Convergence in capacity

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1. Introduction

Th purpose of this paper is to study convergence of Monge-Ampère measures associated to sequences of plurisubharmonic functions defined on a hyperconvex subdomain Ω of $\mathbb{C}^n$.

The concept of convergence in capacity was introduced in [X], where it was proved that for any uniformly bounded sequence φ_j of plurisubharmonic functions that converges in $\mathbb{C}^n$-capacity we have that $(dd^c\varphi_j)^n$ converges weak* to $(dd^c\varphi)^n, j \to +\infty$.

We generalize this result:

Theorem 1.1. *Assume $\mathcal{F} \ni u_0 \leq u_j \in \mathcal{F}$ and that u_j converges to u in $\mathbb{C}^n$-capacity. Then $(dd^c u_j)^n$ converges weak* to $(dd^c u)^n, j \to +\infty$.*

We first recall some definitions. See [C1] and [C2] for details.

The class $\mathcal{F}$ consists of all plurisubharmonic functions φ on Ω such that there is a sequence $\varphi_j \in \mathcal{E}_0, \varphi_j \searrow \varphi, j \to +\infty$ and $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$, where $\mathcal{E}_0$ is the class of bounded plurisubharmonic functions ψ such that $\lim_{z\to\xi} \psi(z) = 0, \forall \xi \in \partial\Omega$ and $\int_\Omega (dd^c\psi)^n < +\infty$.

The following definition was introduced in [X]: A sequence $\varphi_j \in \mathcal{F}$ converges to φ in $\mathbb{C}^n$-capacity if

$$\text{cap}(\{z \in k : |\varphi - \varphi_j| > \delta\}) \to 0, \ j \to +\infty \quad \forall\, k \subset\subset \Omega, \ \forall\, \delta > 0.$$

For $\omega \subset\subset \Omega, \text{cap}(\omega) = \int_\Omega (dd^c h_\omega^*)^n$ where h_ω^* is the smallest upper semicontinuous majorant of $h_\omega(z) =$

$$\sup\{\varphi(z); \varphi \in \xi_0; \varphi|_\omega \leq -1\}.$$

Finally, we write $u_j \rightsquigarrow u, \ j \to +\infty$ if u_j converges weak* to $u, j \to +\infty$.

2. Proofs

Lemma 2.1. *Suppose μ is a positive measure on Ω which vanishes on all pluripolar sets and $\mu(\Omega) < +\infty$. If $u_0, u_j \in \mathcal{F}$, $u_0 \leq u_j \rightsquigarrow u$, $j \to +\infty$ and if $\int u_0 d\mu > -\infty$, then $\lim\limits_{j\to+\infty} \int u_j d\mu = \int u d\mu$.*

Proof. Denote by dV the Lebesgue measure and choose $\tilde{u}_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$, $\tilde{u}_j \geq u_j$ such that $\int\limits_{\Omega}(\tilde{u}_j - u_j)(d\mu + dV) < \frac{1}{j}$. Then $\tilde{u} \rightsquigarrow u$, $\lim\limits_{j\to+\infty} \int u_j d\mu - \int \tilde{u}_j d\mu = 0$ so it is enough to prove that

$$\lim_{j\to+\infty} \int \tilde{u}_j d\mu = \int u d\mu.$$

Thus we can assume $u_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$.

By Theorem 6.3 in [C1] there is a $\psi \in \mathcal{E}_0$, $f \in L^1\big((dd^c\psi)^n\big)$ with

$$\mu = f(dd^c\psi)^n$$

so by lemma 5:2 in [C1], for every $p < +\infty$,

$$\lim_{j\to+\infty} \int u_j d\mu_p = \int u d\mu_p$$

there $\mu_p = \min(f, p)(dd^c\psi)^n$.

Now

$$\begin{aligned}
\lim_{j\to+\infty} \int u_j d\mu &= \lim_{j\to+\infty} \int u_j d\mu_p + \\
&+ \lim_{j\to+\infty} \int u_j\big(f - \min(f,p)\big)(dd^c\psi)^n \geq \\
&\geq \int u d\mu_p + \int u_0\big(f - \min(f,p)\big)d\mu \\
&\to \int u d\mu, \ p \to +\infty
\end{aligned}$$

by monotone convergence. On the other hand, by Fatous lemma,

$$\overline{\lim}_{j\to+\infty} \int u_j d\mu \leq \int u d\mu$$

which gives the desired conclusion. □

Proof. Proof of the theorem.

We prove that

$$\lim_{j\to+\infty} \int h(dd^c u_j)^n = \int h(dd^c u)^n, \ \forall\, h \in \mathcal{E}_0,$$

which is enough by Lemma 3:1 in [C1].

Suppose $\omega_1, \omega_2, \ldots, \omega_{n-1} \in \mathcal{F}$, $h \in \mathcal{E}_0$. It follows from the assumption that $u_j \rightsquigarrow u$, $j \to +\infty$ so $\lim\limits_j \int \omega_1\, dd^c u_j \wedge dd^c\omega_2 \wedge \cdots \wedge dd^c h = \int \omega_1 dd^c u \wedge \cdots \wedge dd^c h$ by the lemma.

