Noncommutative symmetric functions and quasi-symmetric functions with two and more parameters

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Abstract

We define two-parameter families of noncommutative symmetric functions and quasi-symmetric functions, which appear to be the proper analogues of the Macdonald symmetric functions in these settings.

1 Introduction

During the past twenty years, the theory of symmetric functions underwent remarkable developments, including the discovery of Macdonald's symmetric functions, and of various kinds of orthogonal polynomials in several variables.

It also appeared that the classical theory of symmetric functions had to be supplemented with the so-called noncommutative symmetric functions and quasi-symmetric functions. These notions, originally introduced for combinatorial purposes [6, 5, 9], are now known to describe the representations of Hecke algebras and quantum groups of type A in a degenerate case, which is only apparent in a certain parametrization (in the usual convention, the quantum parameter q stands for the point $[q : q^{-1}]$ of a complex projective line, and the relevant degeneracy occurs at the origin $[0 : 1]$ and at the point at infinity $[1 : 0]$) [3, 10, 11, 2].

Quasi-symmetric functions, whose algebra is denoted by $QSym$, and noncommutative symmetric functions, denoted by **Sym**, are two graded Hopf algebras in natural duality. It has been found that almost all interesting objects of the classical theory find an analogue on one side or the other. There exist "Schur functions" in both set-ups. Elementary and complete functions fall on the noncommutative side, while monomial functions have only a quasi-symmetric analogue. Noncommutative power sums are not unique and turned out to be most interesting from a combinatorial point of view. There are analogues of the Littlewood-Richardson rule, of the Robinson-Schensted-Knuth correspondence, of the internal product, of the Frobenius characteristic map, of the Weyl and Demazure character formulae, and so on.

It is therefore quite natural to push the analogy further, and to look for analogues of the most interesting objects of the "modern" theory of symmetric functions, i.e., Hall-Littlewood and Macdonald functions.

Although we have (at the time of writing) no analogue of the Hall algebra to motivate the introduction of quasi-symmetric and noncommutative Hall-Littlewood functions, recent developments connecting these objects to Hecke algebras and Kazhdan-Lusztig polynomials allows one to look for a definition involving a a quasi-symmetrizing action of the Hecke algebra. This has been achieved in [8]. The resulting quasi-symmetric Hall-Littlewood functions, and their noncommutative dual basis have been shown to share many properties with their classical analogues, which makes more than plausible the pertinence of their definition.

In this note, we propose a combinatorial definition of noncommutative and quasi-symmetric analogues of Macdonald's symmetric functions. As we shall see, the naive analogues of the definitions by triangularity properties do not work (although such a definition is in fact possible), but the constraints that we should recover Hall-Littlewood functions at $q = 0$, and that the four-fold symmetry of the classical (q, t)-Kostka matrix has to be retained (because of a conjectural representation theoretical interpretation) are sufficient to suggest a general pattern. Once the noncommutative analogues have been found, the quasi-symmetric ones can be defined by duality.

The noncommutative theory is not expected to yield much information about the classical Macdonald functions. What is expected, but not proved yet, is that both represent different projections of some higher level object, which remains to be discovered. To support this hypothesis, we derive a few properties of the noncommutative and quasi-symmetric analogues which are direct analogues of known properties of the classical Macdonald functions.

Details, proofs and other results will appear in a subsequent paper. Our notation is as in [5] and [13]

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2 Noncommutative analogues of the Macdonald functions

We shall start with the definition of noncommutative analogues of the symmetric functions

$$\tilde{H}_\mu(X;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(X) = t^{n(\mu)}J_\mu\left(\frac{X}{1-t^{-1}};q,t^{-1}\right) . \qquad (1)$$

It has been conjectured by Garsia and Haiman [4], and recently proved by Haiman [7] that these functions were the bigraded Frobenius characteristics of certain realizations of the regular representations of the symmetric group.

