MODULAR HYPERGEOMETRIC RESIDUE SUMS OF ELLIPTIC SELBERG INTEGRALS

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ABSTRACT. It is shown that the residue expansion of an elliptic Selberg integral gives rise to an integral representation for a multiple modular hypergeometric series. A conjectural evaluation formula for the integral then implies a closed summation formula for the series, generalizing both the multiple basic hypergeometric ${}_8\Phi_7$ sum of Milne-Gustafson type and the (one-dimensional) modular hypergeometric ${}_8\mathcal{E}_7$ sum of Frenkel and Turaev. Independently, the modular invariance ensures the asymptotic correctness of our multiple modular hypergeometric summation formula for low orders in a modular parameter.

1. Introduction

Modular or elliptic hypergeometric series first made their appearance via the theory of exactly solvable statistical models in the construction of elliptic solutions of the Yang-Baxter equation [D-O1, D-O2]. Their mathematical theory was developed by Frenkel and Turaev in the seminal paper [FT], and since then has found further applications in the study of novel biorthogonal rational functions on elliptic grids [SZ1, SZ2]. A principal result of [FT] is a modular hypergeometric generalization of the celebrated very-well-poised balanced terminating basic hypergeometric ${}_8\Phi_7$ sum of Jackson [GR]. It is well-known that the classical basic hypergeometric summation formulas are intimately connected—via residue calculus—to the Askey-Wilson and Nassrallah-Rahman generalized beta integrals [AW, NR, Ra, GR]. Recently, one of us found an elliptic beta integral built of Ruijsenaars' elliptic gamma function that generalizes the Askey-Wilson and Nassrallah-Rahman integrals [Sp] and reproduces the Frenkel-Turaev sum by residue calculus [DS1].

In the present paper a multidimensional generalization of the Frenkel-Turaev sum is derived, by means of residue calculus, starting from a conjectural elliptic Selberg type integration formula first presented and partially proved in [DS2]. Our main tool is an integral representation for the multiple modular hypergeometric series following from the elliptic Selberg integral. The series under consideration can be seen as a modular counterpart of the multiple (basic) hypergeometric series studied by Milne and Gustafson *et al* [Mi, G1, DG, LM, ML, K]. It is important to emphasize, though, that these are not the only multidimensional generalizations of the classical (basic) hypergeometric series appearing in the literature. Other (related) types of multiple (basic) hypergeometric series were for instance considered by Aomoto, Ito and Macdonald [A, I1, I2, Ma, D] and by Schlosser [Sc]. Modular analogs of those

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types of series were discussed in [W, DS1, Ro] (Aomoto-Ito-Macdonald type) and [W, Ro] (Schlosser type), respectively.

The material is organized as follows. In section 2 we present a residue formula for a Selberg-type multivariate integral composed of Ruijsenaars' elliptic gamma functions. It is shown in Section 3 that the residue formula in question leads to an integral representation for a multiple modular hypergeometric series of Milne-Gustafson type. Combined with a partially proved evaluation formula for the elliptic Selberg integral from [DS2], this then leads us to a closed summation formula for the series given in Section 4. We close by studying the modular properties of the summation formula at issue in Section 5. More specifically, it turns out that both sides of our multiple modular hypergeometric summation formula are given by Jacobi modular functions on $SL_2(\mathbb{Z})$ (in the sense of Eichler-Zagier [EZ]). This enables an independent verification of the correctness of the summation formula for low orders in $\log(q)$ (up to order 10) via the theory of modular forms [Se].

2. Residue Calculus for an Elliptic Selberg Integral

Ruijsenaars' elliptic gamma function is defined by [Ru, FV]

$$\Gamma(z;p,q) = \frac{(pqz^{-1};p,q)_\infty}{(z;p,q)_\infty}, \tag{2.1}$$

where

$$(a;p,q)_\infty = \prod_{j,k=0}^{\infty}(1-ap^jq^k), \quad \text{with } |p|,|q|<1. \tag{2.2}$$

It satisfies the difference equations

$$\Gamma(qz;p,q)=\theta(z;p)\Gamma(z;p,q), \quad \Gamma(pz;p,q)=\theta(z;q)\Gamma(z;p,q) \tag{2.3a}$$

and the reflection equation

$$\Gamma(z;p,q)\Gamma(z^{-1};p,q)=\frac{1}{\theta(z;p)\theta(z^{-1};q)}, \tag{2.3b}$$

where

$$\theta(z;p)=(z;p)_\infty\,(pz^{-1};p)_\infty \tag{2.4}$$

(with $(a;p)_\infty \equiv (a;p,0)_\infty = \prod_{j=0}^{\infty}(1-ap^j)$). By forming quotients of elliptic gamma functions, one ends up with elliptic Pochhammer symbols:

$$\theta(z;p;q)_m = \frac{\Gamma(zq^m;p,q)}{\Gamma(z;p,q)} = \prod_{j=0}^{m-1}\theta(zq^j;p), \quad m\in\mathbb{N} \tag{2.5}$$

(with $\theta(z;p;q)_0=1$). To ease the notation we will often employ the following short-hand conventions for multiple products of gamma functions, theta functions and Pochhammer symbols

$$\Gamma(a_1,\ldots,a_l;p,q) = \textstyle\prod_{r=1}^{l}\Gamma(a_r;p,q), \tag{2.6a}$$

$$\theta(a_1,\ldots,a_l;p) = \textstyle\prod_{r=1}^{l}\theta(a_r;p), \tag{2.6b}$$

$$\theta(a_1,\ldots,a_l;p;q)_m = \textstyle\prod_{r=1}^{l}\theta(a_r;p;q)_m. \tag{2.6c}$$

