Geometrical theory of dynamical systems and fluid flows

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Dynamical systems can be formulated in general on the basis of the Riemannian geometry and Lie algebra, provided that a dynamical system has a group symmetery, namely it is invariant under group transformations, and further that the group manifold is endowed with a Riemannian metric. The basic ideas and tools are described, and their applications are presented to the following five problems. (a) Free rotation of a rigid body, which is presented as an illustrarative example. This is well-known in physics and one of the simplest systems of finite degrees of freedom. (b) Derivations of a geodesic equation on a group of diffeomorphisms of a circle and KdV equation on its extended group, in which one of the soliton equations is derived on a geometric framework. (c) Geometrical analysis of chaos of a Hamiltonian system, which is a self-gravitating system of a finite number of point masses. (d) Geometrical formulation of hydrodynamics of an incompressible ideal fluid, which gives not only geometrical characterization of flows but also interpretation of the origin of Riemannian curvatures of the fluid flow. (e) Derivation of a geodesic equation on a loop group leads to the local induction equation of the motion of a vortex filament, and the equation on its extended group is found to be equivalent to the equation derived by Fukumoto and Miyazaki (1991) for a vortex filament with an axial flow along it. It is remarkable that the present geometrical formulations are successful for all the problems considered here and give insight into deep background common to the diverse physical systems. Further, the geometrical formulation opens a new approach to various dynamical systems, which is rewarded with new results.

Key words: *Fluid flows, Dynamical systems, Riemannian geometry, Lie group and Lie algebra, Geodesic equation, Connection, Jacobi equation* and *Riemannian curvatures.*

1 Introduction

Various dynamical systems have often common geometrical structures and can be formulated on the basis of Riemannian geometry and Lie group theory. In this note, the mathematical features are illustrated and physical aspects are exemplified by five systems: (a) Rotation of a rigid body, (b) KdV equation and geodesic equation on the group of diffeomorphisms of a circle, (c) A self-gravitating system (a conventional Hamiltonian dynamical system) of a finite number of point masses, (d) Flow of an incompressible ideal fluid and geometrical characterization of volume-preserving flows, and (e) Motion of a vortex filament and geodesic equations on a loop group and its extended group. Before describing the details of particular dynamical systems, mathematical concepts are presented first and reviewed briefly.

In the chapter 2, an introductory review is given for flows and diffeomorphisms, vectors and forms, Lie group and Lie algebra. In the subsequent chapter 3, the theory of Riemannian differential geometry is reviewed and basic concepts are presented: the first and second fundamental forms, affine connection, geodesic equation, Jacobi field, and Riemannian curvatures. Full accounts are found in the textbooks by Frankel (1998) [1], Arnold (1978) [2], and Abraham and Marsden (1978) [4].

In the chapters $4 \sim 8$, typical five dynamical systems are reformulated according to the mathematical framework presented in the preceding chapters. The governing equation of each system is now obtained as a geodesic equation on a group manifold associated with the individual system. It is to be noted that, although the governing equations are derived already in the physics, the present derivations are new and based on very general setup and concepts of metric, connection and Lie algebra in the Riemannian differential geometry. Chapters 4, 5 and 6 are reviews of published works: Suzuki *et al.* (1998) [8] (Ch.4); Ovsienko & Khesin (1987) [9], Misiołek (1997) [10], and Kambe (1998) [7] (Ch.5); and Cerruti-Sola & Pettini(1996) [13] (Ch.6), respectively.

One of the aims of this note is to give a geometrical framework to the description of flows of an ideal incompressible fluid and an iterpretation to the origin of Riemannian curvatures of the flows in chapter 7, based on Misiołek (1993) [16], Nakamura *et al.* (1992) [20], and Hattori & Kambe (1994) [19]. Further, the chapter 8 describes a new formulation for the geodesic equations of motion of a vortex filament on the basis of the theory of loop group and its extension. This gives a new interpretation to the local induction equation and the equation of Fukumoto & Miyazaki (1991) [23] from a geometrical point of view.

2 Flows, Diffeomorphisms and Lie group (Reviews)

2.1 Differentiable map and Diffeomorphisms

A *manifold* M^n is an n-dimensional topological space that is locally R^n, namely, in terms of local coordinates, a point $p \in M^n$ is represented as $p = (x_p^1, \cdots, x_p^n)$.

Let $F : M^n \to V^r$ be a map from a manifold M^n to another V^r. In local coordinates $x = (x^1, \cdots, x^n)$ in the neighborhood of the point $p \in M^n$ and $y = (y^1, \cdots, y^r)$ in the neighborhood of $F(p)$ on V^r, the map F is described by r functions $F^i(x)$, $(i = 1, \cdots, r)$ of n variables, abbreviated to $y = F(x)$ or $y = y(x)$, where F^i are differentiable functions of x^j $(j = 1, \cdots, n)$, and such functions are continuous.

When $n = r$, we say that the map F is a *diffeomorphism*, provided F is differentiable (thus continuous), one-to-one, onto, and in addition F^{-1} is differentiable. Such an F is a *differentiable homeomorphism* with a differentiable inverse F^{-1}. If the inverse F^{-1} does exist and the Jacobian determinant does not vanish, then the inverse function theorem would assure us that the inverse is differentiable. In the next section, the fluid flow is described to be a smooth sequence of diffeomorphisms of particle configuration (of infinite dimension).

2.2 Vector fields and Flows

Given a flow of a fluid in R^3, one can construct a 1-parameter family of maps,

$$\phi_t \ : \ R^3 \to R^3 \, ,$$

where ϕ_t takes a fluid particle located at p when $t = 0$ to the position of the same fluid particle $\phi_t(p) = x_t(p)$ at a later time $t > 0$. The maps are the so-called *Lagrangian* representation of motion of fluid particles. In terms of local coordinates, the j-th coordinate of the particle is written as $x^j \circ \phi_t(p) = x^j(\phi_t(p)) = x_t^j(p)$.

Associated with any such flow, we have a velocity at p,

$$v(p) = \frac{\mathrm{d}}{\mathrm{d}t} x_t(p)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \phi_t(p)\Big|_{t=0} \, .$$

In terms of coordinates we have

$$v^j(p) = \frac{\mathrm{d}\, x_t^j(p)}{\mathrm{d}t}\Big|_{t=0} \, .$$

Taking a smooth function $f : \ R^3 \to R$ and differentiating $f(x_t(p)) = f(\phi_t(p))$ with respect to t, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t(p))\Big|_{t=0} &= \sum_j \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} \\ &= \sum_j v^j(p) \frac{\partial}{\partial x^j} f = v_p f \end{aligned} \tag{1}$$

where $v_p = v(p)$. Thus, one obtains

$$v_p f = \frac{\mathrm{d}}{\mathrm{d}t} f(\phi_t(p)) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \phi_t(p) . \tag{2}$$

Conversely, to each vector field $v(x) = (v^j)$ in R^3, one may associate a flow $\{\phi_t\}$ having v as its velocity field. The map $\phi_t(p)$ with t a continuous parameter can be found by solving the system of ordinary differential equations, $\mathrm{d}x^j/\mathrm{d}t = v^j(x^1(t), x^1(t), x^3(t))$ with the initial condition, $x(0) = p$. Thus one finds the integral curves, which is a one-parameter family of maps $\phi_t(p)$ for any $p \in R^3$, called a flow *generated* by the vector field v, where $v = \dot{x}_t = \dot{\phi}_t$. The map $\phi_t(p)$ is a diffeomorphism because $\phi_t(p)$ is differentiable, one-to-one, onto and F^{-1} is differentiable, with respect to every point p. This is assured by the physical property that each fluid particle is a physical entity which keeps its identity during the motion and two particles do not occupy the same point simultaneously.

In general, on a manifold M^n, we consider a vector v tangent to the parametrized curve ϕ_t at p. The vector field $v(x)$ at a point $x \in M^n$ is called a *tangent* vector, or a vector in the *tangent* space $T_x M$ to the manifold M at a point x.

Remark: Continuous distribution of fluid particles in a space has infinite degrees of freedom. Therefore, the velocity field of all the particles as a whole is regarded to be of infinite dimensional. In this context, the set of diffeomorphisms ϕ_t forms an infinite dimensional manifold $D^{(\infty)}$ and a point $\eta = \phi_t \in D^{(\infty)}$ represents a configuration (as a whole) of all particles at a given time t.

A flow $\phi_t(p) = (x_t^j(p))$ on an n-dimensional manifold M^n is described by the system of ordinary differential equations,

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = v^j \left(x^1(t), \cdots, x^n(t) \right), \qquad (j = 1, \cdots, n) \tag{3}$$

with the initial condition, $\phi_0 = p$. The one-to-one correspondence between the tangent vector $v = (v^j)$ to M^n at p and the first-order differential operator $v^j(p)\partial/\partial x^j$, mediated by (1)~(3), implies the following correspondence,

$$v = \sum_j v^j(x) \frac{\partial}{\partial x^j} , \tag{4}$$

which defines the vector field v as a differential operator. See Arnold [2] (1978).

Usually, in the differential geometry, no distinction is made between a vector and its associated differential operator. Then each one of the n operators $\partial/\partial x^j$ defines a vector, written $\boldsymbol{\partial}/\boldsymbol{\partial} x^j$, at each p. The j-th vector $\boldsymbol{\partial}/\boldsymbol{\partial} x^j$ $(v^j = 1)$ is the tangent vector to the j-th coordinate curve parameterized by x^j.

The tangent space to M^n at the point $x \in M^n$, written $T_x M$, is a vector space consisting of all tangent vectors to M^n at x. The n vectors

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^1}, \cdots\cdots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}$$

form a basis of the vector space, and this base is called a *coordinate* basis. The basis vector $\boldsymbol{\partial}/\boldsymbol{\partial} x^j$ is simply written as $\boldsymbol{\partial}_j$.

If $\boldsymbol{r} = (r^1, \cdots, r^n)$ is a position vector in the space R^n and $M^n \subset R^n$, then the vector $\boldsymbol{\partial}/\boldsymbol{\partial} x^j$ is understood as

$$\boldsymbol{\partial}_j \equiv \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^j} = \frac{\partial \boldsymbol{r}}{\partial x^j} = \frac{\partial}{\partial x^j}(r^1, \cdots, r^n)$$

A tangent vector X is written in general as

$$X = X^j \, \boldsymbol{\partial}_j \ , \qquad \text{or} \qquad X_x = X^j(x) \, \boldsymbol{\partial}_j \ ,$$

In the case of **time-dependent** velocity field, an additional coordinate x^0 is to be introduced, and the n equations (3) are replaced by the following $(n+1)$ equations,

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = X^j\big(x^0(t), x^1(t), \cdots, x^n(t)\big), \quad \text{with} \quad X^0 = 1 \tag{5}$$

for $j = 0, 1, \cdots, n$. Naturally the additional equation is reduced to $x^0 = t$. Correspondingly, the tangent vector in the time-dependent case, denoted by the *hat* symbol, is written as

$$\hat{X} = \hat{X}^i \, \partial_i = X^0 \partial_0 + X^\alpha \partial_\alpha = \partial_t + X^\alpha \partial_\alpha \ , \tag{6}$$

where the index α denotes the spatial components $(\alpha = 1, \cdots, n)$.

2.3 Covector and Inner product

2.3.1 Covector (1-forms):

We regard the differential of a function f defined by $\mathrm{d}f[v] = vf$ as a linear functional $M^n \to R$ for any vector $v \in M^n$. This is a basis-independent definition. In local coordinates, using (1) and (4), we have

$$\mathrm{d}f[v] = \mathrm{d}f[v^j \partial_j] = \sum_j v^j(x) \, \frac{\partial f}{\partial x^j} \ . \tag{7}$$

Therefore, the differential $\mathrm{d}f[v^j \partial_j]$ is linear with respect to the scalar coefficient v^j. In particular, if f is the coordinate function x^i, we obtain, replacing f by x^i,

$$\mathrm{d}x^i[v] = \mathrm{d}x^i[v^j \partial_j] = v^j \, \mathrm{d}x^i \left[\frac{\partial}{\partial x^j} \right] = v^j \, \frac{\partial x^i}{\partial x^j} = v^j \, \delta^i_j = v^i \ .$$

Thus the operator $\mathrm{d}x^i$ reads off the $i-$th component of any vector v. It is seen that

$$\mathrm{d}x^i[\partial_j] = \delta^i_j \ . \tag{8}$$

Thus the n functionals $\mathrm{d}x^i$ $(i = 1, \cdots, n)$ yield the dual bases corresponding to the coordinate bases $(\partial_1, \cdots, \partial_n)$ of a vector space E^n. The dual bases $(\mathrm{d}x^1, \cdots, \mathrm{d}x^n)$ form a dual space $(E^n)^*$.

The most general linear functional, $\alpha : M^n \to R$, is expressed in coordinates as

$$\alpha = a_1 \mathrm{d}x^1 + \cdots + a_n \mathrm{d}x^n \,. \tag{9}$$

The α is called a *covector*, or *covariant vector*, or (differential) *1-form.* When the coefficients a_i are smooth functions $a_i(x)$, the α is a 1-form *field.* Given a vector $v = v^j \partial_j$, the 1-form $\alpha[v]$ takes the value,

$$\alpha[v] = \sum_i a_i \, \mathrm{d}x^i[v^j \partial_j] = \sum_i a_i \, v^j \;.$$

The differential of a function f (without $[\cdot]$) is a typical example of the covector (1-form):

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i \,,$$

where $\mathrm{d}x^i$ is a basis covector and $\partial f / \partial x^i$ is its component. This form holds in any manifold. In the next subsection, a vector $\mathrm{grad}\, f$ is defined as one corresponding to the covector $\mathrm{d}f$. For transformations of a vector and a covector, see (17) and (25) below.

2.3.2 Inner (Scalar) product:

Let the vector space E^n be endowed with an inner (scalar) product $\langle \cdot, \cdot \rangle$. For each pair of vectors $X, Y \in E^n$, the inner product $\langle X, Y \rangle$ is a real number, and it is bilinear and symmetric with respect to X and Y. Furthermore, the $\langle X, Y \rangle$ is *non-degenerate* in the sense that $\langle X, Y \rangle = 0$ for any Y only if $X = 0$. Writing $X = X^i \partial_i$ and $Y = Y^j \partial_j$, the inner product is given by

$$\langle X, Y \rangle \;=\; g_{ij} \, X^i \, Y^j \tag{10}$$

where

$$g_{ij} \;=\; \langle \partial_i, \, \partial_j \rangle = g_{ji} \tag{11}$$

is called the *metric tensor.*

By definition, the inner product $\langle A, X \rangle$ is linear with respect to X when the vector A is held fixed. Then the following operation on X,

$$\alpha[X] = \langle A, \, X \rangle \,,$$

is a linear functional. In other words, to each vector $A = (A^1, \cdots, A^n) \in E^n$, one may associate a covector α. By definition, one has $\alpha[X] = g_{ij} \, A^j X^i = (g_{ij} \, A^j) X^i$. On the other hand, one has from (9)

$$\alpha[X] = a_i \, \mathrm{d}x^i[X] = a_i \, X^i \;,$$

in terms of the basis $\mathrm{d}x^i$. Thus one obtains

$$a_i = g_{ij} \, A^j = A_i \;, \tag{12}$$

which defines A_i. Thus the component a_i is given by $g_{ij} \, A^j$ and written as A_i using the same letter A. The covector $\alpha = A_i \mathrm{d}x^i = (g_{ij} \, A^j) \mathrm{d}x^i$ is called the covariant version of the vector $A = A^j \partial_j$. In tensor analysis, one says that the upper index j is lowered by means of the metric tensor g_{ij} in (12). In other words, the covector is obtained by lowering the upper index of the vector by means of g_{ij}.

On the other hand, a vector A^j is obtained by raising the lower index of the covector A_i:

$$A^j = g^{ji} A_i \,, \tag{13}$$

which is equivalent to solving the equation (12) to obtained A^j. This is verified by reminding that the metric tensor matrix $g = (g_{ij})$ is assumed non-degenerate, therefore that the inverse matrix g^{-1} must exist and is symmetric. The inverse is written as $g^{-1} = (g^{ji})$ in the above equation using the same letter g. Thus we have the expression of the vector grad f as

$$(\mathrm{grad}\, f)^j = g^{ji} \frac{\partial f}{\partial x^i} \,. \tag{14}$$

2.4 Mapping of vectors and covectors

Let $\phi : M^n \to V^r$ be a smooth map and $\phi_* : M_x \to V_y$ be the corresponding differential map. In local coordinates, the map ϕ is represented by the function $y = F(x)$, where $x \in M^n$ and $y \in V^r$.

Let $p(t)$ be a curve on M with $p(0) = p$ and $\dot{p}(0) = X$ (velocity vector), where $X \in T_pM^n$. Then, the differential map $\phi_* X$ is defined by $Y = \phi_* X = F_* X = \mathrm{d}/\mathrm{d}t(F(p(t))|_{t=0}$. Thus, the vector X is mapped to Y, whose component is given as $Y^k = (\phi_* X)^k = (\partial F^k/\partial x^j)X^j = (\partial y^k/\partial x^j)X^j$, by definition. The transformation ϕ_* is linear with respect to the (scalar) coefficient X^j, and also written as

$$\begin{aligned} \phi_* X &= \phi_*\Big[X^j \frac{\partial}{\partial x^j}\Big] = X^j \phi_*\Big[\frac{\partial}{\partial x^j}\Big] = X^j \frac{\partial y^k}{\partial x^j}\frac{\partial}{\partial y^k} && (15)\\ &= Y^k \frac{\partial}{\partial y^k} = Y \,. && (16) \end{aligned}$$

This is called a *push-forward* transformation of the velocity vector X to the vector Y (the velocity vector of the image curve at $F(p)$). The component of Y is given by

$$Y^k = \frac{\partial y^k}{\partial x^j} X^j \,. \tag{17}$$

This is also written as $Y = JX$, where the transformation matrix is the Jacobian,

$$J = \frac{\partial(y)}{\partial(x)} = \frac{\partial(y^1, \cdots, y^r)}{\partial(x^1, \cdots, x^n)} \,. \tag{18}$$

In particular, setting that $X^k = 1$ (for an integer k) and others are zero in (15), it is found that the bases $(\partial/\partial x^k)$ are transformed as

$$\phi_* \Big[\frac{\partial}{\partial x^k}\Big] = \frac{\partial y^j}{\partial x^k}\frac{\partial}{\partial y^j} \tag{19}$$

This is also written as

$$\phi_* \left[\frac{\partial}{\partial x^k}\right] f = \frac{\partial y^j}{\partial x^k}\frac{\partial f}{\partial y^j} = \frac{\partial}{\partial x^k} f(\phi(x)) \equiv \frac{\partial}{\partial x^k} f \circ \phi(x) \ . \tag{20}$$

Writing as $X = X^j(x)\partial/\partial x^j$,

$$\phi_* X[f] = X[f \circ \phi] \ . \tag{21}$$

Corresponding to the *push-forward* ϕ_*, one can define the *pull-back* ϕ^*, which is the linear transformation taking covectors at y to covectors at x, *i.e.* $\phi^* : \ V(y)^* \to M(x)^*$. Suppose that a vector X at $x \in M$ is transformed to $Y = \phi_*(X)$ at $y = \phi(x) \in V$, then the pull-back ϕ^* of a covector α is defined, using the push-forward $\phi_*(X)$, by

$$(\phi^* \alpha)[X] \equiv \alpha[\phi_*(X)] \ , \tag{22}$$

for any covector $\alpha = A_i \, \mathrm{d}y^i$. Note that, owing to (8), one has

$$\alpha \left[\frac{\partial}{\partial y^k}\right] = A_i \mathrm{d}y^i \left[\frac{\partial}{\partial y^k}\right] = A_k \ . \tag{23}$$

Writing as

$$\phi^* \alpha = a_i \, \mathrm{d}x^i \ , \tag{24}$$

one obtains $a_i = \phi^* \alpha[\partial/\partial x^i]$, and further one can derive the following transformation of the components of covectors by using (19) and (23):

$$\begin{aligned} a_k &= \phi^* \alpha\Big[\frac{\partial}{\partial x^k}\Big] = \alpha\Big[\phi_* \frac{\partial}{\partial x^k}\Big] = \alpha\Big[\frac{\partial y^j}{\partial x^k}\frac{\partial}{\partial y^j}\Big] \\ &= \frac{\partial y^j}{\partial x^k} \alpha \left[\frac{\partial}{\partial y^j}\right] = A_j \frac{\partial y^j}{\partial x^k} \ . \end{aligned} \tag{25}$$

Substituting the expression $A_i \, \mathrm{d}y^i$ for α in (24) and using (25), we have

$$\phi^*(A_i \, \mathrm{d}y^i) = A_i \frac{\partial y^i}{\partial x^j} \mathrm{d}x^j \ . \tag{26}$$

Setting $A_k(\text{only}) = 1$ as before (for an integer k), it is found that the bases $(\mathrm{d}y^k)$ are transformed as

$$\phi^*[\mathrm{d}y^k] = \frac{\partial y^k}{\partial x^j} \mathrm{d}x^j \ . \tag{27}$$

The definition (22) is understood as the *invariance* of the pull-back transformation. In fact, the value of the covector $\alpha = Y_i \mathrm{d}y^i$ with the vector $Y = \phi_* X$ (in V) is equal to the value of the pull-back covector $\phi^* \alpha$ with the original vector X (in M). If one sets $A_i = \partial f/\partial y^i$, the equation (22) expresses invariance of the differential: $\phi_*[(\partial f/\partial y^i)\mathrm{d}y^i] = (\partial f/\partial y^i)(\partial y^i/\partial x^j)\mathrm{d}x^j = (\partial f/\partial x^j)\mathrm{d}x^j$. (See also the next section for $M = V$.)