If we want to define noncommutative analogues $\tilde{\mathrm{H}}_J(A;q,t)$, to be called noncommutative Macdonald functions, labelled by compositions J, as

$$\tilde{\mathrm{H}}_J(A;q,t) = \sum_I \tilde{k}_{IJ}(q,t)R_I(A) \qquad (2)$$

where R_I are the noncommutative ribbon Schur functions, it is natural to invoke the representation theoretical interpretation of the ribbons. It is known that they are the characteristics of the indecomposable projective modules of the 0-Hecke algebra $H_n(0)$, each of them occuring with multiplicity one in the decomposition of the regular representation. Hence, if we expect (2) to describe the bigraded characteristic of a regular representation of $H_n(0)$, the $k_{IJ}(q,t)$ have to be monomials q^it^j. This will be our first requirement.

Our second requirement is that

$$\tilde{\mathrm{H}}_J(A;0,t) = t^{\binom{l(J)}{2}}H_J(A;t^{-1}) \qquad (3)$$

where in the right-hand side, H_J is the noncommutative Hall-Littlewood function of [8].

Finally, the structure of the projective modules of $H_n(0)$ leads us to expect that the (q,t)-Kostka monomials should possess the symmetries

$$\tilde{k}_{I\bar{J}^\sim}(q,t) = \tilde{k}_{IJ}(t,q) , \qquad (4)$$

$$\tilde{k}_{IJ}(q,t)\tilde{k}_{\bar{I}^\sim J}(q,t) = q^{\binom{n+1-l(J)}{2}}t^{\binom{l(I)}{2}} , \qquad (5)$$

and that $\tilde{k}_{(n),J}(q,t)$ is always equal to 1.

These requirements are sufficient to determine the first matrices, whose transposes (denoted by K_n) are reproduced below.

$$K_2 = \begin{pmatrix} 2 & 1 & q \\ 11 & 1 & t \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 3 & 1 & q^2 & q & q^3 \\ 21 & 1 & t & q & tq \\ 12 & 1 & q & t & tq \\ 111 & 1 & t^2 & t & t^3 \end{pmatrix}$$

$$K_4 = \begin{pmatrix} 4 & 1 & q^3 & q^2 & q^5 & q & q^4 & q^3 & q^6 \\ 31 & 1 & t & q^2 & tq^2 & q & tq & q^3 & tq^3 \\ 22 & 1 & q^2 & t & tq^2 & q & q^3 & tq & tq^3 \\ 211 & 1 & t^2 & t & t^3 & q & t^2q & tq & t^3q \\ 13 & 1 & q^2 & q & q^3 & t & tq^2 & tq & tq^3 \\ 121 & 1 & t^2 & q & t^2q & t & t^3 & tq & t^3q \\ 112 & 1 & q & t^2 & t^2q & t & tq & t^3 & t^3q \\ 1111 & 1 & t^3 & t^2 & t^5 & t & t^4 & t^3 & t^6 \end{pmatrix}$$

These matrices have an apparent 2×2-block structure. Actually, K_{n-1} is the submatrix of K_n formed by even rows and odd columns. The rest of the matrix is determined by its second column, corresponding to the composition $I = (n-1, 1)$. The monomials in this column are completely determined by our requirements and it is found that $\tilde{k}_{(n-1,1),J}$ is equal to $q^{n-l(J)}$ if J is an odd row, and to $t^{l(J)-1}$ if J labels an even row. Then, each pair of consecutive rows $(1,2), (3,4), \ldots$ is labelled by compositions (L', L''), where L is a composition of $n-1$, L' is obtained from L by incrementing its last part of 1 and L'' by adding at the end of L a part 1, e.g., $(21)' = (22)$ and $(21)'' = (211)$. Let

$$A = \begin{pmatrix} 1 & q^{l(L)} \\ 1 & t^{n-1-l(L)} \end{pmatrix}$$

be the block in the firsts two columns. Then, the block in the columns (M', M'') is $\tilde{k}_{L,M}(q,t)A$. This description will be taken as the definition of K_n for general n.

