometry (see, for example, [22]–[24]).

Example 2. Let us consider the case of the algebra $U'_q(\mathrm{so}_4)$. It is generated by the elements I_{21}, I_{32} and I_{43}. We create the elements

$$I_{31} = [I_{21}, I_{32}]_q, \quad I_{42} = [I_{32}, I_{43}]_q, \quad I_{41} = [I_{21}, I_{42}]_q. \tag{12}$$

Then the elements I_{ij}, $i > j$, satisfy the following set of relations

$$[I_{21}, I_{32}]_q = I_{31}, \quad [I_{32}, I_{31}]_q = I_{21}, \quad [I_{31}, I_{21}]_q = I_{32}.$$

$$[I_{32}, I_{43}]_q = I_{42}, \quad [I_{43}, I_{42}]_q = I_{32}, \quad [I_{42}, I_{32}]_q = I_{43}.$$

$$[I_{31}, I_{43}]_q = I_{41}, \quad [I_{43}, I_{41}]_q = I_{31}, \quad [I_{41}, I_{31}]_q = I_{43}.$$

$$[I_{21}, I_{42}]_q = I_{41}, \quad [I_{42}, I_{41}]_q = I_{21}, \quad [I_{41}, I_{21}]_q = I_{42}.$$

$$[I_{21}, I_{43}] = 0, \quad [I_{32}, I_{41}] = 0, \quad [I_{42}, I_{31}] = (q - q^{-1})(I_{21}I_{43} - I_{32}I_{41}). \tag{13}$$

At $q = 1$ these relations define just the Lie algebra $\mathrm{so}(4,\mathbb{C})$. Each of the sets (I_{21}, I_{32}, I_{31}), (I_{32}, I_{43}, I_{42}), (I_{31}, I_{43}, I_{41}), (I_{21}, I_{42}, I_{41}) determine a subalgebra isomorphic to $U'_q(\mathrm{so}_3)$.

3. The isomorphism $U'_q(\mathrm{so}_n) \to U_q(\mathrm{sl}_n)$

The algebra $U'_q(\mathrm{so}_n)$ can be embedded into the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{sl}_n)$ (see [9]). The last algebra is generated by the elements E_i, F_i, $K_i^{\pm 1} \equiv q^{\pm H_i}$, $i = 1, 2, \cdots, n-1$, satisfying the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j,$$

$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$E_i^2E_{i\pm 1} - (q + q^{-1})E_iE_{i\pm 1}E_i + E_{i\pm 1}E_i^2 = 0,$$

$$F_i^2F_{i\pm 1} - (q + q^{-1})F_iF_{i\pm 1}F_i + F_{i\pm 1}F_i^2 = 0,$$

$$[E_i, E_j] = 0, \quad [F_i, F_j] = 0 \quad \text{for} \quad |i - j| > 1,$$

where a_{ij} are elements of the Cartan matrix of the Lie algebra $\mathrm{sl}(n, \mathbb{C})$.

In order to prove Theorem 3 below, we note that there exists a one-to-one correspondence between the basis elements of the algebra $U'_q(\mathrm{so}_n)$ from Theorem 2 and the basis elements of the subalgebra $\mathfrak{N}^-$ of the quantum algebra $U_q(\mathrm{sl}_n)$, generated by $F_1, F_2, \cdots, F_{n-1}$. The last basis elements are constructed by means of the following ordering of positive roots of the Lie algebra $\mathrm{sl}(n, \mathbb{C})$:

$$\beta_{12}, \beta_{13}, \cdots, \beta_{1n}, \beta_{23}, \cdots, \beta_{2n}, \cdots, \beta_{n-2,n-1}, \beta_{n-2,n}, \beta_{n-1,n}, \tag{14}$$

where $\beta_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ and α_k are simple roots. (This ordering is the same as in Theorem 2.) To every root of this set there corresponds the element $F_{\beta_{ij}} \in \mathfrak{N}^-$ (see, for example, [4]). Then according to the Poincaré–Birkhoff–Witt theorem for the algebra $\mathfrak{N}^-$ (see, [4], subsection 6.2.3) the elements

$$F_{\beta_{12}}^{m_{12}}F_{\beta_{13}}^{m_{13}} \cdots F_{\beta_{1n}}^{m_{1n}} \cdots F_{\beta_{n-1,n}}^{m_{n-1,n}}, \qquad m_{ij} = 0, 1, 2, \cdots, \tag{15}$$

(the order of β_{ij} is the same as in (14)) form a basis of $\mathfrak{N}^-$. Then the mapping

$$F_{\beta_{12}}^{m_{12}}F_{\beta_{13}}^{m_{13}} \cdots F_{\beta_{n-1,n}}^{m_{n-1,n}} \to {I_{21}^-}^{m_{12}}{I_{31}^-}^{m_{13}} \cdots {I_{n,n-1}^-}^{m_{n-1,n}} \tag{16}$$

is the one-to-one correspondence between basis elements in $\mathfrak{N}^-$ and in $U'_q(\mathrm{so}_n)$ which will be denoted by $\mathcal{T}$.

Similarly, to every root β_{ij} from (14) there corresponds the element $E_{\beta_{ij}}$ of the subalgebra $\mathfrak{N}^+ \subset U_q(\mathrm{sl}_n)$, generated by $E_1, E_2, \cdots, E_{n-1}$. The elements

$$E_{\beta_{12}}^{m_{12}}E_{\beta_{13}}^{m_{13}} \cdots E_{\beta_{1n}}^{m_{1n}} \cdots E_{\beta_{n-1,n}}^{m_{n-1,n}}, \qquad m_{ij} = 0, 1, 2, \cdots,$$

(the order of β_{ij} is the same as in (14)) form a basis of $\mathfrak{N}^+$.

The formulas

$$\deg\,(F_{\beta_{12}}^{m_{12}}F_{\beta_{13}}^{m_{13}} \cdots F_{\beta_{n-1,n}}^{m_{n-1,n}}) = -(m_{12}\beta_{12} + m_{13}\beta_{13} + \cdots m_{n-1,n}\beta_{n-1,n}),$$

$$\deg\,(E_{\beta_{12}}^{m_{12}}E_{\beta_{13}}^{m_{13}} \cdots E_{\beta_{n-1,n}}^{m_{n-1,n}}) = m_{12}\beta_{12} + m_{13}\beta_{13} + \cdots m_{n-1,n}\beta_{n-1,n},$$

$$\deg\,(H_1^{m_1} \cdots H_{n-1}^{m_{n-1}}) = 0$$

establish a gradation in $U_q(\mathrm{sl}_n)$ (see [4], subsection 6.1.5).

Let us introduce the elements

$$\tilde{I}_{j,j-1} = F_{j-1} - qq^{-H_{j-1}}E_{j-1}, \quad j = 2, 3, \cdots, n,$$

of $U_q(\mathrm{sl}_n)$. It is proved in [9] that there exists the algebra homomorphism $\varphi : U_q'(\mathrm{so}_n) \to U_q(\mathrm{sl}_n)$ uniquely determined by the relations $\varphi(I_{i+1,i}) = \tilde{I}_{i+1,i}$, $i = 1, 2, \cdots, n-1$. The following theorem states that this homomorphism is an isomorphism.

Theorem 3. *The homomorphism $\varphi : U_q'(\mathrm{so}_n) \to U_q(\mathrm{sl}_n)$ determined by the relations $\varphi(I_{i+1,i}) = \tilde{I}_{i+1,i}$, $i = 1, 2, \cdots, n-1$, is an isomorphism of $U_q'(\mathrm{so}_n)$ to $U_q(\mathrm{sl}_n)$.*

Proof. In [18] the authors of that paper state that this homomorphism is an isomorphism and say that it can be proved by means of the Diamond Lemma. However, we could not restore their proof and found another one. It use the above Poincaré–Birkhoff–Witt theorem for the algebra $U_q'(\mathrm{so}_n)$. Namely, we use the explicit expressions from [18] for the elements $\tilde{I}_{ij} \equiv \varphi(I_{ij}) \in U_q(\mathrm{sl}_n)$ in terms of the elements of the L-functionals of the quantum algebra $U_q(\mathrm{sl}_n)$:

$$\tilde{I}_{ji} = (q - q^{-1})^{-1} c_i K_{ji}^-, \quad j > i, \tag{17}$$

where c_i is equal to q^s with an appropriate $s \in \mathbb{Z}$ and

$$K^- \equiv (K_{ji}^-)_{i,j=1}^n = (L^+)^t J L^-. \tag{18}$$

Here $J = \mathrm{diag}\,(q^{n-1}, q^{n-2}, \cdots, 1)$ and explicit expressions for matrix elements l_{ij}^+ and l_{ij}^- of the matrices L^+ and L^- are given by formulas from [9] (see also [4], subsection 8.5.2). In particular, $l_{ij}^+ = l_{ji}^- = 0$ if $i > j$ and l_{ij}^+ (resp. l_{ji}^-) is expressed in terms of $E_{\beta_{ij}}$ (in terms of $F_{\beta_{ij}}$) if $i < j$. We have

$$\deg l_{ij}^+ = \beta_{ij}, \quad \deg l_{ij}^- = -\beta_{ij}.$$

The elements $l_{jj}^{\pm}$ belong to the subalgebra $\mathfrak{H}$ generated by $K_1, K_2, \cdots, K_{n-1}$. By (18) for $j > i$ we obtain

$$K_{ji}^- = \sum_{s=i}^{j} c_s' l_{sj}^+ l_{si}^-, \quad j > i, \tag{19}$$

where $c_s' = (q - q^{-1})q^r$ with an appropriate $r \in \mathbb{Z}$. The summands in (19) have different degrees and the lowest degree has the only summand $c_j' l_{jj}^+ l_{ji}^-$.