Main object of study in this section is the following Selberg-type integral built of elliptic gamma functions. The integrand is of the form

$$\Delta_{n,m}(\mathbf{z};\mathbf{t}) = \frac{1}{(2\pi i)^n} \prod_{1\le j<k\le n} \Gamma^{-1}(z_jz_k, z_jz_k^{-1}, z_j^{-1}z_k, z_j^{-1}z_k^{-1};p,q) \times \prod_{j=1}^{n} \frac{\prod_{r=1}^{m}\Gamma(t_rz_j, t_rz_j^{-1};p,q)}{\Gamma(z_j^2, z_j^{-2}, z_j\prod_{r=1}^{m}t_r, z_j^{-1}\prod_{r=1}^{m}t_r;p,q)}, \tag{2.7}$$

where $z_1,\ldots,z_n$ and $t_1,\ldots,t_m$ denote respectively the integration variables and parameters of the integral. Our canonical integration domain is over a torus formed by the n-fold product of the unit circle $|z|=1$, with parameters taken inside the open punctured unit disc $0<|t|<1$. The following residue theorem describes the behavior of the integral with respect to a small deformation of this torus and the parameter domain. For our purposes it suffices to restrict to the case that $m=2n+3$.

Theorem 1 (Residue Formula). *Let $\Delta_{n,2n+3}(\mathbf{z};\mathbf{t})$ be of the form in Equation (2.7) and let t_0 be an auxiliary dependent parameter determined by the balancing condition $q^{-1}\prod_{r=0}^{2n+3}t_r=1$. Then, for parameters such that $|t_1|,\ldots,|t_n|>1>|t_{n+1}|,\ldots,|t_{2n+3}|>0$, with generic argument values in the sense that $\#\{\arg(t_r),\arg(t_r^{-1}) \mid r=0,\ldots,2n+3\}=4n+8$, and with $0<q<1$ and $0<p<\min\{|t_0^{-1}|,\ldots,|t_n^{-1}|\}$, we have that*

$$\int_{C_t^n}\Delta_{n,2n+3}(\mathbf{z};\mathbf{t})\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} = \tag{2.8}$$

$$\sum_{J\subset\{1,\ldots,n\}} 2^{c(J)}c(J)! \sum_{\substack{\lambda_j\ge 0 \text{ s.t. } |t_jq^{\lambda_j}|>1\\ j\in J}} \int_{T^{n-c(J)}} \mu_J(\mathbf{z};\lambda;\mathbf{t})\frac{dz_1}{z_1}\cdots\frac{dz_{n-c(J)}}{z_{n-c(J)}},$$

where

$$\mu_J(\mathbf{z};\lambda;\mathbf{t}) = \kappa_J\,\nu_J(\lambda)\,\delta_J(\lambda,\mathbf{z})\,\Delta_{n-c(J),2n+3}(\mathbf{z};\mathbf{t}),$$

with

$$\kappa_J = \frac{1}{(p;p)_\infty^{c(J)}(q;q)_\infty^{c(J)}} \prod_{\substack{j,k\in J\\ j<k}} \frac{\Gamma(t_jt_k;p,q)}{\Gamma(t_j^{-1}t_k^{-1};p,q)} \times \prod_{j\in J}\frac{\prod_{\substack{1\le r\le 2n+3\\ r\notin J}}\Gamma(t_rt_j, t_rt_j^{-1};p,q)}{\Gamma(t_j^{-2}, t_j\prod_{r=1}^{2n+3}t_r, t_j^{-1}\prod_{r=1}^{2n+3}t_r;p,q)},$$

$$\nu_J(\lambda) = q^{\sum_{j\in J}n_j(J)\lambda_j} \prod_{\substack{j,k\in J\\ j<k}} \frac{\theta(t_jt_kq^{\lambda_j+\lambda_k}, t_jt_k^{-1}q^{\lambda_j-\lambda_k};p)}{\theta(t_kt_j, t_kt_j^{-1};p)} \times \prod_{j\in J}\left(\frac{\theta(t_j^2q^{2\lambda_j};p)}{\theta(t_j^2;p)}\prod_{r=0}^{2n+3}\frac{\theta(t_rt_j;p;q)_{\lambda_j}}{\theta(qt_r^{-1}t_j;p;q)_{\lambda_j}}\right),$$

and

$$\delta_J(\lambda,\mathbf{z}) = \prod_{\substack{j\in J\\ 1\le k\le n-c(J)}} \theta(t_j^{-1}z_k, t_j^{-1}z_k^{-1};q)\,\theta(t_jq^{\lambda_j}z_k, t_jq^{\lambda_j}z_k^{-1};p).$$

Here $c(J)$ denotes the cardinality of $J \subset \{1,\ldots,n\}$ and $n_j(J)$ counts the position of j in J when the elements are ordered from small to large. Furthermore, the integration contour T represents the unit circle with positive orientation and the contour $C_{\mathbf{t}}$ denotes a smooth positively oriented Jordan curve around zero such that (i) every half-line parting from zero intersects $C_{\mathbf{t}}$ just once (i.e. the interior is star-shaped), (ii) $C_{\mathbf{t}}^{-1} := \{z \in \mathbb{C}_{\mathbf{t}} \mid z^{-1} \in C_{\mathbf{t}}\} = C_{\mathbf{t}}$ (i.e. the contour respects the symmetry with respect to inversion), (iii) $C_{\mathbf{t}}$ separates the poles in z_j at $\{t_r p^l q^{l'}\}_{l,l'\in\mathbb{N}}$, $r = 1,\ldots,2n+3$, and at $\{p^{l+1}q^{l'+1}\prod_{s=1}^{2n+3} t_s^{-1}\}_{l,l'\in\mathbb{N}}$ (all in the interior of $C_{\mathbf{t}}$) from those related these by inversion (all in the exterior of $C_{\mathbf{t}}$).

