Projectively equivalent Riemannian metrics on the torus

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Abstract

Let Riemannian metrics g and $\bar{g}$ on M^n have the same geodesics, and suppose the eigenvalues of one metric with respect to the other are different at least at one point. We show that then the first Betti number $b_1(M^n)$ is no greater than n, and that if there exists a point where the eigenvalues of one metric with respect to the other are not all different then the first Betti number $b_1(M^n)$ is less than n. In particular, if M^n is homeomorphic to the torus T^n then the eigenvalues of one metric with respect to the other are different at each point. This allows us to classify such metrics on the torus.

1 Introduction

1.1 Metrics with the same geodesics

Definition 1. Two metrics g and $\bar{g}$ on M^n are called *projectively equivalent*, if they have the same geodesics considered as unparameterised curves. The metrics g and $\bar{g}$ are said to be *strictly non-proportional* at $x \in M^n$, if the eigenvalues of g with respect to $\bar{g}$ are all different at x.

Projectively equivalent metrics is a very classical material. In 1865, Beltrami [1] found the first examples of projectively equivalent metrics and formulated a problem of finding all pairs of projectively equivalent metrics. Locally, near the points where the eigenvalues of one metric with respect to the other do not bifurcate, this problem has been solved by Dini [3] for surfaces and Levi-Civita [6] for manifolds of arbitrary dimension. Later, projectively equivalent metrics were considered by Weyl, Eisenhart, E. Cartan, Thomas, Lichnerowicz, Venzi, Voss, Pogorelov, Mikes, Aminova, Sinjukov, Solodovnikov. They found a lot

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of beautiful tensor properties of projectively equivalent metric, see the review paper [14] for details.

However, the global behaviour of projectively equivalent metrics is not understood completely. Most known global results on projectively equivalent metrics require additional strong geometrical assumptions. For example, for Einstein or (hyper)Kahlerian metrics beautiful results were obtained by Lichnerowicz [7], Venzi [17], Mikes [14] and Hasegawa and Fujimura [4].

1.2 Results

Theorem 1. *Suppose M^n is a connected closed manifold. Let Riemannian metrics g and $\bar g$ on M^n be projectively equivalent and strictly non-proportional at least at one point. Then:*

1. *The first Betti number $b_1(M^n)$ is no greater then n.*
2. *If in addition there exists a point where the metrics are not strictly non-proportional then the first Betti number $b_1(M^n)$ is less then n.*

The first Betti number $b_1(T^n)$ of the $n-$torus is precisely n.

Corollary 1. *Let Riemannian metrics g and $\bar g$ on T^n be projectively equivalent and strictly non-proportional at least at one point. Then they are strictly non-proportional at each point.*

As it has been shown in [12], the converse of Corollary 1 is also true:

Theorem 2 ([12]). *Let M^n be closed connected. Let g, $\bar g$ on M^n be projectively equivalent. Suppose they are strictly non-proportional at each point of the manifold. Then the manifold can be covered by the torus.*

In Section 5 we use Corollary 1 to to describe (Theorem 7) and, in a certain sense, to classify (Theorem 8) all projectively equivalent Riemannian metrics on the torus, which are strictly non-proportional at least at one point. It is the first classification result on projectively equivalent metrics on closed $n-$dimensional manifold. Recall that, for surfaces, in view of two-dimensional version of Theorem 3 proved in [10], the classification of projectively equivalent metrics follows immediately from the classification of quadratically-integrable geodesic flows obtained in [2, 5].

In Section 6, we will generalise Corollary 1 for the three-dimensional case: we will show that the number of different eigenvalues of one metric with respect to the other is constant for projectively equivalent metrics on the three-torus, see Theorem 9.

1.3 Methods and ideas of proofs

The new technique that allows us to prove Theorem 1 came from theory of integrable geodesic flows. The connection between projectively equivalent metrics and integrable geodesic flows is established by the following theorem.