Let a be a basis element $I_{21}^{-\,m_{21}} I_{31}^{-\,m_{31}} \cdots I_{n,n-1}^{-\,m_{n,n-1}}$ of the algebra $U_q'(\mathrm{so}_n)$ from Theorem 2. Then

$$\varphi(a) = (\tilde{I}_{21}^-)^{m_{21}} (\tilde{I}_{31}^-)^{m_{31}} \cdots (\tilde{I}_{n,n-1}^-)^{m_{n,n-1}}.$$

Substituting here expressions for $\tilde{I}_{ji}^-$ from (17) and (19), we obtain $\varphi(a)$ in form of a sum with a single summand of the lowest degree. This summand of lowest degree is $c' F_{\beta_{21}}^{m_{21}} F_{\beta_{31}}^{m_{31}} \cdots F_{\beta_{n,n-1}}^{m_{n,n-1}}$ with nonvanishing coefficient c'. The expression at c' is just the basis element of $\mathfrak{N}^- \subset U_q(\mathrm{sl}_n)$ corresponding under the mapping $\mathcal{T}$ to the basis element a of $U_q'(\mathrm{so}_n)$.

Similarly, if $a \in U_q'(\mathrm{so}_n)$ is a linear combination of the basis elements $I_{21}^{-\,m_{21}} I_{31}^{-\,m_{31}} \cdots I_{n,n-1}^{-\,m_{n,n-1}}$ from Theorem 2, then we substitute into $\varphi(a)$ expressions (19) for each K_{ji}^-. As a result, we express $\varphi(a)$ in form of a sum, containing the same linear combination of products $c' F_{\beta_{21}}^{m_{21}} F_{\beta_{31}}^{m_{31}} \cdots F_{\beta_{n,n-1}}^{m_{n,n-1}}$. This linear combination contains a subsum of a (fixed) lowest degree and this subsum cannot be concelled with other summands in $\varphi(a)$. Therefore, $\varphi(a) \neq 0$ and φ is an isomorphism from $U_q'(\mathrm{so}_n)$ to $U_q(\mathrm{sl}_n)$. Theorem is proved.

This theorem has the following important corollary formulated in [15].

Corollary. *Finite dimensional irreducible representations of $U'_q(\mathrm{so}_n)$ separate elements of this algebra, that is, for any $a \in U'_q(\mathrm{so}_n)$ there exists a finite dimensional irreducible representation T of $U'_q(\mathrm{so}_n)$ such that $T(a) \neq 0$.*

Proof. If q is not a root of unity, then the assertion of the theorem follows from Theorem 3 and from the theorem on separation of elements of the algebra $U_q(\mathrm{sl}_n)$ by its representations (see subsection 7.1.5 in [4]) if to take into account the fact (proved in [18]) that a restriction of any finite dimensional irreducible representation of $U_q(\mathrm{sl}_n)$ onto the subalgebra $U'_q(\mathrm{so}_n)$ decomposes into a direct sum of its irreducible representations.

Let now q be a root of unity, that is, $q^k = 1$. We denote by N a positive integer such that every irreducible representation of the quantum algebra $U_q(\mathrm{sl}_n)$ has dimension less than N (see section 8 below). Let a be any nonvanishing element of $U'_q(\mathrm{so}_n)$. Then there exists an irreducible representation T of $U_q(\mathrm{sl}_n)$ such that $T(\varphi(a^N)) \neq 0$. (Note that $a^N \neq 0$ since $U_q(\mathrm{sl}_n)$ has no divisors of zero.) Let $\tilde{T}$ be the restriction of T to the subalgebra $U'_q(\mathrm{so}_n)$. Then $\tilde{T}$ is reducible. For simplicity we suppose that $\tilde{T}$ contains only two irreducible representations of $U'_q(\mathrm{so}_n)$. (If $\tilde{T}$ contains more irreducible constituents, then the proof is the same as for two ones.) Generally speaking, $\tilde{T}$ is not completely reducible, that is in some basis the representation $\tilde{T}$ is of the form

$$\begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix},$$

where T_1 and T_2 are irreducible representations of $U'_q(\mathrm{so}_n)$. Since $\tilde{T}(a^N) \neq 0$, then $\tilde{T}(a) \neq 0$ and

$$\tilde{T}(a) = \begin{pmatrix} T_1(a) & * \\ 0 & T_2(a) \end{pmatrix}.$$

If $T_1(a) \neq 0$ or $T_2(a) \neq 0$, then irreducible representations of $U'_q(\mathrm{so}_n)$ separate the element a. Let $T_1(a) = 0$ and $T_2(a) = 0$. Then

$$\tilde{T}(a) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

In this case $\tilde{T}(a)$ is a nilpotent matrix and $\tilde{T}(a)^N = \tilde{T}(a^N) = 0$. This contradict the assupmtion $\tilde{T}(a^N) \neq 0$. Therefore, the case $T_1(a) = 0$ and $T_2(a) = 0$ is not possible. Corollary is proved.

In fact, a more strong assertion can be proved when q is not a root of unity (see [10]):

Proposition 1. *If q is not a root of unity, then finite dimensional irreducible representations of the classical type separate elements of $U'_q(\mathrm{so}_n)$.*

Note that irreducible representations of the classical type are defined below.

4. Other properties of the algebra $U'_q(\mathrm{so}_n)$

Let give some other properties of the algebra $U'_q(\mathrm{so}_n)$. In this algebra can be separated a maximal commutative subalgebra generated by the elements $I_{21}, I_{43}, I_{65}, \cdots, I_{n-1,n-2}$ (or $I_{n,n-1}$). So, this subalgebra is generated by $\lfloor n/2 \rfloor$ elements, where $\lfloor n/2 \rfloor$ is an integral part of the number $n/2$. However, there exist no root elements in the algebra $U'_q(\mathrm{so}_n)$ with respect to this commutative subalgebra. This leads to the fact that properties of $U'_q(\mathrm{so}_n)$ are not similar to those of the Drinfeld–Jimbo algebra $U_q(\mathrm{so}_n)$.

The algebra $U'_q(\mathrm{so}_n)$ has $\lfloor n/2 \rfloor$ independent central elements [18]. They can be construced as follows. We form the elements

$$J^{\pm}_{k_1 k_2 \cdots k_{2r}} = q^{\mp\{r(r-1)\}/2} \sum_{s \in S_{2r}}{}' \epsilon_{q^{\pm 1}}(s) I^{\pm}_{k_{s(2)} k_{s(1)}} I^{\pm}_{k_{s(4)} k_{s(3)}} \cdots I^{\pm}_{k_{s(2r)} k_{s(2r-1)}},$$

of the algebra $U'_q(\mathrm{so}_n)$, where $1 < k_1 < k_2 < \cdots < k_{2r} \le n$ and summation runs over all permutations s of indices $k_1, k_2, \cdots, k_{2r}$ such that

$$k_{s(2)} > k_{s(1)}, \quad k_{s(4)} > k_{s(3)}, \quad ,\cdots, k_{s(2r)} > k_{s(2r-1)}, \qquad k_{s(2)} < k_{s(4)} < \cdots < k_{s(2r)}.$$

The symbol $\epsilon_{q^{\pm 1}}(s)$ is defined as follows: $\epsilon_{q^{\pm 1}}(s) = (-q^{\pm})^{\ell(s)}$, where $\ell(s)$ is the length of the permutation s. Then the elements

$$C_n^{(2r)} = \sum_{1\le k_1<k_2<\cdots<k_{2r}\le n} J^+_{k_1k_2\cdots k_{2r}} J^-_{k_1k_2\cdots k_{2r}},$$

where $r = 1, 2, \cdots \lfloor n/2 \rfloor$, are independent central elements of $U'_q(\mathrm{so}_n)$. If n is even, then the elements $J^+_{1,2,\cdots,n}$ and $J^-_{1,2,\cdots,n}$ also belongs to the center of $U'_q(\mathrm{so}_n)$.

If q is a root of unity, then there exist aditional central elements in $U'_q(\mathrm{so}_n)$. They are described below.

The algebra $U'_q(\mathrm{so}_n)$ acts on the n-dimensional quantum vector space. This space is defined as follows. Let

$$\mathcal{A} \equiv \mathbb{C}[x_1, x_2, \cdots, x_n]$$

be the associative algebra (with unity) generated by elements $x_1, x_2, \cdots, x_n$ satisfying the defining relations

$$x_i x_j = q x_j x_i, \qquad i < j.$$

This algebra is called the algebra of functions on the n-dimensional quantum vector space. Elements of $\mathcal{A}$ are called polynomials on this quantum vector space and are denoted by $p \equiv p(x_1, x_2, \cdots, x_n) \equiv p(\mathbf{x})$. The elements $x_1, x_2, \cdots, x_n$ are called quantum coordinates on the quantum vector space.

We define on $\mathcal{A}$ the q-differentiations ∂_i and ∂'_i which are linear operators acting as $\partial_i p = \partial'_i p = 0$ on monomials p, not contaning x_i, and as

$$\partial_i = \check{x}_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}, \qquad \partial'_i = \hat{x}_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}} \tag{20}$$

on monomials containing x_i, where $\hat{x}_i$ and $\check{x}_i$ are the operators of left and right multiplication by x_i, respectively, and

$$\gamma_i p(x_1, \cdots, x_n) = p(x_1, \cdots, x_{i-1}, q x_i, x_{i+1}, \cdots, x_n).$$

$$\gamma_i^{-1} p(x_1, \cdots, x_n) = p(x_1, \cdots, x_{i-1}, q^{-1} x_i, x_{i+1}, \cdots, x_n).$$

We have $\partial_i \hat{x}_j = \hat{x}_j \partial_i$, $i \ne j$, and

$$\partial_i \partial_j = q^{-1} \partial_j \partial_i, \quad \partial_i \check{x}_j = q \check{x}_j \partial_i, \quad i < j, \quad \gamma_i \hat{x}_j = q^{\delta_{ij}} \hat{x}_j \gamma_i, \quad \gamma_i \partial_j = q^{-\delta_{ij}} \partial_j \gamma_i.$$

A q-analogue of the Fischer scalar product is defined on $\mathcal{A}$ (see [5]). It is given as

$$\langle p_1, p_2 \rangle = p_1(\partial'_1, \cdots, \partial'_n) p_2^* |_{x=0}, \tag{21}$$

where p_2^* is the polynomial p_2 in which numerical coefficients are replaced by complex conjugate ones, $p_1(\partial'_1, \cdots, \partial'_n)$ means the q-differential operator obtained from the polynomial p by replacement of x_i by ∂'_i, $i = 1, 2, \cdots, n$, and the symbol $p|_{x=0}$ means a constant term of the polynomial p.

