SCHAUDER ESTIMATES FOR FULLY NONLINEAR ELLIPTIC DIFFERENCE OPERATORS

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ABSTRACT. In this paper we are concerned with discrete Schauder estimates for solution of fully nonlinear elliptic difference equations. Our estimates are discrete versions of second derivative Hölder estimates of Evans, Krylov, and Safonov for fully nonlinear elliptic partial differential equations. They extend previous results of Holtby for the special case of functions of pure second order differences on cubic meshes. As with Holtby's work, the fundamental ingredients are the pointwise estimates of Kuo-Trudinger for linear difference schemes on general meshes.

1. INTRODUCTION

In this paper, we derive Schauder estimates for solutions of fully nonlinear elliptic *difference* equations. Letting E denote a mesh, which is a discrete subset of n-dimensional Euclidean space $\mathbb{R}^n$, and $u : E \to \mathbb{R}$, a mesh function, we consider nonlinear difference equations of the form

$$F[u] := F(x, \tilde{L}u(x)) = 0, \tag{1.1}$$

where $F : E \times \mathbb{R}^K \to \mathbb{R}$ and $\tilde{L} = (L_1 , \dots , L_K)$, is a system of linear difference operators given by

$$L_j u(x) = \sum_{x+y \in E} a_j(x,y)(u(x+y) - u(x)) \tag{1.2}$$

with coefficients, $a_j : E \times E \to \mathbb{R}$, $j = 1, \cdots , K$, having finite support in y, for each $x \in E$. The operators L_j are assumed to be *monotone*, that is

$$a_j(x,y) \geq 0 \,, \tag{1.3}$$

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for all x, $x + y \in E$, and *balanced*, that is

$$\sum a_j(x, y)y = 0 \tag{1.4}$$

for all x, $x+y \in E$. Conditions (??) and (??) mean that L_j corresponds to a pure second order degenerate elliptic partial differential operator $\mathfrak{L}_j$ given by (refer to [?])

$$\mathfrak{L}_j u(x) = \frac{1}{2} \sum_{x+y \in E} \sum_{r,s} a_j(x, y) y_r y_s D_{r,s} u(x). \tag{1.5}$$

Concerning the function F, we will assume that F is Hölder continuous with respect to $x \in E$ and concave with respect to $q = \widetilde{L} \in \mathbb{R}^K$, satisfying structure conditions

$$\lambda \leq F_{q_j}(x, q) \leq \Lambda \tag{1.6}$$

$$|F(x, q) - F(z, q)| \leq \mu(1 + |q|)|x - z|^\gamma \tag{1.7}$$

for all $z \in E$, $q \in \mathbb{R}^K$ and fixed positive constants λ, Λ, μ and γ. In this paper we will assume that the operators $L_1, \cdots, L_K$ have constant coefficients, that is $a(x, y) = a(y)$ for all $x, x + y \in E$ and E is an additive group. Accordingly, we can write L_j in the form

$$L_j u(x) = \sum_{y \in E} a_j(y)(u(x + y) - u(x)). \tag{1.8}$$

The difference operator F will correspond to a degenerate elliptic differential operator $\mathfrak{F}$ given by

$$\begin{aligned} \mathfrak{F}[u] &= F(x, \widetilde{\mathfrak{L}} u(x)) \\ &:= G(x, D^2 u(x)) \end{aligned} \tag{1.9}$$

where $\widetilde{\mathfrak{L}} = (\mathfrak{L}_1, \cdots, \mathfrak{L}_K)$ and G is concave with respect to D^2u and Hölder continuous with respect to x. We will impose non-degeneracy conditions on F by requiring that the sum,

$$L := \sum_{j=1}^{K} L_j \tag{1.10}$$

satisfies the non-degeneracy conditions invoked by us in our treatment of local pointwise estimates for linear difference operators on general meshes in [?], [?]. Namely, setting

$$a = \sum_{j=1}^{K} a_j, \tag{1.11}$$

we define a finite set $Z \subset \mathbb{R}^n$ by

$$Z = \left\{ a(x, y)\, y \;\middle|\; y \in E \right\}. \tag{1.12}$$

Note that the condition that L is balanced implies that Z is centered at the origin, 0. Letting $\widehat{Z}$ denote the convex hull of Z, we then assume that there exists a ball, of center 0 and radius ρ, B_ρ satisfying

$$B_\rho \subset \widehat{Z}. \tag{1.13}$$

Next, following [?], we need to assume the mesh points are effectively linked through the operator L. That is, we assume for any two points $x, z \in E$, there exist points $x_0 = x$, x_1, x_2, $\cdots$, $x_\ell = z$ in E such that