Consider the (1,1)-tensor field L given by the formula

$$L^i_j \stackrel{\text{def}}{=} \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j} \tag{1}$$

Theorem 3 ([8, 16]). *Let Riemannian metrics g, $\bar g$ be projectively equivalent. For any $t \in R$, consider the $(1,1)$- tensor field*

$$S_t \stackrel{\text{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}. \tag{2}$$

Let us identify the tangent and cotangent bundles of M^n by g. Consider the standard Poisson structure on T^*M^n. Then for any t_1, t_2, the functions

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(\xi) \stackrel{\text{def}}{=} g(S_{t_i}(\xi), \xi) \tag{3}$$

are commuting integrals for the geodesic flow of g.

Remark 1. *Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor field S_t, and, therefore, the function I_t, is well-defined for any t. Moreover, as it will be clear from Section 2, S_t is a polynomial (in t) of degree $n-1$ with coefficients being (1,1)-tensor fields.*

In Section 2, we will show (Corollary 2) that if the metrics are strictly non-proportional at one point of a connected complete manifold then it is so at almost each point.

If the metric are real-analytic, the first statement of Theorem 1 already follows from [15]:

Theorem 4 (Taimanov, [15]). *If a real-analytic closed manifold M^n with a real-analytic metric satisfies at least one of the conditions:*

a) $\pi_1(M^n)$ is not almost commutative

b) $dim H_1(M^n; \mathbf{Q}) > dim M^n$,

then the geodesic flow on M^n is not analytically integrable.

The first statement of Theorem 1 follows directly from the second statement and Theorem 2; for self-containedness, we will prove Theorem 2 in Section 3.

The idea using in the proof of the second statement of Theorem 1 is borrowed from [15].

We will show that each element of the fundamental group $\pi_1(M^n)$ can be realised on one of a finite number of subsets of M^n; each of these subsets has the first Betti number less than n; then the first homology group $H_1(M^n; Z)$ is the unity of a finite number of commutative subgroups of rank less than n; then the rank of $H_1(M^n; Z)$ (which is precisely the first Betti number of M^n) must be less than n.

The subsets are given in the terms of the eigenvalues of the tensor (1); in Section 2 we show that they are globally ordered (Theorem 5) which, together with classical Levi-Civita's theorem (Theorem 6), guaranties that the subsets are well-definite.

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2 The eigenvalues of L

Let g and $\bar{g}$ be projectively equivalent Riemannian metrics on complete connected M^n. Consider the tensor L given by (1). The main goal of this section is to prove the following theorem: for each point $x \in M^n$, denote by

$$\lambda_1(x) \leq \lambda_2(x) \leq ... \leq \lambda_n(x)$$

the eigenvalues of L at the point x.

Theorem 5. *For any $i \in \{1, ..., n-1\}$, for any $x, y \in M^n$:*

1. *$\lambda_i(x) \leq \lambda_{i+1}(y)$.*
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$ then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

This theorem has been announced in [12]; for three-dimensional manifolds, the theorem has been proven in [13].

Corollary 2. *If the eigenvalues of L are all different at one point of M^n then they are all different at almost each point of M^n.*

Corollary 2 was announced in [9] and proved by a different method in [11].
Proof of Theorem 5: By definition, the tensor L is self-adjoint. Then, for any $x \in M^n$, there exists a basis in T_xM^n such that the metric g is given by the matrix $\mathrm{diag}(1, 1, ..., 1)$ and the tensor L is given by the matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. Then the tensor (2) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(P_1(t), P_2(t), ..., P_n(t)), \end{aligned}$$

where the polynomials $P_i(t)$ are given by the formula

$$P_i(t) \stackrel{\mathrm{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t).$$

Then, for any fixed $\xi = (\xi_1, \xi_2, ..., \xi_n) \in T_xM^n$, the function (3) is the following polynomial in t:

$$I_t = \xi_1^2 P_1(t) + \xi_2^2 P_2(t) + ... + \xi_n^2 P_n(t). \tag{4}$$

Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x, \xi) \leq t_2(x, \xi) \leq ... \leq t_{n-1}(x, \xi).$$

Lemma 1.