Proof. With the aid of the reflection equation (2.3b) for the elliptic gamma function the integrand $\Delta_{n,m}(\mathbf{z};\mathbf{t})$ (2.7) can be rewritten as

$$\Delta_{n,m}(\mathbf{z};\mathbf{t}) = \frac{1}{(2\pi i)^n} \prod_{1\leq j<k\leq n} \theta(z_j z_k, z_j z_k^{-1}; p)\,\theta(z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q) \times \prod_{j=1}^{n} \frac{\theta(z_j^2; p)\,\theta(z_j^{-2}; q) \prod_{r=1}^{m} \Gamma(t_r z_j, t_r z_j^{-1}; p,q)}{\Gamma(z_j \prod_{s=1}^{m} t_s, z_j^{-1}\prod_{s=1}^{m} t_s; p,q)}.$$

From this expression it is manifest that, as a function of z_j, the integrand has poles at $\{t_r p^l q^{l'}\}_{l,l'\in\mathbb{N}}$ $(r = 1,\ldots,m)$, at $\{p^{l+1}q^{l'+1}\prod_{s=1}^{m} t_s^{-1}\}_{l,l'\in\mathbb{N}}$, and at the points related to these by inversion (as a consequence of the $z_j \to z_j^{-1}$ reflection-invariance of the integrand). By deforming the integration contour for z_j from $C_{\mathbf{t}}$ to T (without destroying the contour's $z_j \to z_j^{-1}$ symmetry) we cross over the poles at $z_j = t_r q^l$ (leaving the interior) and at $z_j = t_r^{-1} q^{-l}$ (entering the interior), where $r = 1,\ldots,n$ and $l \in \mathbb{N}$ such that $|t_r q^l| > 1$. The conditions on the parameters guarantee that these poles are simple and, furthermore, that the remaining poles lie outside the symmetric difference of $\mathrm{Int}(C_{\mathbf{t}})$ and $\mathrm{Int}(T)$. A straightforward residue computation moreover entails the recurrence

$$\begin{aligned} \mathrm{Res}_{z_{n-j}=t_{j+1}q^{\lambda_{j+1}}}\{\mu_{\{1,\ldots,j\}}(\mathbf{z};\lambda;\mathbf{t})\} &= -\mathrm{Res}_{z_{n-j}=t_{j+1}^{-1}q^{-\lambda_{j+1}}}\{\mu_{\{1,\ldots,j\}}(\mathbf{z};\lambda;\mathbf{t})\} \\ &= \mu_{\{1,\ldots,j+1\}}(\mathbf{z};\lambda;\mathbf{t}). \end{aligned} \tag{2.9}$$

The theorem now follows by subsequent deformation of the cycles of the integral $\int \Delta_{n,2n+3}(\mathbf{z};\mathbf{t})\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$ in the variables $z_n,\ldots,z_1$ from $C_{\mathbf{t}}$ to T, upon iterated application of the Cauchy residue theorem and the residue evaluation (2.9), while exploiting the permutation-invariance in the variables $z_1,\ldots,z_n$. □

Remark 1. The combinatorial factor $2^{c(J)}c(J)!$ in the residue formula of Theorem 1 stems from the $S_n \ltimes \mathbb{Z}_2^n$ Weyl-group symmetry of the integral. (Here the group S_n acts on the variables $z_1,\ldots,z_n$ by permutation and the $\mathbb{Z}_2$-action corresponds to the inversion $z_j \to z_j^{-1}$.) This combinatorial factor decomposes as the product of $c(J)!$ (the number of ways to order the integration variables of the cycles from which residues are picked up) and $2^{c(J)}$ (originating from the $z_j \to z_j^{-1}$ reflection-invariance, which implies that each time a residue is picked up the cycle actually moves over a *pair* of poles with opposite residue: one entering and one leaving the interior of the contour).

3. Integral Representation for Very-Well-Poised Balanced Multiple Modular Hypergeometric Series

An important consequence of the residue formula of Theorem 1 is the following integral representation for a terminating multiple modular hypergeometric series of Milne-Gustafson type.

