ON EVOLUTIONARY LAWS, INDUCING CONVEX SURFACES TO SHRINK INTO A POINT

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Abstract. The paper presents general point of view on evolutionary laws, forcing convex surfaces to shrink into a point in finite time. Collection of examples, which fail traditional treatment, illustrates nontriviality of suggested generalization.

1. Introduction

We consider evoulution $\{\Gamma_t, t>0\}$ of some closed surface Γ_0 under the law

$$E[\Gamma_t] := E(v,\mathbf{k})[\Gamma_t] = f, \quad t>0, \tag{1.1}$$

where $v(M)$ is the normal component of velocity at the point $M \in \Gamma_t, \mathbf{k}(M) = (k_1, ..., k_n)(M_t)$ is the vector of principal curvatures of Γ_t, E=E(s,S), f=f(t) are given functions, $(s,S) \in D \subset R^1 \times \mathrm{Sym}(n), n \geq 2, \mathrm{Sym}(n)$ is the space of n-th order symmetric matrices. In this context $\mathbf{k}$ ought to be considered as diagonal matrix as well as s below.

Since only the normal component of velocity involved in (1.1), desirable evolutions can exist if $v \neq 0$ and function E monotone in both variables. Without loss of generality we keep to agreements $v>0$, curvatures of the spheres are positive.

This paper concerns description of the laws, which guarantee infinite expansions of Γ_0 in infinite time or its shrinking into a point in finite time. Such general point of view on expansions was developed in the recent paper [8] and we just formulate relevant results from there later on.

Many important examples of geometric contractions, with flow by mean curvature [4] as starting point, were considered long since. However, turnes out that contractions also admit general approach and its description is the actual subject of this paper.

As it follows from modern development of the theory of fully nonlinear second-order differential equations, monotonicity of function E in both variables over some convex domain $D \subset R^1 \times \mathrm{Sym}(n)$ such that if $(s,S) \in D$, then $(s+\sigma\mathrm{sgn}(s), S + \xi \times \xi) \in D$ for $\sigma \geq 0, \xi \in R^n$, is one of the necessary restrictions. In fact, we need strict monotonicity, i.e., assume

$$E(s+\sigma, S+\xi\times\xi) > E(s,S), \quad \sigma \geq 0, \quad \xi \in R^n \quad \sigma + |\xi| > 0 \tag{1.2}$$

for $D \subset R^+ \times \mathrm{Sym}(n)$ and

$$E(s, S+\xi\times\xi) > E(s-\sigma, S), \tag{1.3}$$

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if $D \subset R^- \times \mathrm{Sym}(n)$. Here R^+, R^- are the sets of positive and negative numbers respectively. Our agreement $v > 0$ means that either s=v, or s=-v and (1.2), (1.3) present opposite types of monotonicity in v, while always positive monotone in S. In further proceeding we separate (1.2) from (1.3) saying that variables v, S are E-cooperating or E-competing respectively. Now we can say right away, what kind of evolution the law (1.1) prescribes.

Proposition 1.1. *If $v, \mathbf{k}$ are E-cooperating, then equation (1.1) prescribes expansion in the sense that relevant parabolic problem is correct, if they are E-competing, then it can only be contraction.*

Indeed, Proposition 1.1 separates correct from inverse parabolic problems, what can be easily seen in some local parametrization.

Monotonicity requirements (1.2), (1.3) carry out the notions of E-admissible surfaces, E-admissible evolutions, which are the analogs of relevant notions from the theory of fully nonlinear elliptic and parabolic equations [2], [6], [7].

Definition 1.2. *A surface Γ is E-admissible if it is closed C^3-surface and there exists positive function $v = v(M) \in C^1$ such that $(s(v), \mathbf{k})(M) \in D, M \in \Gamma$. An evolution $\{\Gamma_t, t \in [t_0; t_1)\}$ is E-admissible if the sufaces Γ_t are E-admissible for every $t \in [t_0; t_1)$. A solution to (1.1) is admissible if it appears to be E-admissible evolution.*

Here and below s(v) equals v or -v depending on D. If a set of E-admissible surfaces belongs to the set of strictly convex surfaces, the law of evolution can be prescribed in the form

$$G(v, \mathbf{r})[\Gamma_t] = g, \tag{1.4}$$

with $\mathbf{r}[\Gamma_t] = (r_1, ..., r_n)[\Gamma_t]$ to be the vector of principal radiuses of curvature. The notion of E-admissibility admits obvious reformulation in terms of $(v, \mathbf{r})$ and if so, we will speak about G_r-admissibility. The form (1.4) allows to consider contractions of convex surfaces as cooperative case, what will mostly be the basis of our proceeding. Associated with (1.4) version with competing variables will be taken in the form

$$\hat{G}(-v, \mathbf{k}) = -G(v, (\mathbf{k})^{-1}).$$

Note that we always keep positive in second variable type of monotonicity.

We admit only orthogonal invariant couples (D, E). Namely,if $(s, S) \in D$, then $(s, BSB^T) \in D, E(s, S) = E(s, BSB^T)$ for $B \in O(n)$. For diagonal matrices s it reduces to the invariance under permutation of diagonal elements.

Due to the strict monotonicity of E in v, equation (1.1) can be rewritten in the form

$$v = \hat{F}(\mathbf{k}) \tag{1.5}$$

what actually was the case in all previous papers but [8], concerning general description of the laws of evolution [1], [3], [10]. It is remarkable that in [1] right hand side of (1.5) was taken as $\hat{F} = F(\mathbf{k})$, while in [3], [10], $\hat{F} = 1/F(\mathbf{k})$. In both cases F was assumed to be positive monotone in $\mathbf{k}$ over some convex domain in R^n. This difference in setting agrees with Proposition 1.1 and also separates contractions from expansions.

General form (1.1) of evolutionary laws introduces new sources of information, namely, domain D, its boundary, function f(t) and allows to include into consideration more examples.

One of particular problems in the theory of fully nonlinear equation is to keep under control admissibility of solutions. Some general situations, when right hand side of equations regulates the latter, were first considered in the paper [2] for fully nonlinear differential second order elliptic Hessian equations. Relevant generalizations on parabolic case were suggested in [6], [7]. To adapt the idea from there to equation (1.1) we, as was done in [8] for expansions, relate to the couple $\{D, G\}$ two numbers, which may turn out infinite:

$$\underline{g} = \sup_{\partial D} \lim_{(s,S)\to(s^0,S^0)} G(s,S), \quad \overline{g} = \lim_{a\to\infty} G(a, aI).$$

The cases of interest are

$$\underline{g} < \overline{g}. \tag{1.6}$$

To formulate principal results we introduce notations:

$$E^0 = \frac{\partial E}{\partial s}, \quad E^{ij} = \frac{\partial E}{\partial s_{ij}}, \quad S = (s_{ij}).$$

Note that $E^0, (E^{ij})$ are always positive for assumed type of monotonicity of E. Denote also by $\mathrm{Sym}^+(n)$ the set of positive definite matrices. For the sake of simplicity we assume all given functions and surfaces to be as smooth as necessary.