1. *For any* $\xi \in T_x M^n$,

$$\lambda_i(x) \leq t_i(x,\xi) \leq \lambda_{i+1}(x).$$

In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$ *then* $t_i(x,\xi) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. *If* $\lambda_i(x) < \lambda_{i+1}(x)$ *then for any constant* τ *the Lebesgue measure of the set*

$$V_\tau \subset T_x M^n, \quad V_\tau \stackrel{\text{def}}{=} \{\xi \in T_x M^n : t_i(x,\xi) = \tau\},$$

is zero.

Proof of Lemma 1: Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that ξ_i are non-zero. For any $\alpha \neq i$, we evidently have $P_\alpha(\lambda_i) \equiv 0$. Then

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \xi_\alpha^2 P_\alpha(\lambda_i) = \xi_i^2 P_i(\lambda_i).$$

Hence I_{λ_i} and $I_{\lambda_{i+1}}$ have different signs and, therefore, the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial I_t. The degree of the polynomial I_t equals $n-1$; we have $n-1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of Lemma 1. Suppose $\lambda_i < \lambda_{i+1}$. Let first $\lambda_i < \tau < \lambda_{i+1}$. Then the set

$$V_\tau \stackrel{\text{def}}{=} \{\xi \in T_x M^n : t_i(x,\xi) = \tau\},$$

consists of the points ξ where the function $I_\tau(x,\xi) \stackrel{\text{def}}{=} (I_t(x,\xi))_{|t=\tau}$ is zero; then it is a quadric in $T_x M^n \equiv R^n$ and its measure is zero.

Let τ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i. Then any coefficient $P_\alpha(t)$ of the quadratic form (4) has a factor $(\lambda_i - t)^{k-1}$. Therefore, for any fixed $\xi \in T_x M^n$, the function

$$\tilde{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is well-definite and is a polynomial in t so that $\tilde{I}_\tau$ is a nontrivial quadratic form. Evidently, for any point $\xi \in V_\tau$, we have $\tilde{I}_\tau(\xi) = 0$ so that the set V_τ is a subset of a quadric in $T_x M^n$ and its measure is zero. **Lemma 1 is proved.**

The first statement of Theorem 5 follows immediately from the first statement of Lemma 1: let us join the points $x, y \in M^n$ by a geodesic

$$\gamma : R \to M^n, \quad \gamma(0) = x, \quad \gamma(1) = y.$$

Consider the one-parametric family of integrals $I_t(x,\xi)$ and the roots

$$t_1(x,\xi) \le t_2(x,\xi) \le ... \le t_{n-1}(x,\xi).$$

By Corollary 3, each root t_i is constant on each orbit $(\gamma, \dot\gamma)$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \le t_i(\gamma(0), \dot\gamma(0)) \quad \text{and} \quad t_i(\gamma(1), \dot\gamma(1)) \le \lambda_{i+1}(\gamma(1)).$$

Therefore, $\lambda_i(\gamma(0)) \le \lambda_{i+1}(\gamma(1))$. **The first statement of Theorem 5 is proved.**

Let us prove the second statement of Theorem 5. Suppose $\lambda_i(x) < \lambda_{i+1}(x)$. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for any point y of some subset $V \subset U(\gamma(1))$. Then by the first statement of Theorem 5, the value of λ_i is a constant (independent of $y \in V$). Denote this constant by C. Let us prove that $\lambda_i(x) = \lambda_{i+1}(x) = C$. Let us join the point x with every point of V by all possible geodesics. Consider the set $V_C \subset T_xM^n$ of the initial velocity vectors (at the point x) of these geodesics.

By the first statement of Lemma 1, for any geodesic γ_1 passing through any point of V, the value $t_i(\gamma_1, \dot\gamma_1)$ is equal to C. Then, by the second statement of Lemma 1, the measure of the set V_C is zero and, therefore, the measure of the set V is also zero. **Theorem 5 is proved.**

3 Levi-Civita's theorem and projectively equivalent metrics that are strictly non-proportional at each point

A local description of projectively equivalent Riemannian metrics near the points where the eigenvalues of the tensor L given by (1) do not bifurcate has been obtained by Levi-Civita [6]. Here we formulate Levi-Civita's theorem assuming that the eigenvalues of L are different; then they automatically do not bifurcate.