Theorem 2 (Integral Representation). *Let p, q and t_r $(r = 1, \ldots, 2n+3)$ be parameters in the domain determined by the conditions in Theorem 1 such that $q^{-N_j} < |t_j| < q^{-N_j-1}$ with $N_j \in \mathbb{N}$ for $j = 1, \ldots, n$. Then, by letting t_{n+j} tend to $t_j^{-1}q^{-N_j}$ for $j = 1, \ldots, n$, while simultaneously deforming the contour $C_{\mathbf{t}}$ so as to maintain the conditions (i)-(iii) in Theorem 1 satisfied, the residue formula goes over into the integral representation*

$$\begin{aligned}
&\lim_{\substack{t_{n+j}\to t_j^{-1}q^{-N_j}\\ j=1,\ldots,n}} \sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\ldots,n}} q^{\sum_{j=1}^n j\lambda_j} \prod_{1\le j<k\le n} \frac{\theta(t_jt_kq^{\lambda_j+\lambda_k}, t_jt_k^{-1}q^{\lambda_j-\lambda_k};p)}{\theta(t_jt_k, t_jt_k^{-1};p)} \\
&\qquad \times \prod_{1\le j\le n}\left(\frac{\theta(t_j^2q^{2\lambda_j};p)}{\theta(t_j^2;p)} \prod_{0\le r\le 2n+3} \frac{\theta(t_jt_r;p;q)_{\lambda_j}}{\theta(qt_jt_r^{-1};p;q)_{\lambda_j}}\right) \qquad (3.1)\\
&= \lim_{\substack{t_{n+j}\to t_j^{-1}q^{-N_j}\\ j=1,\ldots,n}} \frac{1}{2^n\, n!\, \kappa} \int_{C_{\mathbf{t}}^n} \Delta_{n,2n+3}(\mathbf{z};\mathbf{t}) \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n},
\end{aligned}$$

where

$$\begin{aligned}
\kappa &= \frac{1}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{1\le j<k\le n} \frac{\Gamma(t_jt_k;p,q)}{\Gamma(t_j^{-1}t_k^{-1};p,q)} \\
&\quad \times \prod_{j=1}^n \frac{\prod_{r=n+1}^{2n+3}\Gamma(t_rt_j, t_rt_j^{-1};p,q)}{\Gamma(t_j^{-2}, t_j\prod_{r=1}^{2n+3}t_r, t_j^{-1}\prod_{r=1}^{2n+3}t_r;p,q)}
\end{aligned}$$

and t_0 is determined from the balancing relation $q^{-1}\prod_{r=0}^{2n+3} t_r = 1$.

Proof. Division of the residue formula (2.8) by $2^n n! \kappa_{\{1,\ldots,n\}}$ and letting t_{n+j} tend to $t_j^{-1}q^{-N_j}$, for $j = 1, \ldots, n$, immediately entails the stated integral representation. Indeed, we have that

$$\lim_{\substack{t_{n+j}\to t_j^{-1}q^{-N_j}\\ j=1,\ldots,n}} \frac{\kappa_J}{\kappa_{\{1,\ldots,n\}}} = 0$$

for $J \subsetneq \{1, \ldots, n\}$, due to the pole of the elliptic gamma function at negative integral powers of q. This implies that, in the limit, we pick up the term corresponding to $J = \{1, \ldots, n\}$ from the residue formula. □

Remark 2. For $n = 1$ the integral representation (3.1) reduces to

$$\begin{aligned}
&\lim_{t_2\to t_1^{-1}q^{-N}} \sum_{\lambda=0}^{N} q^\lambda \frac{\theta(t_1^2q^{2\lambda};p)}{\theta(t_1^2;p)} \prod_{r=0}^{5} \frac{\theta(t_1t_r;p;q)_\lambda}{\theta(qt_1t_r^{-1};p;q)_\lambda} = \qquad (3.2)\\
&\qquad \lim_{t_2\to t_1^{-1}q^{-N}} \frac{1}{4\pi i\kappa} \int_{C_{\mathbf{t}}} \frac{\prod_{r=1}^5 \Gamma(zt_r, z^{-1}t_r;p,q)}{\Gamma(z^2, z^{-2}, z\prod_{r=1}^5 t_r, z^{-1}\prod_{r=1}^5 t_r;p,q)} \frac{dz}{z},
\end{aligned}$$

where

$$\kappa = \frac{\prod_{r=2}^{5}\Gamma(t_rt_1, t_rt_1^{-1}; p, q)}{(q;q)_\infty (p;p)_\infty \Gamma(t_1^{-2}, t_1\prod_{r=1}^{5} t_r, t_1^{-1}\prod_{r=1}^{5} t_r; p, q)}$$

and t_0 is determined from the balancing relation $q^{-1}\prod_{r=0}^{5} t_r = 1$.

4. A Modular Hypergeometric Summation Formula

From Ref. [DS2] we have the following conjecture for the evaluation of our elliptic Selberg integral.

Conjecture (Elliptic Selberg Integral). Let $|p|$, $|q|$ and $|t_r|$ (with $r = 1, \ldots, 2n+3$) be smaller than 1 such that $|pq| < |\prod_{r=1}^{2n+3} t_r|$. Then we have that

$$\int_{T^n} \Delta_{n,2n+3}(\mathbf{z}; \mathbf{t}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{1\le r<s\le 2n+3}\Gamma(t_rt_s; p, q)}{\prod_{r=1}^{2n+3}\Gamma(t_r^{-1}\prod_{s=1}^{2n+3} t_s; p, q)}. \tag{4.1}$$

For $n = 1$ this conjecture reduces to the elliptic beta integral proven by one of us in [Sp]; for $p = 0$ and arbitrary n the integral is due to Gustafson [G2, Theorem 4.1]. When combined with the integral represenation of Theorem 2, the integration formula (4.1) leads to the following summation formula for a multiple modular hypergeometric series of Milne-Gustafson type.