$$a(x_{i+1} - x_i) \geq \lambda_0\,, \qquad i = 0, \cdots, \ell - 1, \tag{1.14}$$

for some positive constant λ_0, with the number $\ell = \ell(x, z)$ uniformly bounded by the distance between x and z, that is

$$\ell(x, z) \leq \frac{\ell_0 |x - z|}{h}\,, \tag{1.15}$$

where ℓ_0 is a positive constant and h is the minimum mesh width at x given by

$$\begin{aligned} h &= \min_{x,z \in E} |x - z| \\ &= \min_{y \in E - \{0\}} |y| \end{aligned} \tag{1.16}$$

To illustrate our conditions, we describe the particular example treated by Holtby in [?]. Here the mesh E is the cubic mesh of width h in $\mathbb{R}^n$, that is

$$E = \Big\{ h(m_1, \cdots, m_n) \in \mathbb{R}^n \,\Big|\, m_i \in \mathbb{Z},\ i = 1, \cdots, n \Big\}, \tag{1.17}$$

and the operators $L_1, \cdots, L_K$ are the second order difference quotients, δ_j^2, $j = 1, \cdots, n$, in the coordinate directions $e_1, \cdots, e_n$, that is

$$\delta_j^2 u(x) = \frac{1}{h^2}\Big\{u(x + 2he_j) - 2u(x + he_j) + u(x)\Big\} \tag{1.18}$$

$j = 1, \cdots, n$. The operator L is the discrete Laplacian, given by

$$Lu(x) = \sum_{j=1}^{n} \delta_j^2 u(x)\,. \tag{1.19}$$

Clearly, $\rho = \frac{1}{h}$, $\lambda_0 = 1$, $\ell_0 = \sqrt{n}$.

The plan of this paper is as follows. In the next section, we establish interior Schauder estimates for second order differences of solutions of equation (??) where F is independent of the x variables, that is the "frozen case". Our main tool here is our discrete weak Harnack inequality in [?], [?], and our overall approach is based on that presented for the continuous case in the monograph [?]. In the last section, we derive the general Schauder estimates, Theorem ??, utilizing the perturbation approach developed by Safonov [?], [?] for the continuous case.

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2. The Frozen Case

In this section, we consider the equation (??), frozen at a mesh point x_0, that is

$$F_0[u] := F_0(\widetilde{L}u) = 0 \tag{2.1}$$

where $F_0 : \mathbb{R}^K \to \mathbb{R}$ is given by $F_0(q) = F(x_0, q)$ and $\widetilde{L} = (L_1, \cdots, L_K)$ is given by (??). By Schmidt [?], the mesh E is a *lattice*, that is there exist linear independent vectors $\zeta_1, \cdots, \zeta_n \in \mathbb{R}^n$ such that

$$E = \Big\{(m_1\zeta_1, m_2\zeta_2, \cdots, m_n\zeta_n) \,\Big|\, m_i \in \mathbb{Z}, \quad i = 1, \dots, n\Big\}. \tag{2.2}$$

The simplest case of a lattice mesh is the cubic mesh (??). Note that $h = \min|\zeta_i|$ and from (??) and (??) we can estimate ρ from below by

$$\rho \geq C_0 \det\left[\frac{\zeta_1}{|\zeta_1|}, \cdots, \frac{\zeta_n}{|\zeta_n|}\right] \lambda_0 h, \tag{2.3}$$

where C_0 is a positive constant depending on n. Setting

$$Y = \{\, y \in E - \{0\} \mid a(y) > 0 \,\}, \tag{2.4}$$

we define the maximum mesh width of E, with respect to L, by

$$\begin{aligned} \overline{h} &= \max_{x, x+y \in E} \Big\{ |y| \,\Big|\, a(x,y) > 0 \Big\} \\ &= \max_{y \in Y} |\, y \,|. \end{aligned} \tag{2.5}$$

Hölder estimates for $\widetilde{L}$

We now proceed from equation (??) to derive Hölder estimates for $\widetilde{L}u$, following the method in [?], [?]. More generally, we can consider an equation of the form

$$F_0(\widetilde{L}u) \;=\; \psi \tag{2.6}$$

where ψ is a given mesh function. By the concavity of F_0, we have for $y \in Y$,

$$\begin{aligned}\psi(x+y) \;-\; \psi(x) \;&=\; F_0(\widetilde{L}u(x+y)) \;-\; F_0(\widetilde{L}u(x)) \\ &\le \sum_{j=1}^{K} \frac{\partial F_0}{\partial q_j}\big(\widetilde{L}u(x)\big)\big(L_ju(x+y) \;-\; L_ju(x)\big)\end{aligned} \tag{2.7}$$