Theorem 6 (Levi-Civita [6]). *Consider two Riemannian metrics on an open subset $U^n \subset M^n$. Consider the tensor L given by (1). Suppose the eigenvalues of L are all different at each point $x \in U^n$.*

Then the metrics are projectively equivalent on U^n if and only if for any point $x \in U^n$ there exist coordinates $x_1, x_2, ..., x_n$ in some neighborhood of the point x such that in these coordinates the metrics have the following model form:

$$ds^2_{g_{model}} = \Pi_1 d(x_1)^2 + \Pi_2 d(x_2)^2 + \cdots + \Pi_n d(x_n)^2, \tag{5}$$

$$ds^2_{\bar g_{model}} = \rho_1\Pi_1 d(x_1)^2 + \rho_2\Pi_2 d(x_2)^2 + \cdots + \rho_n\Pi_n d(x_n)^2, \tag{6}$$

where the functions Π_i and ρ_i are given by

$$\Pi_i \stackrel{\text{def}}{=} (\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\cdots(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)\cdots(\lambda_n - \lambda_i), \tag{7}$$

$$\rho_i \stackrel{\text{def}}{=} \frac{1}{\lambda_1\lambda_2...\lambda_{n-1}}\frac{1}{\lambda_i}.$$

where, for any i, the function λ_i is a smooth function of the variable x_i.

The notations of the eigenvalues of (1) are compatible with the notations in formulae (5,6,7): in the coordinates $x_1, ..., x_n$, the tensor (1) is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$.

Proof of Theorem 2. We assume that M^n is closed and connected, that Riemannian metrics g and $\bar{g}$ on M^n are projectively equivalent and that they are strictly non-proportional at each point of M^n. Then the eigenvalues $\lambda_1 < \lambda_2 < ... < \lambda_n$ of the tensor L given by (1) are all different at any point of M^n. Hence, they are everywhere defined smooth functions on M^n. Therefore, the functions Π_i given by (7) are also smooth on M^n. For any i, at each point of M^n, consider the vector v_i satisfying the conditions

$$\left\{\begin{array}{lcl} Lv_i & = & \lambda_i v_i \\ g(v_i, v_i) & = & \Pi_i. \end{array}\right. \tag{8}$$

The only freedom we have is the sign of the vector. Then we can globally define the vector fields v_i, $i = 1, 2, ..., n$ satisfying (8) on some finite covering $\bar{M}^n$ of M^n. In Levi-Civita's coordinates $x_1, ..., x_n$ from Theorem 6, the vector fields v_i are equal to $\pm\frac{\partial}{\partial x_i}$; then they commute; by definition they never vanish. Then $\bar{M}^n$ must be homeomorphic to the torus T^n. **Theorem 2 is proved.**

4 Proof of Theorem 1

We assume that M^n is closed connected, that Riemannian metrics g and $\bar{g}$ on M^n are projectively equivalent and that they are strictly non-proportional at least at one point.

If the metrics are strictly non-proportional at each point then by Theorem 2 our manifold M^n is covered by the $n-$torus, whose first Betti number is precisely n. Therefore, we have to prove that if there exists a point where the metrics are not strictly non-proportional then the first Betti number $b_1(M^n)$ is less then n.