Theorem 1.3. *Let $D \subset R^+ \times Sym^+(n)$ and Γ_0 be G_r-admissible surface. Assume that for any constant $\beta \in (0;1])$ there exists s_β such that $(s_\beta, \beta I) \in D$,*

$$\overline{g} = \lim_{s\to\infty} G(s, \beta I). \tag{1.7}$$

$$G^0 v \le \mu_1 G^{ii} r_i + \mu_2 \tag{1.8}$$

holds with some constants μ_1, μ_2 on admissible solutions to equation (1.4) and

$$\underline{g} < g(t) < \overline{g}, \quad 0 \le g_t, \quad t \in [0; \overline{T}], \tag{1.9}$$

where $\overline{T}$ is the time of shrinking of the sphere, enclosing Γ_0, into a point.

Assume also G to be concave over D function. Then there exist $T \le \overline{T}$, a point $M \in R^{n+1}$ and the unique G_r-admissible evolution $\{\Gamma_t, t \in [0;T)\}$ satisfying equation (1.4) and in addition $\Gamma_t \to M$, when $t \to T$.

The relevant expansion version from [8] looks as follows.

Theorem 1.4. *Let $D \subset R^+ \times Sym(n), (s,0) \notin D$, G be concave over D, Γ_0 be G-admissible starshaped surface. Assume*

$$\overline{g} = \lim_{a\to\infty} G(s, aS) \quad (s,S) \in D \tag{1.10}$$

and the inequality

(1.11) $$G^0 s \geq G^{ij} s_{ij}, \quad (s,S) \in D.$$

holds. Assume also g to be constant and

(1.12) $$\underline{g} < g < \overline{g}.$$

Then there exists the unique G-admissible evolution $\{\Gamma_t, t \in [0;\infty)\}$, *satisfying equation* $G(v,\mathbf{k} = g$. *Moreover, the rescaled flow* $\{\Gamma_t/R_t\}$ *converges to a sphere, when* $t \to \infty$. *Here* $R_t = r(t), G(r', 1/rI) = c, R_0 = 1$.

Theorem 1.4 has been initially proved for equations in form (1.5) with concave homogeneous function $F = 1/\hat{F}$ in [3], [10], what corrsponds to equality in (1.11). In presented generality the uniqueness and existence were obtained in [8], (see Theorem 1.1). Asymptotic behaviour was also considered in there under assumption (1.11) to be strict inequality (Theorem 1.2). Recently the author has learnt the proof of asymptotic convergence to a sphere under (1.11), discovered by Th.Nehring (private communication).

There exist flows, when strict convexity of initial surface regulates E-admissibility of solution, instead of inequalities (1.9), (1.12), i.e. the whole boundary of D is not of interest and requirement (1.6) gets redundant. For expansions it was observed in [10]. Namely, there was shown that evolution (1.5) with $\hat{F} = 1/F(\mathbf{k})$, F to be homogeneous, positive monotone, concave in $\mathbf{k}$ and convex in $\mathbf{r}$ preserves strict convexity of initial surface, while convexity was not necessary for solutions to be admissible. Here we present sufficient conditions, ensuring such preserving for contractions. To proceed with introduce the numbers

$$\underline{c} = \sup_{s>0} \lim_{\alpha \to 0} G(s, \alpha I), \quad \overline{c} = \inf_{s>0} \lim_{a \to \infty} G(s, aI)$$

instead of $\overline{g}, \underline{g}$. Note that the analog of inequality (1.6) is also required for $\underline{c}, \overline{c}$.

Theorem 1.5. *Let* $D \subset R^+ \times Sym^+(n)$, *G be concave over D and* Γ_0 *be* G_r-*admissible surface. Assume that for any constant* $\beta \in (0;1])$ *there exists* s_β *such that* $(s_\beta, \beta I) \in D$,

$$\overline{c} \leq \lim_{s \to \infty} G(s, \beta I),$$

the inequality (1.8) gets satisfied with some μ_1, μ_2,

(1.13) $$\hat{G}^0 v \leq \hat{G}^{ii} k_i$$

on admissible solutions to equation (1.4) and

(1.9′) $$\underline{c} < g(t) < \overline{c}, \quad g_t > 0.$$

Assume also that function v defined by (1.5) is concave in $\mathbf{k}$. *Then there exist* $T \leq \overline{T}$, *a point* $M \in R^{n+1}$ *and the unique* G_r-*admissible evolution* $\{\Gamma_t, t \in [0;T)\}$ *satisfying equation (1.4) and in addition* $\Gamma_t \to M$, *when* $t \to T$.

Note that boundary of D does not appear in conditions of Theorem 1.5. It means we are free to consider only part of maximal domain of E-admissibility, if have managed to relate numbers $\underline{c}, \overline{c}$ to some $D \subset R^+ \times \mathrm{Sym}^+(n)$. The examples will be presented in Sections 2, 5.

It is of common knowledge that evolution of E-admissible immersed surface under the law (1.1) does exist, if started by E-admissible surface, at least some finite time t_1 and its further existence depends on possibility to establish a priori estimates. In order to ensure convergence of the flow into a point we apply the following idea from [9].

Proposition 1.6. *Let $v[\Gamma_t], k_i[\Gamma_t], i = 1, ..., n$ be a priori uniformly in t bounded from below by positive constant and from above, untill the surfaces Γ_t enclose a ball B_ρ with some $\rho > 0$ and evolution of $\{\Gamma_t\}$ under the law (1.1) stays E-admissible. Then, if $v, \mathbf{k}$ are E-competing, there exist $T \in (0; \overline{T}), \overline{T} < \infty$ and a point M such that admissible solution to equation (1.1) exists for $t < T$ and $\Gamma_t \to M$, when $t \to T$.*

Construction of relevant estimates under conditions of Theorems 1.3, 1.5 is the subject of Sections 4 – 5. Section 3 contains local reduction of geometric flow (1.1) to fully nonlinear second order parabolic equation. In Section 6 we discuss pinching property, imposing some homogenuity requirements on E, and also the case of convex E. Section 2 contains examples to indicate the tendencies. There are also examples in Sections 4 - 6 to illustrate speciality of relevant Section.

This paper resumes author's contemplation over the papers of [1], [9]. It was started in cooperation with Prof.F.Tomi and Dr.Th.Nehring in Heidelberg University, 1999, where the first version of Theorem 1.4 was proved. The whole picture, presented here, was inspired by author's participation in the programme "Nonlinear Partial Differential Equations", Isaac Newton Institute, Cambridge, 2001.

2. Examples

Our examples based on nowdays well known properties of elementary symmetric functions and quotients. Denote

$$H_{m,l}(S) = \frac{\mathrm{tr}_m S}{\mathrm{tr}_l S}, \quad F_{m,l}(S) = H_{m,l}^{\frac{1}{m-l}}(S) \quad 0 \le l < m \le n,$$

where $\mathrm{tr}_i S$ is the sum of all principal i-th order minors of matrix S. When l=0, we omit the second index, i.e., $H_m := H_{m,0}$. Function $F_{m,l}$ is one-homogeneous, positive monotone, concave over the cone C_m,

$$C_m = \{S \in \mathrm{Sym}(n) : H_i(S) > 0, \quad i = 1, ..., m\}.$$

The following simple observation from [8] will also be of use.