The action of the algebra $U'_q(\mathrm{so}_n)$ on the space $\mathcal{A}$ is determied by the formula

$$I_{j+1,j} = \check{x}_{j+1} \gamma_j^{-1} \partial_j - \check{x}_j \gamma_{j+1} \partial_{j+1} \tag{22}$$

(see [18] and [19]). At $q = 1$ it turns into well known formula of action of the Lie algebra so(n) on the Euclidean space.

5. Irreducible finite dimensional representations

The algebra $U'_q(\mathrm{so}_n)$ has two types of irreducible finite dimensional representations:

(a) representations of the classical type;
(b) representations of the nonclassical type.

Irreducible representations of the classical type are q-deformations of the irreducible finite dimensional representations of the Lie algebra so(n). So, there is a one-to-one correspondence between irreducible representations of the classical type of the algebra $U'_q(\mathrm{so}_n)$ and irreducible finite dimensional representations of the Lie algebra so(n). Moreover, formulas for representations of the classical type of $U'_q(\mathrm{so}_n)$ turn into the corresponding formulas for the representations of Lie algebra so(n) at $q \to 1$.

There exists no classical analogue for representations of the nonclassical type. Operators $T(a)$, $a \in U'_q(\mathrm{so}_n)$, have singularities at $q = 1$. Let us describe irreducible finite dimensional representations of both types.

6. Irreducible representations of the classical types

In this section we describe (in the framework of the q-analogue of Gel'fand–Tsetlin formalism) irreducible finite dimensional representations of the algebras $U'_q(\mathrm{so}_n)$, $n \geq 3$, which are q-deformations of the finite dimensional irreducible representations of the Lie algebra so(n). They are given by sets $\mathbf{m}_n$ of $\lfloor n/2 \rfloor$ numbers $m_{1,n}, m_{2,n}, ..., m_{\lfloor n/2 \rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes integral part of $n/2$) which are all integral or all half-integral and satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq ... \geq m_{p,2p+1} \geq 0,$$

$$m_{1,2p} \geq m_{2,2p} \geq ... \geq m_{p-1,2p} \geq |m_{p,2p}|$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. We take the q-analogue of the Gel'fand–Tsetlin basis in the representation space, which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U'_q(\mathrm{so}_{n-1})$, $U'_q(\mathrm{so}_{n-2})$, $\cdots$, $U'_q(\mathrm{so}_3)$, $U'_q(\mathrm{so}_2) := U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by Gel'fand–Tsetlin tableaux

$$\{\xi_n\} \equiv \left\{ \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{matrix} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}, \qquad (23)$$

where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq ... \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1},$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq ... \geq m_{p-1,2p-1} \geq |m_{p,2p}|.$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\{\xi_n\}\rangle$ or simply as $|\xi_n\rangle$.

It is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j,$$

for the numbers $m_{i,k}$. In particular, $l_{1,3} = m_{1,3} + 1$ and $l_{1,2} = m_{1,2}$. The operator $T_{\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\mathbf{m}_n}$ of $U_q'(\mathrm{so}_n)$ acts upon Gel'fand–Tsetlin basis elements, labelled by (23), by the formula

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} \frac{A_{2p}^j(\xi_n)}{q^{l_{j,2p}} + q^{-l_{j,2p}}}|(\xi_n)_{2p}^{+j}\rangle - \sum_{j=1}^{p} \frac{A_{2p}^j((\xi_n)_{2p}^{-j})}{q^{l_{j,2p}} + q^{-l_{j,2p}}}|(\xi_n)_{2p}^{-j}\rangle \tag{24}$$

and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B_{2p-1}^j(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1}]}|(\xi_n)_{2p-1}^{+j}\rangle$$

$$-\sum_{j=1}^{p-1} \frac{B_{2p-1}^j((\xi_n)_{2p-1}^{-j})}{[2l_{j,2p-1} - 1][l_{j,2p-1} - 1]}|(\xi_n)_{2p-1}^{-j}\rangle + \mathrm{i}\, C_{2p-1}(\xi_n)|\xi_n\rangle. \tag{25}$$

In these formulas, $(\xi_n)_k^{\pm j}$ means the tableau (6) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. The coefficients A_{2p}^j, B_{2p-1}^j, C_{2p-1} in (10) and (11) are given by the expressions

$$A_{2p}^j(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p+1} + l_{j,2p}][l_{i,2p+1} - l_{j,2p} - 1] \prod_{i=1}^{p-1}[l_{i,2p-1} + l_{j,2p}][l_{i,2p-1} - l_{j,2p} - 1]}{\prod_{i\neq j}^{p}[l_{i,2p} + l_{j,2p}][l_{i,2p} - l_{j,2p}][l_{i,2p} + l_{j,2p} + 1][l_{i,2p} - l_{j,2p} - 1]} \right)^{1/2}, \tag{26}$$

and

$$B_{2p-1}^j(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p} + l_{j,2p-1}][l_{i,2p} - l_{j,2p-1}] \prod_{i=1}^{p-1}[l_{i,2p-2} + l_{j,2p-1}][l_{i,2p-2} - l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}][l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]} \right)^{1/2}, \tag{27}$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}] \prod_{s=1}^{p-1}[l_{s,2p-2}]}{\prod_{s=1}^{p-1}[l_{s,2p-1}][l_{s,2p-1} - 1]}, \tag{28}$$

where numbers in square brackets mean q-numbers defined by

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}}.$$

In particular,

$$T_{\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{m_{1,2}} + q^{-m_{1,2}}}(([l_{1,3} + m_{1,2}][l_{1,3} - m_{1,2} - 1])^{1/2}|(\xi_n)_2^{+1}\rangle -$$

$$-([l_{1,3} + m_{1,2} - 1][l_{1,3} - m_{1,2}])^{1/2}|(\xi_n)_2^{-1}\rangle),$$

$$T_{\mathbf{m}_n}(I_{2,1})|\xi_n\rangle = \mathrm{i}[m_{1,2}]|\xi_n\rangle,$$

It is seen from formula (28) that the coefficient C_{2p-1} vanishes if $m_{p,2p} \equiv l_{p,2p} = 0$.

A proof of the fact that formulas (24)-(28) indeed determine a representation of $U_q'(\mathrm{so}_n)$ is given in [11].

Theorem 4. *The representations $T_{\mathbf{m}_n}$ are irreducible. The representations $T_{\mathbf{m}_n}$ and $T_{\mathbf{m}_n'}$ are pairwise nonequivalent for $\mathbf{m}_n \neq \mathbf{m}_n'$.*

Example 3. Irreducible representations of the classical type of the algebra $U_q'(\mathrm{so}_3)$ are given by nonnegative integral or half-integral number l and act on vector spaces $\mathcal{H}_l$ with a basis $|l, m\rangle$,

$m = -l, -l+1, \cdots, l$. We denote these representations by T_l. For the operators $T_l(I_{21})$ and $T_l(I_{32})$ we have the formulas $T_l(I_{21})l, m\rangle = \mathrm{i}[m]|l, m\rangle$ and

$$T_l(I_{32})l, m\rangle = \frac{1}{q^m + q^{-m}}([l-m]|l, m+1\rangle - [l+m]l, m-1\rangle),$$

where $[a]$ denotes a q-number.

7. Irreducible representations of the nonclassical types

Irreducible finite dimensional representations of the nonclassical type are given by sets $\epsilon := (\epsilon_2, \epsilon_3, \cdots, \epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $\mathbf{m}_n$ consisting of $\lfloor n/2 \rfloor$ **half-integral** numbers $m_{1,n}, m_{2,n}, \cdots, m_{\lfloor n/2 \rfloor, n}$ that satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq \ldots \geq m_{p,2p+1} \geq 1/2,$$

$$m_{1,2p} \geq m_{2,2p} \geq \ldots \geq m_{p-1,2p} \geq m_{p,2p} \geq 1/2$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\epsilon,\mathbf{m}_n}$.

For a basis in the representation space we use the analogue of the basis of the previous section. Its elements are labelled by tableaux

$$\{\xi_n\} \equiv \left\{ \begin{array}{c} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{array} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}, \tag{29}$$

where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \ldots \geq m_{p,2p+1} \geq m_{p,2p} \geq 1/2,$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \ldots \geq m_{p-1,2p-1} \geq m_{p,2p}.$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$.

As in the previous section, it is convenient to introduce the l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j.$$

The operator $T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,\mathbf{m}_n}$ of $U_q(\mathrm{so}_n)$ acts upon our basis elements, labelled by (29), by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \delta_{m_{p,2p},1/2} \frac{\epsilon_{2p+1}}{q^{1/2} - q^{-1/2}} D_{2p}(\xi_n)|\xi_n\rangle +$$

$$+ \sum_{j=1}^{p} \frac{A_{2p}^j(\xi_n)}{q^{l_{j,2p}} - q^{-l_{j,2p}}} |(\xi_n)_{2p}^{+j}\rangle - \sum_{j=1}^{p} \frac{A_{2p}^j((\xi_n)_{2p}^{-j})}{q^{l_{j,2p}} - q^{-l_{j,2p}}} |(\xi_n)_{2p}^{-j}\rangle, \tag{30}$$

where the summation in the last sum must be from 1 to $p-1$ if $m_{p,2p} = 1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\epsilon,\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B_{2p-1}^j(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1}]_+} |(\xi_n)_{2p-1}^{+j}\rangle -$$

$$- \sum_{j=1}^{p-1} \frac{B_{2p-1}^j((\xi_n)_{2p-1}^{-j})}{[2l_{j,2p-1} - 1][l_{j,2p-1} - 1]_+} |(\xi_n)_{2p-1}^{-j}\rangle + \epsilon_{2p} \hat{C}_{2p-1}(\xi_n)|\xi_n\rangle, \tag{31}$$

where $[a]_+ = (q^a + q^{-a})/(q - q^{-1})$. In these formulas, $(\xi_n)_k^{\pm j}$ means the tableau (29) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. Matrix elements A_{2p}^j and B_{2p-1}^j are given by the same formulas as in (24) and (25) (that is, by the formulas (26) and (27)) and

$$\hat{C}_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]_+ \prod_{s=1}^{p-1}[l_{s,2p-2}]_+}{\prod_{s=1}^{p-1}[l_{s,2p-1}]_+[l_{s,2p-1}-1]_+}, \qquad D_{2p}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p+1}-\frac12]\prod_{i=1}^{p-1}[l_{i,2p-1}-\frac12]}{\prod_{i=1}^{p-1}[l_{i,2p}+\frac12][l_{i,2p}-\frac12]}.$$

For the operators $T_{\epsilon,\mathbf{m}_n}(I_{3,2})$ and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})$ we have

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{m_{1,2}} - q^{-m_{1,2}}}(([l_{1,3}+m_{1,2}][l_{1,3}-m_{1,2}-1])^{1/2}|(\xi_n)_2^{+1}\rangle -$$

$$-([l_{1,3}+m_{1,2}-1][l_{1,3}-m_{1,2}])^{1/2}|(\xi_n)_2^{-1}\rangle)$$

if $m_{1,2} \neq \frac12$,

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{1/2} - q^{-1/2}}(\epsilon_3[l_{1,3}-1/2]|(\xi_n)\rangle + ([l_{1,3}+1/2][l_{1,3}-3/2])^{1/2}|(\xi_n)_2^{+1}\rangle)$$

if $m_{1,2} = \frac12$, and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})|\xi_n\rangle = \epsilon_2[m_{1,2}]_+|\xi_n\rangle$.