Based on this invariance, the general pull-back formula is defined for the integral of a form (covector) α over a curve σ as

$$\int_{\phi(\sigma)} \alpha = \int_\sigma \phi^* \alpha \ , \tag{28}$$

where $\phi : \ \sigma \subset M \to \phi(\sigma) \subset V$. In words, the integral of a form α over the image $\phi(\sigma)$ of a curve σ is the integral of the pull-back $\phi^* \alpha$ over the original σ.

2.5 Lie group and Invariant vector fields

A set G of smooth transformations (maps) of a manifold M into itself is called a group, provided that (i) with two maps $g, h \in G$, the product $gh = g \circ h$ belongs to G: $G \times G \to G$, (ii) for every $g \in G$, there is the inverse map $g^{-1} \in G$. From (i) and (ii), it follows that the group contains the identity map e (unity): $gg^{-1} = g^{-1}g = e$.

A *Lie group* is a group which is a differentiable manifold, for which the operations (i) and (ii) are differentiable. A Lie group always has two families of diffeomorphisms, the left and right translations. Every element $g \in G$ defines the *left* translation of the group onto itself,

$$L_g : \; G \to G\,, \qquad i.e. \quad L_g(h) = gh \quad \text{for any} \quad h \in G\,,$$

and similarly the *right* translation is as follows, $R_g : \; G \to G\,,$ *i.e.* $R_g(h) = hg$ for any $h \in G$. The operation inverse to L_g (or R_g) is simply $L_{g^{-1}}$ (or $R_{g^{-1}}$), respectively.

Given a tangent vector X_e to G at the identity e, one may right-translate or left-translate X_e to every point of G by

$$X_g^{\mathrm{R}} = R_{g*}X_e = X_e \circ g \qquad \text{(right translation)}\,, \tag{29}$$

$$X_g^{\mathrm{L}} = L_{g*}X_e = g_*\, X_e \qquad \text{(left translation, push forward)}\,, \tag{30}$$

respectively. The right-translation of the vector field $X(x)$ (at the identity e) by the map g is denoted as $Xg(x) \equiv X \circ g(x)$ and understood as follows: the map g acts to the point x first and then the vector X is considered at the point $g(x)$.

It is said that a vector field X on G is *right invariant* (or *left invariant*) if it is invariant under all right-translations (or left-translations), namely if

$$R_{h*}X_g^{\mathrm{R}} = X_{gh}^{\mathrm{R}} \qquad \Big(\text{or} \quad L_{h*}X_g^{\mathrm{L}} = X_{hg}^{\mathrm{L}}\ , \text{respectively}\Big).$$

(In mathematics it is common to define a vector field right-invariant, however there is an example of left-invariant field in physics in §4.) It should be readily seen that the transformation (29) gives a right-invariant field generated by X_e at e. Similarly, the transformation (30) gives a left-invariant field.

For two right-invariant tangent vectors X_g^{R} and Y_g^{R}, the metric (10) is called *right-invariant* if

$$\langle X_g^{\mathrm{R}}, Y_g^{\mathrm{R}} \rangle = \langle X_e, Y_e \rangle\ .$$

Similarly, the metric is *left-invariant* if $\langle X_g^{\mathrm{L}}, Y_g^{\mathrm{L}} \rangle = \langle X_e, Y_e \rangle$ for left-invariant tangents, $X_g^{\mathrm{L}}, Y_g^{\mathrm{L}}$.

2.6 Lie algebra and Lie derivative

2.6.1 Adjoint operator and Lie bracket

Every pair of vector fields defines a new vector field called the *Lie bracket* $[\,\cdot\,,\,\cdot\,]$. More precisely, the tangent space T_eG at the identity e of a Lie group G, equipped with the bracket operation $[\,\cdot\,,\,\cdot\,]$ of bilinear skew-symmetric pairing, is called the *Lie algebra* $\mathbf{g}$ of the group G, where $[\,\cdot\,,\,\cdot\,] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$, if the bracket satisfies the Jacobi identity :

$$[[X,\,Y],\,Z] + [[Y,\,Z],\,X] + [[Z,\,X],\,Y] = 0\;, \tag{31}$$

for any triplet of $X, Y, Z \in \mathbf{g} = T_eG$.

Provided that an element $\xi \in G$ acts as a linear transformation on $^\forall Y \in \mathbf{g}$ (*i.e.* its own Lie algebra element) in the following way,

$$Ad_\xi\, Y \equiv L_{\xi *} \circ R_{\xi^{-1}*}\, Y = \xi Y \xi^{-1}\,,$$

then the operator Ad_G is called the *adjoint* representation.

Consider a curve $\xi_t\ :\ t \to G$ with the tangent $\dot{\xi}_0 = X$. The differential of $Ad_{\xi_t}\, Y$ with respect to t at the identity ($t = 0$) is a linear transformation from Y to $ad_X\, Y$:

$$ad_X\, Y \equiv \frac{\mathrm{d}}{\mathrm{d}t}\, \xi_t\, Y\, \xi_t^{-1} = [\,X,\,Y\,]\;. \tag{32}$$

The Lie bracket is skew-symmetric: $[\,X,\,Y\,] = -[\,Y,\,X\,]$, as shown below. The ad_X is a linear transformation from $\mathbf{g} \to \mathbf{g}$, *i.e.* $Y \to [X,\,Y]$. The bracket operation is usually called the *commutator*. The operator ad_X stands for the image of an element X under the action of the linear map ad. The representation of $[X,\,Y]$ depends on each dynamical system, as shown below.

As a first example, consider the rotation group $G = SO(3)$, of which an element A is represented by a 3×3 orthogonal matrix of $\det A = 1$. Let $\xi(t)$ be a curve issuing from e with the velocity $\mathbf{a}$ on $SO(3)$. Then one has $\xi(t) = e + t\,\mathbf{a} + O(t^2)$ for an infinitesimal time parameter t, where $\mathbf{a} = \dot{\xi}(0)$ is an element of the algera $\mathbf{g} = \mathbf{so}(3)$ and represented by a skew-symmetric matrix due to the orthogonality property and $\det \xi(t) = 1$. The algebra element $\mathbf{a}$ is called a generator of $\xi(t)$. Then, for $\mathbf{a}$, $^\forall\mathbf{b} \in \mathbf{so}(3)$, the operator $ad_{\mathbf{a}} : \mathbf{g} \to \mathbf{g}$ is given by

$$ad_{\mathbf{a}}\,\mathbf{b} = [\mathbf{a},\,\mathbf{b}] = \mathbf{ab} - \mathbf{ba} \equiv \mathbf{c}\;, \tag{33}$$

where the commutator $\mathbf{ab} - \mathbf{ba} = \mathbf{c}$ is in the form of another skew-symmetric matrix. The well-known representation of a 3×3 skew-symmetric matrix $\mathbf{a}$ in terms of a three-component (axial) vector $\hat{\mathbf{a}}$ leads to the rule of cross product,

$$\hat{\mathbf{c}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}}\;. \tag{34}$$

Let $s \mapsto \eta(s)$ be another curve with the initial velocity $\dot{\eta}(0) = \mathbf{b}$. Then,

$$\begin{aligned} \xi(t)\,\eta(s)\,\xi(t)^{-1} &= (e + t\,\mathbf{a} + O(t^2))\,(e + s\,\mathbf{a} + O(s^2))\,(e - t\,\mathbf{a} + O(t^2)) \\ &= e + s(\mathbf{b} + t\,(\mathbf{ab} - \mathbf{ba}) + O(t^2)) + O(s^2) \end{aligned} \tag{35}$$

as $t, s \to 0$. This verifies the equation (33).

2.6.2 Lie derivative and Lagrange derivative

Suppose that we are given a vector field $X = X^i\partial_i$ on a manifold M^n. As described in §2.2, with every such vector field, we associate a flow, or one-parameter group of *diffeomorphisms* ξ_t: $M \to M$, for which $\xi_0 = id$ and $(\mathrm{d}/\mathrm{d}t)\xi_t\boldsymbol{x}|_{t=0} = X(\boldsymbol{x})$, where id denotes the identity map.[1] In addition, a first-order differential operator $\mathcal{L}_X$ is defined by

$$\mathcal{L}_X f(\boldsymbol{x}) \equiv X^i \frac{\partial}{\partial x^i} f(\boldsymbol{x}) = \frac{\mathrm{d}}{\mathrm{d}t} f(\xi_t \boldsymbol{x})\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \xi_t^* f(\boldsymbol{x})\Big|_{t=0} . \tag{36}$$

The operator $\mathcal{L}_X$ is called the *Lie derivative.* According to the definition of the pull-back (22), the Lie derivative $\mathcal{L}_X f = X^i\partial_i f$ of a function f (a zero-form) is interpreted as the time derivative of its pull-back $\xi_t^* f$ at $t = 0$. The point $\xi_t\boldsymbol{x}$ moves forward in accordance with the flow. Relatively observing, the pull-back $\xi_t^* f$ is estimated at $\boldsymbol{x}$, and its time derivative defines the Lie derivative. This is sometimes called as a *derivative of a fisherman* [5] sitting at a fixed place $\boldsymbol{x}$. [2] This refers to the rightmost expression $(\mathrm{d}/\mathrm{d}t)\, \xi_t^* f(\boldsymbol{x})$.

In the fluid dynamics however, the same derivative is called the *Lagrange* derivative, which refers to the middle expression,

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t} f(\xi_t\boldsymbol{x}) .$$

This is understood as denoting the time derivative observed by the fluid particle at $\xi_t\boldsymbol{x}$ moving with the flow.

In order to consider the derivative of vectors, suppose that we are given another vector field $Y = Y^i\partial_i$, and denote the flow generated by Y as η_s with $\eta_0 = id$. The commutator of $\mathcal{L}_X$ and $\mathcal{L}_Y$ defines the *Poisson bracket* $\{\,\cdot,\,\cdot\,\}$: [5]

$$[\,\mathcal{L}_X,\, \mathcal{L}_Y\,] = \mathcal{L}_X\, \mathcal{L}_Y - \mathcal{L}_Y\, \mathcal{L}_X = \mathcal{L}_{\{X,Y\}} . \tag{37}$$

Expression of $\{X, Y\}$ in local coordinates will be given just below.

Now, let us consider the Lie derivative of a vector field $Y(\boldsymbol{x})$. The flow is transporting the vector $Y(\boldsymbol{x})$ in front of the fisherman sitting at a fixed point p. After an infinitesimal time t, the point $\xi_t^{-1}p$ will arrive at p and the man will see the vector Y transported forward by the map ξ_t from there, that is $\xi_t\, Y(\xi_t^{-1}p)$ (precisely, $\xi_{t*}\, Y(\xi_t^{-1}p)$). Its time derivative is nothing but $ad_X\, Y$ according to the definition (32). Thus we have

$$\mathcal{L}_X\, Y = ad_X\, Y = [\,X,\, Y\,] = \frac{\mathrm{d}}{\mathrm{d}t}\, \xi_t\, Y\, \xi_t^{-1} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\, \xi_t\, \eta_s\, \xi_t^{-1} , \tag{38}$$

where $[X,\, Y]$ is the *Lie bracket.* The two diffeomorphisms ξ_t and η_s corresponding to the vector fields X and Y respectively can be written in the form:

$$\xi_t \;:\; x \mapsto x + t\,X(x) + O(t^2), \qquad t \to 0 \tag{39}$$

$$\eta_s \;:\; x \mapsto x + s\,Y(x) + O(s^2), \qquad s \to 0 . \tag{40}$$

[1] The *id* may be also written as e. The *id* is used here in order to emphasize that this is a *map.*

[2] The Lie derivative $\mathcal{L}_X$ also acts on any form fields in the same way. [1]

The inverse ξ_t^{-1} is given by $x \mapsto x - t\,X(x) + O(t^2)$. Reminding that $\eta_s\,\xi_t^{-1}\,x = \eta_s(\xi_t^{-1}(x))$, we have

$$\begin{aligned}\eta_s\,\xi_t^{-1} \;:\quad x &\mapsto x - t\,X(x) + O(t^2) + s\,Y(x - t\,X(x) + O(t^2)) + O(s^2) \\ &= x - t\,X(x) + s\,(Y(x) - t\,X^j\partial_j\,Y + O(t^2)) + O(t^2, s^2)\;.\end{aligned}$$

Finally, [5]

$$\xi_t\,\eta_s\,\xi_t^{-1} \;:\quad x \mapsto x + s\Big(Y(x) - t\,\Big(X^j\frac{\partial Y}{\partial x^j} - Y^j\frac{\partial X}{\partial x^j}\Big)\Big) + O(t^2, s^2)\;. \tag{41}$$

According to the definition (38), the Lie derivative of a vector field Y is given as $\mathcal{L}_X Y = -\{X,\,Y\}$, where $\{X,\,Y\}$ is given by

$$\{\,X,\,Y\,\} = \{\,X,\,Y\,\}^k\partial_k = \sum_j\Big(X^j\,\frac{\partial Y^k}{\partial x^j} - Y^j\,\frac{\partial X^k}{\partial x^j}\Big)\,\partial_k\;. \tag{42}$$

In summary, it is found that

$$\mathcal{L}_X Y = [\,X,\,Y\,] = -\,\{X,\,Y\} = -\,\mathcal{L}_{\{X,Y\}} = -\,[\,\mathcal{L}_X,\,\mathcal{L}_Y\,]\;. \tag{43}$$

Note the minus sign in front of $\{X,\,Y\}$ in (43), which is characteristic of the right-invariant field such as the diffeomorphisms (see also §5 and §7). This is in contrast with the left–invariant field such as the case considered in the previous subsection a and §4 for the rotation group. Compare the equations (35) and (41).

In the point of view of the fluid dynamics, the Lagrange derivative would be given by

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big|_{t=0} Y^k(\xi_t x)\,\partial_k \;=\; X^j(x)\,\frac{\partial Y^k(x)}{\partial x^j}\,\partial_k\;, \tag{44}$$

which corresponds to only the first term on the right hand side of (42).

In order to see the significance of the commutator $[\mathcal{L}_X,\,\mathcal{L}_Y]$, let us consider the multiplication of two maps $\xi_t\,\eta_s$, where a point x is mapped to another point $\xi_t\,\eta_s(x)$, which is equivalently represented as $\xi_t(\eta_s(x))$. The two flows ξ_t and η_s do not in general commute: $\xi_t\,\eta_s \neq \eta_s\,\xi_t$. A measure of the degree of non-commutativity is given by the difference of values of a differentiable function $f(\cdot)$ at two points $\xi_t\,\eta_s(x)$ and $\eta_s\,\xi_t(x)$, defined by $\Delta(t,s;x) = f(\eta_s\,\xi_t(x)) - f(\xi_t\,\eta_s(x))$. Clearly, this function vanishes for both of $t=0$ and $s=0$, and therefore the first term of $\Delta(t,s;x)$ different from 0 in Taylor series in s and t is the term proportional to $s\,t$, the other terms of second order vanishing. It can be shown [2] that

$$\frac{\partial^2}{\partial s\,\partial t}\Big|_{s=0,t=0}\Big(f(\eta_s\,\xi_t(x)) - f(\xi_t\,\eta_s(x))\Big) = \mathcal{L}_X\Big(\mathcal{L}_Y\,f\Big)(x) - \mathcal{L}_Y\Big(\mathcal{L}_X\,f\Big)(x)\,. \tag{45}$$

In fact, the first term on the right hand side is written as

$$\mathcal{L}_X\Big(\mathcal{L}_Y\,f\Big)\Big|_{id} = \partial_t\Big(\mathcal{L}_Y\,f\,\circ\xi_t\Big)\Big|_{id} = \partial_t\,\partial_s\,f\circ\eta_s\circ\xi_t\Big|_{id}\,,$$

by applying the operation (2) twice, and similarly we have $\mathcal{L}_Y\left(\mathcal{L}_X f\right)\big|_{id} = \partial_s\, \partial_t\, f \circ \xi_t \circ \eta_s\big|_{id}$. Thus the above equation (45) is verified. Its right hand side is the commutator of $\mathcal{L}_X$ and $\mathcal{L}_Y$.

For $f = id$, we obtain $\mathcal{L}_X\left(\mathcal{L}_Y(id)\right)|_{id} = \partial_t\, \partial_s\, \eta_s \circ \xi_t|_{id}$. Therefore,

$$[\mathcal{L}_X, \mathcal{L}_Y] = \Big(\partial_t\, \partial_s\, \eta_s \circ \xi_t - \partial_s\, \partial_t\, \xi_t \circ \eta_s\Big)\Big|_{id} \tag{46}$$

As a particular case, it is to be noted that the coordinate bases commute. In fact, substituting $X = \partial_\alpha$, $Y = \partial_\beta$, we have

$$[\partial_\alpha, \partial_\beta] = \partial_\alpha\, \partial_\beta - \partial_\beta\, \partial_\alpha = 0\ . \tag{47}$$

which states that each pair of coordinate curves intersects by definition.

Remark: A vector field Y defined along the integral curve of X is said to be *invariant* if $Y_{\xi_t x} = \xi_{t*} Y_x$. Then, putting $x = \xi_t^{-1} p$ and using (42)∼(44), the equation (38) reduces to

$$0 = \mathcal{L}_X\, Y(x) = -\Big(\frac{\mathrm{D}Y^i}{\mathrm{D}t} - Y^j\,\frac{\partial X^i}{\partial x^j}\Big)\,\partial_i\,. \tag{48}$$

Note the minus sign in front of $\mathrm{D}Y/\mathrm{D}t$, suggesting that appropriate time evolution is given by $-\mathcal{L}_X Y(x)$. In the fluid dynamics, this form of equation (48) is often called an equation of *frozen field.* The Jacobi field Y ($= J$ introduced later) satisfies this equation (see §7.3.4 d.). Its solution is given by the Cauchy's solution, $Y^\alpha(t) = Y^j(0)\partial y^\alpha/\partial x^j$, which is nothing but the equation (15).

2.7 Diffeomorphism of a circle S^1

Diffeomorphism of the manifold S^1 (circle of periodicity T) is represented by a map g : $x \in S^1 \to g(x) \in S^1$ and $g(x+T) = g(x) + T$. Such maps constitute a group $D(S^1)$ of diffeomorphisms with the composition law:

$$h = g \circ f\,, \qquad i.e.\quad h(x) = g(f(x)) \in D(S^1)\,,$$

for $f, g \in D(S^1)$. The diffeomorphism is a map of infinite degrees of freedom, *i.e.* having *pointwise* degrees of freedom. In §5, the diffeomorphism is assumed to be orientation-preserving in the sense that $g'(x) > 0$, where the prime denotes $\partial/\partial x$.

Associated with a flow $\xi_t(x)$ that is a smooth sequence of diffeomorphisms with the parameter t (see (39)), its tangent field at t is defined by

$$\dot{\xi}_t(x) = \frac{\mathrm{d}}{\mathrm{d}t}\,\xi_t(x)\,\Big|_t = \lim_{\tau\to 0}\frac{\xi_\tau(x) - 1}{\tau}\,\xi_t(x) = u(x) \circ \xi_t(x)$$

in a right-invariant form. The tangent field at the identity is given by

$$u(x) = \frac{\mathrm{d}\xi_t(x)}{\mathrm{d}t}\Big|_{t=0}\,.$$

Put it in another way, an element X in the **tangent space** $T_{id}D(S^1)$ at the identity is represented as

$$X = u(x)\,\partial_x\ .$$

For the two diffeomorphisms ξ_t and η_t corresponding to the vector fields X and Y respectively, the *Lie bracket* (commutator) is given by (38) with using (43) and (42):

$$[X, Y] = -\,(u\,v' - v\,u')\,\partial_x\,. \tag{49}$$

for $X = u(x)\partial_x,\; Y = v(x)\partial_x \in T_x D(S^1)$. The Lie algebra is sometimes called *Witt algebra.*

3 Riemannian Geometry (Reviews)

3.1 Riemannian metric

On a Riemannian manifold M^n, a positive definite inner product $\langle \cdot, \cdot \rangle$ is defined on the tangent space $T_x M^n$ at $x = (x^1, \cdots, x^n) \in M$ with a differentiable fashion. For two tangent fields $X = X^i(x)\partial_i$, $Y = Y^j(x)\partial_j \in TM^n$, the *Riemannian metric* is given by,

$$\langle X, Y \rangle(x) = g_{ij} X^i(x) Y^j(x) ,$$

as already defined in (10), where the metric tensor, $g_{ij}(x) = \langle \partial_i, \partial_j \rangle = g_{ji}(x)$, is symmetric and differentiable with respect to x^i. This bilinear quadratic form is called the *first fundamental form.* The line element ds is given by $ds^2 = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$.