Proposition 2.1. *Let $\{G_i\}$ be positive monotone, concave functions over $D \subset R^+ \times Sym(n)$, ϕ_i, defined on the range of G_i respectively, be strictly increasing, concave functions and λ_i be positive constants. Then function $G := \sum_i \lambda_i \phi_i \circ G_i$ succeeds properties of positive monotonicity and concavity over D.*

Proposition 2.1 allows to increase the amount of examples, subjected to Theorem 1.3. For instance, to include into consideration the following function from [8]

$$G(v, S) = \log s + \beta \log F_n - \frac{1}{sF_1^\gamma}, \quad \beta, \gamma > 0. \tag{2.1}$$

with $D = D^+ := R^+ \times \mathrm{Sym}^+(n)$.

We start our collection of examples with

$$v[\Gamma_t] = F_{m,l}^q(\mathbf{k})[\Gamma_t]. \tag{2.2}$$

Proposition 2.2. *Any strictly convex closed surface Γ_0 starts strictly convex, positive monotone in t evolution by (2.2), which converges into a point in finite time T, if either of conditions fulfilled:*

(i) $$q \in (0;1) \cup (1;\infty), \quad l = 0;$$

(ii) $$q = 1, \quad 0 < l < m.$$

Moreover, if $-1 \leq q < 0$, then any starshaped surface Γ with

$$\mathbf{k}(M) \in C_m, \quad M \in \Gamma \tag{2.3}$$

starts the unique expansion $\{\Gamma_t, t \in (0:\infty)\}$, satisfying (2.2) and (2.3). In addition, evolution $\{\Gamma_t\}$ asymptotically tends to a sphere.

To prove Proposition 2.2, when (i), we rewrite equation (2.2) on D^+ in equivalent form

$$G(v,\mathbf{r}) = -\frac{1}{vF^q_{n,n-m}(\mathbf{r})}[\Gamma_t] = -1$$

and see that conditions of Theorem 1.3 are fulfilled with $D = D^+, \underline{g} = -\infty, \overline{g} = 0, \mu_1 = 1/q, \mu_2 = 0$, due to Proposition 1.1 and properties of functions $F_{n,n-m}$.

If (ii), associate with equation (2.2) relations

$$G(v,\mathbf{r}) = \frac{-1}{v}F_{n-m,n-l}(\mathbf{r}) = -1, \quad \hat{G}(-v,\mathbf{k}) = \frac{1}{v}F_{m,l}(\mathbf{k}) = 1$$

and apply Theorem 1.5 with $\underline{c} = -\infty, \overline{c} = 0, \mu_1, \mu_2$ to be the same as above. Inequality (1.13) turnes into equality and v is concave in $\mathbf{k}$, hence all conditions of this Theorem are fulfilled and contraction part of Proposition 2.2 gets valid.

As to expansions, we just rewrite equation (2.2) in the form

$$G(v,\mathbf{k}) := -\frac{1}{vF^{|q|}_{m,l}(\mathbf{k})} = -1$$

and check that all conditions of Theorem 1.4 are satisfied. In fact, this example was taken from [8].

We see of particular importance equation (2.2) with $1 \leq l < m \leq n-1$. ¿From general point of view we should relate to such equations $\hat{D} = R^- \times C_m$, i.e. admit nonconvex surfaces Γ_t as admissible to our problem. It is rather remarkable that rewriting of these equations in terms of $(v,\mathbf{r})$ neither helps because formally it brings out as natural the domain $D \in R^+ \times C_{n-l}$ and we can not control convexity of admissible surfaces by right hand side of (1.1). That was the reason to speak about "preserving convexity" flows .

To our knowledge, Proposition 2.2 has no analogs for $l \geq 1, m < n$ in previous investigation of contractions. For expansions this phenomenon was discovered in [10].

It may be shown that evolutionary laws from Theorem 1.5 always admit expression (1.1) with homogeneous in $v, \mathbf{k}$ function E. It is far not so with Theorem 1.3. For instance, the equation

$$v = H_m(\mathbf{k} + \gamma I), \quad \gamma > 0, \tag{2.4}$$

can not be reduced to homogeneous in $\mathbf{k}$ form but Theorem 1.3 covers it. To show the latter we include (2.4) as particular case in the following set of evolutionary laws

$$\sum_0^n \frac{a_i}{v^{q_i}} H_i(\mathbf{k}) = c, \quad a_i \geq 0, \quad \sum_1^n a_i > 0, \quad q_i > 0. \tag{2.5}$$

Proposition 2.3. *Assume $c = c(t), c_t \leq 0$ in (2.5). Then arbitrary strictly convex surface shrinks into a point in finite time T under the law (2.5). Moreover, $\{\Gamma_t, t \in [0;T)\}$ is strictly convex monotone solution to (2.5).*

To prove Proposition 2.3 we apply Theorem 1.3 to the following, equivalent on convex surfaces to (2.5), equation

$$G(v,\mathbf{r}) = -\sum_0^n \frac{a_i}{v^{q_i}} H_{n-i,n}(\mathbf{r}) = -c.$$

Here $D = D^+, \underline{g} = -\infty, \overline{g} = 0$,

$$\mu_1 = \max_i q_i = \overline{q}, \quad \mu_2 = \frac{\overline{q}a_0}{v_0^{q_0}}, \quad v_0 = \min_{\Gamma_0} v. \tag{2.6}$$

Since Γ_0 has been given G-admissible surface, μ_2 is well defined. Due to Proposition 2.1 and concavity of $F_{n,n-i}$, function G is concave in D^+. Hence, equation (2.5) and (2.4), in particular, satisfy assumptions of Theorem 1.3.

We resume indication of tendencies by

Proposition 2.4. *Let Γ_0 be strictly convex closed surface and G be given by (2.1) with constant g. Then, if $\beta, \gamma \in (0;1]$, equation $G(v,\mathbf{k}) = g$ defines the unique G-admissible infinite expansion of Γ_0, asymptotically converging into a sphere, while $G(v,\mathbf{r}) = g$ defines the unique G-admissible evolution of Γ_0 shrinking into a point in finite time whatever positive β, γ be.*

Assertions of Proposition 2.6 follows from Theorems 1.4, 1.3 respectively.

We remark also that, by Theorem 1.4, evolutionary law

$$v^m = H_m(\mathbf{r} + \gamma I), \quad 0 < m \leq n,$$

uniquely defines infinite expansion of arbitrary strictly convex surface and, properly rescaled, it converges to a sphere.

3. Reduction of geometric evolutionary problems to the problems for fully nonlinear parabolic equations

The variety of parametrizations of hypersurfaces suggests variety of reductions of our problems to relevant pieces in the theory of second order parabolic equations. We choose two simplest local parametrization, one will treat the case of competing and the second cooperating variables (v,S).