The fact that the above operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U_q'(\mathrm{so}_n)$ is proved in the following way. We take the formulas (25)–(28) for the classical type representations $T_{\mathbf{m}_n}$ of $U_q'(\mathrm{so}_n)$ with half-integral $m_{i,n}$ and replace there every $m_{j,2p+1}$ by $m_{j,2p+1} - \mathrm{i}\pi/2h$, every $m_{j,2p}$, $j \neq p$, by $m_{j,2p} - \mathrm{i}\pi/2h$ and $m_{p,2p}$ by $m_{p,2p} - \epsilon_2\epsilon_4\cdots\epsilon_{2p}\mathrm{i}\pi/2h$, where each ϵ_{2s} is equal to $+1$ or -1 and h is defined by $q = e^h$. Repeating almost word by word the reasoning of the paper [11], we prove that the operators given by formulas (25)–(28) satisfy the defining relations (1)–(3) of the algebra $U_q'(\mathrm{so}_n)$ after this replacement. Therefore, these operators determine a representation of $U_q'(\mathrm{so}_n)$. We denote this representation by $T'_{\mathbf{m}_n}$. After a simple rescaling, the operators $T'_{\mathbf{m}_n}(I_{k,k-1})$ take the form

$$T'_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} \frac{A_{2p}^j(\xi_n)}{q^{l_{j,2p}} - q^{-l_{j,2p}}}|(\xi_n)_{2p}^{+j}\rangle - \sum_{j=1}^{p} \frac{A_{2p}^j((\xi_n)_{2p}^{-j})}{q^{l_{j,2p}} - q^{-l_{j,2p}}}|(\xi_n)_{2p}^{-j}\rangle,$$

$$T'_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B_{2p-1}^j(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]_+}|(\xi_n)_{2p-1}^{+j}\rangle -$$

$$-\sum_{j=1}^{p-1} \frac{B_{2p-1}^j((\xi_n)_{2p-1}^{-j})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]_+}|(\xi_n)_{2p-1}^{-j}\rangle + \epsilon_{2p}\hat{C}_{2p-1}(\xi_n)|\xi_n\rangle,$$

where A_{2p}^j, B_{2p-1}^j and $\hat{C}_{2p-1}$ are such as in the formulas (30) and (31). The representations $T'_{\mathbf{m}_n}$ are reducible. We decompose these representations into subrepresentations in the following way. We fix p ($p = 1, 2, \cdots, \lfloor (n-1)/2 \rfloor$) and decompose the space $\mathcal{H}$ of the representation $T'_{\mathbf{m}_n}$ into direct sum of two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$, $\epsilon_{2p+1} = \pm 1$, spanned by the basis vectors

$$|\xi_n\rangle_{\epsilon_{2p+1}} = |\xi_n\rangle - \epsilon_{2p+1}|\xi_n'\rangle, \qquad m_{p,2p} \geq 1/2,$$

respectively, where $|\xi_n'\rangle$ is obtained from $|\xi_n\rangle$ by replacement of $m_{p,2p}$ by $-m_{p,2p}$. A direct verification shows that two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ are invariant with respect to all the operators $T'_{\mathbf{m}_n}(I_{k,k-1})$. Now we take the subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ and repeat the same procedure for some s, $s \neq p$, and decompose each of these subspaces into two invariant subspaces. Continueing this procedure further we decompose the representation space $\mathcal{H}$ into a direct sum of $2^{\lfloor (n-1)/2 \rfloor}$ invariant subspaces. The operators $T'_{\mathbf{m}_n}(I_{k,k-1})$ act upon these subspaces by the formulas (30) and (31). We denote the corresponding subrepresentations on these subspaces by $T_{\epsilon,\mathbf{m}_n}$. The above reasoning shows that the operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$.

Theorem 5. *The representations $T_{\epsilon,\mathbf{m}_n}$ are irreducible. The representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon',\mathbf{m}'_n}$ are pairwise nonequivalent for $(\epsilon,\mathbf{m}_n)\neq(\epsilon',\mathbf{m}'_n)$. For any admissable $(\epsilon,\mathbf{m}_n)$ and $\mathbf{m}'_n$ the representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent.*

The algebra $U'_q(\mathrm{so}_n)$ has non-trivial one-dimensional representations. They are special cases of the representations of the nonclassical type. They are described as follows.

Let $\epsilon := (\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i=\pm1$, and let $\mathbf{m}_n=(m_{1,n},m_{2,n},\cdots,m_{\lfloor n/2\rfloor,n})=(\frac12,\frac12,\cdots,\frac12)$. Then the corresponding representations $T_{\epsilon,\mathbf{m}_n}$ are one-dimensional and are given by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{k+1,k})|\xi_n\rangle=\frac{\epsilon_{k+1}}{q^{1/2}-q^{-1/2}}.$$

Thus, to every $\epsilon := (\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i=\pm1$, there corresponds a one-dimensional representation of $U'_q(\mathrm{so}_n)$.

Example 4. Let us describe irreducible representations of the nonclassical type of the algebra $U'_q(\mathrm{so}_3)$ These representations are given by numbers m,ϵ_1,ϵ_2, where m is a positive half-integer and $\epsilon_1,\epsilon_2=\pm1$. We replace m by the number $k=m+1/2$. Then k runs over positive integers. The corresponding representation of $U'_q(\mathrm{so}_3)$ is denoted by $R_k^{\epsilon_1,\epsilon_2}$. The basis vectors of the representation space are $|r\rangle$, $r=1,2,\cdots,k$. For the operators of the representation $R_k^{\epsilon_1,\epsilon_2}$ we have

$$R_k^{\epsilon_1,\epsilon_2}(I_{21})|r\rangle=\epsilon_1\frac{q^{r-1/2}+q^{-r+1/2}}{q-q^{-1}}|r\rangle,$$

$$R_k^{\epsilon_1,\epsilon_2}(I_{32})|1\rangle=\frac{1}{q^{1/2}-q^{-1/2}}(\epsilon_2[k]_q|1\rangle+\mathrm{i}[k-1]_q|2\rangle),$$

$$R_k^{\epsilon_1,\epsilon_2}(I_{32})|r\rangle=\frac{1}{q^{r-1/2}-q^{-r+1/2}}(\mathrm{i}[k-r]_q|r+1\rangle+\mathrm{i}[k+r-1]_q|r-1\rangle).$$

It is easy to see from these formulas that $\mathrm{Tr}\,R_k^{\epsilon_1,\epsilon_2}(I_{21})\neq0$ and $\mathrm{Tr}\,R_k^{\epsilon_1,\epsilon_2}(I_{32})\neq0$. Note that for representations of the classical type to the elements I_{21} and I_{32} there correspond operators with vanishing trace.

8. Finite dimensionality of representations for q a root of unity

Everywhere below in this and in the following sections we assume that q is a root of unity. Moreover, we consider that $q^k=1$ and k is an odd integer.

We shall need an information on the center of the algebra $U'_q(\mathrm{so}_n)$. Central elements of the algebra $U'_q(\mathrm{so}_n)$ for any value of q are described in section 4. They are given in the form of homogeneous polynomials of elements of $U'_q(\mathrm{so}_n)$. If q is a root of unity, then there exist additional central elements of $U'_q(\mathrm{so}_n)$ which are given by the following theorem, proved in [16].

Theorem 6. *Let $q^k=1$ for $k\in\mathbb{N}$ and $q^j\neq1$ for $0<j<k$. Then the elements*

$$C^{(k)}(I_{rl}^+)=\sum_{j=0}^{\lfloor (k-1)/2\rfloor}\binom{k-j}{j}\frac{1}{k-j}\Big(\frac{i}{q-q^{-1}}\Big)^{2j}I_{rl}^{+\,k-2j},\qquad r>l,$$

where $\lfloor (k-1)/2\rfloor$ is the integral part of the number $(k-1)/2$, belong to the center of $U'_q(\mathrm{so}_n)$.

It is well-known that a Drinfeld–Jimbo algebra $U_q(g)$ for q a root of unity ($q^k=1$) is a finite dimensional vector space over the center of $U_q(g)$. The same assertion is true for the algebra $U'_q(\mathrm{so}_n)$. In fact, by Theorem 6 any element $(I_{ij}^+)^s$, $s\geq k$, can be reduced to a linear combination of $(I_{ij}^+)^r$, $r < k$, with coefficients from the center $\mathcal{C}$ of $U'_q(\mathrm{so}_n)$. Now our assertion follows from this sentence and from Poincaré–Birkhoff–Witt theorem for $U'_q(\mathrm{so}_n)$.