Theorem 3 (Modular Milne-Gustafson type Sum). *Let $|p| < 1$ and let q be generic such that it is not an integral power of p. Then the integration formula* (4.1) *implies that*

$$\sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\ldots,n}} q^{\sum_{j=1}^n j\lambda_j} \prod_{1\le j<k\le n} \frac{\theta(t_jt_kq^{\lambda_j+\lambda_k}, t_jt_k^{-1}q^{\lambda_j-\lambda_k}; p)}{\theta(t_jt_k, t_jt_k^{-1}; p)} \tag{4.2}$$

$$\times \prod_{1\le j\le n}\left(\frac{\theta(t_j^2q^{2\lambda_j}; p)}{\theta(t_j^2; p)} \prod_{0\le r\le 2n+3} \frac{\theta(t_jt_r; p; q)_{\lambda_j}}{\theta(qt_jt_r^{-1}; p; q)_{\lambda_j}}\right)$$

$$= \theta(qa^{-1}b^{-1}, qa^{-1}c^{-1}, qb^{-1}c^{-1}; p; q)_{N_1+\cdots+N_n}$$

$$\times \prod_{1\le j<k\le n} \frac{\theta(qt_jt_k; p; q)_{N_j}\theta(qt_jt_k; p; q)_{N_k}}{\theta(qt_jt_k; p; q)_{N_j+N_k}}$$

$$\times \prod_{1\le j\le n} \frac{\theta(qt_j^2; p; q)_{N_j}}{\theta(qt_ja^{-1}, qt_jb^{-1}, qt_jc^{-1}, q^{1+N_1+\cdots+N_n-N_j}t_j^{-1}a^{-1}b^{-1}c^{-1}; p; q)_{N_j}}$$

as a meromorphic identity in the parameters $t_0, \ldots, t_{2n+3}$ subject to the relations

$$q^{-1}\prod_{r=0}^{2n+3} t_r = 1, \qquad \textit{(balancing condition)},$$
$$q^{N_j}t_jt_{n+j} = 1, \qquad j = 1, \ldots, n, \qquad \textit{(truncation conditions)},$$

where $N_j \in \mathbb{N}$ ($j = 1, \ldots, n$) and $a \equiv t_{2n+1}$, $b \equiv t_{2n+2}$, $c \equiv t_{2n+3}$.

Proof. Evaluation of the r.h.s. of integral representation (3.1) by means of the integration formula (4.1)—upon deformation of the integration contour from T to

$C_{\mathbf{t}}$ (cf. Remark 3 below)—entails the following expression for the value for the multiple modular hypergeometric series:

$$\lim_{\substack{t_{n+j}\to t_j^{-1}q^{-N_j}\\ j=1,\dots,n}} \left(\frac{\prod_{n+1\le r<s\le 2n+3}\Gamma(t_rt_s;p,q)\prod_{1\le r\le s\le n}\Gamma(t_r^{-1}t_s^{-1};p,q)}{\prod_{\substack{1\le r\le n\\ n+1\le s\le 2n+3}}\Gamma(t_r^{-1}t_s;p,q)} \times \frac{\prod_{1\le r\le n}\Gamma(t_r\prod_{s=1}^{2n+3}t_s;p,q)}{\prod_{n+1\le r\le 2n+3}\Gamma(t_r^{-1}\prod_{s=1}^{2n+3}t_s;p,q)}\right).$$

This expression simplifies to the product formula on the r.h.s. of the stated summation formula (4.2) upon conversion to the elliptic Pochhammer symbols (2.5). □

For $n=1$ the summation formula in (4.2) simplifies to

$$\sum_{\lambda=0}^{N} q^{\lambda}\frac{\theta(t_1^2q^{2\lambda};p)}{\theta(t_1^2;p)}\prod_{r=0}^{5}\frac{\theta(t_1t_r;p;q)_\lambda}{\theta(qt_1t_r^{-1};p;q)_\lambda} = \frac{\theta(qt_1^2;p;q)_N\prod_{3\le r<s\le 5}\theta(qt_r^{-1}t_s^{-1};p;q)_N}{\theta(qt_1^{-1}t_3^{-1}t_4^{-1}t_5^{-1};p;q)_N\prod_{r=3}^{5}\theta(qt_1t_r^{-1};p;q)_N}, \tag{4.3}$$

with $q^{-1}\prod_{r=0}^{5}t_r=1$ and $q^Nt_1t_2=1$. This $n=1$ sum is due to Frenkel and Turaev [FT]. It constitutes a modular generalization of the celebrated very-well-poised balanced ${}_8\Phi_7$ sum of Jackson [GR].

For $p=0$ and arbitrary n the sum in (4.2) degenerates to the multiple basic hypergeometric summation

$$\begin{aligned}\sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\dots,n}} q^{\sum_{j=1}^n j\lambda_j}&\prod_{1\le j<k\le n}\left(\frac{1-t_jt_kq^{\lambda_j+\lambda_k}}{1-t_jt_k}\frac{1-t_jt_k^{-1}q^{\lambda_j-\lambda_k}}{1-t_jt_k^{-1}}\right)\\ &\times\prod_{1\le j\le n}\left(\frac{1-t_j^2q^{2\lambda_j}}{1-t_j^2}\prod_{0\le r\le 2n+3}\frac{(t_jt_r;q)_{\lambda_j}}{(qt_jt_r^{-1};q)_{\lambda_j}}\right)\\ =(qa^{-1}b^{-1},qa^{-1}c^{-1}&,qb^{-1}c^{-1};q)_{N_1+\dots+N_n}\\ \times&\prod_{1\le j<k\le n}\frac{(qt_jt_k;q)_{N_j}(qt_jt_k;q)_{N_k}}{(qt_jt_k;q)_{N_j+N_k}}\\ \times&\prod_{1\le j\le n}\frac{(qt_j^2;q)_{N_j}}{(qt_ja^{-1},qt_jb^{-1},qt_jc^{-1},q^{1+N_1+\dots+N_n-N_j}t_j^{-1}a^{-1}b^{-1}c^{-1};q)_{N_j}}\end{aligned} \tag{4.4}$$

(with parameters subject to the relations in Theorem 3). This latter sum boils down to the multiple very-well-poised balanced ${}_8\Phi_7$ summation formula of Denis and Gustafson [DG, Theorem 4.1].