Consider the tensor L given by (1) and its eigenvalues

$$\lambda_1(x) \le \lambda_2(x) \le ... \le \lambda_n(x).$$

By Theorem 5, there exist numbers $\tau_1, \tau_2, ..., \tau_{n-1} \in R$ such that, for any $i \in \{1, 2, ..., n-1\}$ and for any point x of the manifold,

$$\lambda_i(x) \le \tau_i \le \lambda_{i+1}(x).$$

For any $1 \le i \le n-1$, consider the following subsets of M^n:

$$V_i^- \stackrel{\text{def}}{=} \{x \in M^n : \quad \lambda_i(x) < \tau_i\};$$

$$V_i^+ \stackrel{\text{def}}{=} \{x \in M^n : \quad \lambda_{i+1}(x) > \tau_i\}.$$

Some of the sets V_i^+, V_i^- can be empty. For example, if $\lambda_1(x)$ is constant and $\tau_1 = \lambda_1$ then the set V_1^- is empty. By assumptions, the eigenvalue λ_i is a globally defined continuous function on M^n. Therefore, the sets V_i^+ and V_i^- are open. Below $V_i^\pm$ will denote either V_i^+ or V_i^-. Consider the sets

$$V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm.$$

There is a finitely many (no greater than 2^{n-1}) of such sets; any of them is open.

Let us take a point $x \in M^n$ where

$$\lambda_1(x) < \lambda_2(x) < ... < \lambda_n(x).$$

For any set $V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm$ containing the point x, we denote by

$$(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$$

its connected component containing the point x. Let us show that the first Betti number of any of the sets $(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$ is less then n.

Let us fix one of the sets $(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$ and denote it by V; V is not empty. At each point of V, the eigenvalues of L are all different. Then they are smooth functions on V. Therefore, the functions Π_i given by (7) are also smooth on V.

For any i, at each point of V, consider the vector v_i satisfying conditions (8). The only freedom we have is the sign of the vector. Then we can globally define the vector fields v_i, $i = 1, 2, ..., n$, satisfying conditions (8) on the universal covering $\tilde{V}$ of V.

We see that in Levi-Civita's coordinates $x_1, x_2, ..., x_n$ from Theorem 6, the vector fields v_i are equal to $\pm\frac{\partial}{\partial x_i}$. Then the vector fields v_i commute and for any $j \neq i$ the eigenvalue λ_j is constant on the integral curves of v_i.

We can globally define Levi-Civita's coordinates on the universal covering $\tilde{V}$: Choose an origin P_0. Join any point P with P_0 by a curve. Then the coordinate x_i is equal to the action of the vector field v_i along the curve. Evidently, the definition is independent on the curve.

By definition of the set V, if the universal covering $\tilde{V}$ contains the points with the coordinates $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ and $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$, then it contains the whole parallelepiped

$$\{(x_1, ..., x_n) : \ \min(\bar{x}_i, \hat{x}_i) \leq x_i \leq \max(\bar{x}_i, \hat{x}_i), \quad i = 1, 2, .., n\}.$$

Then the coordinates uniquely define the point, and the universal covering $\tilde{V}$ is homeomorphic to the band

$$\{(x_1, x_2, ..., x_n) \in R^n : \ \alpha_1 < x_1 < \beta_1, \alpha_2 < x_2 < \beta_2, ..., \alpha_n < x_n < \beta_n\},$$

where $\alpha_i, \beta_i \in R_\infty$ (so that they can be either real numbers or $\pm\infty$.).

The fundamental group of $\pi_1(V)$ acts on $\tilde{V}$. The action preserves the metric g and the tensor L. Therefore, if γ is an element of the fundamental group then for each vector field v_i either $\gamma(v_i) = v_i$ or $\gamma(v_i) = -v_i$. Consider the subgroup

$H \subset \pi_1(V)$ of the elements that preserve the directions of the vector fields. The subgroup H has finite index in the fundamental group $\pi_1(V)$.

Evidently, in coordinates $x_1, x_2, ..., x_n$, the group H acts by parallel translations. Then the group H is commutative and free; since it has finite index, it is isomorphic to the free part of the first homology group $H_1(V; Z)$ and its rank is precisely the first Betti number of V. Thus, our goal is to show that the rank of H is less than n.

Evidently, if either α_i or β_i is finite then the group H preserves the coordinate number i. Consider the numbers $i_1, i_2, ..., i_k$ such that $\alpha_{i_j} = -\infty$ and $\beta_{i_j} = \infty$.