Hence, letting $\lambda_j(x) = \frac{\partial F_0}{\partial q_j}(\widetilde{L}u(x))$, for each $j = 1\,, \cdots\,, K$, we have

$$\begin{aligned}L_j\psi(x) \;&=\; \sum_{y\in Y} a_j(y)\Big((\psi(x+y) \;-\; \psi(x)\Big) \\ &\le \sum_{i=1}^{K} \lambda_i(x)\left(\sum_{y\in Y} a_i(y)\big(L_ju(x+y) \;-\; L_ju(x)\big)\right) \\ &= \sum_{i=1}^{K} \lambda_i(x) L_i(L_ju)(x)\end{aligned} \tag{2.8}$$

Fix j and let $v = L_ju$, $\phi = L_j\psi$ to get, from (??), the linear inequality

$$Lv \;:=\; \sum_{i=1}^{K} \lambda_i L_i v \;\ge\; \phi\,. \tag{2.9}$$

We now invoke the discrete weak Harnack inequality [?]. Let L be any linear operator of the form,

$$Lu(x) \;=\; \sum_{x+y\in E} a(x,y)\big(u(x+y) \;-\; u(x)\big) \tag{2.10}$$

which is balanced, monotone and satisfies the non-degeneracy conditions (??), (??), (??). Letting $B_R(z)$ denote the ball of center z and radius R in $\mathbb{R}^n$ and $E_R(z) = B_R(z) \cap E$ the corresponding mesh ball, we assume that u is a non-negative mesh function, satisfying the difference inequality, $Lu \le f$, in $E_R = E_R(z)$, for some mesh function f. The weak Harnack inequality asserts the existence of a constant $p > 0$, depending on $n, \overline{h}/h$, $a_0/\rho h$, $\tilde{a}/\lambda_0$, where

$$a_0 \;=\; \sum a(y)|y|^2\,, \qquad \tilde{a} \;=\; \sum a(y) \tag{2.11}$$

such that for any $\tau < 1$ and $(1-\tau) < \overline{h}/R$

$$\left\{ \sum_{E_{\tau R}} \left(\frac{h}{R}\right)^n u^p \right\}^{1/p} \leq C \left\{ \min_{E_{\tau R}} u + R \left(\sum_{E_R} \left| \frac{f}{\rho} \right|^n \right)^{1/n} \right\} \tag{2.12}$$

where C is a constant, depending additionally on τ. Returning to (??), we set for $\sigma \leq 1$,

$$M_\sigma = \sup_{E_{\sigma R}(x_0)} v \,, \qquad m_\sigma = \inf_{E_{\sigma R}(x_0)} v \,,$$

and apply (??) to the function $M_1 - v$, thereby obtaining

$$\begin{aligned} \left\{ \left(\frac{h}{R}\right)^n \sum_{E_{\tau R}} \left(M_1 - v\right)^p \right\}^{1/p} & \\ &\leq C \left\{ \min_{E_{\tau R}} \left(M_1 - v\right) + R \left(\sum_{E_R} \left| \frac{\phi}{\rho} \right|^n \right)^{1/n} \right\} \\ &\leq C \left\{ M_1 - M_\tau + R \left\| \frac{\phi}{\rho} \right\|_{n;E_R} \right\} \end{aligned} \tag{2.13}$$

where p depends on $n, \overline{h}/h, \ell_0, a_0/h\rho, \widetilde{a}/\lambda_0$, λ and Λ, C depends additionally on τ, and

$$\| f \|_{n;E_R} = \left(\sum_{E_R} |f|^n \right)^{1/n}$$

Following [?], to conclude a Hölder estimate for v from (??), we need a corresponding inequality for $-v$, which we obtain by considering (??) as a functional relationship between $L_j u$, $j = 1, \cdots, K$. In fact, using the concavity of F_0 again, we have for any $x, z \in E_R$,

$$\sum_{i=1}^K \lambda_i(z)\Big(L_i u(z) - L_i u(x)\Big) \leq \psi(z) - \psi(x) \,. \tag{2.14}$$