The entries of the matrix K_n can be directly described in terms of the geometry of composition diagrams .

Given a composition I of n, let $\mathrm{Des}\,(I) \subseteq \{1, \ldots, n-1\}$ be the descent set of I. For $1 \le k \le n-1$, put

$$d(I,k) = \#\{k' < k,\ k' \in \mathrm{Des}\,(I)\}$$

and

$$v(I,k) = \begin{cases} t^{1+d(I,k)} & \text{if } k \in \mathrm{Des}\,(I)\,, \\ q^{k-d(I,k)} & \text{if } k \notin \mathrm{Des}\,(I)\,. \end{cases}$$

Then,

$$K_n(I,J) = \prod_{k \in \mathrm{Des}\,(J)} v(I,k)\,. \tag{6}$$

For $I = (1,2,2,4,1,1,3)$ and $J = (2,3,1,2,3,1,2)$ for example, one has

$$\mathrm{Des}\,(I) = \{1,3,5,9,10,11\}, \mathrm{Des}\,(J) = \{2,5,6,8,11,12\}$$

and thus

$$K_{14}(I,J) = q^{2-1}\,t^{1+2}\,q^{6-3}\,q^{8-3}\,t^{1+5}\,q^{12-6} = q^{15}\,t^{9}\,.$$

The determinant of the matrix $K_n(q,t)$ can be computed, and the result is very similar to what is obtained in the classical case:

$$\det K_n(q,t) = \prod_{m=1}^{n-1}\prod_{k=1}^{m}\left(t^{m+1-k} - q^k\right)^{2^{n-1-m}\binom{m-1}{k-1}}\,. \tag{7}$$

The condition that the $\tilde{\mathrm{H}}$ functions should reduce to Hall-Littlewood functions at $q = 0$ imply the specialization

$$\tilde{\mathrm{H}}_J(A;0,1) = S^J(A) \tag{8}$$

and the symmetries of the matrix imply that

$$\tilde{\mathrm{H}}_J(A;1,0) = S^{J^\sim}\,. \tag{9}$$

Also, setting $t = q^{-1}$ and clearing the denominators, we have

$$q^{\binom{l(J)-1}{2}}\tilde{\mathrm{H}}_J(A;q,q^{-1}) \equiv R_J(A) \pmod{q\mathcal{L}} \tag{10}$$

where $\mathcal{L}$ is the $\mathbb{Z}[q]$-lattice spanned by the ribbons $R_I(A)$ in **Sym**.

The expansions of the $\tilde{\mathrm{H}}$ functions in some other bases of **Sym** can also be given in closed form. Given a composition J of n, we associate to the k-th box (where $1 \le k \le n-1$) of the diagram of J the polynomials

$$e(J,k) = \begin{cases} t^i - 1 & \text{if } k \in \mathrm{Des}\,(J)\,, \\ q^j - 1 & \text{if } k \notin \mathrm{Des}\,(J)\,. \end{cases} \tag{11}$$

if this box is located in row i and column j of the diagram. Then, the expansion of $\tilde{\mathrm{H}}_J$ on the basis Λ^I (products of elementary functions) is given by

$$\tilde{\mathrm{H}}_I = \sum_I \prod_{k \notin \mathrm{Des}\,(I)} e(I,k)\Lambda^I \,. \tag{12}$$

For example, the filling of the diagram of $J = (2,2)$ is

$q-1$	$t-1$	
	q^2-1	$\times$

so that

$$\begin{aligned}\tilde{\mathrm{H}}_{22} = \; & \Lambda^{1111} + (q-1)\Lambda^{211} + (t-1)\Lambda^{121} + (q^2-1)\Lambda^{112} \\ & +(q-1)(t-1)\Lambda^{31} + (q-1)(q^2-1)\Lambda^{22} + (t-1)(q^2-1)\Lambda^{13} \\ & +(q-1)(t-1)(q^2-1)\Lambda^4 \,.\end{aligned}$$

The expansion on the basis (S^I) is, up to powers of q and t, given by the same formula.