The group H is a discrete subgroup of the group $(R^k, +)$; then its rank can be maximum k. If $k < n$ then the Betti number of V is no greater than k and is automatically less than n. Let k be equal to n so that the group $(R^n, +)$ freely acts on $\tilde{V}$ by parallel translations. Then the the group H is a discrete subgroup of the group $(R^n, +)$. If the rank of H is n then the factorspace $R^n/_H$ is homeomorphic to the torus and, therefore, is compact; this factorspace naturally covers V so that V is also compact. Since V is open by definition, it coincides with the whole manifold M^n so that in each point of M^n the eigenvalues of L are all different which contradicts the assumptions.

Thus, the first Betti number of any of the sets $(V_1^{\pm} \cap V_2^{\pm} \cap ... \cap V_{n-1}^{\pm})_x$ is less then n.

Let us now show that each element of the fundamental group $\pi_1(M^n)$ can be realised on one of the sets $(V_1^{\pm} \cap V_2^{\pm} \cap ... \cap V_{n-1}^{\pm})_x$. By Hopf-Rinow theorem, any element of the fundamental group can be realised by a geodesic loop γ, $\gamma(0) = \gamma(1) = x$. We consider this geodesic loop as a curve $(\gamma, \dot{\gamma})$ on TM^n. The values of the roots t_i of the polynomial $I_t(\gamma, \dot{\gamma})$ are constant on the curve $(\gamma, \dot{\gamma})$. If $t_i(\gamma, \dot{\gamma}) \neq \tau_i$ for any i then by Lemma 1 the geodesic loop γ already lies in one of the sets $(V_1^{\pm} \cap V_2^{\pm} \cap ... \cap V_{n-1}^{\pm})_x$. Suppose $t_i(\gamma, \dot{\gamma}) = \tau_i$ for some numbers i. Let us slightly perturb the initial velocity vector $\dot{\gamma}(0)$ and consider a geodesic γ_ϵ such that $\gamma_\epsilon(0) = \gamma(0)$, $|\dot{\gamma}_\epsilon(0) - \dot{\gamma}(0)| = \epsilon << 1$ and $t_i(\gamma_\epsilon, \dot{\gamma}_\epsilon) \neq \tau_i$ for any i. The geodesic γ lies then in one of the sets $(V_1^{\pm} \cap V_2^{\pm} \cap ... \cap V_{n-1}^{\pm})_x$. If ϵ is small then the geodesic segment $\gamma_\epsilon(T)$, $0 \leq T \leq 1$, lies in a thin regular neighborhood of the geodesic loop γ and the point $\gamma_\epsilon(1)$ lies in a small disk neighborhood of x and we can connect the points $\gamma_\epsilon(1)$ and x by a segment in this disk. Then the curve which is made from this segment and from the geodesic segment $\gamma_\epsilon(T)$, $0 \leq T \leq 1$, represents the same homotopy class as the geodesic loop γ so that any element of the fundamental group can be realized on one of the sets $(V_1^{\pm} \cap V_2^{\pm} \cap ... \cap V_{n-1}^{\pm})_x$.

Finally, the first homology group $H_1(M^n; Z)$ of M^n is a unity of a finite number of subgroups; each of these subgroups has rank less than n; then the first Betti number $b_1(M^n)$ is less than n. **Theorem 1 is proved.**

5 Projectively equivalent metrics on the torus

Consider R^n with the standard coordinates $x_1, x_2, ..., x_n$. Consider an $n-$lattice G on R^n. By an $n-$lattice we mean the set of the vectors $k_1 v_1 + k_2 v_2 + ... + k_n v_n$, where $v_1, ..., v_n \in R^n$ are linearly independent vectors and $k_1, k_2, ..., k_n \in Z$.

Let λ_i, $i = 1, ..., n$, be smooth functions on R^n satisfying the following three conditions:

(i) For any i, the function λ_i depends on the the variable x_i only.

(ii) $0 < \lambda_i(x) < \lambda_j(y)$ for any $i < j$ and for any x, y.