Suppose now that

$$\lim_j \int \omega_1 (dd^c u_j)^p \wedge dd^c\omega_{p+1} \wedge \cdots \wedge dd^c h = \int \omega_1 (dd^c u)^p \wedge \cdots \wedge dd^c h.$$

for $1 \le q \le p \le n-2$. We claim

$$\lim_j \int \omega_1 (dd^c u_j)^{p+1} \wedge dd^c\omega_{p+2} \wedge \cdots \wedge dd^c\omega_{n-1} \wedge dd^c h =$$

$$= \int \omega_1 (dd^c u)^{p+1} \wedge dd^c\omega_{p+2} \wedge \cdots \wedge dd^c\omega_{n-1} \wedge dd^c h.$$

Given $\varepsilon > 0$ choose $k \subset\subset \Omega$ such that $\{z \in \Omega; h < -\varepsilon\} \subset k$ and then a subsequence u_{j_t} such that

$$\int_\Omega \left(dd^c \sum_{t=1}^{\infty} h^*_{\{z\in k; |u-u_{j_t}|>\varepsilon\}} \right)^n < 1$$

and denote by

$$h_N = \max\left(\sum_{t=N}^{\infty} h^*_{\{z\in k; |u-u_{j_t}|>\varepsilon\}}, -1 \right).$$

Then $h_N \to 0, N \to +\infty$ outside a pluripolar set and

$$\int \omega_1 (dd^c u_{j_k})^{p+1} \wedge \cdots \wedge dd^c h - \int \omega_1 (dd^c u_{j_k})^p dd^c u \wedge \cdots \wedge dd^c h =$$

$$= \int (u_{j_k} - u)\big(dd^c u_{j_k}\big)^p \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c h =$$

$$= \int -h_N (u_{j_k} - u)\big(dd^c u_{j_k}\big)^p \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c h +$$

$$+ \int (1+h_N)(u_{j_k} - u)\big(dd^c u_{j_k}\big)^p \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c (h - h_\varepsilon) +$$

$$+ \int (1+h_N)(u_{j_k} - u)\big(dd^c u_{j_k}\big)^p \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c h_\varepsilon = I_{j_k} + II_{j_k} + III_{j_k}.$$

where $h_\varepsilon = \max(h, -\varepsilon)$.
Thus

$$|I_{j_k}| \le 2 \int (-h_N)(-u_0)(dd^c u_{j_k})^p \wedge \cdots \wedge dd^c h \le$$

$$\le \int \big[-u_0 + \max(u_{0_1} - R)\big] (dd^c u_{j_k})^p \wedge \cdots \wedge dd^c h +$$

$$+ R \int -h_N \big(dd^c u_{j_k}\big)^p \wedge \cdots \wedge dd^c h$$

$$\to \int \big[-u_0 + \max(u_0, -R)\big] (dd^c u)^p \wedge \cdots \wedge dd^c h +$$

$$+ R \int (-h_N)(dd^c u)^p \wedge \cdots \wedge dd^c h, \; j_k \to +\infty.$$

Now, the first term on the right hand side is small if R is large, the second is small if N is even larger.

$$|III_{j_k}| \le \int -u_0(dd^c u_{j_k})^p \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c h_\varepsilon \le$$
$$\le \int -u_0(dd^c u_0)^p \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c h_\varepsilon \le$$
$$\le \int -h_\varepsilon(dd^c u_0)^{p+1} \wedge dd^c\omega_1 \wedge \cdots \wedge dd^c\omega_{n-1} \le$$
$$\varepsilon \int (dd^c u_0)^{p+1} \wedge dd^c\omega_1 \cdots \wedge dd^c\omega_{n-1}$$

so it remains to estimate

$$|II_{j_k}| \le 2\varepsilon\Big(\int (dd^c u_{j_k})^p \wedge dd^c\omega_1 \cdots \wedge dd^c(h+h_\varepsilon)\Big) \le$$
$$\le 4\varepsilon \int (dd^c u_0)^p \wedge dd^c\omega_1 \cdots \wedge dd^c h.$$

which proves the claim.

Using the claim and repeating the chain of inequalities above, we conclude that

$$\int h(dd^c u_{j_k})^n - \int h(dd^c u)^n =$$
$$\int (u_{j_k} - u)(dd^c u_{j_k})^{n-1} \wedge h +$$
$$\int u(dd^c u_{j_k})^{n-1} \wedge h - \int u(dd^c u)^{n-1} \wedge h \to 0, j_k \to +\infty.$$

This proves the theorem, since we have proved that every subsequence of u_j contains a subsequence u_{j_t} such that $(dd^c u_{j_t})^n$ converges weak* to $(dd^c u)^n, t \to +\infty$.

□

References

[C1] Cegrell, U., *Pluricomplex energy*. Acta Math. **180:2**, 187–217.

[C2] Cegrell, U., *The general definition of the complex Monge-Ampère operator*. Manuscript, Umeå and Sundsvall, 2001.

[X] Xing, Y., *Continuity of the complex Monge-Ampère operator*. Proc. AMS., **124:2** (1996), 457–467.