The inner product is *non-degenerate*, for $^\forall Y \in TM$, if

$$\langle X, Y \rangle = 0 \qquad \text{only when} \quad X = 0 \, . \tag{50}$$

If the inner product is only non-degenerate rather than positive definite, the resulting structure on M^n is called a *pseudo*-Riemannian.

3.2 Examples of metric tensor

3.2.1 Finite dimensions

Consider a dynamical system of N degrees of freedom in a *gravitational* field with the potential $V(\bar{q})$ and the kinetic energy

$$T = \frac{1}{2} a_{ij}\, \dot{q}^i\, \dot{q}^j \, , \qquad \text{where} \quad \bar{q} = (q^i), \qquad (i = 1, \cdots, N).$$

The metric is defined by $g_{ij}\, \dot{q}^i\, \dot{q}^j$, where

$$g_{ij} = g^J_{ij}(\bar{q}) = 2\big(E - V(\bar{q})\big)\, a_{ij} \qquad \text{for} \quad i, j = 1, \cdots, N$$

is the *Jacobi's* metric tensor [12]. In chapter 6, we consider a physical system in terms of another metric called the Eisenhart metric g^E_{ij}.

3.2.2 Infinite dimensions

Metric on the group $D(S^1)$ of diffeomorphisms (§2.7) is defined for the right-invariant tangent fields $U_\xi(x) = u \circ \xi(x)$ and $V_\xi(x) = v \circ \xi(x)$ in the following invariant way (also see §7.2) :

$$\langle U, V \rangle_\xi = \int_{S^1} (U_\xi \circ \xi^{-1}, V_\xi \circ \xi^{-1})_x\, \mathrm{d}x = \int_{S^1} u(x)\, v(x)\, \mathrm{d}x = \langle X, Y \rangle \, , \tag{51}$$

for $X = u(x)\, \partial_x$, $Y = v(x)\, \partial_x \in T_{id} D(S^1)$.

3.3 Connection (Covariant derivative)

An affine connection is an operator ∇ that satisfies the following relations, at $x \in M$:

$$\left.\begin{array}{llcl} \text{(i)} & \nabla_\xi (aX + bY) & = & a\nabla_\xi X + b\nabla_\xi Y \\ \text{(ii)} & \nabla_{a\xi+b\eta} X & = & a\nabla_\xi X + b\nabla_\eta X \\ \text{(iii)} & \nabla_\xi f(x)X & = & f(x)\nabla_\xi X + \xi(f)X \end{array}\right\} \tag{52}$$

for any vector fields X, Y and a function $f(x)$ with a, $b \in R$ and for vectors ξ and η at x. The connection $\nabla_X Y$ is also called the *covariant derivative* of the vector field $Y(x)$ in the direction of the vector X. Using the representations $X = X^i\,\partial_i$ and $Y = Y^j\,\partial_j$ and applying the above properties (i)$\sim$(iii), we obtain

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i\partial_i} Y^j\partial_j = X^i\nabla_{\partial_i}(Y^j\partial_j) \\ &= (X^i\partial_i Y^j)\partial_j + X^i Y^j\,\Gamma_{ij}^k\,\partial_k\,, \end{aligned} \tag{53}$$

where Γ_{ij}^k is the Christoffel symbols defined by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k\,. \tag{54}$$

Therefore,

$$(\nabla_X Y)^i = \left[\frac{\partial Y^i}{\partial x^j} + \Gamma_{jk}^i Y^k\right] X^j \tag{55}$$

Most dynamical systems are *time-dependent* and every tangent vector is written in the form, $\hat{X} = \hat{X}^i\,\partial_i = \partial_t + X^\alpha\partial_\alpha$ (see (6)). Correspondingly, the connection is written as

$$\begin{aligned} \nabla_{\hat{X}}\hat{Y} &= \nabla_{\hat{X}^i\partial_i}\hat{Y}^j\partial_j \\ &= \nabla_{\partial_t}\hat{Y}^j\partial_j + X^\alpha\nabla_{\partial_\alpha}(\hat{Y}^j\partial_j) \end{aligned}$$

where $\hat{Y} = \partial_t + Y^\alpha\partial_\alpha$ and α denotes the indices of spatial components ($\alpha = 1, \cdots, n$). For the spatial part $Y = Y^\alpha\partial_\alpha$ (with $X = X^\alpha\partial_\alpha$), we have

$$\nabla_{\hat{X}} Y = \frac{\partial}{\partial t} Y + \nabla_X Y\,, \quad (\text{where} \quad \nabla_{\partial_t} Y = \partial_t Y \quad \text{and} \quad \nabla_{\partial_t}\partial_\alpha = 0). \tag{56}$$

3.4 Riemannian connection

There is a unique connection ∇ on a Riemannian manifold called *Riemannian* connection or *Levi-Civita* connection that satisfies

$$\text{(i)} \quad \nabla_X Y - \nabla_Y X = [X, Y] \quad \text{(torsion free)} \tag{57}$$

$$\text{(ii)} \quad Z\langle X, Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad \text{(compatibility with metric)} \tag{58}$$

for vector fields X, Y and Z. The torsion-free requires the symmetry, $\Gamma_{ij}^k = \Gamma_{ji}^k$, with respect to i and j. One consequence of the second compatibility condition with metric is given at the end of the next section §3.5.

Owing to the two properties, the Riemannian connection satisfies the following identity,

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\quad + \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \,, \end{aligned} \tag{59}$$

where $X\langle \cdot, \cdot\rangle = X^j\partial_j \langle \cdot, \cdot\rangle$. The last equation defines the connection ∇ in terms of the inner product $\langle \cdot\,, \cdot\rangle$ and the commutator $[\cdot\,, \cdot]$.

In most dynamical systems studied below, the metrics are defined invariant (with respect to either right or left translation). Thus, if X, Y, Z are invariant vector fields, the first three terms on the right hand side of (59) vanishes identically. Hence on the Riemannian manifold of *invariant vector fields*, the equation (59) reduces to

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \,. \tag{60}$$

3.5 Covariant derivative along a parameterized curve

Consider a curve $x(t)$ on M passing through p whose tangent at p is given by $T = \mathrm{d}x/\mathrm{d}t = \dot{x}$, and let Y be a tangent vector field defined along the curve $x(t)$. According to (55), the covariant derivative $\nabla_T Y$ is also written as

$$\nabla_T Y = \frac{\boldsymbol{\nabla} Y}{\mathrm{d}t} = \left[\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj} Y^j\right] \dot{x}^k\, \boldsymbol{\partial}_i = \left[\frac{\mathrm{d}}{\mathrm{d}t} Y^i + \Gamma^i_{kj}\dot{x}^k Y^j\right] \boldsymbol{\partial}_i \,. \tag{61}$$

The second expression $\boldsymbol{\nabla} Y/\mathrm{d}t$ emphasizes the derivative along the curve $x(t)$ parameterized with t.

On the manifold M endowed with the connection $\boldsymbol{\nabla}$, one can consider *parallel* displacement of a vector Y along a parameterized curve $x(t)$, which is defined by vanishing covariant derivative:

$$\frac{\boldsymbol{\nabla} Y}{\mathrm{d}t} = \nabla_T Y = 0 \,.$$

Thus, Y is translated parallel along the curve $x(t)$ when $\dot{x}^k\,(\partial Y^i/\partial x^k) + \Gamma^i_{kj}\dot{x}^k Y^j = 0$.

It is readily seen from (58) that the scalar product is invariant, $T\,\langle X,\, Y\rangle = 0$, for the vector fields translated parallel along the curve:

$$\langle\, X,\, Y\,\rangle = \textit{constant} \qquad \text{(under parallel translation)}.$$

3.6 Curvature tensors

The vector field given by

$$R(X,\, Y)\, Z \;=\; \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}\, Z \tag{62}$$

$$\;\equiv\; (R^i_{jkl} Z^j X^k Y^l)\, \partial_i \tag{63}$$

defines a linear transformation, $Z \to R(X,\, Y)\, Z$, called the curvature transformation for a pair of vector fields X, Y, where R^i_{jkl} is the *Riemannian curvature tensor.* Then we have

$$\langle W,\, R(X,\, Y)\, Z\rangle = R^\alpha_{jkl}\, \langle \partial_i,\, \partial_\alpha\rangle\, W^i Z^j X^k Y^l = R_{ijkl}\, W^i Z^j X^k Y^l \tag{64}$$

according to the traditional ordering of suffices of the tensor R_{ijkl}, where $R_{ijkl} = g_{i\alpha} R^{\alpha}_{jkl}$ and $g_{i\alpha} = \langle \partial_i, \partial_\alpha \rangle$. From its definition, one may write

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} ,$$

which clearly shows the anti-symmetry $R(X, Y) = -R(Y, X)$. Correspondingly, we have $R^i_{jkl} = -R^i_{jlk}$. In particular, for $X = \partial_\alpha$, $Y = \partial_\beta$ and $Z = \partial_\gamma$, we have

$$R(\partial_\alpha, \partial_\beta)\, \partial_\gamma = \nabla_{\partial_\alpha}(\nabla_{\partial_\beta} \partial_\gamma) - \nabla_{\partial_\beta}(\nabla_{\partial_\alpha} \partial_\gamma) = R^i_{\gamma\alpha\beta}\, \partial_i \tag{65}$$

where the third term $\nabla_{[\partial_\alpha, \partial_\beta]}$ disappeared in the middle expression because $[\partial_\alpha, \partial_\beta] = 0$. Using (54), we obtain

$$\begin{aligned} R(\partial_\alpha, \partial_\beta)\, \partial_\gamma &= \nabla_{\partial_\alpha}(\Gamma^k_{\beta\gamma}\partial_k) - \nabla_{\partial_\beta}(\Gamma^k_{\alpha\gamma}\partial_k) \\ &= \partial_\alpha \Gamma^l_{\beta\gamma}\partial_l - \partial_\beta \Gamma^l_{\alpha\gamma}\partial_l + \Gamma^k_{\beta\gamma}\Gamma^l_{\alpha k}\partial_l - \Gamma^k_{\alpha\gamma}\Gamma^l_{\beta k}\partial_l \end{aligned}$$

This leads to the formula of the curvature tensor in terms of Γ^k_{ij}:

$$R^i_{\gamma\alpha\beta} = \partial_\alpha \Gamma^i_{\beta\gamma} - \partial_\beta \Gamma^i_{\alpha\gamma} + \Gamma^i_{\alpha k}\Gamma^k_{\beta\gamma} - \Gamma^i_{\beta k}\Gamma^k_{\alpha\gamma} \tag{66}$$

The Christoffel symbol Γ^k_{ij} is in turn represented in terms of the metric tensor $g = (g_{ij})$ by the following:

$$\Gamma^k_{ij} = \frac{1}{2}\, g^{km}(\partial_i\, g_{jm} + \partial_j\, g_{mi} - \partial_m\, g_{ij}) \tag{67}$$

where g^{km} denotes the component of the inverse g^{-1}, that is $g^{km} = (g^{-1})^{km}$, satisfying the relations $g^{km} g_{ml} = g_{lm} g^{mk} = \delta^k_l$ (Kronecker's delta). Note that $\Gamma^k_{ij} = \Gamma^k_{ji}$ since $g_{ij} = g_{ji}$. The formula (67) can be shown by using (11), (54) and (58), and noting that

$$\partial_m\, g_{ij} = \partial_m\, \langle \partial_i, \partial_j \rangle = \langle \nabla_{\partial_m} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_m} \partial_j \rangle = \Gamma^k_{mi} g_{kj} + \Gamma^k_{mj} g_{ki}$$

and that $\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij} = 2 g_{km} \Gamma^k_{ij}$.

The curvature tensor $\langle \partial_\delta, R(\partial_\alpha, \partial_\beta)\, \partial_\gamma \rangle = R^i_{\gamma\alpha\beta} \langle \partial_\delta, \partial_i \rangle \equiv R_{\delta\gamma\alpha\beta}$ is anti-symmetric with respect to γ and δ as well as α and β. In fact, using (65) and (58), we have

$$\begin{aligned} \langle \partial_\delta, R(\partial_\alpha, \partial_\beta)\, \partial_\gamma \rangle &= \langle \partial_\delta, \nabla_\alpha(\nabla_\beta\, \partial_\gamma) \rangle - \langle \partial_\delta, \nabla_\beta(\nabla_\alpha\, \partial_\gamma) \rangle \\ &= \partial_\alpha \langle \partial_\delta, \nabla_\beta\, \partial_\gamma \rangle - \langle \nabla_\alpha\, \partial_\delta, \nabla_\beta\, \partial_\gamma \rangle - \partial_\beta \langle \partial_\delta, \nabla_\alpha\, \partial_\gamma \rangle + \langle \nabla_\beta\, \partial_\delta, \nabla_\alpha\, \partial_\gamma \rangle \end{aligned}$$

where $\nabla_\alpha = \nabla_{\partial_\alpha}$. The third term $-\partial_\beta \langle \nabla_\alpha\, \partial_\gamma, \partial_\delta \rangle$ in the last expression can be written as

$$-\partial_\alpha \langle \nabla_\beta \partial_\delta, \partial_\gamma \rangle + \langle \nabla_\beta \nabla_\alpha \partial_\delta, \partial_\gamma \rangle - \langle \partial_\delta, \nabla_\beta \nabla_\alpha \partial_\gamma \rangle - \langle \partial_\delta, R(\partial_\alpha, \partial_\beta)\, \partial_\gamma \rangle$$

Thus, $2\, \langle \partial_\delta, R(\partial_\alpha, \partial_\beta)\, \partial_\gamma \rangle$ is given by

$$\begin{aligned} \partial_\alpha \langle \partial_\delta, \nabla_\beta\, \partial_\gamma \rangle &- \partial_\alpha \langle \nabla_\beta \partial_\delta, \partial_\gamma \rangle - \langle \nabla_\alpha\, \partial_\delta, \nabla_\beta\, \partial_\gamma \rangle + \langle \nabla_\beta\, \partial_\delta, \nabla_\alpha\, \partial_\gamma \rangle \\ + \langle \nabla_\beta \nabla_\alpha \partial_\delta, \partial_\gamma \rangle &- \langle \partial_\delta, \nabla_\beta \nabla_\alpha \partial_\gamma \rangle \end{aligned}$$

which is obviously anti-symmetric with respect to γ and δ.

3.7 Induced connection and Second fundamental form

Let $V^m \subset M^n$ be a submanifold of a Riemannian manifold M with the metric g_{ij}. Let us consider the restriction of the Riemannian metric g_{ij} to the tangent vectors to V. This action induces the Riemannian metric (an induced metric) for V. An arbitrary vector field Z in M can be decomposed into two orthogonal components: $Z = Z_V + Z_N$, where $Z_V = \mathtt{P}\{Z\}$ is the projected component to V and $Z_N = \mathtt{Q}\{Z\}$ the component perpendicular to V. The symbols $\mathtt{P}$ and $\mathtt{Q}$ denote the orthogonal projections onto the space V and the space orthogonal to it.

Let $\hat{\nabla}$ be the Riemannian connection for M^n, and define a new connection $\bar{\nabla}$ for V^m $(m < n)$ as follows. Consider a vector X tangent to V and a vector field Z defined along V, but Z is not necessarily tangent to V. Then the $\bar{\nabla}$ is defined by

$$\bar{\nabla}_X Z \equiv \hat{\nabla}_X Z - \mathtt{Q}\{\hat{\nabla}_X Z\} = \mathtt{P}\{\hat{\nabla}_X Z\} \,, \tag{68}$$

where the right hand side is the projection of $\hat{\nabla}_X Z$ onto the tangent space of V. It can be checked that the operator $\bar{\nabla}$ satisfies the properties (52) and said to be an induced connection.

Let X, Y and Z be tangent to V, hence one has $\mathtt{Q}\{X\} = 0$, *etc.* and further $\mathtt{Q}\{\,[X,\, Y]\,\} = 0$ by (42) and (43). Then we extend the vectors X and Y to be vector fields on M. By the torsion-free of the Riemannian connection $\hat{\nabla}$, one has $\mathtt{Q}\{\hat{\nabla}_X Y - \hat{\nabla}_Y X\} = \mathtt{Q}\{\,[X,\, Y]\,\} = 0$. Thus, the connection $\bar{\nabla}$ is also torsion-free:

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X,\, Y] \,. \tag{69}$$

Thus it is verified that the connection $\bar{\nabla}$ is also Riemannian (the condition (57) is satisfied).

Now, *the second fundamental form* $S(X, Y)$ is introduced by the definition:

$$\hat{\nabla}_X Y = \bar{\nabla}_X Y + S(X,Y) \,, \tag{70}$$

which is called the *Gauss' formula.* It is not difficult to see that the function $S(X, Y)$ satisfies the following relation, which is found to be symmetric with respect to X and Y,

$$S(X,Y) \;(= \hat{\nabla}_X Y - \bar{\nabla}_X Y) \; = \mathtt{Q}\{\hat{\nabla}_X Y\} = \mathtt{Q}\{\hat{\nabla}_Y X\} = S(Y,X) \,. \tag{71}$$

This is a Riemannian generalization of the *Weingarten equation.*

Corresponding to $\hat{\nabla}$ and $\bar{\nabla}$, we have two kinds of curvature tensors, $\hat{R}(X,Y)Z$ and $\bar{R}(X,Y)Z$, respectively. Using the definition (62) of $R(X,Y)Z$ and the above relations (70) and (71), one can show the following *Gauss-Codazzi equation*:

$$\langle W,\, \hat{R}(X,\, Y)\, Z\rangle = \langle W,\, \bar{R}(X,\, Y)\, Z\rangle + \langle S(X,Z),\, S(Y,W)\rangle - \langle S(X,W),\, S(Y,Z)\,\rangle, \tag{72}$$

where X, Y, Z, W are tangent to V and hence $\langle W,\, S([X,Y],Z)\rangle$ vanishes. See Frankel [1] or Abraham & Marsden [4] for more details.

3.8 Geodesic equation

3.8.1 Local coordinates representation

A parameterized curve $\gamma(s)$ is *geodesic* if its tangent $T = \mathrm{d}\gamma/\mathrm{d}s = \dot{\gamma}$ is displaced in parallel :

$$\frac{\boldsymbol{\nabla} T}{\mathrm{d}s} = \boldsymbol{\nabla}_T T = \frac{\boldsymbol{\nabla}}{\mathrm{d}s}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right) = 0 \, .$$

In local coordinates $\gamma = (x^i)$,

$$\begin{aligned} T &= T^i\, \partial_i = \frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{\mathrm{d}x^i}{\mathrm{d}s}\, \partial_i \, , && (73) \\ \frac{\boldsymbol{\nabla} T}{\mathrm{d}s} &= \boldsymbol{\nabla}_T T = \left[\frac{\mathrm{d}T^i}{\mathrm{d}s} + \Gamma^i_{jk} T^j T^k\right] \partial_i = 0 \, . && (74) \end{aligned}$$

Thus we obtain the *geodesic equation* :

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0 \, . \qquad (75)$$

3.8.2 Group-theoretic representation

On the Riemannian manifold of invariant metric, another formulation of the geodesic equation is possible, because most dynamical systems considered below are equipped with invariant metrics (with respect to either right or left translation). In such cases, the following derivation would be useful.

In terms of the adjoint operator $ad_X\, Z = [X, Z]$ introduced in (32), let us define the *coadjoint* operator by

$$\langle ad^*_X\, Y, Z\rangle = \langle Y, ad_X\, Z\rangle = \langle Y, [X, Z]\rangle \, . \qquad (76)$$

Then the equation (60) is transformed to $2\langle \nabla_X Y, Z\rangle = \langle ad_X Y, Z\rangle - \langle ad^*_Y X, Z\rangle - \langle ad^*_X Y, Z\rangle$. The non-degeneracy of the inner product given in (50) leads to

$$\nabla_X\, Y = \frac{1}{2}\left(ad_X\, Y - ad^*_X Y - ad^*_Y X\right) \, . \qquad (77)$$

Thus, the another form of the geodesic equation is given by $\nabla_X X = -ad^*_X X = 0$, since $ad_X X = [X, X] = 0$. In particular, the geodesic equation of a time-dependent problem is represented as

$$\nabla_{\hat{X}}\, X = \partial_t\, X + \nabla_X\, X = \partial_t\, X - ad^*_X\, X = 0 \, , \qquad (78)$$

for the spatial part X from (56). It should be noted that this is valid for the left-invariant field such as the rotation group considered in §2.6.1**a** and §4. There is difference by the sign $\pm$ in the relation between the commutator of the Lie algebra and the Poisson bracket whether the vector fields are left-invariant or right-invariant, as illustrated in §2.6. [5]

In the case of the right-invariant field, it was shown in §2.6.2**b** that $[X, Y] = \mathcal{L}_X Y = -\{(\mathrm{d}Y^i/\mathrm{d}t) - Y^j(\partial X^i/\partial x^j)\}\, \partial_i$, where $\mathcal{L}$ is the Lie derivative. When the time evolution of such system is concerned, the negative of $\mathcal{L}_X Y = [X,\, Y]$ is appropriate to derive the connection. This requires that

$$\nabla^{(\mathrm{R})}_X Y \equiv -\nabla_X Y$$

should be used instead of (77), and that the time-dependent geodesic equation takes the form,

$$\partial_t X + \nabla^{(\mathrm{R})}_X X = \partial_t X + ad^*_X X = 0\ , \tag{79}$$

instead of (78). This describes a curve with the time parameter t whose tangent is displaced parallel along itself in the right-invariant way.