3 Quasi-symmetric analogues of the Macdonald functions

Quasi-symmetric Macdonald functions can now be defined by duality. The dual basis of $(\tilde{\mathrm{H}}_J)$ in $QSym$ will be denoted by $(\tilde{\mathrm{G}}_I)$. We have

$$\tilde{\mathrm{G}}_I(X;q,t) = \sum_J \tilde{g}_{IJ}(q,t)F_J(X) \tag{13}$$

where the coefficients are given by the transposed inverse of the Kostka matrix: $(\tilde{g}_{IJ}) = (\tilde{k}_{IJ})^{-1}$. Remarkably, there is for each I a polynomial $D_I(q,t)$ such that

$$\tilde{g}_{IJ}(q,t) = (-1)^{l(I)-l(J)} \frac{t^{a(I,J)}q^{b(I,J)}}{D_I(q,t)} \tag{14}$$

It will be convenient to get rid of the common denominator and to introduce the polynomials

$$\tilde{\mathrm{P}}_I(X;q,t) = D_I(q,t)\tilde{\mathrm{G}}_I(X;q,t)\,. \tag{15}$$

The denominators are also common to pairs of consecutive columns. In the notation of the previous section, if L is a composition of $n-1$,

$$D_{L'}(q,t) = D_{L''}(q,t)\,. \tag{16}$$

If $I = L'$ or $I = L''$, the formula is

$$D_I(q,t) = \prod_{(i,j)\in \mathrm{Diagr}\,(L)} (t^i - q^j)\,, \tag{17}$$

where Diagr (L) is the ribbon diagram of the composition L, the cells being labelled from top to bottom and left to right, i.e., i is the row number and j the column number, as in a matrix. For example,

Diagr $(3,1,2) =$

$t_1 - q_1$	$t_1 - q_2$	$t_1 - q_3$	
		$t_2 - q_3$	
		$t_3 - q_3$	$t_3 - q_4$

The numerators can be described as follows. Define

$$u(I,k) = \begin{cases} q^{k-d(I,k)} & \text{if } k \in \mathrm{Des}\,(I)\,, \\ t^{d(I,k)} & \text{if } k \notin \mathrm{Des}\,(I)\,. \end{cases}$$

Then,

$$t^{a(I,J)}q^{b(I,J)} = \prod_{k \notin \mathrm{Des}\,(J)} u(I,k)$$

For example, with the same I and J as before, one has

$$\{1,\ldots,13\} \setminus \mathrm{Des}\,(J) = \{1,3,4,7,9,10,13\}$$

and the numerator of $\tilde{g}_{IJ}$ is

$$(-1)^{6-7}(q^{1-0}\,q^{3-1}\,t^{1+2}\,t^{1+3}\,q^{9-3}\,q^{10-4}\,t^{1+6}) = -t^{14}\,q^{15}$$

For $n = 3$, the coefficient of F_J in $\tilde{\mathrm{P}}_I$ is in row J and column I of the matrix

$$\begin{pmatrix} 3 & t^2 & -tq^2 & -t^2q & q^2 \\ 21 & -t & t & q & -q \\ 12 & -t & q^2 & t^2 & -q \\ 111 & 1 & -1 & -1 & 1 \end{pmatrix}$$