Proposition 2. *If q is a root of unity, then any irreducible representation of $U'_q(\mathrm{so}_n)$ is finite dimensional.*

Proof. Let q be a root of unity, that is, $q^k = 1$. Let T be an irreducible representation of $U'_q(\mathrm{so}_n)$ on a vector space $\mathcal{V}$. Then T maps central elements into scalar operators. Since the linear space $U'_q(\mathrm{so}_n)$ is finite dimensional over the center $\mathcal{C}$ with the basis $I_{21}^{+\,m_{21}} I_{31}^{+\,m_{31}} \cdots I_{n,n-1}^{+\quad m_{n,n-1}}$, $m_{ij} < k$, then for any $a \in U'_q(\mathrm{so}_n)$ we have $T(a) = \sum_{m_{ij}<k} c_{\{m_{ij}\}} T(I_{21}^{+\,m_{21}} I_{31}^{+\,m_{31}} \cdots I_{n,n-1}^{+\quad m_{n,n-1}})$, where $c_{\{m_{ij}\}}$ are numerical coefficients. Hence, if $\mathbf{v}$ is a nonzero vector of the representation space $\mathcal{V}$, then $T(U'_q(\mathrm{so}_n))\mathbf{v} = \mathcal{V}$ since T is an irreducible representation. Since $T(a)$ is of the above form for any $a \in U'_q(\mathrm{so}_n)$, then $\mathcal{V}$ is finite dimensional. Proposition is proved.

It follows from this proof that there exists a fixed positive integer r such that dimension of any irreducible representation of $U'_q(\mathrm{so}_n)$ at q a root of unity does not exceed r. Of course, the number r depends on k.

9. Irreducible representations at q a root of unity

Let us consider irreducible representations of $U'_q(\mathrm{so}_n)$ for q a root of unity ($q^k = 1$ and k is a smallest positive integer with this property). We also assume that k is odd. If k would be even, then almost all below reasoning is true, if to replace k by $k' = k/2$ (as in the case of irreducible representations of the quantum algebra $U_q(\mathrm{sl}_2)$ for q a root of unity in [4], chapter 3).

There many series of irreducible representations of $U'_q(\mathrm{so}_n)$ in this case. We describe the main series of such representations. We fix complex numbers $m_{1,n}, m_{2,n}, ..., m_{\lfloor n/2\rfloor,n}$ (here $\lfloor n/2\rfloor$ denotes an integral part of $n/2$) and c_{ij}, h_{ij}, $j = 2, 3, \cdots, n-1$, $i = 1, 2, \cdots, \lfloor j/2\rfloor$ such that no of the numbers

$$m_{in},\quad h_{ij},\quad h_{ij} - h_{sj},\quad h_{ij} - h_{s,j\pm 1},\quad h_{ij} + h_{sj},\quad h_{ij} + h_{s,j\pm 1},\quad h_{b,n-1} - m_{sn},\quad h_{b,n-1} + m_{sn}$$

belongs to $\frac{1}{2}\mathbb{Z}$. (We also suppose that $c_{ij} \neq 0$.) The set of these numbers will be denoted by ω:

$$\omega = \{\mathbf{m}_n, \mathbf{c}_{n-1}, \mathbf{h}_{n-1}, \cdots, \mathbf{c}_2, \mathbf{h}_2\},$$

where $\mathbf{m}_n$ is the set of the numbers $m_{1,n}, m_{2,n}, ..., m_{\lfloor n/2\rfloor,n}$, and $\mathbf{c}_j$ and $\mathbf{h}_j$ are the sets of numbers c_{ij}, $i = 1, 2, \cdots, \lfloor j/2\rfloor$, and h_{ij}, $i = 1, 2, \cdots, \lfloor j/2\rfloor$, respectively. (Thus, ω contains $r = \dim \mathrm{so}_n$ complex numbers.) Let V be a complex vector space with a basis labelled by the tableaux

$$\{\xi_n\} \equiv \left\{ \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \cdots \\ \mathbf{m}_2 \end{matrix} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}, \qquad (32)$$

where the set of numbers $\mathbf{m}_n$ consists of $\lfloor n/2\rfloor$ numbers $m_{1,n}, m_{2,n}, \cdots, m_{\lfloor n/2\rfloor,n}$ given above, and for each $s = 2, 3, \cdots, n-1$, $\mathbf{m}_s$ is a set of numbers $m_{1,s}, \cdots, m_{\lfloor s/2\rfloor,s}$ and each $m_{i,s}$ runs independently the values $h_{i,s}, h_{i,s}+1, \cdots, h_{i,s}+k-1$. *Thus,* $\dim V$ *coincides with* k^N, *where* N *is the number of positive roots of* so_n. It is convenient to use for the numbers $m_{i,s}$, $s = 2, 3, \cdots, n$, the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j.$$

To the set of numbers ω there corresponds the irreducible finite dimensional representation T_ω of the algebra $U'_q(\mathrm{so}_n)$. The operators $T_\omega(I_{2p+1,2p})$ of the representation T_ω act upon the basis elements, labelled by (32), by the formula

$$T_\omega(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} c_{j,2p}\frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}}+q^{-l_{j,2p}}}\,|(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p} c^{-1}_{j,2p}\frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}}+q^{-l_{j,2p}}}\,|(\xi_n)^{-j}_{2p}\rangle \tag{33}$$

and the operators $T_\omega(I_{2p,2p-1})$ of the representation T_ω act as

$$T_\omega(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} c_{j,2p-1}\frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]}\,|(\xi_n)^{+j}_{2p-1}\rangle -$$

$$-\sum_{j=1}^{p-1} c^{-1}_{j,j,2p-1}\frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]}\,|(\xi_n)^{-j}_{2p-1}\rangle + \mathrm{i}\, C_{2p-1}(\xi_n)\,|\xi_n\rangle, \tag{34}$$

where numbers in square brackets mean q-numbers. In these formulas, $(\xi_n)^{\pm j}_s$ means the tableau (32) in which j-th component $m_{j,s}$ in $\mathbf{m}_s$ is replaced by $m_{j,s} \pm 1$. If $m_{j,s}+1 = h_{j,s}+k$ (resp. $m_{j,s}-1 = h_{j,s}-1$), then we set $m_{j,s}+1 = h_{j,s}$ (resp. $m_{j,s}-1 = h_{j,s}+k-1$). The coefficients A^j_{2p}, B^j_{2p-1}, C_{2p-1} in (33) and (34) are given by the expressions

$$A^j_{2p}(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}][l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]}\right)^{1/2},$$

$$B^j_{2p-1}(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]\prod_{i=1}^{p-1}[l_{i,2p-2}+l_{j,2p-1}][l_{i,2p-2}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}][l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]}\right)^{1/2},$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]\prod_{s=1}^{p-1}[l_{s,2p-2}]}{\prod_{s=1}^{p-1}[l_{s,2p-1}][l_{s,2p-1}-1]}.$$

The fact that the operators $T_\omega(I_{j,j-1})$, given above, satisfy the defining relations (1)–(3) is proved in the same way as in the case of irreducible representations of $U'_q(\mathrm{so}_n)$ when q is not a root of unity (see [15]).

As in the case of finite dimensional irreducible representations of the Lie algebra so_n, the form of the basis elements of the above representation space V and the formulas for the operators $T_\omega(I_{j,j-1})$ allow us to decompose the restriction of the representation T_ω, $\omega = \{\mathbf{m}_n, \mathbf{c}_{n-1}, \mathbf{h}_{n-1}, \cdots, \mathbf{c}_2, \mathbf{h}_2\}$, to the subalgebra $U'_q(\mathrm{so}_{n-1})$. We have

$$T_\omega|_{U'_q(\mathrm{so}_{n-1})} = \bigoplus_{\omega_{n-1}} T_{\omega_{n-1}},$$

where $\omega_{n-1} = \{\mathbf{m}_{n-1}, \mathbf{c}_{n-2}, \mathbf{h}_{n-2}, \cdots, \mathbf{c}_2, \mathbf{h}_2\}$ and $\mathbf{m}_{n-1}$ runs over the vectors

$$(h_{1,n-1}+a_1, h_{2,n-1}+a_2, \cdots, h_{s,n-1}+a_s), \quad s = \lfloor (n-1)/2 \rfloor, \quad a_j = 0,1,2,\cdots,k-1,$$

and $\mathbf{c}_j$ and $\mathbf{h}_j$ are such as in ω.

Theorem 7. *Representations T_ω with the domain of values of representation parameters, as described above, are irreducible.*

A proof is given by induction and can be found in [15].

There are equivalence relations in the set of irreducible representations T_ω. In order to extract a subset of pairwise nonequivalent representations from the entire set, we introduce some domains on the complex plane. The set

$$D = \{x \in \mathbb{C} \mid |\mathrm{Re}\, x| < k/4 \text{ or } \mathrm{Re}\, x = -k/4,\ \mathrm{Im}\, x \le 0 \text{ or } \mathrm{Re}\, x = k/4,\ \mathrm{Im}\, x \ge 0\}$$

is a maximal subset of $\mathbb{C}$ such that for all $x, y \in D$, $x \neq y$, we have $[x] \neq [y]$. The set

$$D^{\pm} = \{x \in \mathbb{C} \mid 0 < \mathrm{Re}\, x < k/4 \text{ or } \mathrm{Re}\, x = 0,\ \mathrm{Im}\, x \geq 0 \text{ or } \mathrm{Re}\, x = k/4,\ \mathrm{Im}\, x \geq 0\}$$

is a maximal subset of $\mathbb{C}$ such that for all $x, y \in D^{\pm}$, $x \neq y$, we have $[x] \neq \pm[y]$. We need also the sets

$$D_h = \{x \in \mathbb{C} \mid |\mathrm{Re}\, x| < 1/4 \text{ or } \mathrm{Re}\, x = -1/4,\ \mathrm{Im}\, x \leq 0 \text{ or } \mathrm{Re}\, x = 1/4,\ \mathrm{Im}\, x \geq 0\},$$

$$D_h^{\pm} = \{x \in \mathbb{C} \mid 0 < \mathrm{Re}\, x < 1/4 \text{ or } \mathrm{Re}\, x = 0,\ \mathrm{Im}\, x \geq 0 \text{ or } \mathrm{Re}\, x = 1/4,\ \mathrm{Im}\, x \geq 0\}.$$

We introduce an ordering in the set $D^{\pm}$ (resp. $D_h^{\pm}$) as follows: we say that $x \succ y$, $x, y \in D^{\pm}$ (resp. $x, y \in D_h^{\pm}$) if either $\mathrm{Re}\, x > \mathrm{Re}\, y$ or both $\mathrm{Re}\, x = \mathrm{Re}\, y$ and $\mathrm{Im}\, x > \mathrm{Im}\, y$.