However, the curvature tensor and the Jacobi equation considered in the next section are not changed because those include the nabla twice or its equivalent of multiplying $(-1)^2$.

3.9 Jacobi equation

Let $C_0 : \gamma_0(s)$ be a geodesic curve with the length parameter $s \in [0, L]$, and $C_\alpha : \gamma(s, \alpha)$ a varied geodesic curve where $\alpha \in (-1, +1)$ is a variation parameter and $\gamma_0(s) = \gamma(s, 0)$ with s being the arc length for $\alpha = 0$. Because $\gamma(s, \alpha)$ is a geodesic, we have $\nabla(\partial_s\gamma)/\partial s = 0$ for all α. The function $\gamma(s, \alpha)$ is a differentiable map $\gamma :\ U \subset R^2 \to M^n$ with the property $[\partial/\partial s,\ \partial/\partial\alpha] = 0$ in U. In this circumstance, the following two identities are known,

$$\left(\frac{\nabla}{\partial s}\right)\partial_\alpha\gamma \;=\; \left(\frac{\nabla}{\partial \alpha}\right)\partial_s\gamma\,, \tag{80}$$

$$\frac{\nabla}{\partial s}\left(\frac{\nabla Z}{\partial \alpha}\right) - \frac{\nabla}{\partial \alpha}\left(\frac{\nabla Z}{\partial s}\right) \;=\; R(\partial_s\gamma, \partial_\alpha\gamma)\, Z\,, \tag{81}$$

(see Frankel [1]), where $\partial_s\gamma = \partial\gamma/\partial s$ and $\partial_\alpha\gamma = \partial\gamma/\partial\alpha$. Along the reference geodesic $\gamma_0(s)$, let us use the notation $T = \partial_s\gamma$ for the tangent to the geodesic and $J = \partial_\alpha\gamma$ $(\alpha = 0)$ for the variation vector. Using $\nabla T/\partial s = 0$ and the above identities with $Z = T$, we have

$$\begin{aligned} 0 = \frac{\nabla}{\partial\alpha}\frac{\nabla T}{\partial s} &= \frac{\nabla}{\partial s}\frac{\nabla T}{\partial\alpha} - R(T, J)\, T \\ &= \frac{\nabla}{\partial s}\frac{\nabla}{\partial\alpha}\partial_s\gamma + R(J, T)\, T = \frac{\nabla}{\partial s}\frac{\nabla}{\partial s}\partial_\alpha\gamma + R(J, T)\, T \\ &= \frac{\nabla}{\partial s}\frac{\nabla}{\partial s} J + R(J, T)\, T\ . \end{aligned}$$

Thus we have obtained the *Jacobi equation* for the geodesic variation J,

$$\frac{\nabla}{\partial s}\frac{\nabla}{\partial s} J + R(J, T)\, T = 0\ . \tag{82}$$

The variation vector field J is called *Jacobi field.*

Defining $\|\, J \,\|^2 \equiv \langle J,\ J\rangle$ and differentiating it two times with respect to s and using (82) and (58), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\|\, J \,\|^2}{2} = \|\, \nabla_T J \,\|^2\ -\ K(T, J)\ , \tag{83}$$

where $\nabla_T J = \nabla J/\partial s$, and

$$K(T,J) \equiv \langle J,\, R(J,T)T\rangle = R_{ijkl}\, J^i T^j\, J^k T^l \tag{84}$$

is a sectional curvature factor associated with the two-dimensional section spanned by J and T. The *sectional curvature* is defined by $K(T,J)/(\| T \|^2 \| J \|^2 - \langle T,\, J\rangle^2)$, which reduces to $K(T,J)$ when T and J are orthonormal. Writing $J = \| J \|\; \boldsymbol{e}_J$ where $\| \boldsymbol{e}_J \| = 1$, the equation (83) is transformed to

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \| J \| = \Big(\| \nabla_T\, \boldsymbol{e}_J \|^2 \; - K(T,\, \boldsymbol{e}_J) \Big) \; \| J \| \; . \tag{85}$$

The equation (82) provides the link between the stability of geodesic curves and the Riemannian curvature, and one of the bases for geometrical description of dynamical systems considered below.

3.10 Arc length

Arc length of the curve C_α is given by

$$L(\alpha) = \int_0^L \Big\|\frac{\partial\gamma(s,\alpha)}{\partial s}\Big\| \,\mathrm{d}s = \int_0^L \left\langle \frac{\partial\gamma(s,\alpha)}{\partial s},\; \frac{\partial\gamma(s,\alpha)}{\partial s} \right\rangle^{1/2} \mathrm{d}s = \int_0^L \langle\, T(s,\alpha),\, T(s,\alpha)\,\rangle^{1/2}\, \mathrm{d}s \,,$$

and its variation is

$$L'(\alpha) = \int_0^L \frac{\partial}{\partial\alpha} \left\langle \frac{\partial\gamma}{\partial s},\, \frac{\partial\gamma}{\partial s} \right\rangle^{1/2} \mathrm{d}s = \int_0^L \Big\|\frac{\partial\gamma(s,\alpha)}{\partial s}\Big\|^{-1} \left\langle \frac{\nabla}{\partial\alpha}\frac{\partial\gamma}{\partial s},\, \frac{\partial\gamma}{\partial s}\right\rangle \mathrm{d}s \; . \tag{86}$$

When $\alpha = 0$, we have $\|\partial\gamma(s,0)/\partial s\| = 1$ and

$$L'(0) = \int_0^L \frac{\partial}{\partial s}\left\langle \frac{\partial\gamma}{\partial\alpha},\, \frac{\partial\gamma}{\partial s}\right\rangle \mathrm{d}s - \int_0^L \left\langle \frac{\partial\gamma}{\partial\alpha},\, \frac{\nabla}{\partial s}\frac{\partial\gamma}{\partial s}\right\rangle \mathrm{d}s \; .$$

where the identity (80) is applied to (86). Thus, the first variation of arc length is given by

$$L'(0) = < J, T >_{\mathrm{Q}} - < J, T >_{\mathrm{P}} - \int_0^L \left\langle J,\, \frac{\nabla}{\partial s} T \right\rangle \mathrm{d}s \; ,$$

where P$= \gamma(0,0)$, Q$= \gamma(L,0)$ and $T = \partial_s\gamma(s,0)$, $J = \partial_\alpha\gamma(s,0)$.

Thus, *another definition* of the geodesic is as follows. The curve $C_0 : \gamma_0(s)$ is said to be a *geodesic* if $L'(0) = 0$ for all variations J that vanish at the end points P and Q, *i.e.* $J = 0$ at P and Q. Then we have

$$\left\langle J,\, \frac{\nabla}{\partial s} T \right\rangle = 0 \qquad \text{for} \quad 0 < s < L \; ,$$

for every vector J tangent to M along the geodesic C_0. Thus the vector $\nabla T/\partial s$ must be normal to the manifold M. The geodesic is a curve on which the component of $\nabla_T T$ tangent to M vanishes. Denoting $\nabla T/\partial s$ as $\nabla T/\mathrm{d}s$, we have the equation for a geodesic as

$$\frac{\nabla T}{\mathrm{d}s} = \nabla_T T = 0 \; ,$$

by the non-degeneracy of the metric.

4 Free Rotation of a Rigid Body

We now consider a physical problem, that is, an application of the geometrical theory formulated in the previous two sections to one of the simplest dynamical system: Free rotation of a rigid body without action of external torque. The basic idea is that the governing equation is the geodesic equation over the manifold space of a group of transformations $SO(3)$ (a Lie group), which describes the motion of the physical problem. We begin with this simplest system in order to illustrate the underlying geometrical ideas. This chapter is based on Arnold (1978) [2], Kambe (1998) [7], and Suzuki *et al.* (1998) [8].

In the mechanics of rigid bodies, *free* rotation is described by the Euler's equation,

$$J_1 \frac{\mathrm{d}\Omega^1}{\mathrm{d}t} - (J_2 - J_3)\Omega^2\Omega^3 = 0, \quad J_2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}t} - (J_3 - J_1)\Omega^3\Omega^1 = 0, \quad J_3 \frac{\mathrm{d}\Omega^3}{\mathrm{d}t} - (J_1 - J_2)\Omega^1\Omega^2 = 0 \tag{87}$$

in the body frame (*i.e.* in the moving frame of reference), where $\Omega = (\Omega^1, \Omega^2, \Omega^3)$ is the angular velocity vector in the frame of the principal axes (x^1, x^2, x^3) of the body's *moment of inertia* $J = (J_{\alpha\beta}) = \mathrm{diag}(J_1, J_2, J_3)$, J_α being the diagonal elements. The moment of inertia is defined by $J_{\alpha\beta} = \int x^\alpha x^\beta \rho\, \mathrm{d}^3\boldsymbol{x}$, where ρ is the mass density assumed constant.

The angular momentum is given by $M = (M_\alpha) = J\Omega = (J_{\alpha\beta}\Omega^\beta)$, and the kinetic energy K is expressed as

$$K = \frac{1}{2}(\, M,\ \Omega\,)_{R^3} = \frac{1}{2} M_\alpha \Omega^\alpha = \frac{1}{2}(\, J\Omega,\ \Omega\,)_{R^3}\,, \tag{88}$$

which is invariant during the motion, where

$$(\, A\,,\ B\,)_{R^3} = A_\alpha\, B^\alpha = A_1 B^1 + A_2 B^2 + A_3 B^3 \tag{89}$$

is the scalar product in R^3. Stability is considered by deriving the Jacobi equation for the geodesic variation vector J.

4.1 Rotation as elements of $SO(3)$

Rotation of a rigid body is regarded as a smooth sequence of transformations of the body (*e.g.* body's principal axes), which are elements of the group of special orthogonal transformation of dimension 3: $SO(3)$. By a transformation matrix $A \in SO(3)$, a point $\mathbf{x}$ (fixed to the body) is moved (mapped) to $\mathbf{x}' = A\mathbf{x}$ where $\det A = 1$. The group $G = SO(3)$ is a Lie group and consists of all orientation-preserving rotations (*i.e.* $\det A = +1$) of a rigid body.

An element g of the group G corresponds to a position of the body with its motion arriving at g from the initial position e (the identity). A motion of the body is described by a curve $\mathrm{C}\colon\ t \to g_t$ on the manifold $SO(3)$ with t the time parameter $(g_0 = e)$: [2, 3]

$$g_t = A(t), \qquad A(t) \in SO(3), \quad t \in R\,.$$

An infinitesimal transformation $\bar{\Omega}$ is defined by $\delta t \bar{\Omega} \cdot A(t) = A(t + \delta t) - A(t)$ for an infinitesimal time increment δt at g_t, where $\bar{\Omega}$ is shown to be an anti-symmetric matrix.

The time derivative $\dot{g}_t$ is the angular velocity of the body in the (fixed) inertial space and given by $\dot{g}_t = \bar{\Omega} g_t$. The angular velocity Ω in the (moving) body frame is obtained by the left translation of $\dot{g}_t$ by $(g_t^{-1})_* \dot{g}_t = (g_t^{-1})_* \bar{\Omega} g_t$, the axial vectors equivalent to Ω or $\dot{g}_t$ being denoted as $\hat{\Omega}$ or $\hat{\dot{g}}_t$. The $\hat{\Omega}$ is a *tangent* vector at the identity e of the group G and an element of the Lie algebra $\mathbf{so(3)}$. The space of such vectors is denoted by $T_eG = \mathbf{so}(3)$. According to (33) and (34), the commutator of this algebra is given by the vector product in R^3:

$$[X,\ Y] = X \times Y\,, \qquad \text{for } X,\ Y \in T_eG = \mathbf{so}(3)\,. \tag{90}$$

The kinetic energy of the body motion is given by the scalar product of the angular velocity vector $\hat{\dot{g}}_t$ and the angular momentum $\bar{J}\hat{\dot{g}}_t$ multiplied by $\frac{1}{2}$, where $\bar{J}$ is the moment of inertia in the fixed space. The kinetic energy is a scalar of frame-independent. In other words, it does not depend on the coordinate change of left-translation mentioned above. Hence the energy K gives a left-invariant Riemannian metric on the group:

$$K = \frac{1}{2}(\bar{J}\hat{\dot{g}}_t,\ \hat{\dot{g}}_t) = \frac{1}{2}(J\hat{\Omega},\ \hat{\Omega}), \tag{91}$$

where $J = (\,J_{ij}) = (g_t)^{-1}\bar{J}g_t$ is a diagonal matrix (in the principal axes) with positive elements $J_\alpha(>0)$. The angular momentum $M = J\,\Omega$ (in the body system) is a covector (an element in the cotangent space T_e^*G to the group at e).

Now one can define the metric $<\cdot,\cdot>$ on T_eG by

$$< X,\ Y > \equiv (\,JX,\ Y\,) \qquad \text{for } X,\ Y \in T_eG. \tag{92}$$

Then the kinetic energy is given by $E = \frac{1}{2} < \Omega, \Omega >$. The group G is a Riemannian manifold endowed with the left-invariant metric (92).

4.2 Geometry of the rigid body motion

Let us consider the geodesic equation on the manifold $SO(3)$. We have already introduced the commutator (90) and the inner product (92) together with the moment of inertia J in diagonal form. Further, the metric is left-invariant, that is, the metric (92) is conserved by the left-translation on the Lie group $SO(3)$. In such a case of invariant metric, the connection satisfies the equation (60), and in terms of the operators ad and ad^*, we have the expression (77) of $\nabla_X Y$ for $X,\ Y \in \mathbf{so(3)}$. By the definition (32), $ad_{\mathbf{X}}Y = [X,\ Y] = X \times Y$. Then the $ad_X^*\,Y$ satisfies

$$\langle ad_X^*\,Y,\ Z\rangle = \langle Y, ad_X\,Z\rangle = (JY, X\times Z)_{R^3} = (JY\times X,\ Z)_{R^3} = \langle J^{-1}(JY\times X),\ Z\rangle\,.$$

Hence, the non-degeneracy of the metric leads to

$$ad_X^*\,Y = J^{-1}(JY \times X)\;.$$

Thus it is found from (77) that

$$\begin{aligned} \nabla_X Y &= \frac{1}{2}J^{-1}\Big(J(X\times Y) - (JX)\times Y - (JY)\times X\Big) \\ &= \frac{1}{2}J^{-1}\,(\tilde{K}X \times Y)\,, \end{aligned} \tag{93}$$

[8], where $\tilde{K}$ is a diagonal matrix with the diagonal elements, $\tilde{K}_\alpha \equiv -J_\alpha + J_\beta + J_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ or its cyclic permutation (all $\tilde{K}_\alpha > 0$ owing to the fundamental definition of the moment of inertia).

In the case of the left-invariant field, it is sufficient to give expressions at the identity e, and the tangent vector at e is the angular velocity Ω. The geodesic equation of the time-dependent problem is given by (78) with $X = \Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega - J^{-1}\Big((J\Omega) \times \Omega\Big) = 0\,, \tag{94}$$

which is also written as $(\mathrm{d}/\mathrm{d}t)J\Omega = (J\Omega) \times \Omega$. This is nothing but the Euler's equation (87). Equation of the *Jacobi* field Y along the geodesic generated by X is derived as [8]

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + X \times \frac{\mathrm{d}Y}{\mathrm{d}t} - F\Big(X, \frac{\mathrm{d}Y}{\mathrm{d}t}\Big) + \frac{1}{2}F\big(X, X\big) \times Y - F\big(X, X \times Y\big) = 0\,,$$

where

$$F\big(X, Y\big) = J^{-1}\big(JX \times Y + JY \times X\big)\,.$$

To calculate the curvature tensor, we apply the expression of the connection ∇ of (93) repeatedly in the formula (62) together with the definition of the commutator (90). Finally it is found that

$$R(X,\,Y\,)\,Z \;=\; -\frac{1}{4J_1J_2J_3}\,\Big(\tilde{M}\,(X \times Y)\Big) \times JZ\,, \tag{95}$$

where $\tilde{M}$ =diag$(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3)$ is a diagonal matrix of third order with the diagonal element $\tilde{M}_\alpha = -3J_\alpha^2 + (J_\beta - J_\gamma)^2 + 2J_\alpha(J_\beta + J_\gamma)$ with $(\alpha, \beta, \gamma) = (1, 2, 3)$ and its cyclic permutation.

Defining the unit basis vectors as $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ in the principal-axes system and writing the sectional curvature $K(\mathbf{e}_i, \mathbf{e}_j)$ (see (84)) as K_{ij} in short, one obtains

$$K_{12} = \frac{1}{J_3}\,\tilde{M}_3, \quad K_{23} = \frac{1}{J_1}\,\tilde{M}_1, \quad K_{31} = \frac{1}{J_2}\,\tilde{M}_2. \tag{96}$$

It is not difficult to see that all the three sectional curvatures are positive for an ellipsoidal rigid body whose deviation from a sphere is sufficiently small (by using the definition of the moment of inertia). On the other hand, if the rigid body is flat like a plate, then one of the sectional curvatures will be negative.

For the stable steady rotation, it is found that the curvature K_{ij} take either positive values always, or both positive and negative values in oscillatory manner, depending on the tensor J. However, it is found that the time average $\bar{K}$ is always positive for any J in the linearly stable case, while there exist J's which make $\bar{K}$ negative in the case of linear instability [8]. The results are consistent with the known properties of rotating rigid bodies in Mechanics.

5 Geodesic equation on $D(S^1)$ and KdV equation on $\hat{D}(S^1)$

Now we consider a second example of application of the geometrical theory. This is a dynamical system of smooth mapping along a circle and found to be a fundamental problem in physics. The geodesic is a curve over the manifold of orientation-preserving diffeomorphisms of a circle S^1 (with a periodicity T) noted in §2.7. Two systems are considered below: the first one is the geodesic equation over a group of diffeomorphisms $D(S^1)$, describing a simple diffeomorphic flow on S^1, and the second one is the KdV equation, which is the geodesic equation over an extended group $\hat{D}(S^1)$. The manifold S^1 is spatially one-dimensional, but its diffeomorphism has infinite degrees of freedom because the pointwise mapping describes arbitrary (but orientation-preserving) deformation of the circle. Thus we consider infinite-dimensional Lie groups $D(S^1)$ and $\hat{D}(S^1)$, including an infinite-dimensional algebra called the Virasoro algebra [6]. This part is based on Ovsienko & Khesin (1987) [9], Misiołek (1997) [10], and Kambe (1998) [7].

5.1 Dispersionless KdV

Consider a group $D(S^1)$ of diffeomorphisms of a circle S^1 of periodicity $T = 1$, equipped with a *right-invariant* metric (see §3.2). Because of this metric invariance, the Riemannian connection ∇ is given by the expression (77): $\nabla_X Y = \frac{1}{2}(ad_X Y - ad^*_Y X - ad^*_X Y)$, for two elements X and Y of the corresponding Lie algebra, $T_{id}D(S^1)$.

Using the tangent fields $X = u(x)\partial_x$, $Y = v(x)\partial_x$, $Z = w(x)\partial_x \in TD(S^1)$ of the S^1 diffeomorphisms (see §2.7) and the definition (49) of Witt algebra, we have $ad_X Y = [X, Y] = -(u\,v' - v\,u')\,\partial_x$, where $u' = \partial_x u = u_x$. Then the $ad^*_X Y$ is given by

$$\langle ad^*_X Y,\, Z\rangle = \langle Y, ad_X Z\rangle = -\int_{S^1} v\,(u\,w' - w\,u')\,\mathrm{d}x = \int_{S^1}(u\,v' + 2v\,u')\,w\,\mathrm{d}x\,,$$

where the definition of metric (51) is used and integration by parts is performed with the periodic boundary conditions $u(x+1) = u(x)$, *etc.* Hence, one obtains

$$ad^*_X Y = (u\,v' + 2v\,u')\,\partial_x\,.$$

It is found from (77) that the Riemannian connection on the group manifold $D(S^1)$ is given by

$$\nabla_X Y = -\,(2uv' + vu')\,\partial_x\,.$$

Remind that the present system is right-invariant (§2.7, §3.2). Then, the nabla $\nabla^{(\mathrm{R})}_X Y = -\nabla_X Y$ must be used for the connection in the time evolution and the geodesic equation is given by the form (79) :

$$\partial_t X + \nabla^{(\mathrm{R})}_X X = 0\,.$$

Thus it is found that the geodesic equation on the manifold $D(S^1)$ is

$$u_t + 3\,u\,u_x = 0\,.$$

Compared with the KdV equation of the form, $\partial\bar{u}/\partial t + \bar{u}\,\partial\bar{u}/\partial x + \kappa\bar{u}_{xxx} = 0$, this equation has no third-order dispersion term $\kappa\bar{u}_{xxx}$, where κ is a constant and $\bar{u} = 3u$. The third-order derivative term is only introduced by the central extension considered in the next section. The above equation would be termed as the one governing a *simple diffeomorphic* flow.