and the denominators are

$$D_3 = D_{21} = (t-q)(t-q^2)$$

$$D_{12} = D_{111} = (t-q)(t^2-q)$$

For $n = 4$, the matrix is

$$\begin{pmatrix} 4 & t^3 & -t^2q^3 & -t^3q^2 & tq^4 & -t^4q & t^2q^3 & t^3q^2 & -q^3 \\ 31 & -t^2 & t^2 & tq^2 & -tq^2 & t^2q & -t^2q & -q^2 & q^2 \\ 22 & -t^2 & tq^3 & t^3 & -tq^2 & t^2q & -q^3 & -t^3q & q^2 \\ 211 & t & -t & -t & t & -q & q & q & -q \\ 13 & -t^2 & tq^3 & t^2q^2 & -q^4 & t^4 & -t^2q^2 & -t^3q & q^2 \\ 121 & t & -t & -q^2 & q^2 & -t^2 & t^2 & q & -q \\ 112 & t & -q^3 & -t^2 & q^2 & -t^2 & q^2 & t^3 & -q \\ 1111 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

and the denominators are

$$\begin{aligned} D_4 = D_{31} &= (t-q)(t-q^2)(t-q^3) \\ D_{22} = D_{211} &= (t-q)(t-q^2)(t^2-q^2) \\ D_{13} = D_{121} &= (t-q)(t^2-q)(t^2-q^2) \\ D_{112} = D_{1111} &= (t-q)(t^2-1)(t^3-q) \end{aligned}$$

4 Multiparameter versions and the multiplication rule

It is possible to replace q^i and t^j by independent indeterminates q_i, t_j in the combinatorial rule (6), and it turns out that most of the previous formulas remain valid for the new functions. Moreover, the multiplication rule is more transparent on the multiparameter version.

Let $Z = \{z_0 = 1, z_1, z_2, \ldots\}$ be a sequence of commuting indeterminates. We set

$$\mathbf{K}_n(A;Z) = \sum_{|I|=n} \left(\prod_{d \in \mathrm{Des}\,(I)} z_d \right) R_I \,. \tag{18}$$

Given a composition J of n, let

$$\tilde{v}(J,k) = \begin{cases} t_{1+d(J,k)} & \text{if } k \in \mathrm{Des}\,(J)\,, \\ q_{k-d(J,k)} & \text{if } k \notin \mathrm{Des}\,(J)\,. \end{cases}$$

and let

$$Z(J) = \{z_0 = 1, z_1 = \tilde{v}(J,1), z_2 = \tilde{v}(J,2), \ldots, z_{n-1} = \tilde{v}(J,n-1)\}\,.$$

We now define the *multiparameter noncommutative Macdonald functions* by:

$$\widetilde{\mathbf{H}}_J(A;Q,T) = \mathbf{K}_n(A;Z(J))\,.$$

Thus $\tilde{\mathrm{H}}_J$ is the image of $\widetilde{\mathbf{H}}_J$ under $t_j \mapsto t^j$, $q_j \mapsto q^j$, $j = 1, 2, \ldots, n-1$. The multiparameter Kostka monomials $\tilde{\mathbf{k}}_{IJ}(Q,T)$ are defined as the coefficients in the expansion of $\widetilde{\mathbf{H}}_J$ in terms of ribbons

$$\widetilde{\mathbf{H}}_J(A;Q,T) = \sum_I \tilde{\mathbf{k}}_{IJ}(Q,T) R_I(A)\,. \tag{19}$$

When $Q = 0$, one obtains in particular multiparameter Hall-Littlewood functions. In the commutative case, such multiparameter functions have been defined in [12], in the case of rectangular partitions, and their representation theoretical meaning has been explained in [1]. The noncommutative multiparameter Hall-Littlewood function corresponding to the column composition has been introduced in [9], and is used in [12] to give a closed expression for the commutative one.