Let $(M_0) \in \Gamma_{t_0}$ be the origin of Euclidian coordinate system $\{Y = (y, y^{n+1}), y = (y_1, ..., y_n)\}, \nu(M,t)$ be interior normal to Γ_t at M and $\nu(M_0, t_0) = (0, ..., 1)$. There are vector-functions Y(t) such that in some vicinity of (M_0, t_0)

$$\Gamma_t = \{y(t), y^{n+1} = u(y,t)\}, \quad |u_y(0,t_0)| = 0, \quad v = (Y_t, \nu) \tag{3.1}$$

with

$$(3.2) \qquad \nu = -\frac{(u_1, ..., u_n, -1)}{\sqrt{1+u_y^2}}.$$

Here and below lower indices are differential. Namely,

$$u_i = \frac{\partial u}{\partial y^i}, \quad |u_y|^2 = \sum_1^n u_i^2, \quad u_t = \frac{\partial u}{\partial t}.$$

We also notate Hesse matrix of u by $u_{yy} = (u_{ij})$. When competing variables, i.e., $(v, \mathbf{k})$ are of interest, parametrization (3.1) will be fixed such way that vector y is independent on t and matrix $u_{yy}(0, t_0)$ gets diagonal. Consider symmetric matrix $\tau[\Gamma_t]$,

$$\tau = \sqrt{g^{ij}}, \quad g^{ij} = \frac{1}{\sqrt{1+u_y^2}}(\delta_j^i - \frac{u_i u_j}{1+u_y^2}),$$

and let $u_{(yy)} := \tau u_{yy} \tau$. Then

$$v = \frac{u_t}{\sqrt{1+u_y^2}} := u_{(t)},$$

principal curvatures of Γ_t are the eigenvalues of $u_{(yy)}$ and, due to orthogonal invariance of E, in our vicinity of (M_0, t_0) evolutionary geometric equation (1.1) reduces to partial parabolic second-order differential equation

$$(3.3) \qquad E(v, \mathbf{k}) = \hat{G}(-u_{(t)}, u_{(yy)}) = \hat{g}.$$

The case of cooperating variables $(v, \mathbf{r})$ will be attended by different parametrization. To describe it we introduce X(t) as a position vector of strictly convex surfaces (3.1) with origin enclosed by Γ_t. Let P(t) be Legendre transformation of $X(t) = X_0 + Y(t)$, i.e.,

$$p = u_y, \quad h = (x_0 + y, p) - (x_0^{n+1} + u).$$

Then locally $\Gamma_t = \{p, p_{n+1} = h(p,t)\}, \nu = \nu(p)$. In contrast to previous specialization we fix P-parametrization of Γ_t with vector p independent on t in some neighborhood if (M_0, t_0). Now, the radiuses of curvature of Γ_t are the eigenvalues of the matrix $h_{(pp)} := \eta h_{pp} \eta$ with $\eta = \eta(p)$ been the inverse to τ,

$$\eta^2 = (g_{ij})(p) = \sqrt{1+p^2}(\delta_j^i + p_i p_j).$$

In this situation

$$v = (Y_t, \nu) = (X, \nu)_t = \frac{-h_t}{\sqrt{1+p^2}} := h_{(t)}$$

and partial differential equation

$$(3.5) \qquad G(-h_{(t)}, h_{(pp)}) = g,$$

with G associated with $\hat{G}$, gets locally equivalent to (1.1).

To construct a priori estimates we use the following linearizations of fully nonlinear operators from (3.3), (3.5).

$$(3.6) \qquad L[w] := -E^0 w_{(t)} + E^{ij} w_{(ij)},$$

where the meaning of round brackets depends on the type of parametrization, and actually line (3.6) presents two different linear parabolic operators.

The following obvious generalization of common approach to maximum principle will be involved somtimes in the later reasoning.

Proposition 3.1. *Let $w', w'' \in C^2(\Omega), \Omega \subset R^n, w'' > 0$. Assume $w = w'/w''$ attains its maximum at some interior point from Ω. Then at this point*

$$L[w'] - wL[w''] \leq 0. \tag{3.7}$$

Theorem 1.5 contains the requirement for v, implicitly given by $\hat{G}(-v, \mathbf{k}) = \hat{g}$ as function of $\mathbf{k}$, to be concave. Applied to parametrized equation (3.3), it leads to conclusion

Proposition 3.2. *If equation $\hat{G}(-v, S) = c$ defines concave in S function v, then*

$$\frac{\partial^2 \hat{G}}{\partial v^2} v_1^2 + 2\frac{\partial^2 \hat{G}}{\partial v \partial u_{(ij)}} v_1 u_{(ij)1} + \frac{\partial^2 \hat{G}}{\partial u_{(ij)} \partial u_{(kl)}} u_{(ij)1} u_{(kl)1} \leq 0. \tag{3.8}$$

4. Estimation, when cooperating v, r. Proof of Theorem 1.3

Geometric evolutionary equation (1.4) is under consideration here and we start the proceeding with

Lemma 4.1. *Let $\Gamma_t, t \in [0; t_1]$ be admissible solution to (1.4). Assume*

$$g_t \geq 0, \quad t \in (0; t_1]. \tag{4.1}$$

Then

$$v[\Gamma_t] \geq \min_{\Gamma_0} v[\Gamma_0] > 0. \tag{4.2}$$

If in addition G is concave over D, then

$$r_i[\Gamma_t] \leq \max_{i, \Gamma_0} r_i[\Gamma_0], \quad i = 1, ..., n. \tag{4.3}$$

Proof. Fix some point (M,t) and rewrite equation (1.4) as (3.5) in vicinity of this point. Due to the choice of parametrization,

$$v_t = -h_{tt}, \quad v_{(ii)} = -h_{iit} - v \tag{4.4}$$

at this point. Differentiating equation (3.5) in t, we arrive to inequality

$$L[v] = -g_t - v\sum_1^n G^{ii} < 0$$

at (M,t), what proves assertion (4.2).

To obtain (4.3) assume there are $t' \in (0; t_1], M' \in \Gamma_{t'}$ such that $r_1(M', t') \geq r_i[\Gamma_t], t \in (0; t_1], i = 1, ..., n$. Relate again eguation (3.5) to this point and note that $h_{(11)}$ also attains maximum at $(0, t')$. Since at our point

$$h_{(ii)t} = h_{tii}, \quad h_{(ii)(jj)} - h_{(jj)(ii)} = h_{ii} - h_{jj}, \quad i, j = 1, ..., n, \tag{4.5}$$

(4.4) and relevant differentiation of equation (3.5) brings out the inequality

$$L[h_{(11)}] \geq G^0 v + G^{ii}(h_{11} - h_{ii}) > 0.$$

But then $h_{(11)}$ can not attain maximum for $t' > 0$. This contradiction proves (4.3).

The uniqueness of admissible solution to equation (1.4) follows, for instance, from Inclusion Principle for G_r-admissible evolutions.