Remark 3. If we assume that $0<p,q<1$ and that $t_1,\dots,t_{2n+3}$ are generic such that $\#\{\arg(t_r),\arg(t_r^{-1})\mid r=0,\dots,2n+3\}=4n+8$ (where t_0 is determined from the balancing condition $q^{-1}\prod_{r=0}^{2n+3}t_r=1$), then we may deform the contour in the integration formula (4.1) from the unit circle T to any (smooth) positively oriented Jordan curve $C_{\mathbf{t}}\subset\mathbb{C}$ around zero such that (i) the interior is star shaped around the origin: every half-line parting from zero intersects $C_{\mathbf{t}}$ just *once*, (ii) $C_{\mathbf{t}}^{-1}:=\{z\in\mathbb{C}\mid z^{-1}\in C_{\mathbf{t}}\}=C_{\mathbf{t}}$, and (iii) $C_{\mathbf{t}}$ separates the poles in z_j at $\{t_rp^lq^{l'}\}_{l,l'\in\mathbb{N}}$ ($r=1,\dots,2n+3$) and $\{p^{l+1}q^{l'+1}\prod_{s=1}^{2n+3}t_s^{-1}\}_{l,l'\in\mathbb{N}}$ (all in the interior of $C_{\mathbf{t}}$) from those related to it by inversion (all in the exterior of $C_{\mathbf{t}}$). Indeed, the conditions on $C_{\mathbf{t}}$ guarantee that one does not cross over poles when deforming from T to $C_{\mathbf{t}}$, so the value of the integral remains unchanged. This observation permits an extension of the parameter domain of (4.1) (assuming the above reality and genericity conditions) through analytic continuation. Indeed, we can perform a radial dilation of one or more parameters t_r from the interior of the unit circle to the exterior while simultaneously deforming the integration contour $C_{\mathbf{t}}$ so as to maintain the above conditions (i)–(iii) satisfied.

Remark 4. Let $0 < p, q < 1$ and let $t_1, \ldots, t_{2n+2}$ be complex parameters such that $0 < |t_r| < 1$ for $r = 1, \ldots, 2n+1$ and with $t_{2n+2} = \prod_{r=1}^{2n+1} t_r^{-1}$. Let us furthermore assume that argument values of t_r are generic in the sense that $\#\{\arg(t_r), \arg(t_r^{-1}) \mid r = 1, \ldots, 2n+2\} = 4n+4$. Then the integration formula (4.1) implies that

$$\int_{C_{\mathbf{t}}^n} \prod_{1\leq j<k\leq n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q) \times \prod_{j=1}^{n} \frac{\prod_{r=0}^{2n+1} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}; p, q)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = 0, \tag{4.5}$$

where the contour $C_{\mathbf{t}} \subset \mathbb{C}$ is a positively oriented Jordan curve around zero such that (i) the interior is star shaped around the origin, (ii) $C_{\mathbf{t}}^{-1} := \{z \in \mathbb{C} \mid z^{-1} \in C_{\mathbf{t}}\} = C_{\mathbf{t}}$, and (iii) the points t_r ($r = 0, \ldots, 2n+1$) all lie in the interior of $C_{\mathbf{t}}$. Indeed, we arrive at the formula in (4.5) from the formula in (4.1) by deforming the parameters such that t_{2n+3} tends to $\prod_{r=1}^{2n+3} t_r$ while simultaneously performing a deformation of the integration contour of the type detailed in Remark 3 above. In the limit at issue the r.h.s. of the integration formula (4.1) tends to zero because of the pole of factor $\Gamma(t_{2n+3}^{-1} \prod_{r=1}^{2n+3} t_r; p, q)$ (appearing in the denominator) at $t_{2n+3} = \prod_{r=1}^{2n+3} t_r$. In [DS2] it was shown that, reversely, the conjectural integration formula (4.1) follows from the vanishing of the integral in (4.5). In other words: the vanishing of the elliptic Selberg integral for parameters on the hypersurface $\prod_{r=1}^{2n+2} t_r = 1$ necessarily extends to the conjectural evaluation formula (4.1) on the full parameter space.