5.2 Central extension

An extension of the group D, denoted by the *hat* symbol, is defined as

$$\hat{f} = (f,\, a), \quad \hat{g} = (g,\, b) \quad \in \hat{D}(S^1)\,,$$

for $f, g \in D(S^1)$ and $a, b \in R$, where $\hat{D}(S^1) = D(S^1) \oplus R$. The group operation is given by

$$\hat{g} \circ \hat{f} = \big(g \circ f,\ a + b + B(g, f)\big) \tag{97}$$

where

$$B(g, f) = \frac{1}{2}\int_{S^1} \ln\,\partial_x\big(g \circ f\big)\ \mathrm{d}\ln\,\partial_x f \tag{98}$$

is the Bott cocycle (Bott 1977). It can be readily shown that the following subgroup $\hat{D}_0$ is a *center* of the extended group $\hat{D}$, where $\hat{D}_0$ is defined by $\{\hat{f}_0\ :\ \hat{f}_0 = (id,\ a),\ \mid a \in R\}$, and $id(x) = x$.

5.3 KdV equation as a geodesic equation on $\hat{D}(S^1)$

We now consider the geodesic equation on the extended manifold $\hat{D}(S^1)$ [9]. Let $t \mapsto \hat{\xi}_t$ be a flow starting at $\hat{\xi}_0 = (id, 0) \equiv \hat{id}$ in the direction $\dot{\hat{\xi}}_0 = (u(x)\partial_x, \alpha)$, and the second flow $t \mapsto \hat{\eta}_t$ starting at $\hat{id}$ in the direction $\dot{\hat{\eta}}_0 = (v(x)\partial_x, \beta)$ (see §5.5). Non-trivial central extension of $T_{id}D(S^1)$ to $T_{\hat{id}}\hat{D}(S^1)$ is the *Virasoro algebra* [6], in which the extended component is the general 2-cocycle found by Gelfand-Fuchs [11]. For any two tangent fields

$$\hat{u} = (\,u(x,t),\ \alpha), \quad \hat{v} = (\,v(x,t),\ \beta) \quad \in\ T_{id}\hat{D}(S^1),$$

the *commutator* is given by

$$[\,\hat{u},\ \hat{v}\,] = -\,(\,u\,\partial_x v - v\,\partial_x u\,,\quad c(u,v)\,) \tag{99}$$

where

$$c(u,v) = \int \partial_x^2 u\,\partial_x v\,dx = -c(v,u) \tag{100}$$

(see §5.5 for the cocycle $c(u,v)$). The *metric* is defined by

$$\langle\,\hat{u},\ \hat{v}\,\rangle = \int u(x)\,v(x)\,dx + \alpha\,\beta\ , \tag{101}$$

Following the procedure of §5.1, the covariant derivative is derived for $\hat{u}$ and $\hat{v}$,

$$\nabla^{\mathrm{R}}_{\hat{u}}\,\hat{v} = -\,\nabla_{\hat{u}}\,\hat{v} = \Big(w\,\partial_x,\ \frac{1}{2}\int_{S^1} u_{xx}v_x\,\mathrm{d}x\Big)$$

where $$w = 2uv_x + vu_x + \frac{1}{2}(\alpha v_{xxx} + \beta u_{xxx})$$

The geodesic equation is written as $\partial \hat{u}/\partial t + \nabla^{\mathrm{R}}_{\hat{u}} \hat{u} = 0$, which leads to the following two equations:

$$\begin{aligned} u_t + 3uu_x + \alpha u_{xxx} &= 0 \\ \partial_t \, \alpha &= 0 \end{aligned} \tag{102}$$

Thus, the *KdV equation* is derived, where the constant α is called a *central charge.*

5.4 Sectional curvatures of KdV system

The geometrical theory leads to a relationship between the stability of geodesic curves on a Riemannian manifold and its curvature. The link is expressed by the Jacobi equation for geodesic variation J in §3.9. An evolution equation for the norm $\| J \|$ is given by Eq. (83):

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \frac{\| J \|^2}{2} = \| \nabla_T J \|^2 - K(J, T) \,, \tag{83}$$

where the second term on the right-hand side $K(J, T)$ is the sectional curvature associated with the two-dimensional section spanned by J and T. If $K(J, T)$ is negative, the right-hand side is positive. Then exponential growth of the magnitude of the variation $\| J \|$ is predicted, which is understood that the geodesics are *unstable.*

In this context, the sectional curvature of the KdV system is estimated [7, 10] for the section spanned by the two tangent vectors (with a common central charge α), $\hat{u} = (\, u(x,t),\, \alpha)$ and $\hat{v} = (\, v(x,t),\, \alpha)$:

$$K(\hat{u}, \hat{v}) = \frac{1}{4}F - 9G - \frac{3}{4}H^2$$

where

$$\begin{aligned} F &= \int_{S^1} \Big(a(u''' - v''') + 2(vu' - uv') \Big)^2 \, \mathrm{d}x, \\ G &= \alpha \int_{S^1} u'v'(u'' + v'') \, \mathrm{d}x, \\ H &= \int_{S^1} u''v' \, \mathrm{d}x \,. \end{aligned}$$

For the sinusoidal fields $\hat{u}_n = (a_n \sin nx \, \partial_x, \, \alpha)$ and $\hat{v}_n = (a_n \cos nx \, \partial_x, \, \alpha)$, it is found that

$$\begin{aligned} K(\hat{v}_1, \, \hat{u}_n) &= \frac{\pi}{4}\Big(\alpha^2 b_1^2 + \alpha^2 (a_n n^3)^2 + 2(b_1 a_n)^2 (1 + n^2)\Big) > 0 \quad \text{for} \quad n \geq 3 \,, \\ K(\hat{v}_1, \, \hat{v}_n) &= \frac{\pi}{4}\Big(\alpha^2 b_1^2 + \alpha^2 (b_n n^3)^2 + 2(b_1 b_n)^2 (1 + n^2)\Big) > 0 \quad \text{for} \quad n \geq 3. \end{aligned}$$

Therefore, both of the sectional curvatures $K(\hat{v}_1, \, \hat{u}_n)$ and $K(\hat{v}_1, \, \hat{v}_n)$ are *positive* for $n \geq 3$. Thus, most sectional curvatures are positive, however there are some sections which are not always positive. In fact,

$$K(\hat{v}_1, \, \hat{u}_1) = \frac{\pi}{4}(a_1 b_1)^2 \Big(-3\pi + 8 + \alpha^2 \frac{a_1^2 + b_1^2}{a_1^2 \, b_1^2} \Big).$$

Similarly it can be shown that $K(\hat{v}_n, \, \hat{u}_n)$ is not always positive for any integer n as well.

5.5 Note on the cocycles of the extended algebra

Here a note is given about the relation between the group cocycle $B(g, f)$ of (98) and the algebra cocycle $c(u, v)$ of (100). On the extended space $\hat{D}(S^1)$, we consider two flows $\hat{\xi}_t$ and $\hat{\eta}_s$ generated by $\hat{U} = (u\,\partial_x,\ \alpha)$ and $\hat{V} = (v\,\partial_x,\ \beta)$ respectively, defined by

$$t \mapsto \hat{\xi}_t \qquad \text{where} \quad \hat{\xi}_0 = (id,\ 0), \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\hat{\xi}_t = \hat{U}\,,$$

$$s \mapsto \hat{\eta}_s \qquad \text{where} \quad \hat{\eta}_0 = (id,\ 0), \quad \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\hat{\eta}_s = \hat{V}$$

(see §2.6.2). Lie bracket of the two tangent vectors $\hat{U}$ and $\hat{V}$ which is defined by

$$[\hat{U},\ \hat{V}\,](f) \;=\; \hat{U}\left(\hat{V}\,(f)\right)\Big|_{(id,\ 0)} - \hat{V}\left(\hat{U}\,(f)\right)\Big|_{(id,\ 0)}$$

is represented, according to Eq.(46), as

$$[\hat{U},\ \hat{V}\,] \;=\; \Big(\,\partial_t\,\partial_s\,\hat{\eta}_s \circ \hat{\xi}_t - \ \partial_s\,\partial_t\,\hat{\xi}_t \circ \hat{\eta}_s\,\Big)\Big|_{(id,\ 0)}\,.$$

Denoting only the extended component of the product $\hat{\eta} \circ \hat{\xi}$ of (97) as $\mathrm{Ext}\{\hat{\eta}_s \circ \hat{\xi}_t\}$, we have

$$\mathrm{Ext}\{\hat{\eta}_s \circ \hat{\xi}_t\} \quad = \quad a_t + b_s + B(\eta_s, \xi_t)\ .$$

Therefore,

$$\mathrm{Ext}\{\partial_t\ \partial_s\ \hat{\eta}_s \circ \hat{\xi}_t\} = \partial_t\ \partial_s\ B(\eta_s, \xi_t)\ .$$

and

$$\mathrm{Ext}\{\ [\hat{U},\ \hat{V}\,]\ \} \;=\; \partial_t\ \partial_s\ B(\eta_s, \xi_t) - \partial_s\ \partial_t\ B(\xi_t, \eta_s)\Big|_{(id,\ 0)}$$

Carrying out the calculation, we have

$$\partial_s\ B(\eta_s, \xi_t) = \partial_s \int_{S^1} \ln \partial_x\big(\eta_s \circ \xi_t\big)\ \mathrm{d}\ln\,\partial_x\xi_t = \int_{S^1} \frac{\partial_x\big(v \circ \eta_s \circ \xi_t\big)}{\partial_x\big(\eta_s \circ \xi_t\big)}\ \mathrm{d}\ln\,\partial_x\xi_t\ .$$

Since the Taylor expansion of ξ_t is $\xi_t = id + t\ u(x) + O(t^2)$, its x-derivative is given as $\partial_x\,\xi_t = 1 + t\ \partial_x\,u + O(t^2)$. Hence, one obtains

$$\begin{aligned}
\ln\,\partial_x\,\xi_t &= \ln\,(1 + t\ \partial_x\,u + O(t^2)\ ) = t\ \partial_x\,u + O(t^2) \\
\mathrm{d}\ \ln\,\partial_x\,\xi_t &= t\ u_{xx}\ \ \mathrm{d}x + O(t^2)\ \mathrm{d}x
\end{aligned}$$

and further

$$\begin{aligned}
\eta_s \circ \xi_t &= (x + s\ v(x) + \cdots) \circ (x + t\ u(x) + \cdots) \\
&= x + t\ u(x) + s\ v(x + t\ u(x) + \cdots) + \cdots \\
&= x + t\ u(x) + s\ v(x) + O(s^2, st, t^2) \\
\partial_x\big(\eta_s \circ \xi_t\big) &= 1 + t\ u_x + s\ v_x + O(s^2, st, t^2)
\end{aligned}$$

Therefore one finds

$$\partial_s\ B(\eta_s, \xi_t) = \int_{S^1} \frac{\partial_x\, v\,(x + O(s,t))}{1 + t\ u_x + s\ v_x + O(s^2, st, t^2)}\ (\ t\ u_{xx} + O(t^2)\ )\ \mathrm{d}x$$

Differentiating with respect to t and setting $t = 0$ and $s = 0$,

$$\partial_t\, \partial_s\ B(\eta_s, \xi_t)\Big|_{s=0,t=0} = \int_{S^1} v_x\ u_{xx}\ \ \mathrm{d}x\ .$$

Thus finally one finds

$$\begin{aligned} c(u,v) &= \mathrm{Ext}\{\ [\hat{U},\ \hat{V}\,]\ \} = \int_{S^1} v_x\ u_{xx}\ \mathrm{d}x - \int_{S^1} u_x\ v_{xx}\ \mathrm{d}x \\ &= 2\int_{S^1} u_{xx}\ v_x\ \mathrm{d}x = -c(v,u)\,. \end{aligned}$$

This form of $c(u,v)$ is the Gelfand-Fuchs cocycle [11]. The anti-symmetry of $c(u,v)$ can be shown by performing integration by parts and using the periodicity.

6 Geometrical theory of chaos of a Hamiltonian system

A self-gravitating system of N point masses is one of the typical dynamical systems studied in the conventional analytical dynamics. As a third example, the differential geometric formulation is given to this system of finite degrees of freedom (M. Pettini [12] (1993)). A simplest non-trivial case is the Hénon-Heiles system, a two-degrees-of-freedom Hamiltonian system, which is well known to be a chaotic system. Within the present framework, stability of the trajectories of the dynamical system is studied when the Riemannian curvatures of the manifold are known. This leads to the geometric characterization of *Hamiltonian chaos* (M. Cerruti-Sola and M. Pettini [13] (1996)).

It has recently been revealed that the phenomenon of phase transitions is related at a deep level to a change in the topology of configuration space of the system. Fluctuations of the configuration-space curvature exhibit a singular behavior at the phase transition.

In this section, only the former chaos analysis is presented. As to the latter subject of geometrical theory of phase transition, only the following reference is given here: Casetti, Pettini and Cohen (2000) [14].

6.1 A dynamical system with self-gravitation

Consider a dynamical system described by the Lagrangian function,

$$L(\bar{q},\dot{\bar{q}}) = T - V = \frac{1}{2}\,a_{ij}(\bar{q})\dot{q}^i\dot{q}^j - V(\bar{q})\ , \tag{103}$$

where $\bar{q} = (q^1,\cdots,q^N)$ and $\dot{\bar{q}} = (\dot{q}^1,\cdots,\dot{q}^N)$ are the coordinates and velocities of the N degrees of freedom system, and $V(\bar{q})$ is the potential of self-gravitation. The first term $T = (1/2)a_{ij}\dot{q}^i\dot{q}^j$ represents the kinetic energy and a_{ij} $(i,j = 1,\cdots,N)$ are the mass tensors. We consider only the case of $a_{ij} = \delta_{ij}$ (Kronecker's delta). The Hamiltonian H is represented as $H = p_\alpha q^\alpha - L(\bar{q},\dot{\bar{q}}) = (1/2)\,a^{\alpha\beta}p_\alpha p_\beta + V(\bar{q}) = T + V$ where $p_\alpha = a_{\alpha i}\dot{q}^i$, and $(\,a^{\alpha\beta}\,)$ is the inverse of $(\,a_{ij}\,) = \underline{\underline{a}}$, *i.e.* $(\,a^{\alpha\beta}\,) = \underline{\underline{a}}^{-1}$.

Here the *Eisenhart* metric g^E is defined [12] by introducing two additional coordinates q^0 $(= t)$ and q^{N+1}. Defining $Q = (q^0,\bar{q},q^{N+1})$, the arc length $\mathrm{d}s$ is represented by

$$\mathrm{d}s^2 = g^E_{ij}(Q)\,\mathrm{d}Q^i\,\mathrm{d}Q^j = a_{ij}q^iq^j - 2V(\bar{q})\mathrm{d}q^0\mathrm{d}q^0 + 2\mathrm{d}q^0\mathrm{d}q^{N+1},$$

$(q^0 = t)$, where the metric tensor $g^E = g^E_{ij}(Q)$ is represented as, for $i,j = 0,\cdots,N+1$,

$$g^E = \begin{pmatrix} -2V(\bar{q}) & \underline{0} & 1 \\ \underline{0}^T & \underline{\underline{a}} & \underline{0}^T \\ 1 & \underline{0} & 0 \end{pmatrix}, \qquad \left(g^E\right)^{-1} = \begin{pmatrix} 0 & \underline{0} & 1 \\ \underline{0}^T & \underline{\underline{a}}^{-1} & \underline{0}^T \\ 1 & \underline{0} & 2V(\bar{q}) \end{pmatrix} \tag{104}$$

where $\underline{\underline{a}} = (\,a_{ij}\,)$, $\underline{0}$ is the null row vector and $\underline{0}^T$ is its transpose.

The Christoffel symbols Γ^k_{ij} are given by (67) with the metric tensors g^E. Since the matrix elements $a_{ij}(= \delta_{ij})$ are constant, only non-vanishing Γ^k_{ij} are

$$\Gamma^i_{00} = g^{il}\frac{\partial V}{\partial q^l} = \partial_i V\,, \quad \Gamma^{N+1}_{0i} = \Gamma^{N+1}_{i0} = -g^{0\,N+1}\frac{\partial V}{\partial q^i} = -\partial_i V \qquad (i = 1,\cdots,N)\,. \tag{105}$$

The natural motions are obtained as the projection on the space-time configuration space $(t, \bar{q})$ and given by the geodesics satisfying $\mathrm{d}s^2 = k^2 \mathrm{d}t^2$ and $\mathrm{d}q^{N+1} = (k^2/2 - L(\bar{q}, \dot{\bar{q}}))\mathrm{d}t$. The constant k can be always set as $k = 1$, leading to $\mathrm{d}s^2 = \mathrm{d}t^2$. [14]

The covariant derivative $\boldsymbol{\nabla} Y/\mathrm{d}s$ is defined by the form (61). The geodesic equation is represented as $(\boldsymbol{\nabla}/\mathrm{d}s)\mathrm{d}Q/\mathrm{d}s = 0$, which is given by the form (75) in local coordinates:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} q^i \;+\; \Gamma^i_{00} \frac{\mathrm{d}q^0}{\mathrm{d}s}\frac{\mathrm{d}q^0}{\mathrm{d}s} = 0 \qquad (i = 1, \cdots, N) \; ,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} q^0 \;=\; 0 , \qquad\qquad \frac{\mathrm{d}^2}{\mathrm{d}s^2} q^{N+1} + 2\, \Gamma^{N+1}_{0i} \frac{\mathrm{d}q^0}{\mathrm{d}s}\frac{\mathrm{d}q^i}{\mathrm{d}s} = 0 \; .$$

Using (105) and $\mathrm{d}s = \mathrm{d}t$ (and $\underline{\underline{a}} = \underline{\underline{a}}^{-1} = (\delta_{ij})$), we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} q^i &= -\frac{\partial V}{\partial q^i} \quad (i = 1, \cdots, N), \qquad\qquad \frac{\mathrm{d}q^0}{\mathrm{d}t} = 1, \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} q^{N+1} &= 2\, \frac{\partial V}{\partial q^i} \frac{\mathrm{d}q^i}{\mathrm{d}t} = -\frac{\mathrm{d}L}{\mathrm{d}t} \qquad\qquad (\text{since } \; \mathrm{d}T/\mathrm{d}t = -\mathrm{d}V/\mathrm{d}t) \; . \end{aligned} \tag{106}$$

Choosing arbitrary constants appropriately, we have $q^0 = t$ and $\mathrm{d}q^{N+1}/\mathrm{d}t = 1/2 - L$. The equation (106) is the Newton's equation of motion. Thus it is found that the geometric machinery works for the present dynamical system too. The Eisenhart metric (Newtonian limit metric of the general relativity) is chosen here because it is seen just below to have very simple curvature properties, although there is another metric known as the Jacobi metric.

The link between the stability of trajectories and the geometrical characterization of the manifold $(M(\bar{q}) \times R^2,\, g^E)$ is expressed by the Jacobi equation (82) (rewritten):

$$\left(\frac{\boldsymbol{\nabla}}{\mathrm{d}s}\right)^2 J + R(J, \dot{Q})\, \dot{Q} = 0 \; . \tag{107}$$

From (66), the non-vanishing components of the curvature tensors are

$$R^i_{0j0} = -R^i_{00j} = \partial_i \partial_j V \; . \tag{108}$$

The Ricci tensor, defined by $R_{kj} \equiv R^l_{klj}$, has only nonzero component $R_{00} = R^l_{0l0} = \Delta V$. The scalar curvature, defined by $R \equiv g^{ij} R_{ij} = g^{00} R_{00}$, vanishes identically since $g^{00} = 0$ by (104).

It is interesting to find that the Jacobi equation (82) is equivalent to the equation of tangent dynamics, that is the evolution equation of infinitesimal variation vector $\xi(t)$ along the reference trajectory $\bar{q}_0(t)$. In fact, writing the perturbed trajectory as $q^i(t) = q^i_0(t) + \xi^i(t)$ and substituting it to the equation of motion, $\mathrm{d}^2 q^i/\mathrm{d}t^2 = -\partial V/\partial q^i$, the linearized perturbation equation with respect to $\xi(t)$ reads

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \xi^i = -\left(\frac{\partial^2 V(\bar{q})}{\partial q^i \partial q^j}\right)_{\bar{q} = \bar{q}_0(t)} \xi^j \; .$$

This is equivalent to the Jacobi equation (82) by using (61), (105) and (108) because, noting $J^0 = 0$, one has $(\boldsymbol{\nabla} J/\mathrm{d}s)^i = \mathrm{d}J^i/\mathrm{d}t + \Gamma^i_{00} J^0 \dot{Q}^0 = \mathrm{d}J^i/\mathrm{d}t$ and $R(J, \dot{Q})\, \dot{Q} = (\partial_i \partial_j V)\, J^j$.