Theorem 4.2. *Let $\Gamma_t, \tilde{\Gamma}_t$ be G-admissible evolutions of $\Gamma_0, \tilde{\Gamma}_0$ respectively and satisfy the relations*

$$G[\Gamma_t] = g, \quad G[\tilde{\Gamma}_t] \leq \tilde{g}, \quad t \in [0;T). \tag{4.6}$$

Assume $\tilde{\Gamma}_0$ encloses Γ_0, g satusfies (4.1) and

$$g(t) \geq \tilde{g}(t), \quad t \in (0;T) \tag{4.7}$$

. Then $\tilde{\Gamma}_t$ encloses Γ_t for all t.

Proof. Let

$$d^\epsilon(t) := \underline{\operatorname{dist}}\{\Gamma_{t+\epsilon}; \tilde{\Gamma}_t\}.$$

Due to (4.2), $d^\epsilon(0) > 0$ for arbitrary small $\epsilon > 0$. Assume there exists

$$0 < \underline{t} = \min_t\{t < T : d^\epsilon(t) = 0\}.$$

Then function $d^\epsilon(t)$ attains its minimum at some t_0 on any interval $(0;t_1], t_1 < \underline{t}$. Also, there is parametrization of $\Gamma_{t+\epsilon}, \tilde{\Gamma}_t$ such that function

$$w = \frac{(\tilde{h} - h^\epsilon)(p,t)}{\sqrt{1+p^2}}$$

attains its minimum at the point $(0, t_0)$, what follows from the fact that in some neighborhood of $(0, t_0)$ w is the difference of normalized support functions of the surfaces $\Gamma_{t+\epsilon}, \tilde{\Gamma}_t$ respectively, i.e., $w = ((\tilde{X}, -\nu) - (X^\epsilon, -\nu))$ with the same normal ν. Since $w_t \leq 0, w_{pp} \geq 0$ at $(0, t_0)$, presentation (3.5), monotonicity of G and speciality of our point bring out relations

$$G(-\tilde{h}_{(t)}, \tilde{h}_{pp}) \geq G(-h^\epsilon_t, h^\epsilon_{pp} + d^\epsilon(t_0)I) > g(t_0 + \epsilon) \geq g(t_0).$$

The latter contradicts to assumptions (4.6), (4.7) and t_0 does not exist. But then also there is no $\underline{t}$, what means $\tilde{\Gamma}_t$ encloses $\Gamma_{t+\epsilon}$, untill both evolutions exist. Since ϵ has been arbitrary, it proves Theorem 4.2.

To estimate velocity from above we adapt one finding from the paper [9] to new enviroments.

Lemma 4.3. *Let $\{\Gamma_t, t \in [0;t_1]\}$ be admissible solution to equation (1.4) and Γ_{t_1} encloses a boll B_ρ. Assume function G is concave over D and satisfies conditions (1.7), (1.8). Then*

$$v[\Gamma_t] \leq C(\rho, \Gamma_0), \quad t \in [0;t_1]. \tag{4.8}$$

Proof. Let v_ρ be solution to equation

$$G(v, \frac{\rho}{2(\mu+1)}) = c, \quad c = \max_{(0;t_1)} g + \frac{\mu_2 + \delta}{1 + \mu_1} \tag{4.9}$$

with some $\delta > 0$. Assumption (1.7) guarantees existence of v_ρ if $c < \overline{g}$. Without loss of generality μ_1 in (1.8) can be taken sufficiently large and δ small to get the latter fulfilled. Denote

$$\overline{v} = \max\{v_\rho; 2\frac{|g_t|R_0}{\delta}\},$$

where R_0 is the radius of a sphere, enclosing Γ_0, and introduce function

$$w = -\frac{v}{(X,\nu) + \frac{\rho}{2}}, \tag{4.10}$$

where X(t) – position vector of Γ_t with origine in the centre of given ball B_ρ. Then, by Theorem 4.2, Γ_t encloses B_ρ for all $t \in [0; t_1]$ and $-(X,\nu) \geq \rho, w > 0$. Let w attains maximum at $(M', t') \in \{\Gamma_t\}$. Relate to (M', t') standard parametrization and equation (3.5). Then function (4.10) looks as

$$w = \frac{-h_t}{h - \frac{\rho}{2}\sqrt{1+p^2}}$$

in some neighborhood of $(0, t')$ and at $(0; t')$

$$0 \geq L[w] = -g_t - w(G^0 v + G^{ii} h_{ii}) + \frac{\rho}{2}\sum_1^n G^{ii}. \tag{4.11}$$

Assume $v > \overline{v}$ in (4.11). Then, due to concavity of G in S, the following line reads true

$$\frac{\mu_2 + \delta}{2(\mu_1+1)} < G(v, \frac{\rho}{2(\mu_1+1)}I) - G(v, h_{(pp)}) \leq -G^{ii}h_{ii} + \frac{\rho}{2(\mu_1+1)}\sum_i G^{ii}. \tag{4.12}$$

But inequality (1.8) makes (4.11), (4.12) incompatible, what means that $v \leq \overline{v}$ at the point of maximum of w. The latter validates (4.8).

We are now in position to deliver

Proof of Theorem 1.3. We constructed all estimates to apply Proposition 1.6 but the estimate for curvatures from above. The latter is equivalent to estimation of radiuses of curvature from below and here the left hand inequality (1.9) gets involved. Indeed, let the reqirements of Lemma 4.3 be satisfied and

$$\underline{r} := r_1(M', t') \leq r_i(M, t), \quad (M,t) \in \{\Gamma_t\}, \quad \overline{r}_0 := \max_{i,\Gamma_0} r_i[\Gamma_0], \quad i = 1, \dots, n.$$

Then, due to (1.9), (4.3), (4.8), the relation

$$\underline{g} < g = G(v, \mathbf{r}) \leq G(C(\rho, R_0), \overline{r}, \dots, \overline{r}, \underline{r})$$

holds valid at (M', t') and gives wanted estimate for $\underline{r}$ from below.

To underline speciality of Theorem 1.3 consider an example of G associated with nontrivial domain D, namely, the equation

$$v^q F_{n,l}(vI - \mathbf{k}) = g^{q+1}, \quad 0 \leq l < n, \quad q \geq 0. \tag{4.13}$$

Rewrite (4.13) as

$$G(v, \mathbf{r}) := (v^q F_{n,l}(vI - \mathbf{r}^{-1}))^{\frac{1}{q+1}} = g$$

and relate to G

$$D = \{(s, S) \in R^+ \times \mathrm{Sym}^+(n) : sI - S^{-1} \in \mathrm{Sym}^+(n)\}.$$

Function G is concave over D, $\underline{g} = 0, \overline{g} = \infty$, inequality (1.7) and (1.8) with $\mu_1 = q, \mu_2 = q \max_t g$ are satisfied. Moreover, arbitrary strictly convex surface is G_r-admissible and capable to start evolution (4.13), which shrinks into a point in finite time, if $g > 0, g_t \geq 0$.