5. Modular Invariance and Asymptotics

We conclude by exhibiting the modular properties of the generalized Milne-Gustafson type series of Theorem 3. Let us to this end set

$$p = e^{2\pi i\tau}, \qquad q = e^{2\pi i\sigma}, \tag{5.1}$$

with $\mathrm{Im}(\tau), \mathrm{Im}(\sigma) > 0$, and let us recall the Jacobi theta function [WW]

$$\theta_1(x|\tau) = 2\sum_{m=0}^{\infty} (-1)^n p^{(2m+1)^2/8} \sin \pi(2m+1)x \tag{5.2a}$$

$$= p^{1/8} i e^{-\pi i x} (p;p)_\infty\, \theta(e^{2\pi i x}; p) \tag{5.2b}$$

(where $\theta(z;p)$ refers to the theta function of Equation (2.4)). With the aid of the Jacobi theta function one defines *elliptic numbers* as [D-O1, D-O2, FT]

$$[x; \sigma, \tau] \equiv \frac{\theta_1(\sigma x|\tau)}{\theta_1(\sigma|\tau)} = q^{(1-x)/2} \frac{(q^x, pq^{-x}; p)_\infty}{(q, pq^{-1}; p)_\infty}. \tag{5.3}$$

The modular symmetries

$$\begin{cases} \theta_1(x|\tau+1) = e^{\pi i/4}\theta_1(x|\tau), \\ \theta_1(\frac{x}{\tau}|-\frac{1}{\tau}) = -i\sqrt{-i\tau}e^{\pi i x^2/\tau}\theta_1(x|\tau) \end{cases}$$

and quasi-periodicity relations

$$\begin{cases} \theta_1(x+1|\tau) = -\theta_1(x|\tau), \\ \theta_1(x+\tau|\tau) = -e^{-\pi i\tau - 2\pi i x}\theta_1(x|\tau) \end{cases}$$

of Jacobi's theta function [WW] give rise to the following modular transformation properties of the elliptic numbers $[x;\sigma,\tau]$:

$$\begin{cases} [x;\sigma,\tau+1] = [x;\sigma,\tau], \\ [x;\sigma/\tau,-1/\tau] = [x;\sigma,\tau]e^{\pi i(x^2-1)\sigma/\tau} \end{cases} \tag{5.4a}$$

and

$$\begin{cases} [x;\sigma+1,\tau] = (-1)^{x-1}[x;\sigma,\tau], \\ [x;\sigma+\tau,\tau] = (-1)^{x-1}[x;\sigma,\tau]e^{\pi i(1-x^2)(\tau+2\sigma)}, \end{cases} \tag{5.4b}$$

where in the second pair of identities it is assumed that x is an integer. From the elliptic numbers it is obvious to construct *elliptic shifted factorials*

$$[x;\sigma,\tau]_m = \prod_{j=0}^{m-1}[x+j;\sigma,\tau], \quad [x;\sigma,\tau]_0 = 1, \tag{5.5a}$$

$$[g_1,\ldots,g_l;\sigma,\tau]_m = \prod_{r=1}^{l}[g_r;\sigma,\tau]_m. \tag{5.5b}$$

We are now in the position to introduce a multiple modular analogue of the very-well-poised balanced terminating basic hypergeometric ${}_{l+1}\Phi_l$ series:

$${}_{l+1}\mathcal{E}_l^{(n)}(g_0,\ldots,g_{2n+l-4};\sigma,\tau) = \sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\ldots,n}} {}_{l+1}\nu_l^{(n)}(\lambda;\sigma,\tau), \tag{5.6a}$$

where

$$\begin{aligned} {}_{l+1}\nu_l^{(n)}(\lambda;\sigma,\tau) &= \prod_{1\le j<k\le n}\frac{[g_j+g_k+\lambda_j+\lambda_k, g_j-g_k+\lambda_j-\lambda_k]}{[g_j+g_k, g_j-g_k]} \\ &\times \prod_{1\le j\le n}\left(\frac{[2g_j+2\lambda_j]}{[2g_j]}\prod_{r=0}^{2n+l-4}\frac{[g_j+g_r]_{\lambda_j}}{[1+g_j-g_r]_{\lambda_j}}\right) \end{aligned} \tag{5.6b}$$

and with parameters $g_0,\ldots,g_{2n+l-4}$ subject to the constraints

$$\begin{aligned} &\textstyle\sum_{r=0}^{2n+l-4} g_r - (l-5)/2 = 0, && && \text{(balancing condition)}, \\ &g_j + g_{n+j} + N_j = 0, && j=1,\ldots,n, && \text{(truncation conditions)}. \end{aligned} \tag{5.6c}$$

To ease the notation of the modular hypergeometric series, we suppressed here the explicit dependence of the elliptic numbers $[x] \equiv [x;\sigma,\tau]$ on the modular parameters σ,τ. For $\tau \to +i\infty$ (i.e. $p \to 0$), the series in Equation (5.6a) reduces to a Milne-Gustafson type very-well-poised multiple basic hypergeometric series associated to the symplectic Lie group $Sp(n,\mathbb{C})$ (i.e. of Type C) [Mi, G1, DG, LM, ML, K]. For $n=1$ and generic τ in the upper half-plane (i.e. $|p|<1$) the series reduces to the standard modular hypergeometric series of Refs. [D-O1, D-O2, FT].

The modular symmetries (5.4a), (5.4b) give rise to the following invariance of the multiple modular hypergeometric series with respect to the action of $SL_2(\mathbb{Z})$ on the modular parameters σ, τ. For $n=1$ this modular invariance is due to Frenkel and Turaev [FT].

Theorem 4 (Modularity). (i). *For generic values of the parameters* $g_0, \ldots, g_{2n+l-4}$ *subject to the balancing and truncation conditions in* (5.6c), *the multiple modular hypergeometric series* ${}_{l+1}\mathcal{E}_l^{(n)}$ (5.6a) *is invariant with respect to the natural action of the group* $SL_2(\mathbb{Z})$ *on the modular parameters* (σ, τ):

$$ {}_{l+1}\mathcal{E}_l^{(n)}\left(\frac{\sigma}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = {}_{l+1}\mathcal{E}_l^{(n)}(\sigma,\tau), \tag{5.7a} $$

where $a,b,c,d \in \mathbb{Z}$ *such that* $ad-bc=1$.