We say that the set of complex numbers $\mathbf{l}_{2p} = (l_{1,2p}, l_{2,2p}, \cdots, l_{p,2p})$ is *dominant* if $l_{1,2p}, l_{2,2p}, \cdots, l_{p-1,2p} \in D^{\pm}$, $l_{p,2p} \in D$, and $l_{1,2p} \succ l_{2,2p} \succ \cdots \succ l_{p-1,2p} \succ l^*_{p,2p}$, where $l^*_{p,2p} = l_{p,2p}$ if $l_{p,2p} \in D^{\pm}$ and $l^*_{p,2p} = -l_{p,2p}$ if $l_{p,2p} \notin D^{\pm}$.

The notion of *dominance* for the set $\mathbf{h}_{2p} = (h_{1,2p}, h_{2,2p}, \cdots, h_{p,2p}) \in \mathbb{C}^p$ is introduced by the replacements $l_{i,2p} \to h_{i,2p}$, $D \to D_h$ and $D^{\pm} \to D_h^{\pm}$ in the previous definition.

We say that the set of complex numbers $\mathbf{l}_{2p+1} = (l_{1,2p+1}, l_{2,2p+1}, \cdots, l_{p,2p+1})$ is *dominant* if $l_{1,2p+1}, l_{2,2p+1}, \cdots, l_{p,2p+1} \in D^{\pm}$ and $l_{1,2p+1} \succ l_{2,2p+1} \succ \cdots \succ l_{p,2p+1}$.

The notion of *dominance* for the set of complex numbers $\mathbf{h}_{2p+1} = (h_{1,2p+1}, h_{2,2p+1}, \cdots, h_{p,2p+1})$ is introduced by the replacements $l_{i,2p+1} \to h_{i,2p+1}$ and $D^{\pm} \to D_h^{\pm}$ in the previous definition.

We say that $\omega = \{\mathbf{m}_n, \mathbf{c}_{n-1}, \mathbf{h}_{n-1}, \cdots, \mathbf{c}_2, \mathbf{h}_2\}$ is *dominant* if every of the sets $\mathbf{l}_n$, $\mathbf{h}_{n-1}, \cdots$, $\mathbf{h}_2$ is dominant and if $0 \leq \mathrm{Arg}\, c_{ij} < 2\pi/k$, $j = 2, 3, \cdots, n-1$; $i = 1, 2, \cdots, \lfloor j/2 \rfloor$.

Theorem 8 [15]. *The representations T_ω of $U'_q(\mathrm{so}_n)$ with dominant ω are pairwise nonequivalent. Any irreducible representation $T_{\omega'}$ is equivalent to some representation T_ω with dominant ω.*

10. Relation to Macdonald symmetric polynomials

It was shown by M. Noumi [9] that the algebra $U'_q(\mathrm{so}_n)$ is related to Macdonald symmetric polynomials. This relation is described in term of zonal spherical functions on the quantum symmetric space which is a quntization of the classical symmetric space $GL(n)/SO(n)$. This quantum space is described in terms of the space of function $\mathcal{F}_q(GL(n)/SO(n))$ which is defined as subspace of the space $\mathcal{F}_q(GL(n))$ of regular functions on the quantum group $GL_q(n)$ left invatiant with respect to the algebra $U'_q(\mathrm{so}_n)$:

$$\mathcal{F}_q(GL(n)/SO(n)) = \{f \in \mathcal{F}_q(GL(n)) \mid a \triangleright f = f,\ a \in U'_q(\mathrm{so}_n)\}.$$

(Note that left action of the algebra $U'_q(\mathrm{so}_n)$ on f correspond to the right action of the quantum group $GL_q(n)$ on $\mathcal{F}_q(GL(n)/SO(n))$ (see [4], chapter 1).

Under left (regular) action of the quantum group $GL_q(n)$ the space $\mathcal{F}_q(GL(n)/SO(n))$ decomposes into direct sum of irreducible representations of this quantum group:

$$\mathcal{F}_q(GL(n)/SO(n)) \sim \bigoplus\nolimits_{\lambda \in P^+_{so(n)}} \mathcal{F}_q(\lambda).$$

where $P^+_{so(n)} = \{\lambda \in P^+ \mid \lambda_i - \lambda_{i+1} \in 2\mathbb{Z}\}$ is the subset of the set $P^+ = \{\lambda = (\lambda_1, \cdots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$ of all highest weights of the group $GL(n)$. The irreducible representations of the quantum group $GL_q(n)$ with highest weights λ are realized on the subspaces $\mathcal{F}_q(\lambda)$, respectively.

Each space $\mathcal{F}_q(\lambda)$ has exactly one-dimensional subspace $\mathbb{C}\varphi_\lambda$ of right invariants with respect to the subalgebra $U'_q(\mathrm{so}_{n-1})$ of the algebra $U'_q(\mathrm{so}_n)$. The function φ_λ is called *zonal sherical function* of the space $\mathcal{F}_q(\lambda)$.

In order to describe these zonal spherical functions we note that the quantum group $GL_q(n)$ has n-dimensional torus $\mathbb{T}^n$ on its main diagonal. We denote by $z = (z_1, z_2, \cdots, z_n)$ canonical coordinates on $\mathbb{T}^n$. It is proved by Noumi [9] that

$$\varphi_\lambda|_{\mathbb{T}^n} = c_\lambda P_\mu(z_1^2, z_2^2, \cdots, z_n^2;\ q^4, q^2)(z_1 z_2 \cdots z_n)^l, \tag{35}$$

where c_λ is an appropriate constant and λ, μ and l are related as

$$\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_n)(2\mu_1, 2\mu_2, \cdots, 2\mu_n) + (l, l, \cdots, l).$$

Here $P_m(x; q, t)$ are well known Macdonald symmetric polynomials (see [25], chapter 6), defined for partitions $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$.

The explicit expression for the constant c_λ is given in [9] for the case when φ_λ is normalized by the condition $\varphi_\lambda(q^\rho) = 1$, where $q^\rho = (q^{\frac{n-1}{2}}, q^{\frac{n-3}{2}}, \cdots, q^{\frac{-n+1}{2}})$.

The following formula has place for the Macdonald polynomials $P_m(x; q, t)$ (see [25]):

$$P_\mu P_\nu = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda.$$

It is noted in [25], p. 343, that the explicit calculation of the coefficients $f_{\mu\nu}^\lambda \equiv f_{\mu\nu}^\lambda(q,t)$ is open problem. Let us give group-theoretical approach to this formula for the Macdonald polynomials from (35) and to the coefficients $f_{\mu\nu}^\lambda$. This approach uses the fact that the zonal functions φ_λ can be realized as the matrix elements

$$\varphi_\lambda = \langle \lambda, 0 | T_\lambda | \lambda, 0 \rangle,$$

where T_λ is the left corepresentation of the Hopf algebra $\mathcal{F}_q(GL(n))$ on $\mathcal{F}_q(\lambda)$ and $|\lambda, 0\rangle$ is the right $U_q'(\mathrm{so}_{n-1})$ invariant vector in $\mathcal{F}_q(\lambda)$.

We consider the tensor product $T_{\lambda_1} \otimes T_{\lambda_2}$ of two corepresentations of the Hopf algebra $\mathcal{F}_q(GL(n))$ such that $\lambda_1, \lambda_2 \in P^+_{so(n)}$. Then

$$T_{\lambda_1} \otimes T_{\lambda_2} \sim \bigoplus\nolimits_{\lambda \in P^+} n_\lambda T_\lambda. \tag{36}$$

Moreover, considering the $*$-structure on $\mathcal{F}_q(GL(n))$ (transforming this Hopf algebra into the Hopf $*$-algebra $\mathcal{F}_q(U(n))$, we may assume that all corepresentations in (36) are unitary. Then there exists a unitary operator C from the representation space of $n_\lambda T_\lambda$ to the representation space of $T_{\lambda_1} \otimes T_{\lambda_2}$ such that

$$T_{\lambda_1} \otimes T_{\lambda_2} = C \left(\bigoplus\nolimits_{\lambda \in P^+} n_\lambda T_\lambda \right) C^*. \tag{37}$$

Now we take basis elements $|\lambda_1, k_1; \lambda_2, k_2\rangle$ in the space of the representation $T_{\lambda_1} \otimes T_{\lambda_2}$ (where k_1 and k_2 numerate basis elements in the spaces which are tensored) and basis elements $|j; \lambda, k\rangle$ in the space of the representation $n_\lambda T_\lambda$ (where j separate multiple representations T_λ). The matrix elements

$$\langle \lambda_1, k_1; \lambda_2, k_2 | C | j; \lambda, k \rangle := C^{\lambda, j}_{\lambda_1 \lambda_1 2}(k_1, k_2, k)$$

are called Clebsch–Gordan coefficients of the tensor product $T_{\lambda_1} \otimes T_{\lambda_2}$. Let $|\lambda_1, 0\rangle$, $|\lambda_2, 0\rangle$ and $|j; \lambda, 0\rangle$ be the normalized right $U_q'(\mathrm{so}_{n-1})$ invariant vectors in the correponding representation spaces. Then

$$\langle \lambda_1, 0; \lambda_2, 0 | C | j; \lambda, k \rangle = 0,$$

if $|j; \lambda, k\rangle \neq |j; \lambda, 0\rangle$. We have

$$\langle \lambda_1, 0; \lambda_2, 0 \mid T_{\lambda_1} \otimes T_{\lambda_2} \mid \lambda_1, 0; \lambda_2, 0 \rangle = \varphi_{\lambda_1} \varphi_{\lambda_2}$$

and

$$\langle j;\lambda,0|T_\lambda|j;\lambda,0\rangle = \varphi_\lambda.$$

Let us introduce the notation

$$\langle \lambda_1,0;\lambda_2,0|C|j;\lambda,0\rangle := C^{\lambda,j}_{\lambda_1\lambda_2}.$$

Taking matrix elements of both sides of the relation (37) between the vectors $\langle\lambda_1,0;\lambda_2,0|$ and $|\lambda_1,0;\lambda_2,0\rangle$ we obtain the relation

$$\varphi_{\lambda_1}\varphi_{\lambda_2} = \sum\nolimits_{\lambda\in P^+_{so(n)}} \left(\sum_{j=0}^{n_\lambda} |C^{\lambda,j}_{\lambda_1\lambda_2}|^2\right)\varphi_\lambda. \tag{38}$$

Introducing the notation

$$\sum_{j=0}^{n_\lambda} |C^{\lambda,j}_{\lambda_1\lambda_2}|^2 = C^{\lambda}_{\lambda_1\lambda_2},$$

we may write down the relation (38) in the form

$$\varphi_{\lambda_1}\varphi_{\lambda_2} = \sum\nolimits_{\lambda\in P^+_{so(n)}} C^{\lambda,j}_{\lambda_1\lambda_2}\varphi_\lambda.$$

It is the relation

$$P_{\mu_1}P_{\mu_2} = \sum_\mu f^\mu_{\mu_1\mu_2}P_\mu$$

for the Macdonald polynomials $P_\mu(x;q^4,q^2)$, written in other notations (see (35)). The Clebsch–Gordan coefficients $C^{\lambda,j}_{\lambda_1\lambda_2}$ are now under calculation.