5. Estimation, when competing $(v, \mathbf{k})$. Proof of Theorem 1.5

Equations (1.1) with $D \subset R^- \times \mathrm{Sym}(n)$ determine contractions and we rewrite it here as

$$E(s,S)[\Gamma_t] = \hat{G}(-v,\mathbf{k})[\Gamma_t] = \hat{g}. \tag{5.1}$$

This time we keep to parametrization (3.1), (3.2) and, instead of (4.4), (4.5), involve

$$v_t = u_{tt}, \quad v_{(ii)} = u_{tii} - vu_{ii}^2, \tag{5.2}$$

$$u_{(ii)(jj)} - u_{(jj)(ii)} = u_{ii}u_{jj}(u_{ii} - u_{jj}) \tag{5.3}$$

at the point $(0, t_0)$ as a substitute to $(M, t) \in \{\Gamma_t\}$. Identities (5.2), (5.3) are ifinitesimal versions of some well known global relations in differential geometry and can be easily derived by differentiation of $u_{(t)}, u_{(yy)}$ in vicinity of zero, [5].

Differentiation of equation (3.3), which is locally equivalent to (5.1), carries out the following line

$$L[v] = -G^0 v_t + G^{ii} v_{(ii)} = \hat{g}_t - vG^{ii}u_{ii}^2 \tag{5.4}$$

and also for v concave in $\mathbf{k}$ over D, due to (3.8),

$$L[u_{(11)}] \geq u_{11}(u_{11}(-\hat{G}^0 v + G^{ii}u_{ii}) - \hat{G}^{ii}u_{ii}^2). \tag{5.5}$$

Lemma 5.1. *Let $\{\Gamma_t, t \in [0;T]\}$ be admissible solution to equation (5.1). Assume $\hat{g}_t \leq 0$. Then*

$$v[\Gamma_t] \geq \min_{\Gamma_0} v[\Gamma_0]. \tag{5.6}$$

Lemma 5.1 is obvious consequence to (5.4). Moreover, line (5.6) formally identical to (4.1). But assertion of Lemma 5.1 serves more general set of equations, there is no requirement $D \subset R^- \times \mathrm{Sym}^+(n)$. For instance, it holds for m-convex solutions to equation (2.1) with $m \neq n$. We don't know if Inclusion Principle holds in such generality.

Further development involves requirement (1.13) for admissible to (3.3) solutions.

Lemma 5.2. *Assume in addition to assumptions of Lemma 5.1 concavity of v in $\mathbf{k}$, (1.13) and also convexity of the surfaces $\Gamma_t, t \in [0; t_1]$. Then*

$$\frac{k_i}{v}[\Gamma_t] \leq \max_{i,\Gamma_0} \frac{k_i}{v}, \quad i = 1, \ldots, n. \tag{5.7}$$

Proof. Let

$$\max_{i,\{\Gamma_t\}} \frac{\exp(-\epsilon t)k_i}{v}[\Gamma_t] = \frac{\exp(-\epsilon t')k_1}{v}(M', t'). \tag{5.8}$$

Relate to (M', t') parametrization (3.1), (3.2). Then function

$$w^\epsilon = \frac{\exp(-\epsilon t)u_{(11)}}{u_t}(y, t)$$

attains maximum at $(0,t')$. If $t'>0$, Proposition 3.1 with

$$w=w^{\epsilon},\quad w'=\exp(-\epsilon t)u_{(11)},\quad w''=u_{(t)}$$

and relations (5.4), (5.5) bring out inequality

$$0\geq-\frac{1}{v}\hat{g}_t+\epsilon G^0+u_{11}(G^{ii}u_{ii}-G^0u_t). \tag{5.9}$$

But assumption (1.13) makes line (5.9) impossible, hence $t'=0$. Since $\epsilon>0$ has been arbitrary, our argument validates inequality (5.7).

Now we are to prove Theorem 1.5. Two functions $G,\hat{G}$ participate in conditions of this Theorem. Both of them originated by our evolutionary equation (1.1) and G will be function from Section 4, associated with $\hat{G}$.

Proof of Theorem 1.5. As the result of above development, we have a priori estimates for curvature from above depending on velocity, (5.7), and from below for very velocity, (5.6). The requirement of convexity of v in $\mathbf{r}$ and strict convexity of Γ_t makes available some results from Section 4, namely, Inclusion Principle, estimate (4.3) for curvatures from below and also (4.8) for velocity from above, if Γ_{t_1} encloses a ball B_ρ. By assumption (1.9′) our solution succeeds G-admissibility of initial surface, i.e., equation does not degenerate. So, we are again in conditions of Proposition 1.6, what proves Theorem 1.5.

Remark. *Requirement of concavity (convexity) of function v in $\mathbf{k}$, when competeing variables, can be compatible with inequality (1.13) or inverse for only homogeneous F, $v=F(\mathbf{k})$. Hence, the law (1.1), subjected to Theorem 1.5, may always be expressed in terms of homogeneous function E.*

Corollary to Theorem 1.5. *Any admissible solution $\{\Gamma_t,t>0\}$ to equation (1.1) shrinks into a point in finite time under conditions of Theorem 1.5, if there existed $t=t_1$ such that Γ_{t_1} turned out to be strictly convex.*

Theorem 1.5 contains implicit assumption that admissible surfaces do exist and this point ought to be attended to for concrete evolutionary equations. Consider, for instance, following example.

$$\hat{G}(-v,\mathbf{k})=-v+F_{m,l}(\mathbf{k})=\hat{g},\quad 0\leq l<m\leq n \tag{5.10}$$

with $\hat{g}=\hat{g}(t)$. We associate to $\hat{G}$ function G from Section 4:

$$G(v,\mathbf{r})=v-F_{n-m,n-l}(\mathbf{r})=g \tag{5.11}$$

with $g=-\hat{g}$. Functions G, $\hat{G}$ are concave, positive monotone, satisfy inequalities (1.8), (1.13) and by assumption $g_t\geq 0$. Hence, Theorem 1.5 can be applied with $\underline{c}=-\infty,\overline{c}=0$ and $g<0$. But requirement of E-admissibility of initial surface Γ_0 presupposes $v_0=F_{m,l}(\mathbf{k})[\Gamma_0]>0$, i.e., Γ_0 ought in addition to satisfy inequality

$$F_{m.l}(\mathbf{k})[\Gamma_0]>\hat{g}(0),\quad M\in\Gamma. \tag{5.12}$$

Therefore, Theorem 1.5 guarantees that contraction (5.10) can be started by strictly convex surface satisfying (5.12).

The case $\hat{g} = 0$ fails the above argument but can be treated by Theorem 1.5 with arbirary strictly convex Γ_0 if, instead of (5.10), (5.11), we rewrite the law of evolution in the form

$$\hat{G}'(v, \mathbf{k}) := -(vF_{l,m}(\mathbf{k}))^{\frac{1}{2}} = -g', \quad G'(v, \mathbf{r}) := (vF_{n-l,n-m}(\mathbf{r}))^{\frac{1}{2}} = g' \tag{5.13}$$

with $g' = 1$. Actually, it is the example from Section 2, where different choice of G was suggested. Reformulation (5.13) satisfies all conditions of Theorem 1.5 but with another $\underline{c}, \overline{c}$. This time $\underline{c} = 0, \overline{c} = \infty$ and $g' = 1$ certainly gets included.