(ii). *If, in addition, the parameters* $g_0, \ldots, g_{2n+l-4}$ *are also integer-valued, then* ${}_{l+1}\mathcal{E}_l^{(n)}(\sigma,\tau)$ *is elliptic in* σ:

$$ {}_{l+1}\mathcal{E}_l^{(n)}(\sigma+k+m\tau,\tau) = {}_{l+1}\mathcal{E}_l^{(n)}(\sigma,\tau) \qquad (k,m\in\mathbb{Z}). \tag{5.7b} $$

Proof. Part *(i)* follows from the modular symmetries in Equation (5.4a) and the fact that the difference between the sums of the squares of the arguments of the elliptic numbers in the numerator and denominator of the terms ${}_{l+1}\nu_l^{(n)}(\lambda;\sigma,\tau)$ (5.6b), which is given by

$$ 2\Big(\sum_{r=0}^{2n+l-4} g_r - (l-5)/2\Big)\sum_{j=1}^{n}\lambda_j(\lambda_j+2g_j), $$

vanishes as a consequence of the balancing condition in Equation (5.6c). Part *(ii)* now follows similarly from the quasi-periodicity relations (5.4b) and the observation that the difference between the sums of the arguments of the elliptic numbers in the numerator and denominator of the terms ${}_{l+1}\nu_l^{(n)}(\lambda;\sigma,\tau)$, given by

$$ 2\sum_{j=1}^{n}\lambda_j\Big(\sum_{r=0}^{2n+l-4} g_r - (l-5)/2 - j\Big), $$

amounts to $-2\sum_{j=1}^{n} j\lambda_j = 0 \mod 2$ (upon once more invoking of the balancing condition). □

After setting $t_r = q^{g_r}$, $r=0,\ldots,2n+3$, and picking p,q from Equation (5.1), the summation formula (4.2) admits recasting as the modular hypergeometric evaluation

$$ \begin{aligned} &{}_8\mathcal{E}_7^{(n)}(g_0,\ldots,g_{2n+3};\sigma,\tau) \\ &= [1-g_a-g_b, 1-g_a-g_c, 1-g_b-g_c]_{N_1+\cdots+N_n} \\ &\quad\times \prod_{1\le j<k\le n}\frac{[1+g_j+g_k]_{N_j}[1+g_j+g_k]_{N_k}}{[1+g_j+g_k]_{N_j+N_k}} \\ &\quad\times \prod_{1\le j\le n}\frac{[1+2g_j]_{N_j}}{[1+\sum_{k=1}^{n}N_k - N_j - g_j - g_a - g_b - g_c]_{N_j}\prod_{r\in\{a,b,c\}}[1+g_j-g_r]}, \end{aligned} \tag{5.8} $$

with

$$ \begin{aligned} &\textstyle\sum_{r=0}^{2n+3} g_r = 1, && \text{(balancing condition)},\\ &g_j+g_{n+j}+N_j=0,\quad j=1,\ldots,n, && \text{(truncation conditions)}. \end{aligned} $$

Theorem 4 states that the series ${}_{l+1}\mathcal{E}_l^{(n)}$ constitutes a Jacobi modular function on $SL_2(\mathbb{Z})$ in the sense of Eichler-Zagier [EZ]. This modular invariance permits us to independently deduce the asymptotic validity of the evaluation formula (5.8) (and thus the summation formula (4.2)) for low orders in σ.

Theorem 5 (Asymptotics for $\sigma \to 0$). *Let $g_0, \ldots, g_{2n+3}$ be generic parameters subject to the balancing and truncation conditions in* (5.6c) *with $l = 7$. Then the Taylor expansion in σ of the difference between both sides of the multiple modular hypergeometric summation formula* (5.8), *around the point $\sigma = 0$, vanishes (at least) up to the terms of order σ^{10}.*

Proof. By checking that the difference between the sums of the squares of the arguments of the elliptic numbers in the numerator and denominator of the r.h.s. of (5.8) is zero, one checks independently that the r.h.s. is also invariant with respect to the action of $SL_2(\mathbb{Z})$ (cf. the proof of Theorem 4). The modular invariance of Equation (5.8) implies that the difference between the l.h.s. and r.h.s. has a Taylor expansion around $\sigma = 0$ of the form

$$\sum_{m \geq 0} c_m(\tau)\sigma^{2m},$$

where $c_m(\tau)$ is a modular form of weight $2m$ on $SL_2(\mathbb{Z})$ (i.e., $c_m(\tau)$ is holomorphic on the upper half plane and $c_m(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^{2m} c_m(\tau)$ for $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$). (Notice that only the even terms in σ are nonzero because $[x; \sigma, \tau] = [x; -\sigma, \tau]$.) Furthermore, since in the limit $\mathrm{Im}(\tau) \to +\infty$ both sides of (5.8) are equal in view of the Denis-Gustafson sum (4.4), we conclude that $\lim_{\tau \to +i\infty} c_m(\tau) = 0$, i.e. $c_m(\tau)$ is a cusp form. Since no nontrivial cusp forms exist with weight below 12 [Se], it follows that the first nonzero term in the above Taylor expansion cannot appear before degree 12. □

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