We can derive two properties of the coefficients $f^\mu_{\mu_1\mu_2}$ starting from properties of Clebsch–Gordan coefficients $C^{\lambda,j}_{\lambda_1\lambda_2}$. Namely, it follows from (38) that $C^{\lambda,j}_{\lambda_1\lambda_2}\ge 0$. It gives *the first property of the coefficients* $f^\mu_{\mu_1\mu_2}$.

Since Clebsch–Gordan coefficients constitute a unitary matrix, then we have

$$\sum_{j,\lambda} |\langle\lambda_1,0;\lambda_2,0|C|j;\lambda,0\rangle|^2 = 1.$$

Therefore,

$$\sum\nolimits_{\lambda\in P^+_{so(n)}} C^{\lambda}_{\lambda_1\lambda_2} = 1.$$

This gives *the second propery of the coefficients* $f^\mu_{\mu_1\mu_2}$. Note that $f^\mu_{\mu_1\mu_2}$ differs from the corresponding $C^{\lambda}_{\lambda_1\lambda_2}$ by the coefficients from (35).

Remark. In was shown in [9] that zonal spherical functions for the quantized algebra of functions $\mathcal{F}_q(GL(2n)/Sp(2n))$ are expressed in terms of the Macdonald polynomials $P_\mu(x;q^2,q^4)$. For these polynomials the above two properties of the coefficients $f^\lambda_{\mu\nu}$ are also true since the above reasoning can be repeated word by word for this case. We only have to take into account coefficients relating the zonal functions and the Macdonald polynomials $P_\mu(x;q^2,q^4)$.

11. Harmonic polynomials on the quantum vector space

In section 4 we defined the action of the algebra $U'_q(\mathrm{so}_n)$ on the algebra $\mathcal{A}$ of polynomials in the elements $x_1,x_2,\cdots,x_n$ such that $x_ix_j = qx_jx_i$, $i<j$. This action determine the representation of $U'_q(\mathrm{so}_n)$ on $\mathcal{A}$, which will be denoted by T.

The algebra $\mathcal{A}$ can be represented as a sum $\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m$, where $\mathcal{A}_m$ denote the subspace of homogeneous polynomials of homogeneity degree m. These subspaces are invariant with respect to action of $U'_q(\mathrm{so}_n)$. We denote the restriction of the representation T to $\mathcal{A}_m$ by $T^{(m)}$.

In general, the representations $T^{(m)}$ of $U'_q(\mathrm{so}_n)$ are reducible. It is checked by a direct computation that the element

$$Q = x_1^2 + q^{-1}x_2^2 + \cdots + q^{-n+1}x_n^2 \in \mathcal{A}_2 \tag{39}$$

is invariant with respect to the representation $T^{(2)}$ (and hence with respect to the representation T), that is, $T^{(2)}(I_{k,k-1})Q = 0$ for $k = 2, 3, \cdots, n$. Similarly, the element $Q^k \in \mathcal{A}_{2k}$ is invariant with respect to the representation $T^{(2k)}$.

To the element (39) there corresponds the operator

$$\hat{Q} = \hat{x}_1^2 + q^{-1}\hat{x}_2^2 + \cdots + q^{-n+1}\hat{x}_n^2$$

on $\mathcal{A}$ which commutes with operators of the representation T. We also consider on $\mathcal{A}$ the operator

$$\Delta_q \equiv \Delta = q^{n-1}\partial_1^2 + q^{n-2}\partial_2^2 + \cdots + \partial_n^2. \tag{40}$$

It is called the *q-Laplace operator* on the quantum vector space.

A polynomial $p \in \mathcal{A}$ is called *q-harmonic* if $\Delta p = 0$. The linear subspace of $\mathcal{A}$ consisting of all q-harmonic polynomials is denoted by $\mathcal{H}$. Let $\mathcal{H}_m = \mathcal{A}_m \cap \mathcal{H}$. Then

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

Similar to the classical case, the space $\mathcal{A}_m$ can be represented in the form of the direct sum (see [18])

$$\mathcal{A}_m = \mathcal{H}_m \oplus Q\mathcal{A}_{m-2}. \tag{41}$$

This decomposition has the following consequence:

$$\mathcal{A}_m = \bigoplus_{0 \le 2j \le m} Q^j \mathcal{H}_{m-2j} \tag{42}$$

(the summation here is over $j = 0, 1, 2, \cdots, \lfloor m/2 \rfloor$, where $\lfloor m/2 \rfloor$ is the integral part of $m/2$).

It follows from (42) that $\mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H}$. This decomposition is a q-analogue of Kostant's theorem on separation of variables for Lie groups in an abstract form.

If $h_m(\mathbf{x}) \in \mathcal{H}_m$ and $h'_s(\mathbf{x}) \in \mathcal{H}_s$, then (since $\hat{Q}^* = q^{-n+1}\Delta$ with respect to the scalar product (21)) we have

$$\langle Q^k h_m, Q^l h'_s \rangle = q^{k(-n+1)} \langle h_m, \Delta^k Q^l h'_s \rangle.$$

It is derived by direct calculation that $\Delta(Q^l h'_s) = Q^{l-1}[2l][2l+n+2s-2]h'_s$. Applying repeatedly this formula we obtain from the previous formula that

$$\langle Q^k h_m, Q^l h'_s \rangle = \delta_{kl} q^{k(-n+1)} [2l]!! \frac{[2k+n+2s-2]!!}{[n+2s-2]!!} \langle h_m, h'_s \rangle. \tag{43}$$

Remark. In an analogy with the classical case, we may consider the scalar product (21) as an integral of the function $p_1 p_2^*$. Then the formula (43) means a fulfilment of "integration" with respect to the q-radial part. Like to the classical case, the scalar product $\langle h_m, h'_s \rangle$ can be treated as "integration" over q-spherical coordinates for q-harmonic polynomials.

Proposition 3 [18]. *The operator Δ commutes with the action T of the algebra $U'_q(\mathrm{so}_n)$.*

It follows from this proposition that restriction of the representation $T^{(m)}$ onto the subspace $\mathcal{H}_m$ is invariant with respect to $U'_q(\mathrm{so}_n)$. We denote the restriction of this representation to $\mathcal{H}_m$ by T_m. Since Q is invariant with respect to $U'_q(\mathrm{so}_n)$, it follows from (42) that

$$T^{(m)} = \bigoplus\nolimits_{0\le 2j\le m} T_{m-2j}.$$

Proposition 4. *The representation T_m of $U'_q(\mathrm{so}_n)$ is irreducible and equivalent to the classical type representation characterized by the integers $(m, 0, \cdots, 0)$.*

The decomposition (41) is orthogonal with respect to the scalar product (21). There exists the projector $H_m : \mathcal{A}_m = \mathcal{H}_m \oplus Q\mathcal{A}_{m-2} \to \mathcal{H}_m$. This projector can be represented in the form

$$H_m p = \sum_{k=0}^{\lfloor m/2\rfloor} \alpha_k \hat{Q}^k \Delta^k p, \qquad \alpha_k \in \mathbb{C}, \quad p \in \mathcal{A}_m, \tag{44}$$

($\lfloor m/2\rfloor$ means the integral part of $m/2$), where the coefficients α_k are given by

$$\alpha_k = \frac{(-1)^k [n+2m-2k-4]!!}{[2k]!![n+2m-4]!!}$$

with $[s]!! = [s]_q[s-2]_q[s-4]_q \cdots [2]_q$ (or $[1]_q$) and $[0]!! = 1$. These coefficients are taken in such a way that $H_m^2 = H_m$. It is clear that the operator H_m commutes with the operators of the representation $T^{(m)}$ of $U'_q(\mathrm{so}_n)$. Considering the scalar product (21) on the space $\mathcal{A}_m$ we have $H^* = H$.

Let us show how to construct, by using the operator H_m, a zonal polynomial (that is, an invariant element with respect to the subalgebra $U'_q(\mathrm{so}_{n-1})$) in the space $\mathcal{H}_m$. In order to do this, we have to take a polynomial $p \in \mathcal{A}_m$ invariant with respect to $U'_q(\mathrm{so}_{n-1})$ and to act upon it by the operator H_m. Since the projector H_m commutes with the action of $U'_q(\mathrm{so}_{n-1})$, a polynomial obtained in this way is a zonal polynomial. Clearly, the polynomial $p(\mathbf{x}) = x_n^m$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, belongs to $\mathcal{A}_m$ and is invariant under the action of $U'_q(\mathrm{so}_{n-1})$. We have

$$\varphi'_m := H_m x_n^m = \sum_{k=0}^{\lfloor m/2\rfloor} \alpha_k \hat{Q}^k \partial_n^{2k} x_n^m = \sum_{k=0}^{\lfloor m/2\rfloor} (-1)^k \frac{[m]![n+2m-2k-4]!!}{[m-2k]![2k]!![n-2m-4]!!} Q^k x_n^{m-2k}. \tag{45}$$

Using the notation $(a;q)_s = (1-a)(1-qa)(1-q^2a)\cdots(1-q^{s-1}a)$, we reduce the zonal polynomial (45) to the form

$$\varphi'_m = \sum_{k=0}^{\lfloor m/2\rfloor} \frac{(q^{-2m};q^4)_k (q^{-2m+2};q^4)_k}{(q^4;q^4)_k (q^{-2n-4m+8};q^4)_k} q^{-k(n-5)} Q^k x_n^{m-2k}.$$

It concides with the formula for a zonal polynomial found by another method in [18].