So, there exists the unique admissible solution, started by strictly convex Γ_0, to equation (5.10) with $\hat{g} \geq 0, \hat{g}_t \leq 0$, if in addition (5.12) holds. Moreover, this solution shrinks into a point in finite time.

If $l = 0$, equation (5.10) is also subject to Theorem 1.3 as particular case of equations (2.3) with q=1. When $l > 0, m < n$, the set of all $\hat{G}$-admissible evolutions to (5.10) contains nonconvex flows and we see here the example of preserving convexity laws.

Equation (5.13) with $g' > 0, g'_t \geq 0$ represents one more example of evolutionary preserving convexity law because maximal domain of E-admissible evolutions is considerably wider, than the set of monotone convex flows if $1 < l < m < n$. Note that, in contrast to (5.10), arbitrary strictly convex surface is E-admissible for (5.13).

6. Homogeneous laws of contractions

We consider here evolutionary laws in the form

$$E[\Gamma_t] := E(s, \mathbf{s})[\Gamma_t] = 0, \quad (s, \mathbf{s}) \in D \subset \overline{R}^- \times \mathrm{Sym}^+(n), \tag{6.1}$$

where E is homogeneous function,

$$s[\Gamma_t] = a_0 - a_1 v, \quad \mathbf{s}[\Gamma_t] = \mathbf{k} - a_2 v I \tag{6.2}$$

with consant $a_i \geq 0, i = 0, 2, a_1 > 0$. Option (6.2) indicates the possibility to involve linear functions of $v, \mathbf{k}$ in capacity of $s, \mathbf{s}$. It also describes, in what sense laws (6.1) are to be homogeneous.

We always assume E to be defined on $D^0 = R^- \times \mathrm{Sym}^+(n)$. Then for homogeneous E relation $(-1, I) \in D$ implies $D^0 \subset D$ and in further development D will be fixed as D^0.

Denote

$$\mathbf{s}^{n-1,1} = (\overline{s}, ..., \overline{s}, \underline{s}), \quad \overline{s} \geq \underline{s}$$

and consider first the case of concave E.

Lemma 6.1. *Let E be concave over D, $a_0 = 0$. Assume that for arbitrary $\alpha > 0, \overline{s} > 1$ there exists $\underline{s}_0 = \underline{s}(\alpha, \overline{s})$ such that $(-\alpha, \mathbf{s}_0^{n-1,1}) \in D$ and*

$$E(-\alpha, \mathbf{s}^{n-1,1}) < 0. \tag{6.3}$$

Then for admissible to equation (6.1), (6.2) solutions $\{\Gamma_t, t \in [0; t_1]\}$ the following inequalities hold

$$\frac{k_i}{v} \leq \max_{\Gamma_0, j} \frac{k_j}{v} = C, \quad i, j = 1, \ldots, n, \tag{6.4}$$

$$0 < \delta(a_1, C) \leq \frac{k_i}{v} - a_2, \quad M \in \Gamma_t. \tag{6.5}$$

Proof. First of all, note that E-admissible surfaces are strictly convex. Moreover, homogenuity of E brings out the relation

$$\frac{\partial E}{\partial v}v + \frac{\partial E}{\partial k_i}k_i = 0$$

being true on admissible solutions. The latter means inequality (1.13) gets valid, conditions of Lemma 5.2 satisfied and assertion (6.4) proved.

To obtain (6.5) we fix some point $(M,t), M \in \Gamma_t, 0 < t \le t_1$ and choose

$$\overline{s} = C - a_2, \quad \underline{s} = \frac{1}{v}\min_i k_i - a_2, \quad \alpha = a_1.$$

Due to (6.4), the following line holds under conditions of Lemma 6.1.

$$0 = \frac{1}{v}E[\Gamma_t](M,t) \le E(-\alpha, \mathbf{s}^{n-1,1}).$$

But because of (6.3) it may only be possible if $\underline{s} > \underline{s}_0$ with $\underline{s}_0 = \underline{s}(a_1, C)$, what proves inequality (6.5).

The immediate sequence to Lemma 6.1 and Proposition 1.6 is the following existence theorem.

Theorem 6.2. *Let assumptions of Lemma 6.1 be fulfilled and Γ_0 be E-admissible surface. Then there exist $T < \infty, M_0 \in R^n$ and the unique E-admissible evolution $\{\Gamma_t, t \in [0;T)\}$ satisfying equation (6.1) such that $\Gamma_T = M_0$.*

Indeed, if our evolution exists for $0 \le t \le t_1$, then it encloses the ball $B_\rho, \rho = 1/\max_{i,\Gamma_{t_1}} k_i$ and inequalities (6.4), (6.5) give estimates for the principal curvatures and velocity in terms of ρ. Inequality (6.5) guarantees that $(s, \mathbf{s})$ can not arrive to the boundary of D, i.e., concerned parabolic problem does not degenerate.

Theorem 6.2 was motivated by the part of Theorem 1.2 from [1], containing the existence of admissible solution shrinking into a point to equation

$$v = F(\mathbf{k}), \quad F > 0 \tag{6.6}$$

for homogeneous, positive monotone, concave functions F, vanishing at $\partial\mathrm{Sym}^+(n)$.

To show the point of our generalization consider the example

$$E[\Gamma_t] = -v + F_{n,l}(\mathbf{s}) = 0, \quad \mathbf{s} = \mathbf{k} - avI. \tag{6.7}$$

If $a = 0$, it is the example to the theorem of B.Andrews. If $a > 0$, then equation (6.7) of course can be reduced to (6.6) but we are loosing vanishing of new F at $\partial\mathrm{Sym}^+(n)$ and existence of admissible evolutions is the subject to Theorem 6.2. We also could not treat equation (6.7) with $a > 0$ by Theorems 1.3, 1.5 because failed to find equivalent setting in $(v, \mathbf{r})$ variables with concave function G.

Theorem 1.2 from [1] also implies existence of admissible solutions shrinking into a point for evolutionary equations (6.6) with positive convex homogeneous functions F without other assumptions but positive monotonicity of F. We survey this situation in terms of function E. Note that convexity of E provides the inequality

$$E(s, S + \xi \times \xi) - E(s, S) \ge E^{ij}\xi^i\xi^j$$

and local positive monotonicity has as a consequence nonlocal monotonicity. This fact facilitates the treatment of equation (6.1).

We begin with the following analog of Lemma 6.1.