6.2 System of two degrees of freedom (Hénon-Heiles model)

In regard to the information of dynamical behaviors, either regular or chaos, all is included in this geometrical characterization. In order to see this, the previous formulation is applied to a two-degrees-of-freedom system described by the Lagrangian,

$$L = \frac{1}{2}\left((\dot{q}_1)^2 + (\dot{q}_2)^2\right) - V(q_1, q_2)$$

where we are using the lower suffices such as q_1 (or J_2) in this section only in order to make a concise definition of its square $(q_1)^2 = q_1^2$. The enlarged coordinate and velocity are $Q = (t, q_1, q_2, q_3)$ and $\dot{Q} = (1, \dot{q}_1, \dot{q}_2, \dot{q}_3)$, and the geodesic separation vector is $J = (J_0 = 0, J_1, J_2, J_3)$. On the constant energy surface and along the geodesic $Q(t)$, one can always assume that J is orthogonal to $\dot{Q}$, *i.e.* $\langle J, \dot{Q}\rangle = g_{ij}J_i\dot{Q}_j = 0$. In fact, expressing as $J = J^{\perp} + c\dot{Q}$ and substituting in Eq.(82), it is readily seen that the terms related to the parallel component $c\dot{Q}(= cT)$ drop out and the equation is nothing but the one for the orthogonal component $J^{\perp}$. Further, the components J_3 and $\dot{q}_3$ are irrelevant because $g_{33} = 0$.

The equation for the norm of geodesic separation $\| J \|$ is Eq.(83): $(\mathrm{d}/\mathrm{d}s)^2 \| J \|^2 /2 = \| \nabla_T J \|^2 - K(J, \dot{Q})$, where the sectional curvature $K(J, \dot{Q})$ is given by

$$\begin{aligned} K(J,\dot{Q}) &= R_{i0j0}\, J_i \frac{\mathrm{d}Q_0}{\mathrm{d}t}\, J_j \frac{\mathrm{d}Q_0}{\mathrm{d}t} = (\partial_i\partial_j V)\, J_i J_j \\ &= \frac{\partial^2 V}{\partial q_1^2} J_1^2 + 2\frac{\partial^2 V}{\partial q_1 \partial q_2} J_1 J_2 + \frac{\partial^2 V}{\partial q_2^2} J_2^2 . \end{aligned}$$

Cerruti-Sola and Pettini [13] chose as $J = (0, \dot{q}_2, -\dot{q}_1, 0)$. Then the normalized curvature $\hat{K}$ is given by

$$\hat{K}(\dot{Q}, Q) \equiv \frac{K(J,\dot{Q})}{\| J \|^2} = \frac{1}{2(E - V(\bar{q}))}\left(\frac{\partial^2 V}{\partial q_1^2}\dot{q}_2^2 - 2\frac{\partial^2 V}{\partial q_1 \partial q_2}\dot{q}_1\dot{q}_2 + \frac{\partial^2 V}{\partial q_2^2}\dot{q}_1^2\right),$$

($E = T + V$, total energy), which is computable on the constant energy surface S_E. They define the integral of negative curvature value $\hat{K}_- = \{\hat{K} : \hat{K}(\dot{Q}, Q) < 0\}$ by

$$\langle \hat{K}_- \rangle = \frac{1}{A(S_E)} \int_{S_E} \hat{K}_-\, \mathrm{d}\bar{q}\, \mathrm{d}\dot{\bar{q}}$$

where $A(S_E)$ is the area of S_E. The quantity $\langle \hat{K}_- \rangle$ was estimated at different energy values E.

In the Hénon-Heiles [15] model, the Hamiltonian is $H = (1/2)(p_1^2 + p_2^2) + V(q_1, q_2)$ and the potential is chosen as

$$V(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3 = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin 3\theta,$$

where $q_1 = r\cos\theta$ and $q_2 = r\sin\theta$. It is shown that the transition from order to chaos is quantitatively described by measuring, on a Poincaré section, the ratio σ of the area covered by the regular trajectories divided by the total area accessible to the motions. For low energies E the whole area is practically covered by regular orbits and hence the ratio σ is almost 1. As E is increased, σ begins to decrease from 1, and drops rapidly to very small values. At $E = 1/6$, the accessible area is marginal because the equi-potential curve $V(q_1, q_2) = 1/6$ is an equilateral triangle (including the origin within it). Beyond $E = 1/6$, the equi-potential curves are open, and the motions are unbounded. Thus the accessible area becomes infinite.

It is shown in Cerruti-Sola and Pettini [13] that for low energies E the integral of the negative curvature $\langle \hat{K}_- \rangle$ is almost zero, but that, at the same E value (≈ 0.1) at which σ begins to decrease, the value $\langle \hat{K}_- \rangle$ starts to increase. The exact coincidence between the critical energy level for the σ decrease below 1 and the one for the $\langle \hat{K}_- \rangle$ increase above 0 is understood that the onset of sharp increase of chaotic domains is detected by the increase of the negative curvature integral $\langle \hat{K}_- \rangle$. Along with this, the fraction of the area $A_-(S_E)$ where $\hat{K} < 0$ is also estimated as a function of E. The transition is again detected by this quantity too.

7 Flows of an inviscid incompressible fluid

Motion of fluid particles of an inviscid incompressible fluid on a bounded domain is described on the basis of the geometrical framework. In the conventional approach, flows of an inviscid fluid are well described already in the fluid dynamics. However, the fluid flows are equivalently expressed by the geodesics on the manifold of all volume-preserving diffeomorphisms. This formulation is based on the Riemannian geometry and Lie group theory, developed first by Arnold [3] (1966). The present approach reveals new aspects which are not studied in the conventional fluid dynamics. For example, behaviors of the geodesics are controlled by Riemannian (sectional) curvatures, which are quantitative characterizations of the flow (in infinite many numbers). In particular, the analysis shows that the curvatures are found to be mostly negative (with some exceptions), which can be related to mixing and ergodicity of the fluid motion in a bounded domain.

It is known that the geodesic equation on a central extension of the group of volume preserving diffeomorphisms is equivalent to the flow of a perfectly conducting fluid. Here, only the following references are referred: Vizman (2001) [24] (and Zeitlin (1992) [25]). The present chapter is based on the works of Misiołek (1993) [16], Nakamura *et al.* (1992) [20], Hattori & Kambe (1994) [19], and Ebin & Misiołek (1997) [17].

7.1 Basic concepts

We consider flows of an inviscid incompressible fluid on a manifold M (which is the flow region): R^2 (or T^2), or R^3 (or T^3). The motivation of the geometrical analysis is the observation that Euler's equation of motion is a geodesic equation on a group of volume-preserving diffeomorphisms with the metric defined by the kinetic energy. The set of all volume-preserving diffeomorphisms of M composes a group manifold $D_\mu(M)$, of which an element $g \in D_\mu(M)$ is a map, $g: M \to M$.

Suppose that a curve $t \to g_t(x)$ denotes a fluid flow, then a point x is mapped to the point $g_t(x)$ at a time t. This is a *Lagrangian* description of flows. Tangent vector field (velocity field) at the time t is represented as

$$U_t(x) = \dot{g}_t(x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(g_{\Delta t} - 1) \circ g_t(x) = u \circ g_t(x) = u(g_t(x)) .$$

By definition, the tangent field $U_t(x) = U_{g_t}$ at g_t is a *right-invariant* field (see the next section). Right translation with g_t^{-1} yields the velocity field at $e = g_0$:

$$u_t(x) = \dot{g}_t \circ g_t^{-1}(x), \qquad u_t \in T_e D_\mu^s(M) .$$

At the identity e (*i.e.* at the initial manifold M_0), the velocity field $u_t(x)$ satisfies the divergence-free property, *i.e.* div $u_t = 0$. The suffix t denotes that the tangent vector u_t is time-dependent. In terms of the fluid-dynamics, $u_t(x)$ represents an *Eulerian* velocity field at a time t, whereas U_{g_t} is its Lagrangian counterpart. In mathematical language, u_t is an element of the Lie algebra $T_e D_\mu(M)$.

It is useful to consider the manifold $D_\mu^s(M)$ which is a subgroup of volume-preserving diffeomorphisms (of M) of Sobolev class H^s, where $s > n/2 + 1$ (n =dim M). The group manifold $D^s_\mu(M)$ is a weak Riemannian submanifold of the group $D^s(M)$ of all Sobolev H^s-diffeomorphisms of M.

An arbitrary vector field $v(x) \in T_e M$ can be decomposed into L^2-orthogonal components of divergence-free part $\bar{v}$ and gradient part (Appendix A2):

$$v = \bar{v} + \operatorname{grad} f\,, \qquad f \in H^{s+1}(M)\,. \tag{109}$$

7.2 Right invariant field

Before presenting the geometrical theory of hydrodynamics, it is essential to introduce the notion of right invariant field for its basis. Consider the tangent field $U_\eta \in T_\eta D^s_\mu(M)$ at any $\eta \in D^s_\mu(M)$, and suppose that U_η is *right-invariant*, that is,

$$U_\eta(x) = U_e \circ \eta(x)\,, \qquad \text{for} \quad U_e \in T_e D^s_\mu(M)\ .$$

Correspondingly, the right-invariant L^2-metric is defined on $D_\mu(M)$ (not on all of $D(M)$) by

$$\langle U,\, V\rangle_\eta \equiv \int_M \left(U_\eta \circ \eta^{-1},\; V_\eta \circ \eta^{-1}\right)_x \mathrm{d}\mu(x) = \ \langle U_e,\, V_e\rangle_e \tag{110}$$

where $U_\eta,\, V_\eta \in T_\eta D^s_\mu(M)$, $\mathrm{d}\mu$ is the volume form and $(\,\cdot,\,\cdot\,)_x$ is the scalar product at each point $x \in M$. Applying a right translation to the middle side of (110) by η and using the formula (141) in the appendix with $f = \eta$, $\eta(\sigma) = M$ and $\sigma = \eta^{-1}(M)$, we have

$$\begin{aligned} \langle U,\, V\rangle_\eta \ &= \ \int_{\eta^{-1}(M)} \left(U_\eta \circ \eta^{-1},\; V_\eta \circ \eta^{-1}\right)_x \circ \eta \quad \eta^* \mathrm{d}\mu \\ &= \ \int_M \left(U_\eta(x),\; V_\eta(x)\right)_{\eta(x)} \mathrm{d}\mu(x)\,. \end{aligned}$$

The second equality is verified by the commutability of d and η^* and the change of variables with the volume-preserving property $\eta^*\mu(\eta^{-1}x) = \mu(x)$. Thus the present L^2-metric (110) is *isometric* with respect to the right translation by any $\eta \in D^s_\mu(M)$.

The metric (110) induces, on $D^s(M)$ and $D^s_\mu(M)$, smooth Riemannian connections $\hat{\nabla}$ and $\bar{\nabla} = \mathrm{P}\hat{\nabla}$ respectively, where the symbol P is the projection operator to the divergence-free part. For any right invariant vector fields $U_\eta,\, V_\eta \in T_\eta D^s_\mu(M)$, we have the *right-invariant* connection,

$$(\hat{\nabla}_{U_\eta} V_\eta)_\eta = (\nabla_{U_e} V_e)_e \circ \eta\ , \tag{111}$$

for $\eta \in D^s_\mu(M)$, where ∇ is the covariant derivative on M (manifold of Eulerian description). Similarly, we have

$$(\bar{\nabla}_{U_\eta} V_\eta)_\eta = \mathrm{P}[\nabla_{U_e} V_e]_e \circ \eta\ . \tag{112}$$

An arbitrary vector field $X_\eta(x)$ on $D^s(M)$ can be decomposed into L^2-orthogonal components of divergence-free part $\bar{X}_\eta$ and gradient part by (109) and the isometry of the L^2-metric:

$$X_\eta = \bar{X}_\eta + \operatorname{grad} f\,, \tag{113}$$

for any $\eta \in D^s_\mu(M)$, where $\bar{X}_\eta \in T_\eta D^s_\mu(M) = \{X \in T_\eta D^s : \mathrm{div}(X \circ \eta^{-1}) = 0\}$ and $f \in H^{s+1}(M) \circ \eta$. We denote by $\mathtt{P}_\eta$ and $\mathtt{Q}_\eta$ the orthogonal projections onto the first and the second terms at η in (113) respectively.

The difference of the two connections $\hat{\nabla}$ and $\bar{\nabla}$ is the second fundamental form S of $D^s_\mu(M)$, given by

$$S(U_\eta, V_\eta) = \hat{\nabla}_{U_\eta} V_\eta - \bar{\nabla}_{U_\eta} V_\eta = \mathtt{Q}_\eta[\, \hat{\nabla}_{U_\eta} V_\eta \,] \tag{114}$$

(see (71)). This is also right-invariant.

For tangent fields $U, V, W, Z \in T_\eta D^s_\mu(M)$, the curvature tensor $\hat{R}$ on $D^s(M)$ and $\bar{R}$ on $D^s_\mu(M)$ are also defined in the right-invariant way. First, the curvature tensor $\hat{R}$ is defined on $D^s(M)$ by

$$\Big(\hat{R}(U, V)\, W\Big)_\eta \;=\; \Big(R(U \circ \eta^{-1}, V \circ \eta^{-1})\, W \circ \eta^{-1}\Big) \circ \eta\,, \tag{115}$$

where the R is the curvature tensor on M for $u,\ v,\ w, \in T_e D^s_\mu(M)$:

$$R(u,\ v)\, w = \nabla_u(\nabla_v\, w) - \nabla_v(\nabla_u\, w) - \nabla_{[u,v]}\, w\ . \tag{116}$$

The curvature tensor $\bar{R}$ on $D^s_\mu(M)$ is given by the right hand side of (115) but ∇ replaced with $\bar{\nabla}$ in (116). Both of the curvature tensors $\hat{R}$ and $\bar{R}$ are related by the following Gauss-Codazzi equation (72):

$$\big\langle \hat{R}(U,\ V)W,\ Z \big\rangle_{L^2} = \big\langle \bar{R}(U,\ V)W,\ Z \big\rangle + \big\langle S(U,\ W),\ S(V,\ Z) \big\rangle - \big\langle S(U,\ Z),\ S(V,\ W) \big\rangle. \tag{117}$$

7.3 Formulation of hydrodynamics

7.3.1 Hydrodynamic connection

Let us consider the geodesic equation on the manifold $D^s_\mu(M)$, the group of volume-preserving diffeomorphisms of M. This is a mathematical derivation of the hydrodynamic equation of an inviscid incompressible fluid, for which the Eulerian representation is given in the manifold M at the identity.

Because of the right-invariance of the metric on $TD^s_\mu(M)$ defined in the previous section, the Riemannian connection satisfies the expression like (60). In the present case, using the symbol $\bar{\nabla}$ for the induced connection, we obtain

$$2\langle\, \bar{\nabla}_u v, w \,\rangle = \langle\, [u, v], w \,\rangle - \langle\, [v, w], u \,\rangle + \langle\, [w, u], v \,\rangle\ , \tag{118}$$

for $u,\ v,\ w \in T_e D^s_\mu(M)$. The commutator of the present problem is given by

$$[u,\ v](s) \;=\; \bar{\nabla}_u v - \bar{\nabla}_v u \tag{119}$$

(see (57)), where the right hand side is divergence-free too.

Introducing the adjoint operator $ad_v\, w = [v, w]$ and the coadjoint operator by $\langle ad^*_v\, u, w\rangle = \langle u, ad_v\, w\rangle = \langle u, [v, w]\rangle$, we obtain from (118)

$$\bar{\nabla}_u\, v = \frac{1}{2}\,(\ [u,\ v] - ad^*_u\, v - ad^*_v\, u\ ) + \mathrm{grad}\ f\,,$$

by the non-degeneracy of the metric (§2.3). The function f is naturally introduced because of the divergence-free of the tangent vector w. In fact, $\langle \operatorname{grad} f,\, w \rangle = 0$ for any scalar function $f(x)$, which is determined so as to satisfy the condition, $\operatorname{div} \bar{\nabla}_u v = 0$.

Because of the right-invariance of the diffeomorphisms, the commutator is $[u,\, v] = \mathcal{L}_u v = -\,\mathrm{P}\{(\mathrm{d}v^i/\mathrm{d}t) - v^j(\partial u^i/\partial x^j)\}\,\partial_i$. Therefore the time evolution is described by the nabla $\nabla^{(\mathrm{R})}_u\, v = -\,\bar{\nabla}_u\, v$ (see §3.8.2**b**) in the right-invariant way.

7.3.2 Formulas in R^3 space

In the space R^3, for $\boldsymbol{u},\, \boldsymbol{v} \in T_e D_\mu(R^3)$,

$$-\,ad_{\boldsymbol{u}}\, \boldsymbol{v} \;\equiv\; -[\,\boldsymbol{u},\, \boldsymbol{v}\,] = [\mathcal{L}_{\boldsymbol{u}},\, \mathcal{L}_{\boldsymbol{v}}] = (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u}\;,$$

(see (42), (43)), where $\boldsymbol{\nabla} = (\partial_1, \partial_2, \partial_3)$, $\boldsymbol{u}\cdot\boldsymbol{\nabla} = u_1\partial_1 + u_2\partial_2 + u_3\partial_3$ and $\nabla\cdot\boldsymbol{u} = 0$, $\nabla\cdot\boldsymbol{v} = 0$. The definition relation $\langle ad_{\boldsymbol{u}}\,\cdot\,,\, \boldsymbol{v}\rangle = \langle\,\cdot\,,\, ad^*_{\boldsymbol{u}}\, \boldsymbol{v}\rangle$ leads to, with integration by parts,

$$\begin{aligned}
-\,ad^*_{\boldsymbol{u}}\, \boldsymbol{v} \;&=\; -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} - v^k \boldsymbol{\nabla} u^k \\
&=\; \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) - \boldsymbol{\nabla} h - \boldsymbol{\nabla} f_{uv}\;,
\end{aligned}$$

for a function f_{uv}, where $h = u^k v^k = \boldsymbol{u}\cdot\boldsymbol{v}$. Thus, we obtain

$$\begin{aligned}
\bar{\nabla}^{(\mathrm{R})}_{\boldsymbol{u}}\, \boldsymbol{v} = -\,\bar{\nabla}_{\boldsymbol{u}}\, \boldsymbol{v} \;&=\; \mathrm{P}\left[\frac{1}{2}\big(\,[\boldsymbol{u},\, \boldsymbol{v}] - ad^*_{\boldsymbol{u}}\boldsymbol{v} - ad^*_{\boldsymbol{v}}\boldsymbol{u}\,\big) + \operatorname{grad} f\right] \\
&=\; \mathrm{P}\left[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{2}\boldsymbol{\nabla} h + \boldsymbol{\nabla} f + \boldsymbol{\nabla} f_{uv} + \boldsymbol{\nabla} f_{vu}\right] \\
&=\; \mathrm{P}\,[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla}\, p]
\end{aligned}$$

where $p = (1/2)h + f + f_{uv} + f_{vu}$ and $\operatorname{div} \bar{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = 0$. In particular setting $\boldsymbol{v} = \boldsymbol{u}$, we have

$$\bar{\nabla}^{(\mathrm{R})}_{\boldsymbol{u}}\, \boldsymbol{u} = \mathrm{P}\,[\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}\, p\,]\,.$$

7.3.3 Geodesic equation

The geodesic equation in the right-invariant time-dependent problem must be considered according to the formulation of §3.8.2.

Consider a curve $t \to \eta_t = \eta$ and its tangent $\dot{\eta}_t$. Using (79) and (111) with $U_e = V_e$ and $U_e = \dot{\eta}\circ\eta^{-1}$, the right-invariant connection of a time-dependent problem is given by

$$\left(\hat{\nabla}^{(\mathrm{R})}_{U_{\eta_t}}\, U_{\eta_t}\right)_{\eta_t} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\eta}\circ\eta^{-1})\circ\eta + (\nabla^{(\mathrm{R})}_{\dot{\eta}\circ\eta^{-1}}\, \dot{\eta}\circ\eta^{-1})\circ\eta\,. \tag{120}$$

A geodesic is a curve g_t whose tangent is translated parallel along itself:

$$0 = \mathrm{P}\Big[\,\left(\hat{\nabla}^{(\mathrm{R})}_{\dot{g}_t}\, \dot{g}_t\right)_{g_t}\,\Big] = \mathrm{P}\Big[\,\partial_t\, u + \nabla^{(\mathrm{R})}_u\, u\,\Big]\circ g_t\,, \tag{121}$$

($U_e = u$ and $\eta_t \to g_t$), where $\operatorname{div} u = 0$. The Euler's equation of motion of an incompressible fluid in R^3 is obtained by right translation g_t^{-1} to the identity $e = g_0$:

$$\partial_t\, \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}\, p = 0\,, \tag{122}$$

where the function p is determined so as to satisfy the condition $\nabla \cdot \boldsymbol{u} = 0$.