(iii) The functions λ_i are invariant modulo the lattice so that for any vector $v = (v^1, v^2, ..., v^n) \in G$, and for any $x_i \in R$

$$\lambda_i(x_i + v^i) = \lambda(x_i).$$

Consider the Riemannian metrics g_{model} and $\bar{g}_{model}$ on R^n given by the formulae (5,6). By Levi-Civita's theorem, the metrics are projectively equivalent; by definitions, they are invariant modulo the lattice so they generate two projectively equivalent metrics on the torus $R^n/_G$. We will call such metrics *model* metrics corresponding to the lattice G and to the functions λ_i.

Theorem 7. *Let Riemannian metrics g and $\bar{g}$ on the torus T^n be projectively equivalent and strictly non-proportional at least at one point. Then there exist an $n-$lattice G, functions $\lambda_1, ..., \lambda_n$ satisfying conditions (i-iii) and a diffeomorphism $\phi : T^n \to R^n/_G$ such that $g = \phi^* g_{model}$, $\bar{g} = \phi^* \bar{g}_{model}$, where g_{model} and $\bar{g}_{model}$ are model metrics with respect to the lattice G and the functions λ_i.*

Thus, any pair of projectively equivalent Riemannian metrics on the $n-$torus which are strictly non-proportional at least at one point is given by an $n-$lattice G and by functions λ_i satisfying conditions (i-iii). The following theorem answers when two of such sets of data define the same pair of metrics:

Theorem 8. *Let G^1 and G^2 be two $n-$lattices on R^n. Suppose the functions $\lambda_1^1, ..., \lambda_n^1$ satisfy conditions (i-iii) with respect to the lattice G^1; suppose the functions $\lambda_1^2, ..., \lambda_n^2$ satisfy conditions (i-iii) with respect to the lattice G^2. Consider the pairs of model projectively-equivalent metrics $g^1_{model}, \bar{g}^1_{model}$ (which are model metrics with respect to the lattice G^1 and the functions λ_i^1) and $g^2_{model}, \bar{g}^2_{model}$ (which are model metrics with respect to the lattice G^2 and the functions λ_i^2).*

Then there exists a diffeomorphism $\phi : R^n/_{G^1} \to R^n/_{G^2}$ such that $g^1_{model} = \phi^* g^2_{model}$ and $\bar{g}^1_{model} = \phi^* \bar{g}^2_{model}$, if and only if there exists $\alpha_1, ..., \alpha_n \in R$ and $\epsilon_1, ..., \epsilon_n \in \{+1, -1\}$ such that the coordinate change

$$x_i \mapsto \epsilon_i x_i + \alpha_i, \quad i = 1, 2, ..., n,$$

takes the lattice G^1 to the lattice G^2 and the functions λ_i^1 to the functions λ_i^2.

Proof of Theorems 7,8: Let g and $\bar{g}$ be projectively equivalent Riemannian metrics on the torus T^n. Suppose they are strictly non-proportional at least at one point of the torus. Then by Corollary 1 they are strictly non-proportional at each point of the torus. As in Section 3, for any i, at each point of the torus, consider a vector v_i satisfying conditions (8). The only freedom we have is the sign of the vector. Then, on the universal covering R^n, we can globally define vector fields v_i satisfying (8). As in Section 4, consider the coordinate system on R^n defined as follows: choose an origin P_0. Join any point P with P_0 by a curve. Then the coordinate x_i is equal to the action of the vector field v_i along the curve. Evidently, the definition is independent of the curve, and the coordinate system is isomorph to the standard one on R^n.