It is known (see [26], chapters 9 and 10) that in the space of classical homogeneous harmonic polynomials there exist different orthonormal bases. They correspond to different separations of variables. Each separation of variables corresponds to a certain chain of subgroups of the rotation group $SO(n)$. A similar picture has place for spaces $\mathcal{H}_m$ of homogeneous q-harmonic polynomials.

In the classical case, the tree method distinguishes different separations of variables or, equivalently, different chains of subgroups of $SO(n)$ (see [26], chapter 10). The same tree method (see [19]) can be used for q-harmonic polynomials, but instead of chains of subgroups of $SO(n)$ we have to take the corresponding chains of subalgebras of the algebra $U'_q(\mathrm{so}_n)$.

Let us show how to construct an orthonormal basis of the space $\mathcal{H}_m$ of homogeneous q-harmonic polynomials which corresponds to the chain

$$U'_q(\mathrm{so}_n) \supset U'_q(\mathrm{so}_{n-1}) \supset \cdots \supset U'_q(\mathrm{so}_3) \supset U'_q(\mathrm{so}_2), \tag{46}$$

where $U'_q(\mathrm{so}_2)$ is the commutative subalgebra generated by the element I_{21}. This basis is a q-analogue of the well known set of associated spherical harmonics which are products of certain Gegenbauer polynomials (see, [26], chapter 9). Let us first note that the following proposition is true [19]:

Proposition 5. *Let $h_s(\mathbf{x}')$ be a homogeneous harmonic polynomial of degree s in $\mathbf{x}' = (x_1, x_2, \cdots, x_{n-1})$. Then for $x_n^{m-s} h_s(\mathbf{x}') \in \mathcal{A}_m$ we have*

$$H_m(x_n^{m-s} h_s(\mathbf{x}')) = \left(\sum_{k=0}^{\lfloor (m-s)/2 \rfloor} \frac{(-1)^k q^{-2sk} [m-s]! [2m+n-2k-4]!!}{[m-s-2k]! [2k]!! [2m+n-4]!!} Q^k x_n^{m-s-2k} \right) h_s(\mathbf{x}'). \quad (47)$$

We denote by $\hat{t}_s^{n,m}(Q, x_n)$ the expression at $h_s(\mathbf{x}')$ on the right hand side of (47):

$$\hat{t}_s^{n,m}(Q, x_n) = \sum_{k=0}^{\lfloor (m-s)/2 \rfloor} \frac{(-1)^k q^{-2sk} [m-s]! [2m+n-2k-4]!!}{[m-s-2k]! [2k]!! [2m+n-4]!!} Q^k x_n^{m-s-2k}. \quad (48)$$

In order to construct an orthonormal basis of $\mathcal{H}_m$, we have to normalize expression (47). Let τ_s^m denote the expression (47). We have (see [19])

$$\langle \tau_s^m, \tau_s^m \rangle = \langle H_m(x_n^{m-s} h_s(\mathbf{x}')), H_m(x_n^{m-s} h_s(\mathbf{x}')) \rangle = \langle x_n^{m-s} h_s(\mathbf{x}'), \tau_s^m \rangle$$
$$= c_m^{(s)} \langle x_n^{m-s} h_s(\mathbf{x}'), x_n^{m-s} h_s(\mathbf{x}') \rangle = c_m^{(s)} q^{-s(m-s)} [m-s]! \langle h_s(\mathbf{x}'), h_s(\mathbf{x}') \rangle,$$

where $c_m^{(s)}$ is the coefficient at x_n^{m-s} in the expression (48). It is given as

$$c_m^{(s)} = \frac{(q^{2(-n-m-s+3)}; q^4)_{(m-s)/2}}{(q^{2(-n-2m+4)}; q^4)_{(m-s)/2}} \quad \text{if} \quad m-s \quad \text{is} \quad \text{even},$$

$$c_m^{(s)} = \frac{(q^{2(-n-m-s+4)}; q^4)_{(m-s-1)/2}}{(q^{2(-n-2m+4)}; q^4)_{(m-s-1)/2}} \quad \text{if} \quad m-s \quad \text{is} \quad \text{odd}.$$

Instead of $\hat{t}_s^{n,m}(Q, x_n)$ we shall use the normalized expression

$$t_s^{n,m}(Q, x_n) = q^{s(m-s)/2} (c_m^{(s)} [m-s]!)^{-1/2} \hat{t}_s^{n,m}(Q, x_n). \quad (49)$$

The space $\mathcal{H}_m$ can be represented as the orthogonal sum

$$\mathcal{H}_m = \bigoplus_{m_{n-1}=0}^{m} t_{m_{n-1}}^{n,m}(Q, x_n) \mathcal{H}_{m_{n-1}}^{(n-1)},$$

where $\mathcal{H}_{m_{n-1}}^{(n-1)}$ is the space of homogeneous q-harmonic polynomials in $x_1, x_2, \cdots, x_{n-1}$. Applying this formula to $\mathcal{H}_{m_{n-1}}^{(n-1)}$, then to $\mathcal{H}_{m_{n-2}}^{(n-2)}$ and so on, we obtain the following decomposition of $\mathcal{H}_m$ into the sum of one-dimensional subspaces:

$$\mathcal{H}_m = \bigoplus_{m_{n-1}, m_{n-2}, \cdots, m_3, m_2} \mathbb{C}\, \Xi_{m, m_{n-1}, m_{n-2}, \cdots, m_2}(\mathbf{x}),$$

where

$$\Xi_{\mathbf{m}}(\mathbf{x}) \equiv \Xi_{m, m_{n-1}, m_{n-2}, \cdots, m_2}(\mathbf{x})$$
$$= t_{m_{n-1}}^{n,m}(Q, x_n) t_{m_{n-2}}^{n-1, m_{n-1}}(Q_{n-1}, x_{n-1}) \cdots t_{m_2}^{3, m_3}(Q_3, x_3) t^{2, m_2}(x_1, x_2), \quad (50)$$

where $Q_k = x_1^2 + q^{-1} x_2^2 + \cdots + q^{-k+1} x_k^2$ and summation is over all integral values of $m_{n-1}, m_{n-2}, \cdots, m_2$ for which

$$m \geq m_{n-1} \geq m_{n-2} \geq \cdots \geq m_3 \geq |m_2|, \quad (51)$$

and $t^{k,m_k}_{m_{k-1}}(Q_k, x_k)$ are given by (49). A complete set of linearly independent harmonic polynomials in x_1 and x_2 coincides with

$$z^{(0)} \equiv 1, \qquad z^{(s)} = (\mathrm{i}x_1 + x_2)(\mathrm{i}x_1 + qx_2)\cdots(\mathrm{i}x_1 + q^{s-1}x_2), \quad s > 0,$$

$$z^{(s)} = (\mathrm{i}x_1 - x_2)(\mathrm{i}x_1 - qx_2)\cdots(\mathrm{i}x_1 - q^{-s+1}x_2), \quad s < 0$$

and $t^{2,m_2}(x_1, x_2) = (c^{(m_2)})^{-1/2} z^{(m_2)}$.

To every set of integers $m_{n-1}, m_{n-2}, \cdots, m_3, m_2$ satisfying the condition (51) corresponds a polynomial (50). (For fixed m_3 the number m_2 takes the values $-m_3, -m_3 - 1, \cdots, m_3$.) A direct calculation shows that the number of these polynomials is equal to the dimension of the space $\mathcal{H}_m$. From the other side, the polynomials (50) are pairwise orthogonal. This means that *the set of all polynomials (50) constitute an orthonormal basis of the space $\mathcal{H}_m$. This basis corresponds to the chain of subalgebras (46).*

Representation of the basis of the space $\mathcal{H}_m$ of solutions of the equation $\Delta p_m = 0$ in the form (50) gives us a q-analogue of separation of variables of the classical analysis. This q-separation of variables corresponds to the chain of subalgebras (46). q-Analogues of other types of separations of variables can be constructed similarly (see [19] for explicit formulas).

It is proved in [19] that the operators $T_m(I_{k,k-1})$, $k = 2, 3, \cdots, n$, of the representation T_m of the algebra $U'_q(\mathrm{so}_n)$ act upon the basis elements $\Xi_{\mathbf{m}} \equiv |\mathbf{m}\rangle$, given by (50), as

$$T_m(I_{k,k-1})|\mathbf{m}\rangle = -([m_k + m_{k-1} + k - 2][m_k - m_{k-1}])^{1/2} A(m_{k-1})|\mathbf{m}^+_{k-1}\rangle$$

$$+([m_k + m_{k-1} + k - 3][m_k - m_{k-1} + 1])^{1/2} A(m_{k-1} - 1)|\mathbf{m}^-_{k-1}\rangle, \quad k \neq 2,$$

$$T_m(I_{21})|\mathbf{m}\rangle = \mathrm{i}[m_2]|\mathbf{m}\rangle,$$

where $m_n \equiv m$, $\mathbf{m}^{\pm}_{k-1}$ denote the set of numbers $\mathbf{m}_{k-1}$ with m_{k-1} replaced by $m_{k-1} \pm 1$, respectively, and

$$A(m_{k-1}) = \left(\frac{[m_{k-1} + m_{k-2} + k - 3][m_{k-1} - m_{k-2} + 1]}{[2m_{k-1} + k - 3][2m_{k-1} + k - 1]} \right)^{1/2}.$$

These formulas give an explicit form of the representation T_m mentioned in Proposition 4.

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