The Jacobi equation (85) is rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \parallel J \parallel = \left(\parallel \nabla_T \, \boldsymbol{e}_J \parallel^2 \; - K(T,\, \boldsymbol{e}_J) \right) \parallel J \parallel , \tag{123}$$

where $T = \partial g / \partial t$ and $J = \partial g / \partial \alpha = \parallel J \parallel \boldsymbol{e}_J$ for a varied family of geodesic curves $g(t, \alpha)$, and

$$K(T, \boldsymbol{e}_J) = < \bar{R}(T, \boldsymbol{e}_J)\, \boldsymbol{e}_J,\, T >= R_{ijkl} T^i e_J^j T^k e_J^l$$

is the sectional curvature in the two-dimensional section spanned by the tangent vector T and the Jacobi vector J.

7.3.4 Jacobi field as a frozen field

Let us consider the Jacobi field from a different point of view. According to the above definitions of T and J, the equation (80) is written as $\bar{\nabla}_T J = \bar{\nabla}_J T$ (see (61)). Therefore, both of the vector fields T and J commute by the torsion-free property (57), and further the Lie derivative vanishes:

$$L_T \, J = [\, T,\, J\,] = \bar{\nabla}_T J - \bar{\nabla}_J T = 0 \tag{124}$$

(see (48)). This suggests also that, when the T-field is a divergence-free field, then the Jacobi field is also a divergence-free tangent field. In addition, the argument just above the equation (69) asserts that the torsion-free is valid not only with the connection $\bar{\nabla}$, but with the connection $\hat{\nabla}$ as well. Thus, we have

$$\hat{\nabla}_T J = \hat{\nabla}_J T \, .$$

In the time-dependent problem in R^3 space, this is rewritten as

$$\partial_t \boldsymbol{J} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{J} = (\boldsymbol{J} \cdot \nabla)\, \boldsymbol{u} \; ,$$

(equivalent to (48)), which is transformed to

$$\partial_t \boldsymbol{J} + \nabla \times (\boldsymbol{J} \times \boldsymbol{u}\,) = 0 \, , \tag{125}$$

due to a familiar vector identity, because $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{J} = 0$. This equation is usually called the equation of *frozen* field, since it describes that the vector field $\boldsymbol{J}$ is carried along with the flow $\boldsymbol{u}$ and behaves as if $\boldsymbol{J}$ was frozen to the carrier fluid. If the flow u_t is represented by the map $\phi_t(x) = y_t(x)$, then the $\boldsymbol{J}(t) = (J^\alpha)$ is given by the Cauchy's solution (see Remark of §2.6.2):

$$J^\alpha(t) = \frac{\partial y_t^\alpha}{\partial x^k} \, J^k(0) \; .$$

It is well-known that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ satisfies the equation of the form (125) with $\boldsymbol{J}$ replaced by $\boldsymbol{\omega}$. The magnetic induction $\boldsymbol{B}$ in the ideal magneto-hydrodynamics is also governed by the equation of frozen field.

7.3.5 Interpretation of the Riemannian curvature

(i) *Time evolution of Jacobi field*

A Jacobi field $J(t)$ is uniquely determined by its value $J(0)$ and the value of $\nabla_T J$ at $t = 0$ on the geodesic g_t. Provided that $J(0) = 0$ and $(\nabla_T J)_{t=0} = a_0$, it can be shown that [18]

$$\frac{J(t)}{a_0} = t - \frac{t^3}{6} \frac{K(T, J)}{\| J \|^2} + O(t^4) \qquad (t > 0)\,, \tag{126}$$

where $\lim_{t\to 0} K(T, J)/ \| J \|^2 = K(T_0, \boldsymbol{e}_J) \equiv K_J(0)$, and $T_0 = T(0)$. Therefore, if $K_J(0) < 0$, then $\| J(t)/a_0 t \| \geq 1$ for sufficiently small t, and if $K_J(0) > 0$, then $\| J(t)/a_0 t \| \leq 1$ for t near zero. Thus, the time development of the Jacobi vector is controlled by the curvature $K(T_0, \boldsymbol{e}_J)$ and in particular by its sign.

An L^2-distance between two corresponding points (having the same time t) on two geodesics $g_t(\boldsymbol{x} : v_1)$ and $g_t(\boldsymbol{x} : v_2)$ starting at the same point $g_0(\boldsymbol{x} : v_1) = g_0(\boldsymbol{x} : v_2) = \boldsymbol{x}$ with different initial velocity fields v_1 and v_2 of flows in a bounded domain D is defined by

$$d(v_1, v_2 : t) = \left(\int_D |g_t(\boldsymbol{x} : v_1) - g_t(\boldsymbol{x} : v_2)|^2 \mathrm{d}\boldsymbol{x} \right)^{1/2} .$$

Evidently, one has $d(v_1, v_2 : 0) = 0$. The *distance* d is the mean L^2-distance between particles starting at the same position but evolving with the different velocity fields. It is shown in Hattori and Kambe [19] that

$$d(v_1, v_2 : t) = 2\varepsilon |v'| \left(t - \frac{t^3}{6} \frac{K(\bar{v}, 2v')}{(2v')^2} \right) + O(\varepsilon^2 t, \varepsilon t^5)\,, \tag{127}$$

where $\bar{v} = (v_1 + v_2)/2$, $\varepsilon v' = (v_1 - v_2)/2$ and ε is an infinitesimal constant. According to the definition of the Jacobi filed, $J(t) = (\partial/\partial\varepsilon)\, g_t(\boldsymbol{x}, : \bar{v} + \varepsilon v')|_{\varepsilon=0}$, one finds that $d \sim \varepsilon J$ for infinitesimally small ε.

It is found that the sectional curvature appears as a factor to the t^3 term with a negative sign and determines the departure from the linear growth of the L^2-distance. This means, if the curvature is negative, that the L^2-distance d grows faster than the linear behavior, and further infers an exponential growth of the distance $d \sim \varepsilon J$ (see (123)). The d is also interpreted as the distance between two neighboring particle in the same flow field. [19]

In the case of fluid motions in a bounded domain D without mean flow, the particle trajectory will be folded within the domain D during its evolution. Thus, there would occur stretching and folding of the Lagrangian segment connecting two neighboring particles which would lead to *mixing* of particles and *ergodicity* of the fluid motions.

(ii) *Second fundamental form*

Let us consider how the fluid motion acquires a curvature and what the curvature is. On the group $D^s_\mu(M) = \{\eta \in D^s \,:\, \eta^*(\mu) = \mu\}$ of all volume preserving diffeomorphisms of M, the Jacobian operator J_x applied to $\eta(x)$ at $x \in M$ takes always the value unity:

$$D^s_\mu(M) = \{\eta \in D^s \,:\, J_x(\eta(x)) = 1, \quad \forall x \in M\}$$

(see Appendix A1), where $J: \ D^s(M) \to H^{(s-1)}(M)$. From the implicit function theorem, $D^s_\mu(M) = \eta = \{J_x^{-1}(1)\}$ is a *closed* submanifold of $D^s(M)$. According to the formulation of §3.7 and §7.2, the difference of the two connections, $\hat{\nabla}$ in $D^s(M)$ and $\bar{\nabla}$ in $D^s_\mu(M)$, is given by the second fundamental form S of (114).

The *curvature* of the closed submanifold $D^s_\mu(M)$ is given by $\langle \bar{R}(U, V)W, Z \rangle$ in the Gauss-Codazzi equation (117). In particular, the sectional curvature of the section spanned by the tangent vectors $X, Y \in T_\eta D^s_\mu(M)$ is given by

$$\begin{aligned} \bar{K}(X, Y)^{D_\mu} &\equiv \left\langle \bar{R}(X, Y)Y, X \right\rangle_{L^2}^{D_\mu} \\ &= \left\langle R(X, Y)Y, X \right\rangle^M + \left\langle S(X, X), S(Y, Y) \right\rangle - \| S(X, Y) \|^2 . \end{aligned}$$

Even when the manifold M is flat, *i.e.* the curvature $\left\langle R(X, Y)Y, X \right\rangle^M$ vanishes, the sectional curvature $\bar{K}(X, Y)^{D_\mu}$ of the closed submanifold $D^s_\mu(M)$ does not necessarily vanish due to the second and third terms (defined as $\bar{K}_S(X,Y)$) associated with $S(X,Y)$, *etc.*. Namely the curvature of a fluid motion in this case *originates* from the $\bar{K}_S$ part,

$$\bar{K}_S(X,Y) = \left\langle S(X, X), S(Y, Y) \right\rangle_{L^2} - \| S(X, Y) \|^2 .$$

for a flow of an incompressible (inviscid) fluid. Thus it is found that the restriction to the volume-preserving gives rise to the above curvature.

Further it is interesting to observe that the second fundamental forms are related to the pressure gradients. In fact, we have

$$S(X, Y) \equiv \hat{\nabla}_X Y - \bar{\nabla}_X Y = \mathsf{Q}[\hat{\nabla}_X Y]$$

In the Appendix A2, it is shown that an arbitrary vector field v can be decomposed orthogonally into a divergence-free part and a gradient part. Taking as $v = \hat{\nabla}_X Y$, one obtains immediately the form, $\mathsf{Q}[\hat{\nabla}_X Y] = \text{grad}\ (F_{\mathrm{D}}(v) + H_{\mathrm{N}}(v))$. Thus it is found that the curvature is related to the 'grad' part of the connection $\hat{\nabla}_X Y$ which is normal to $TD^s_\mu(M)$.

In particular, for $\eta \in D^s_\mu(M)$ and $\dot{\eta} = X$, we have

$$\begin{aligned} S(X, X) &= Q_\eta\, (\hat{\nabla}_{\dot{\eta}}, \dot{\eta}) = Q_\eta \Big(\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\eta} \circ \eta^{-1}) \circ \eta + (\nabla^{(\mathrm{R})}_{\dot{\eta}\circ\eta^{-1}}\, \dot{\eta} \circ \eta^{-1}) \circ \eta \Big) \\ &= -(\,\text{grad}\, p_X\,) \circ \eta\ , \end{aligned}$$

where p_X is the pressure of the velocity field X. Hence, the first term of $\bar{K}_S$ is represented as $\langle S(X, X), S(Y, Y) \rangle = \langle \text{grad}\, p_X, \ \text{grad}\, p_Y \rangle$, which is the correlation of two pressure gradients. Thus the $\bar{K}_S$ part of the curvature is given by

$$\bar{K}_S(X,Y) = \langle\, \text{grad}\, p_X, \ \text{grad}\, p_Y \,\rangle - \| \text{ grad } (F_{\mathrm{D}}(v) + H_{\mathrm{N}}(v)) \|^2 .$$

7.3.6 Space-periodic flows in a cubic space (Fourier representation)

Explicit forms are given for space-periodic flows in a cube by Fourier representation, *i.e.* for flows on the flat 3-torus $M = T^3 = R^3/(2\pi Z)^3$ (Nakamura, Hattori and Kambe[20], Hattori and Kambe[19]). With $\boldsymbol{x} \in T^3$, we have $\boldsymbol{x} = \{(x^1, x^2, x^3);\ \mathrm{mod}\, 2\pi\}$. The manifold T^3 is a bounded manifold without boundary. The elements of the Lie algebra of $D_\mu(T^3)$ can be thought of as real periodic vector fields on T^3 with divergence-free property. Such periodic fields are represented by the real part of corresponding *complex* Fourier forms.

The Fourier bases are denoted by $e_k = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ where $\boldsymbol{k} = (k_i)$ is a wave number covector $(i = 1, 2, 3)$. Now the representations are complexified so that all the fields become linear (or multilinear) in the complex vector space of the complexified Lie algebra. The bases of this vector space are given by the functions e_k $(\boldsymbol{k} \in Z^3, \boldsymbol{k} \neq 0)$. The velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is represented as

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \boldsymbol{u}_k(t)\, e_k\,,$$

where $\boldsymbol{u}_k(t)$ is the Fourier amplitude and also written as $u^i(\boldsymbol{k})$ $(i = 1, 2, 3)$. The amplitude must satisfy the two properties,

$$\boldsymbol{k} \cdot \boldsymbol{u}_k = 0\,, \qquad \boldsymbol{u}_{-k} = \boldsymbol{u}_k^*\,, \tag{128}$$

to describe the solenoidal condition and reality condition respectively, where the asterisk denotes the complex conjugate. It should be noted that $\boldsymbol{u}_k$ has two independent polarization components.

Let us take four tangent fields satisfying (128): $\boldsymbol{u}_k e_k$, $\boldsymbol{v}_l e_l$, $\boldsymbol{w}_m e_m$, $\boldsymbol{z}_n e_n$. Then we have the followings. The scalar product convention such as $(\boldsymbol{u} \cdot \boldsymbol{v}) = u^1 v^1 + u^2 v^2 + u^3 v^3$ is used below. The *metric* is

$$\langle\, \boldsymbol{u}_k e_k,\ \boldsymbol{v}_l e_l \,\rangle = (2\pi)^3 (\boldsymbol{u}_k \cdot \boldsymbol{v}_l)\, \delta_{0,k+l}\,,$$

where $\delta_{0,k+l} = 1$ (if $\boldsymbol{k} + \boldsymbol{l} = 0$) and 0 (otherwise). The *covariant derivative* is

$$\bar{\nabla}_{\boldsymbol{u}_k e_k}\, \boldsymbol{v}_l e_l = i\, (\boldsymbol{u}_k \cdot \boldsymbol{l})\, \frac{\boldsymbol{k}+\boldsymbol{l}}{|\boldsymbol{k}+\boldsymbol{l}|} \times \left(\boldsymbol{v}_l \times \frac{\boldsymbol{k}+\boldsymbol{l}}{|\boldsymbol{k}+\boldsymbol{l}|} \right) e_{k+l}\,, \tag{129}$$

where the amplitude vector on the right hand side is perpendicular to $\boldsymbol{k}+\boldsymbol{l}$. The *commutator* is

$$[\, \boldsymbol{u}_k\, e_k,\ \boldsymbol{v}_l\, e_l \,] = i \Big((\boldsymbol{u}_k \cdot \boldsymbol{l}) \boldsymbol{v}_l - (\boldsymbol{v}_l \cdot \boldsymbol{k}) \boldsymbol{u}_k \Big)\, e_{k+l}\,,$$

the right side being also perpendicular to $\boldsymbol{k}+\boldsymbol{l}$. The *geodesic equation* (122) reduces to

$$\frac{\partial}{\partial t} u^l(\boldsymbol{k}) \;+\; i \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} \sum_{m,n} \left(\delta_{ln} - \frac{k_l k_n}{k^2} \right) k_m\, u^m(\boldsymbol{p})\, u^n(\boldsymbol{q}) = 0\,, \tag{130}$$

by using (129), where δ_{ln} is the Kronecker's delta. From the definitions (115) $\sim$ (117) and (129), the *curvature tensor* is

$$\begin{aligned} \bar{R}_{klmn} &= \langle\, \bar{R}(\boldsymbol{u}_k e_k,\ \boldsymbol{v}_l e_l)\, \boldsymbol{w}_m e_m,\ \boldsymbol{z}_n e_n \,\rangle \\ &= (2\pi)^3 \left(\frac{(\boldsymbol{u}_k \cdot \boldsymbol{m})(\boldsymbol{w}_m \cdot \boldsymbol{k})}{|\boldsymbol{k}+\boldsymbol{m}|} \frac{(\boldsymbol{v}_l \cdot \boldsymbol{n})(\boldsymbol{z}_n \cdot \boldsymbol{l})}{|\boldsymbol{l}+\boldsymbol{n}|} - \frac{(\boldsymbol{v}_l \cdot \boldsymbol{m})(\boldsymbol{w}_m \cdot \boldsymbol{l})}{|\boldsymbol{l}+\boldsymbol{m}|} \frac{(\boldsymbol{u}_k \cdot \boldsymbol{n})(\boldsymbol{z}_n \cdot \boldsymbol{k})}{|\boldsymbol{k}+\boldsymbol{n}|} \right), \end{aligned}$$

for $\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}+\boldsymbol{n}=0$ only and vanishes otherwise (derived from the definitions (114)~(117) and (129)). The cases when the denominator vanishes should be excluded. It is to be noted that the above formulas reduce to those of Arnold [3] when two-dimensionality is imposed.

As an application, a flow with Beltrami property is considered, that is, we assume that the velocity field $U_{\boldsymbol{p}} = \boldsymbol{u}_{\boldsymbol{p}} e_p + \boldsymbol{u}_{-\boldsymbol{p}} e_{-p} = \mathrm{Re}[\boldsymbol{u}_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}]$ satisfies the Beltrami condition, $\nabla \times U_{\boldsymbol{p}} = \lambda U_{\boldsymbol{p}}$, $\lambda \in R$. This eigenvalue problem can be solved with $\lambda^2 = |\boldsymbol{p}|^2$. It is readily shown that $U_{\boldsymbol{p}}$ is a steady-state solution. Let $X = \sum \boldsymbol{v}_l e_l$ be any velocity field satisfying (128). Then one obtains

$$K(U_{\boldsymbol{p}}, X) = \ \{\text{negative terms}\}\ [20].$$

This result of negative sectional curvature is a three-dimensional counterpart of the Arnold's two-dimensional finding [3]. The negative sectional curvature leads to exponential growth of the Jacobi vector $\| \ J \ \|$ according to (123) (see also (126), (127)). This means that the distance between the two neighboring geodesics grows also exponentially.

8 Motion of a vortex filament

The dynamics of an isolated thin vortex filament, embedded in an ideal incompressible fluid, is known to be well-approximated by the *local induction equation* (LIE) [21] when the filament curvature is small. A vortex filament is assumed to be spatially periodic and given by a time-dependent C^∞-curve $\boldsymbol{x}(s,\, t)$ in R^3 with $s \in S^1$ the length parameter and t the time parameter, that is, $\boldsymbol{x} \,:\, S^1 \times R \to R^3$.

As illustrated just below (Suzuki *et al.* [22]), this system is characterized with the rotation group $G = SO(3)$ associated pointwise with the S^1 manifold. The group $G(S^1)$ of smooth mappings, $g \,:\, s(\in S^1) \mapsto g(s) \in G = SO(3)$, equipped with the pointwise composition law, $g''(s) = g'(s) \circ g(s)$ for $g, g', g'' \in G$, is an infinite-dimensional Lie group, *i.e.* a *loop group*. The corresponding loop algebra leads to the Landau-Lifshitz equation as the geodesic equation (§8.2). Further in §8.3, its central extension results in the so-called Kac-Moody algebra [6]. This chapter includes a new formulation for the geodesic equations of motion of a vortex filament on the basis of the theory of loop group and its extension. This gives a new interpretation to the local induction equation and the equation of Fukumoto & Miyazaki (1991) [23] from a geometrical point of view.

8.1 Local induction equation

Suppose that motion of a vortex filament is governed by the LIE, namely,

$$\partial_t \boldsymbol{x} = \partial_s \boldsymbol{x} \times \partial_s^2 \boldsymbol{x}\ . \qquad (131)$$

In the conventional mechanics terms, $\boldsymbol{x}(s,\, t) \in M$ is the position vector of a point on the filament. Mathematically, $\boldsymbol{x}(s,\, t)$ is an element of the C^∞-embeddings of S^1 into a three-dimensional (oriented) manifold M. This system is reconsidered on the basis of the Riemannian geometry.

A well-known local orthonormal system, at each point $\boldsymbol{x}(s)$ on the filament at a time t, is denoted as $(T,\, N,\, B)$, where $T(s)$ is the unit tangent $T(s) = \partial\boldsymbol{x}/\partial s$, $N(s)$ and $B(s)$ are the unit principal normal and binormal respectively. These unit vectors satisfy the Frenet-Serret equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix} \qquad (132)$$

where $\kappa(s)$, $\tau(s) \in R$ are the curvature and torsion of the filament. The Hamiltonian of the system (131) is given by

$$H = \int \kappa^2(s)\, \mathrm{d}s\ . \qquad (133)$$

The motion of the curve $\boldsymbol{x}(s,t)$ is a map $t \mapsto \phi_t$, that is, $x_0(s) \mapsto x_t(s) = \phi_t \circ x_0(s)$, where $x_t(s) = \boldsymbol{x}(s,t)$. Following the motion, the tangent vector $T_t(s)$ on the curve is left-translated, *i.e.* $T_t(s) = \partial x_t(s)/\partial s = \phi_t \circ \partial x_0(s)/\partial s = (\phi_t)_* T_0(s)$. Correspondingly, its derivatives are left-translated, *e.g.* $\partial_s T_t = (\phi_t)_* \partial_s T_0(s)$. Note that $\partial_s T = \kappa(s) N(s)$, the suffix t being omitted here and below. Differentiating the equation (131) with respect to s, one obtains

$$\partial_t T = T \times \partial_s^2 T = -\partial_s^2 T \times T\ , \qquad (134)$$

for the unit tangent vector field T. The equation of the form $\partial_t X = \Omega \times X$ describes "rotation" of the vector X with the angular velocity Ω. Hence, the factor $-\partial_s^2 T$ $(= \Omega$, say) in the above equation is interpreted as the angular velocity at s pointwise, that is, the $\Omega = -T''$ is an element of the Lie algebra $\mathbf{so(3)}$, where the prime denotes the differentiation with respect to s. The commutator of the $\mathbf{so(3)}$ is given by the vector product of two elements of (see §2.6**a** and §4.1). In the sense of the pointwise locality, the group $G(S^1)$ is called the local group. As illustrated just below, the vector $T\,(= -\partial_s^{-2}\Omega)$ itself is interpreted as an element on the dual space $\mathbf{so(3)}^*$.