For $n = 3, 4$, the transposed Kostka matrices are as follows:

$$\mathbf{K}_3 = \begin{pmatrix} 3 & 1 & q_2 & q_1 & q_1\,q_2 \\ 21 & 1 & t_1 & q_1 & q_1\,t_1 \\ 12 & 1 & q_1 & t_1 & q_1\,t_1 \\ 111 & 1 & t_2 & t_1 & t_1\,t_2 \end{pmatrix}$$

$$\mathbf{K}_4 = \begin{pmatrix} 4 & 1 & q_3 & q_2 & q_2\,q_3 & q_1 & q_1\,q_3 & q_1\,q_2 & q_1\,q_2\,q_3 \\ 31 & 1 & t_1 & q_2 & q_2\,t_1 & q_1 & q_1\,t_1 & q_1\,q_2 & q_1\,q_2\,t_1 \\ 22 & 1 & q_2 & t_1 & q_2\,t_1 & q_1 & q_1\,q_2 & q_1\,t_1 & q_1\,q_2\,t_1 \\ 211 & 1 & t_2 & t_1 & t_1\,t_2 & q_1 & q_1\,t_2 & q_1\,t_1 & q_1\,t_1\,t_2 \\ 13 & 1 & q_2 & q_1 & q_1\,q_2 & t_1 & q_2\,t_1 & q_1\,t_1 & q_1\,q_2\,t_1 \\ 121 & 1 & t_2 & q_1 & q_1\,t_2 & t_1 & t_1\,t_2 & q_1\,t_1 & q_1\,t_1\,t_2 \\ 112 & 1 & q_1 & t_2 & q_1\,t_2 & t_1 & q_1\,t_1 & t_1\,t_2 & q_1\,t_1\,t_2 \\ 1111 & 1 & t_3 & t_2 & t_2\,t_3 & t_1 & t_1\,t_3 & t_1\,t_2 & t_1\,t_2\,t_3 \end{pmatrix}$$

The factorisation of the determinant still holds:

$$\det \mathbf{K}_n(Q,T) = \prod_{m=1}^{n-1} \prod_{k=1}^{m} (t_{m+1-k} - q_k)^{2^{n-1-m}\binom{m-1}{k-1}}. \tag{20}$$

When one set of parameters is specialized to 1, e.g. $t_1 = t_2 = \ldots = t_{n-1} = 1$, one has the factorization

$$\widetilde{\mathbf{H}}_J(A;\mathbf{1},Q) = \widetilde{\mathbf{H}}_{j_1}(A;\mathbf{1},Q_1)\widetilde{\mathbf{H}}_{j_2}(A;\mathbf{1},Q_2)\cdots\widetilde{\mathbf{H}}_{j_r}(A;\mathbf{1},Q_r)\,, \tag{21}$$

where $Q_1 = \{q_1,\ldots,q_{j_1-1}\}$, $Q_2 = \{q_{j_1},\ldots,q_{j_1+j_2-1}\}$, and so on.

Moreover, the products

$$\mathbf{K}_n(Q,T)\mathbf{K}_n(Q,\mathbf{1})^{-1} \quad\text{and}\quad \mathbf{K}_n(Q,T)\mathbf{K}_n(\mathbf{1},T)^{-1}$$

are respectively lower and upper triangular matrices, whose diagonal elements can be given explicitly. This can be seen as the noncommutative multiparameter version of the characterization of Macdonald's polynomials $\tilde{\mathrm{H}}_\mu(X;q,t)$ based on the two λ-ring transformations $f(X) \to f((1-q)X)$ and $f(X) \to f((1-t)X)$ (*cf.* [7]). There are several possible tranformations extending $X \to (1-q)X$ in the noncommutative case. The one which is involved here is different from the transformation appearing in [5, 9], even when $q_i = q^i$ or $t_i = t^i$.

Given three integers n, r, c, associate to every composition J of $n+1$ the following rational function read off from the diagram of J (boxes are numeroted from 0 to n and $z_0 = 1$).