Now setting

$$M_{\sigma_i} = \sup_{E_{\sigma R}} L_i u \,, \qquad m_{\sigma_i} = \inf_{E_{\sigma R}} L_i u \,,$$

we obtain, by summing (??), from $i = 1$ to K,

$$\left\{\left(\frac{h}{R}\right)^n \sum_{E_{\tau R}} \left[\sum_{i=1}^{K}\left(M_{1i} - L_i u\right)\right]^p\right\}^{1/p}$$
$$\leq C\left\{\sum_{i=1}\left(M_{1i} - M_{\tau i}\right) + R\sum_{i=1}^{K}\left\|\frac{L_i\psi}{\rho}\right\|_{n;E_R}\right\}, \quad (2.15)$$

where C depends on K, as well as the quantities in (??). Using (??) in (??), we obtain the complementary inequality,

$$\left\{\left(\frac{h}{R}\right)^n \sum_{E_{\tau R}} \left[\sum_{i=1}^{K}\left(L_i u - m_{1i}\right)\right]^p\right\}^{1/p}$$
$$\leq C\left\{\sum_{i=1}\left(M_{1i} - M_{\tau i}\right) + R\sum_{i=1}^{K}\left\|\frac{L_i\psi}{\rho}\right\|_{n;E_R} + R^\gamma|\psi|_{\gamma,E_R}\right\}, \quad (2.16)$$

where $0 < \gamma \leq 1$ and, for any $\Omega \subset E$,

$$[\psi]_{\gamma,\Omega} = \sup_{x,z\in\Omega}\frac{|\psi(x) - \psi(z)|}{|x - z|^\gamma}.$$

Writing,

$$\omega(\tau R) = \sum_{i=1}^{K}\operatorname*{osc}_{E_{\tau R}} L_i u = \sum_{i=1}^{K}\left(M_{\tau i} - m_{\tau i}\right)$$

we then obtain, by adding (??) and (??),

$$\omega(\tau R) \leq \chi\omega(R) + R^\gamma[\psi]_{\gamma,E_R} + R\sum_{i=1}^{K}\left\|\frac{L_i\psi}{\rho}\right\|_{n;E_R} \quad (2.17)$$

where χ, depends on $n, \overline{h}/h, \ell_0, a_0/h\rho, \widetilde{a}/\lambda_0, \lambda, \Lambda, K$ and τ. We then conclude, from Lemma 8.23 [?], the *Hölder estimate*, for any τ, $0 < \tau < 1$,

$$\sum_{i=1}^{K}\operatorname*{osc}_{E_{\tau R}} L_i u \leq C\tau^\alpha\left\{\sum_{i=1}^{K}\operatorname*{osc}_{E_R} L_i u + R^\gamma[\psi]_{\gamma,E_R} + R\left\|\frac{\widetilde{L}\psi}{\rho}\right\|_{n,E_R}\right\} \quad (2.18)$$

where α ($\leq \gamma$) and C are positive constants, depending on $n, \overline{h}/h$, ℓ_0, $a_0/h\rho$, $\widetilde{a}/\lambda_0$, λ, Λ and K.

To pass from the estimate (??) to a Schauder estimate, that is a Hölder estimate for second difference quotients of u, we apply the Schauder estimates of Thomeé [?] to the sum (??). Because E is a lattice, we can transform coordinates so that E is mapped to the cubic mesh $\mathbb{Z}^n$, through the matrix T^{-1} where $T = [\zeta_1, \cdots, \zeta_n]$. We may then express the Thomeé estimates in terms of our original mesh E by defining the forward differences

$$\delta_i u(x) = \frac{u(x+\zeta_i) - u(x)}{|\zeta_i|} \tag{2.19}$$

and for any multi-index $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_i \geq 0$, $i = 1, \cdots, n$,

$$\delta^\beta u(x) = \delta_1^{\beta_1} \delta_2^{\beta_2}, \cdots, \delta_n^{\beta_n} u(x)\,. \tag{2.20}$$

The right hand side of the estimate (??) can be expressed in terms of the operator δ through the following lemma, whose proof we defer until the end of this section.

Lemma 2.1. *Let L be a balanced monotone operator of the form (??) on the lattice mesh E, with*

$$Y = Y_x = \Big\{ y \in E - \{0\} \;\Big|\; a(x,y) > 0 \Big\}. \tag{2.21}$$

Denote by $\widetilde{Y}$ the smallest parallelogram (with axes parallel to $\zeta_1, \cdots, \zeta_n$) containing Y. Then we can write

$$Lu(x) = \sum_{i,j=1}^{n} \sum_{y \in \widetilde{Y}} c_{ij}(y) \delta_i \delta_j u(x+y) \tag{2.22}$$

where the coefficients c_{ij} satisfy bounds,

$$\sum \Big| c_{ij} \Big| \leq C a_0 \left(\frac{\overline{h}}{h} \right)^n, \tag{2.23}$$

where C is a constant depending only on n.