The fundamental group $\pi_1(T^n)$ acts on the universal covering R^n. The action preserves the metrics $g, \bar g$; therefore, it preserves the coordinate net. Then the group $\pi_1(T^n)$ acts by translations and compositions of translations and reflections. Since the fundamental group $\pi_1(T^n)$ is commutative and isomorphic to Z^n, compositions of translations and reflections can not occur so that the fundamental group acts by translations. Since the action is co-compact, free and is a discrete subgroup of the group of all translations, the vectors of such translations form an n−lattice G. Since the action preserves the metrics, it must preserve the eigenvalues of the tensor L given by 1. Then the functions $\lambda_1 < \ldots < \lambda_n$ are invariant modulo the lattice G. By Levi-Civita's theorem, for any i the function λ_i depends on the variable x_i only. The only freedom we have is in choosing the origin P_0 of the coordinate system, which gives us the translation $x_i \mapsto x_i + \alpha_i$, and the freedom in choosing the directions of the vectors v_i, which gives us the changes of the signs of the coordinates. Thus, any pair of projectively equivalent strictly non-proportional Riemannian metrics on the torus is model, and two pairs of model metrics are the same if their data satisfy the conditions in Theorem 8. **Theorems 7, 8 are proved.**

6 The eigenvalues of L for projectively equivalent metrics on the 3-torus never bifurcate

Theorem 9. *Let g and $\bar g$ be projectively equivalent Riemannian metrics on the three-dimensional torus T^3. Then the number of different eigenvalues of the tensor L given by (1) is constant on the torus.*

Proof: Let Riemannian metrics g and $\bar g$ on T^3 be projectively equivalent. By Corollary 1, if they are strictly non-proportional at least at one point then at each point of T^3 the number of different eigenvalues of L is precisely three and, therefore, is constant. If the metrics are proportional at each point of T^3 then at each point of T^3 the number of different eigenvalues is equal to one and, therefore, is constant. So, the only case we need to consider is when there exists a point $x \in T^3$ such that at these point the number of different eigenvalues is equal to two; our goal is to show that then the number of different eigenvalues at any other point can not be equal to one.

Actually, this fact has been essentially proved in [13]. But since in the paper [13] the proof is hidden in the third part of the proof of the main theorem, we will repeat it here.

Let us denote the eigenvalues of L at $x \in T^3$ by $\lambda_1(x), \lambda_2(x), \lambda_3(x)$. By Theorem 5, without loss of generality we can assume that

$$\lambda_1(x) < \lambda_2(x) = \lambda_3(x) = const \stackrel{\text{def}}{=} \lambda$$

for almost each point $x \in T^3$. Suppose the eigenvalues bifurcate so that there exists $y \in T^3$ such that

$$\lambda_1(y) = \lambda_2(y) = \lambda_3(y).$$

Let us show that it is possible only on the sphere S^3 or on the Projective Space RP^3. As in Section 2, at each point $x \in T^3$, we can find a basis of the space T_xT^3 such that in this basis the metric g is given by the diagonal matrix diag$(1, 1, 1)$ and the matrix L is given by the diagonal matrix diag$(\lambda_1(x), \lambda_2(x), \lambda_3(x))$. In this basis, the polynomial $I_t(x, \xi)$ given by (3) reads

$$I_t = (\lambda - t)^2\xi_1^2 + (\lambda_1(x) - t)(\lambda - t)(\xi_2^2 + \xi_3^2).$$

Therefore, for any t, the functions

$$\bar{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda - t)} = (\lambda - t)\xi_1^2 + (\lambda_1(x) - t)(\xi_2^2 + \xi_3^2)$$

is an integral for the geodesic flow of g. Substituting $t = \lambda$ we get that the function

$$\bar{I}_\lambda = (\lambda_1(x) - \lambda)(\xi_2^2 + \xi_3^2)$$

is also an integral

By Lemma 1, for any geodesic γ passing through the point y we have $\bar{I}_\lambda(\gamma, \dot{\gamma}) \equiv 0$. Then, for any $x \in \gamma$ such that $\lambda_1(x) \neq \lambda$ the sum $(\xi_2^2 + \xi_3^2)$ is zero. Therefore, the velocity vector $\dot{\gamma}(x)$ is an eigenvector of L with the eigenvalue $\lambda_1(x)$. Then two geodesics passing through y can transversally intersect only in the points z where $\lambda_1(z) = \lambda$. Then there can be maximum two of such points and T^3 is homeomorphic either to S^3 or to RP^3. The contradiction shows that if number of eigenvalues of L is equal to two at least at one point of T^3 then it is so at each point. **Theorem 9 is proved.**

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