Associated with the Hamiltonian (133), it is useful to observe that $(\partial_s T,\, \partial_s T)_{R^3} = (\kappa N,\, \kappa N) = \kappa^2$, and further that $\Omega = -T'' \in \mathbf{so(3)}$ (Lie algebra). Then the metric of the system is defined as follows:

$$\langle \Omega,\, \Omega \rangle = \langle T'',\, T'' \rangle \equiv \int_{S^1} (T'',\, A T'')_{R^3}\, \mathrm{d}s = \int_{S^1} (\partial_s T,\, \partial_s T)_{R^3}\, \mathrm{d}s = \int \kappa^2\, \mathrm{d}s\,,$$

where $A \equiv -\partial_s^{-2}$, thus $AT'' = -T$, and $(A,\, B)_{R^3} = \delta_{kl} A^k B^l$. Integration by parts is carried out at the last equality. The left-translation of $\partial_s T$ noted above and the invariance of the Hamiltonian H induces the *left-invariant* property of the above metric.

Using $\partial_s T = -\partial_s^{-1}\Omega$, the metric is also written as

$$\langle \Omega,\, \Omega \rangle = \int_{S^1} (\partial_s^{-1}\Omega,\, \partial_s^{-1}\Omega)_{R^3}\, \mathrm{d}s = \int_{S^1} (\Omega,\, A\Omega)_{R^3}\, \mathrm{d}s \tag{135}$$

The operator $A = -\partial_s^{-2}$ is often called the inertia operator (or momentum map) of the system.[3] Thus, it is found that, using the new symbol L for T,

$$\begin{aligned} L'' &\in C^\infty(S^1,\, \mathbf{so(3)}) \\ L(= -A L'') &\in C^\infty(S^1,\, \mathbf{so(3)}^*) \end{aligned}$$

where $C^\infty(S^1, \mathbf{so(3)}) = \mathcal{L}\mathbf{g} = \mathbf{so(3)}[S^1]$ is the *loop algebra* of the loop group, $\mathcal{L}G = SO(3)[S^1]$, and $C^\infty(S^1, \mathbf{so(3)}^*)$ is the dual algebra.

8.2 Loop algebra and Landau-Lifshitz equation

Now let us reformulate the above dynamical system in the following way. Let

$$X,\, Y,\, Z \in \mathcal{L}\mathbf{g} = \mathbf{so(3)}[S^1] = C^\infty(S^1,\, \mathbf{so(3)})$$

be the vector fields. Correspondingly, we define the dual fields:

$$AX,\, AY,\, AZ \in C^\infty(S^1,\, \mathbf{so(3)}^*)\,,$$

where $A = -\partial_s^{-2}$. The left-invariant *metric* is defined by

$$\langle X,\, Y \rangle = \int_{S^1} (\partial_s^{-1} X,\, \partial_s^{-1} Y)_{R^3}\, \mathrm{d}s = \int_{S^1} (X,\, AY)_{R^3}\, \mathrm{d}s\,.$$

[3] According to the theory of elliptic operators, $A = \partial^{-2}$ is defined uniquely for C^∞ functions.

The *commutator* is given by

$$[X,\ Y](s) \ = \ X(s) \ \times \ Y(s)$$

at each s, pointwise.

In the case of the invariant metric, the connection satisfies the equation (60), and in terms of the operators ad and ad^*, we have the expression (77) of $\nabla_X Y = \frac{1}{2}(ad_X Y - ad^*_Y X - ad^*_X Y)$, for $X,\ Y \in \mathbf{so(3)}$. By the definition (32), $ad_X Y = [X,\ Y] = X \times Y$. Then the $ad^*_X Y$ satisfies

$$\langle ad^*_X Y,\ Z\rangle = \langle Y,\ ad_X Z\rangle = \int_{S^1} (AY,\ X \times Z)_{R^3}\, \mathrm{d}s = \int_{S^1} \big(AA^{-1}(AY \times X),\ Z\big)_{R^3}\, \mathrm{d}s\ ,$$

which leads to

$$ad^*_X Y = A^{-1}(AY \times X) = -\partial_s^2\, [\, AY,\ X\,]\ .$$

Thus it is found that the *connection* ∇ of the filament motion is represented as [22]

$$\nabla_X Y = \frac{1}{2}\Big([X,Y] + \partial_s^2[AX,Y] + \partial_s^2[AY,X]\Big)$$

for $X(s),\ Y(s) \in\ \mathcal{L}\mathbf{g} = \mathbf{so(3)}[S^1]$, the loop algebra.

The *geodesic equation* $(\partial_t X + \nabla_X X = 0)$ on the loop group $\mathcal{L}G = SO(3)[S^1]$ is given by

$$\partial_t X + \partial_s^2(AX \times X) = 0\ .$$

Applying the operator $\partial_s^{-2} = -A$, we get a corresponding equation in the dual space,

$$\partial_t L - (L \times L'') = 0\ .$$

where $L = -AX$ and $X = L''$. Thus, we have recovered the equation (134) for the vector $T = L$. This type of equation is called the *Landau-Lifshitz* equation. Further, integrating with respect to s, one gets back to the equation (131).

The Jacobi equation is of the form: $(\mathrm{d}^2/\mathrm{d}s^2)\ \|\ J\ \|^2 = 2\ \|\ \nabla_T J\ \|^2\ - 2K(T,J)$. The *sectional curvature* is defined by

$$K(X,Y) = \langle R(X'',Y'')Y'',\ X''\rangle = \int_{S^1} f(s)\, \mathrm{d}s$$

for $X'',\ Y'' \in\ C^\infty(S^1,\ \mathbf{so(3)})$, where [22]

$$\begin{aligned} f(s) \ &= \ (X \times X'')'' \cdot (Y \times Y'') - \frac{3}{4}\Big(\partial_s^{-1}(X'' \times Y'')\Big)^2 \\ &\quad + \frac{1}{4}\Big(\partial_s(X \times Y'' + Y \times X'')\Big)^2 + \frac{1}{2}\Big(|X''|^2\,|Y'|^2 + |Y''|^2\,|X'|^2\Big) \\ &\quad + \frac{1}{2}(X'' \cdot Y'')\Big((X \cdot Y'') + (X'' \cdot Y)\Big) \end{aligned}$$

The curvature was estimated in two example cases [22]. First one is the section spanned by the tangent vector $X = (0,0,1)$ of a straight-line vortex and an arbitrary tangent field Y, for which the curvature is found to be

$$K(X,Y) = \frac{1}{4}\int_{S^1} (X \times Y''')^2\, \mathrm{d}s \geq 0\,,$$

that is, the curvature is always positive.

The second case is the section spanned by the tangent vector X of a spiral vortex and an arbitrary tangent field Y. If the wave lengths of Y field are shorter than the radius of curvature of the spiral, then the curvature $K(X,Y)$ is positive and large enough to make the right hand side of the Jacobi equation (83) *negative*, thus the spiral vortex being stable. On the other hand, if the wave lengths of Y field is large enough, then the $K(X,Y)$ becomes negative and the spiral vortex is unstable.

8.3 Central extension of the algebra of filament motion

The central extension of the algebra of the previous section is considered in an analogous way to the KdV problem (§5.2, 5.3). Let us introduce an extended algebra denoted as

$$\hat{X},\, \hat{Y},\, \hat{Z} \in \mathbf{so(3)}[S^1] \oplus R$$

where

$$\hat{X} = (X,\, a), \quad \hat{Y} = (Y,\, b), \quad \hat{Z} = (Z,\, c)$$

and $a, b, c \in R$. The new *metric* is defined by

$$\langle \hat{X},\, \hat{Y} \rangle = \int_{S^1} (X,\, AY)_{R^3}\, ds + a\, b$$

where $A = -\partial_s^{-2}$. The *extended algebra* is defined by

$$[\hat{X},\, \hat{Y}] = \Big([X,\, Y](s),\ \ c(X,Y) \Big), \tag{136}$$

where

$$c(X,Y) = \int_{S^1} (X(s),\, Y'(s))_{R^3}\, ds = -c(Y,X)\,,$$

and remarkably the Jacobi identity is satisfied by the new commutator:

$$[\,[\hat{X},\hat{Y}],\, \hat{Z}\,] + [\,[\hat{Y},\hat{Z}],\, \hat{X}\,] + [\,[\hat{Z},\hat{X}],\, \hat{Y}\,] = 0\,.$$

It is not difficult to show that the commutator (136) is equivalent to that of Kac-Moody algebra [6]. The extended *connection* is found to be given by

$$\nabla_{\hat{X}}\hat{Y} = \Big(\nabla_X Y,\ \frac{1}{2}\int_{S^1} (X, \partial_s Y)_{R^3} ds\Big)\,,$$

where

$$\nabla_X Y \equiv \frac{1}{2}\Big([X,Y] + \partial_s^2[AX,Y] + \partial_s^2[AY,X] - \partial_s^2(a\partial_s Y + b\partial_s X)\ \Big)$$

Then the *geodesic equation* $(\partial_t \hat{X} + \nabla_{\hat{X}}\hat{X} = 0)$ for the extended system is obtained as

$$\begin{aligned} \partial_t X + \partial_s^2\Big(AX \times X\Big) - a\,\partial_s^3 X \ &= \ 0\,, \\ \partial_t\, a \ &= \ 0\,. \end{aligned}$$

Applying the operator A, we obtain the equation for $\boldsymbol{x}_s = L = -AX$ $(X = L'')$:

$$\partial_t L - \left(L \times L''\right) - a\,\partial_s^3 L = 0\ .$$

Integrating this with respect to s, we get back to the equation for the **space curve** $\boldsymbol{x}(s,\,t)$:

$$\boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + a\,\boldsymbol{x}_{sss}\ ,$$

in R^3. Reparameterizing s to make it to be the arc-length, one finds

$$\boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + a\left(\boldsymbol{x}_{sss} + \frac{3}{2}\kappa^2\,\boldsymbol{x}_s\right)\ .$$

where $\kappa(s) = (\boldsymbol{x}_{ss}, \boldsymbol{x}_{ss})^{1/2}$ is the curvature of the filament at a point s. The shape of the filament is not changed by the additional term. The *velocity* $\boldsymbol{x}_t$ is also represented as

$$\boldsymbol{x}_t = \kappa\,B + a\left(\frac{1}{2}\kappa^2 T + \kappa' N + \kappa\,\tau\right)$$

where (T, N, B) is the local orthonormal bases and τ is the torsion.

This is equivalent to the equation of Fukumoto and Miyazaki [23], called FM equation here. The equation was originally derived for the motion of a thin vortex tube with an axial flow along it. These are known to be the first two members of the hierarchy of completely integrable equations for the filament motion. First five members of the integral invariants of the system are as follows [26]:

$$I_0 = \int \mathrm{d}s\,, \qquad I_1 = \textstyle\int \tau\ \mathrm{d}s\,, \qquad I_2 = \displaystyle\int \kappa^2\ \mathrm{d}s\,,$$
$$I_3 = \int \kappa^2\,\tau\ \mathrm{d}s\,, \qquad I_4 = \int \left((\kappa')^2 + \kappa^2\tau^2 - (1/4)\kappa^4\right)\ \mathrm{d}s\,,\cdots$$

It was shown in §5 that the KdV equation is a geodesic equation on the diffeomorphism group of a circle S^1 with a central extension. Here it is verified that the motion of a vortex filament governed by the LIE equation is a geodesic on the loop group $\mathcal{L}G = SO(3)[S^1]$ which is $SO(3)$-valued with pointwise multiplication. Further, the infinite-dimensional loop algebra $\mathcal{L}\mathbf{g}$ has non-trivial central extension equivalent to the Kac-Moody algebra. This is a new formulation verifying that the extended system leads to another geodesic equation with an additional third derivative term, which was derived earlier by Fukumoto and Miyazaki [23] and shown to be a completely integrable system. It is remarkable that there is a similarity in the forms between the KdV equation and FM equation. These are two integrable systems defined over the S^1 manifold: one is a geodesic equation over the extended diffeomorphism group $\hat{D}(S^1)$ and the other is the one over the extended loop group $\hat{SO}(3)[S^1]$.

9 Conclusion

A geometrical theory is developed for diverse dynamical systems of both finite and infinite degrees of freedom. Based on the mathematical framework presented in the beginning chapters 2 and 3, and according to the published works by the author and others, five dynamical systems are reformulated geometrically in the subsequent chapters. Although those systems are already studied with the conventional methods in physics, the present formulation provides us a deep insight into the systems and adds new geometrical characterizations of the dynamical evolutions in terms of the geodesic equations, Jacobi fields and Riemannian curvatures. The last chapter for the motion of a vortex filament includes a new formulation on the basis of notion of the loop algebra, disclosing an analogy between the diffeomorphic flows on a circle and the flows over loop groups.

Finally, it is to be remarked that, as noted already in the beginning of the chapter 6 but not included, the geometrical approach can be applied to the phase transition problem. It is found [14] that fluctuations of the configuration-space curvature exhibit a singular behavior at the phase transition. This is an evidence that the scope of geometrical theories are fundamental and very broad.

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Appendix

A1. Forms and Exterior multiplications

A covector is also called 1-form, which is a linear function $\omega^1(v) : R^n \to R$. The set of all 1-forms on R^n constitutes an n-dimensional dual space. Similarly, a 2-form ω^2 is defined as a function on pairs of vectors $\omega^2(v_1, v_2) : R^n \times R^n \to R$, which is bilinear and skew-symmetric with respect to two vectors v_1 and v_2.

A k-form is defined as a function of k vectors $\omega^k(v_1, \cdots, v_k) : R^n \times \cdots \times R^n$ (k-direct-product of R^n) $\to R$, which is k-linear and skew-symmetric in the sense, $\omega^k(v_{a_1}, \cdots, v_{a_k}) = (-1)^\sigma \omega^k(v_1, \cdots, v_k)$ where $\sigma = 0$ (if the permutation $(a_1, \cdots, a_k)$ with respect to $(1, \cdots, k)$ is even) and $\sigma = 1$ (if it is odd), where $v_a = v_a^j \partial_j$.

We now introduce an exterior multiplication of two 1-forms, which associates to every pair $(\omega_\alpha^1, \omega_\beta^1)$ on R^n a 2-form $\omega_\alpha^1 \wedge \omega_\beta^1$ on $R^n \times R^n$ defined by

$$\omega_\alpha^1 \wedge \omega_\beta^1(v_a, v_b) = \omega_\alpha^1(v_a)\omega_\beta^1(v_b) - \omega_\alpha^1(v_b)\omega_\beta^1(v_a) , \tag{137}$$

which is obviously bilinear with respect to v_a and v_b and skew-symmetric. For example, if ω_α^1 and ω_β^1 are differential 1-forms, *i.e.* $\omega_\alpha^1 = \mathrm{d}x^i$ and $\omega_\beta^1 = \mathrm{d}x^j$, then we have

$$\begin{aligned} \mathrm{d}x^i \wedge \mathrm{d}x^j(v_1, v_2) &= \mathrm{d}x^i(v_1)\,\mathrm{d}x^j(v_2) - \mathrm{d}x^i(v_2)\,\mathrm{d}x^j(v_1) \\ &= v_1^i v_2^j - v_2^i v_1^j = \begin{vmatrix} v_1^i & v_1^j \\ v_2^i & v_2^j \end{vmatrix} , \end{aligned} \tag{138}$$

where the last one is the determinant.

In general, a *differential* k-form on R^n can be written in the form,

$$\omega^k = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

Definition of exterior multiplication: The exterior multiplication of an arbitrary k-form ω^k by an arbitrary l-form ω^l is a $(k+l)$-form, and satisfies the following properties:

(i)	$\omega^k \wedge \omega^l$	$=$	$(-1)^{kl}\, \omega^l \wedge \omega^k$	(skew-commutative)
(ii)	$(c_1\omega_1^k + c_2\omega_2^k) \wedge \omega^l$	$=$	$c_1\omega_1^k \wedge \omega^l + c_2\omega_2^k \wedge \omega^l$	(distributive)
(iii)	$\left(\omega^k \wedge \omega^l\right) \wedge \omega^m$	$=$	$\omega^k \wedge \left(\omega^l \wedge \omega^m\right)$	(associative)

For example, let us consider a volume form. Let (x^1, x^2, x^3) be a local orthonormal coordinate system of M^3. Then the volume form *vol* is a 3-form: $vol^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Let $y = f(x)$ be any coordinate transformation in the neighborhood of the point $x = (x^1, x^2, x^3) \in M^3$. The differential is given as $\mathrm{d}x^i = (\partial x^i / \partial y^k)\mathrm{d}y^k$. Then, the length element is

$$\mathrm{d}s^2 = (\mathrm{d}x^i)^2 = \frac{\partial x^i}{\partial y^j}\frac{\partial x^i}{\partial y^k}\mathrm{d}y^j\mathrm{d}y^k \equiv g_{jk}\,\mathrm{d}y^j\mathrm{d}y^k ,$$

where

$$g_{jk} = \frac{\partial x^i}{\partial y^j}\frac{\partial x^i}{\partial y^k} .$$

Hence,

$$\det(g_{jk}) = \left(\det\left(\frac{\partial(x)}{\partial(y)}\right)\right)^2 = g(y)\,.$$

The Jacobian of the transformation at x is given by

$$J_x[y(x)] = \det\left(\frac{\partial(y)}{\partial(x)}\right) = \pm g^{-1/2}\,. \tag{139}$$

Consider the volume form defined by

$$vol^3(y) = o(y)\sqrt{g(y)}\;\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3\,,$$

where $o(y) = \pm 1$ is an orientation factor of the coordinate frame $y = (y^1, y^2, y^3)$. Reminding the definition (22), the transformation (27), and using the above properties $(i) - (iii)$, the pull-back of the volume form is given by

$$f^* vol^3(x) = o(y)\sqrt{g}\;J_x[y(x)]\;\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,, \tag{140}$$

where the orientation $o(y)$ is chosen according to the sign of the Jacobian determinant $J_x(y(x))$, that is, $o(y) = 1$ if the (y^1, y^2, y^3)-frame has the same orientation as the (x^1, x^2, x^3)-frame (assumed to be right-handed usually), and $o(y) = -1$ otherwise.

Let f be a differentiable map of an orientation-preserving diffeomorphism, $f:\ M_1 \to M_2$, from an interior σ of M_1 onto an interior $f(\sigma)$ of M_2. Then, for any differential k-form ω^k on M_2, the following general formula of pull-back integration holds:

$$\int_{f(\sigma)} \omega^k = \int_\sigma f^*\omega^k\,, \tag{141}$$

which is a generalization of the integral formula (28) for 1-form. The integral of a k-form ω^k over the image $f(\sigma)$ is equal to the integral of the pull-back $f^*\omega^k$ over the original subset σ.

A2. Orthogonal decomposition (Helmholtz decomposition, or Hodge decomposition)

An arbitrary vector field v on M can be decomposed orthogonally into divergence-free and gradient parts. In fact, a H^s vector field $v \in T_eM$ is written as

$$\begin{aligned} v &= P_e(v) + Q_e(v) \\ Q_e(v) &= \mathrm{grad}\, F_{\mathrm{D}} + \mathrm{grad}\, H_{\mathrm{N}} = \mathrm{grad}\,(F_{\mathrm{D}} + H_{\mathrm{N}}), \\ P_e(v) &= v - Q_e(v), \end{aligned}$$

where

The scalar functions F_{D} and H_{N} are the solutions of the following Dirichlet problem and Neumann problem, respectively,

$$\begin{aligned} \Delta\, F_{\mathrm{D}}(v) &= \mathrm{div}\, v, && \text{where} \quad \mathrm{supp}\, F_{\mathrm{D}} \subset M \\ \Delta\, H_{\mathrm{N}}(v) &= 0\,, && \text{and} \quad \langle \nabla H_{\mathrm{N}},\, \boldsymbol{n}\rangle = \langle v - \nabla F_{\mathrm{D}},\, \boldsymbol{n}\rangle \end{aligned}$$

where $\boldsymbol{n}$ is the unit normal on the boundary ∂M. There is orthogonality, $\langle \mathrm{grad} F_{\mathrm{D}}, \mathrm{grad}\, H_{\mathrm{N}} \rangle = 0$. Then, it can be shown that

$$\begin{aligned} \mathrm{div}\, P_e(v) &= 0\,, \\ \langle\, P_e(v),\ Q_e(v)\,\rangle &= 0\,. \end{aligned}$$

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