Place the first box of the diagram of J in position (r, c). Fill this box with $1/(t_r - q_c)$, and box number k, for $k = 1, \ldots, n$, in row i and column j, with

$$\begin{cases} \dfrac{t_i - z_{k-1}}{t_i - q_j} & \text{if the box has no other box above itself in its column} \\ \dfrac{z_{k-1} - q_j}{t_i - q_j} & \text{otherwise.} \end{cases}$$

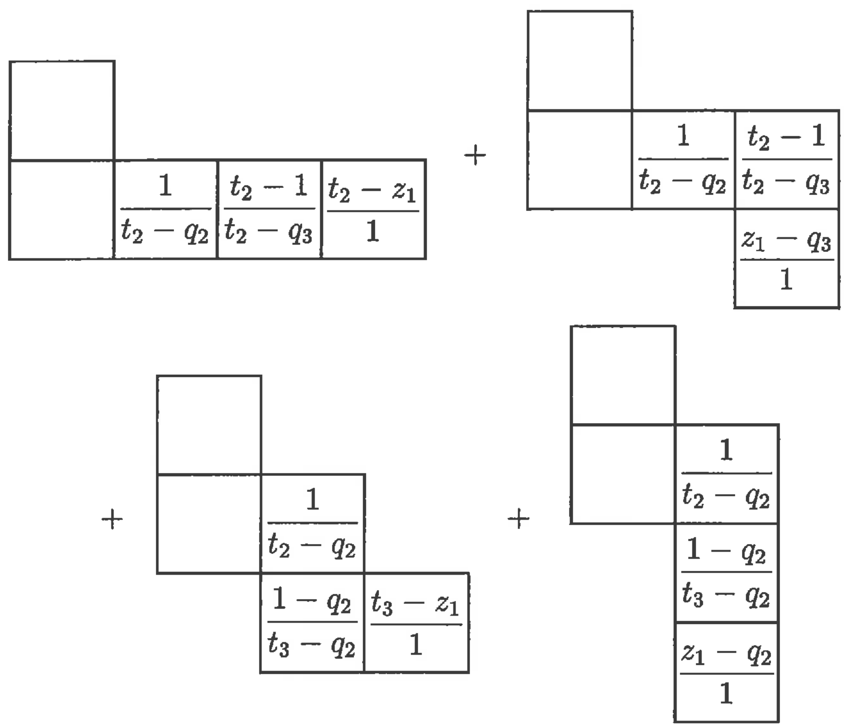
For the last box ($k = n$) omit the denominator. Let $\phi(J, r, c)$ be the product of all the entries contained in the boxes of J.

With this at hand, we can now state the multiplication rule for the $\widetilde{\mathbf{H}}$-functions. Let I be a composition, n an integer. Let (r, c) be the coordinates of the last box of the diagram of I. Then

$$\widetilde{\mathbf{H}}_I(A; Q, T)\, \mathbf{K}_n(A; Z) = \sum_J \phi(J', r, c)\, \widetilde{\mathbf{H}}_J(A; Q, T)\,,$$

where the sum is over the 2^n compositions J obtained by concatenating I with a composition of n, J' being the restriction of J to its last $n + 1$ boxes.

For example, the product $\widetilde{\mathbf{H}}_{12}\, \mathbf{K}_2(A; 1, z_1)$ decomposes into the following sums of $\widetilde{\mathbf{H}}_J(A; Q, T)$ functions (writing the coefficients in the boxes of the diagram of J)



which reads

$$\begin{aligned}\widetilde{\mathbf{H}}_{12}\,\mathbf{K}_2(A;1,z_1) &= \frac{(t_2-1)(t_2-z_1)}{(t_2-q_2)(t_2-q_3)}\,\widetilde{\mathbf{H}}_{14} + \frac{(t_2-1)(z_1-q_3)}{(t_2-q_2)(t_2-q_3)}\,\widetilde{\mathbf{H}}_{131}\\ &+ \frac{(1-q_2)(t_3-z_1)}{(t_2-q_2)(t_3-q_2)}\,\widetilde{\mathbf{H}}_{122} + \frac{(1-q_2)(z_1-q_2)}{(t_2-q_2)(t_3-q_2)}\,\widetilde{\mathbf{H}}_{1211}\end{aligned}$$

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