Lemma 6.3. *Let* $\{\Gamma_t, t \in [0; t_1]\}$ *be admissible to equation (6.1), (6.2) solution,* $a_0, a_2 \geq 0, a_1 > 0$. *Then the following inequalities hold*

$$v \geq \min_{\Gamma_0} v \geq \frac{a_0}{a_1}, \tag{6.8}$$

$$\frac{k_i}{v} - a_2 \geq \min_{\Gamma_0, j} \frac{k_j}{v} - a_2 = \kappa > 0, \tag{6.9}$$

$$\frac{k_i}{v} - a_2 \leq \frac{\mu}{\nu} + \kappa, \quad i, j = 1, \ldots, n, \quad M \in \Gamma_t, \tag{6.10}$$

where

$$\mu = -E(-a_1, \kappa), \quad \nu = \frac{1}{n}\frac{\partial \mu}{\partial \kappa}.$$

Proof. The first of inequalities (6.8) contains two assertions. One of them is already known as the estimate (5.6), whereas relation $v \geq a_0/a_1$ reads as necessary condition for admissibility of solutions to the problem (6.1), (6.2) with homogenous function E.

Estimate (6.9) presents opposite to (6.4) inequality and bases on the opposite to (1.13) inequality

$$E^0 u_t \geq E^{ii} u_{ii}, \tag{6.11}$$

which gets valid due to homogenuity of E in s, s and requirement $a_0 \geq 0$. Appearance of function u in (6.11) indicates that we have again related standard coordinate system to the point under consideration. Similar to the proof of Lemma 5.2, we fix M', t' as the point of maximum of ratio $\exp(-\epsilon t)v/k_i$ and consider

$$w^\epsilon = \frac{\exp(-\epsilon t) u_t}{u_{11}} \leq w(0, t').$$

This time identities (5.2), (5.3), Proposition 3.1, arbitrariness of ϵ and (6.11) together with convexity of E conclude estimate (6.9).

To derive inequality (6.10) we introduce diagonal matrix $\mathrm{s}^{1,n-1}(M, t)$ with

$$\overline{s} = \frac{1}{v} \max_i k_i - a_2, \quad \underline{s} = \kappa$$

and apply the following consequence of convexity of E,

$$\frac{\partial E}{\partial \overline{s}}(-a_1, \mathrm{s}^{1,n-1}) > \nu.$$

Indeed, (6.10) follows from the line

$$0 = \frac{1}{v} E[\Gamma_t] \geq E(-a_1, \mathrm{s}^{1,n-1}) \geq -\mu + \nu(\frac{1}{v} \max_i k_i - a_2 - \kappa).$$

Lemma 6.3 guarantees that, untill admissible solution to equation (6.1), (6.2) exists, it encloses some ball B_ρ and Proposition 1.6 carries out

Theorem 6.4. *Let E be convex over D and Γ_0 be E-admissible surface. Then there exists the unique started by Γ_0 admissible solution to (6.1), (6.2). Moreover, it shrinks into a point in finite time.*

To illustrate Theorem 6.4 we generalize relevant examples from [1] following way,

$$0 = E[\Gamma_t] = a_0 - a_1 v + (\sum_1^n (k_i - a_2 v)^m)^{\frac{1}{m}}, \quad m \geq 1.$$

When $a_0 = a_2 = 0, a_1 = 1$, it is exactly the example from [1] and without loss of generality equation (6.1) can be rewritten in the form (6.6). Also notion of E-admissible surfaces looks rather redundant, when $a_2 = 0$. The set of strictly convex surfaces suits the case.

Assumption $a_2 > 0$ totally changes the situation. Actually, Theorems 6.2, 6.4 are conditional in this case because contain implicit requirement of existence of E-admissible surfaces. We give sufficient condition for E-admissibility of the surface and show that this set is never empty.

To proceed with we relate to function E new parameter σ_0 as the root of equation $E(-\sigma, I) = 0$. The existence of $\sigma_0 > 0$ is necessary for solvability of equation (6.1). Function $f(\sigma) := E(-\sigma, I)$ is positive for $\sigma > \sigma_0$ and negative otherwise. Our further argument will be based on this observation.

Proposition 6.5. *Let E be homogeneous positive monotone over D function and Γ be strictly convex surface with*

$$\underline{k} = \min_{i,\Gamma} k_i, \quad \overline{k} = \max_{i,\Gamma} k_i.$$

Assume that

$$\underline{k} > \frac{a_0 a_2}{a_1} \tag{6.12}$$

and

$$(a_1 + a_2\sigma_0)\underline{k} \geq a_2(a_0 + \sigma_0 \overline{k}). \tag{6.13}$$

Then Γ is E-admissible surface.

Proof. In this Section $D = R^- \times \mathrm{Sym}^+(n)$ and to satisfy Definition 1.2 will be sufficient to prove the existence of v(M) such that $s(v) = a_0 - a_1 v < 0, \quad S(\mathbf{k}, v) = \mathbf{k} - a_2 v I \in \mathrm{Sym}^+(n)$ and

$$0 = E(a_0 - a_1 v, \mathbf{k} - a_2 v I) \quad M \in \Gamma. \tag{6.14}$$

Fix a point $M \in \Gamma$ and consider function of one variable $f_M(v) = E(s(v), S(M))$. The inequality (6.12) is obviously necessary for E-admissibility of Γ and there exist $\delta, \epsilon > 0$ such that

$$\underline{k} = \frac{a_0 a_2}{a_1} + \delta, \quad \underline{v} = \frac{a_0 + \epsilon}{a_1}, \quad \underline{k} - \underline{v} = \delta - \frac{\epsilon}{a_1} > 0, \quad s(\underline{v}) = -\epsilon.$$

Then, due to homogenuity of E,

$$f_M(\underline{v}) \geq E(-\epsilon, (\delta - \frac{\epsilon}{a_1})I) > 0 \tag{6.15}$$

for sufficiently small ϵ.

Assume now $a_2 > 0$ and let

$$\overline{v} = \frac{a_0}{a_1} + \frac{\delta - \epsilon}{a_2}.$$

Requirement (6.13) brings out the line

$$f_M(\overline{v}) \leq E(-\frac{a_1}{a_2}(\delta - \epsilon), (\overline{k} - \underline{k} + \epsilon)I) < 0.$$

The latter together with (6.15) guarantees the existence of $v_M \in (\underline{v}; \overline{v})$ satisfying $f_M(v_M) = 0$, what implies (6.14). To conclude the proof we remark that for $a_2 = 0, \overline{v} = \infty$.

Back to Theorems 6.2, 6.4, we can say that spheres satisfying (6.12) are always E-admissible surfaces, i.e., this set is never empty.

In the theory of geometric flows there appears pinching condition as the first step in investigation of asymptotic behaviour, [1]. Namely, evolution $\{\Gamma_t\}$ satisfies pinching condition if there exists independent in t constant C such that

$$\frac{\overline{k}}{\underline{k}}(t) \leq C. \tag{6.16}$$

Inequalities (6.4), (6.5) and (6.9), (6.10) ensure pinching condition for the flows from Theorem 6.2, 6.4 respectively. We see it as direct consequence to homogenuity of E. It is rather remarkable that for $a_2 > 0$ requirement (6.13) contains restrictions on the value of C in (6.16) for initial surface Γ_0.

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