Next to relate the ellipticity condition of Thomée [?] to our non-degeneracy condition, we see that the characteristic polynomial of any monotone balanced operator L in (??) is given by

$$p(\theta) = p(\theta, x) = \sum_{y \in Y} a(x,y) \left(e^{i\theta \cdot y} - 1 \right), \qquad \Big(|\,\theta\,| \leq \pi \Big), \tag{2.24}$$

Hence, using the balance condition,

$$\begin{aligned} \Big| p(\theta) \Big| &\geq C \sum_{y \in Y} a(x,y) \Big(y \cdot \theta \Big)^2 \\ &\geq C \frac{(\rho\, h)^2}{a_0} |\,\theta\,|^2 \end{aligned} \tag{2.25}$$

by inequality (4.10) in [?], where C is a positive constant. We then conclude from (??),

$$\operatorname*{osc}_{E_{\tau R}} \delta^\beta u \le C\tau^\alpha \left\{ \sup_{|\beta|=2} \operatorname*{osc}_{E_R} \delta^\beta u + R^\gamma \left[\frac{\psi}{\rho h}\right]_{\gamma, E_R} + R \left\|h\, \delta^2 \psi\right\|_{n, E_R} \right\}, \tag{2.26}$$

where

$$\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}_+^n \qquad \text{(the set of non-negative integers)}$$

and

$$|\beta| = \beta_1 + \cdots + \beta_n = 2\,.$$

By interpolation [?],

$$\operatorname*{osc}_{E_{\tau R}} \delta^2 u \le C\tau^\alpha \left\{ \frac{1}{R^2} \max_{E_R} u + R^\gamma \left[\frac{\psi}{\rho h}\right]_{\gamma, E_R} + R \left\|h\, \delta^2 \psi\right\|_{n, E_R} \right\} \tag{2.27}$$

where constants C and α depend on the same quantities as in (??).

Proof of Lemma ??

Lemma ?? will follow from a multi-dimensional Taylor formula. To obtain this, we write

$$y = \sum_{i=1}^n m_i \zeta_i\,,$$

and assume initially that $x = 0$ and $m_i \ge 0$, $i = 1, \cdots, n$. Expanding, as in the one dimensional case, [?], [?], we have

$$u(y) = (u + m_1|\zeta_1|\delta_1 u)(0, y_2, \cdots, y_n) + \sum_{k=0}^{m_1-2} (m_1 - k - 1)|\zeta_1|^2 \delta_1\delta_1 u(k\zeta_1, y_2, \cdots, y_n) \tag{2.28}$$

Expanding the first terms on the right hand side in the ζ_2 direction, we then obtain

$$\begin{aligned} u(y) &= (u + m_1|\zeta_1|\delta_1 u + m_2|\zeta_2|\delta_2 u)(0, 0, y_3, \cdots, y_n) \\ &+ \sum_{k=0}^{m_1-2} (m_1 - k - 1)|\zeta_1|^2 \delta_1\delta_1 u(k\zeta_1, y_2, \cdots, y_n) \\ &+ \sum_{k=0}^{m_2-2} \left[(m_2 - k - 1)|\zeta_2|^2 \delta_2\delta_2 u + m_2|\zeta_1|\,|\zeta_2|\delta_1\delta_2 u\right](0, k\zeta_2, y_3, \cdots, y_n) \end{aligned} \tag{2.29}$$

Continuing this process, we end up with

$$
\begin{aligned}
&u(y) - u(0) \qquad\qquad (2.30)\\
&= \sum_{i=1}^{n} m_i|\zeta_i|\delta_i u(0)\\
&\quad + \sum_{k=0}^{m_1-2}(m_1-k-1)|\zeta_1|^2\delta_1\delta_1 u(k\zeta_1, y_2, \cdots, y_n)\\
&\quad + \cdots\cdots\\
&\quad + \sum_{k=0}^{m_n-2}\left[(m_n-k-1)|\zeta_n|^2\delta_n\delta_n u + \sum_{i=1}^{n-1} m_i|\zeta_i|\,|\zeta_n|\delta_i\delta_n u\right](0,0,\cdots,k\zeta_n)
\end{aligned}
$$

For general m_i, we replace ζ_i by $-\zeta_i$ and m_i by $|m_i|$ in the above formula whenever $m_i < 0$, to obtain,

$$
u(y) - u(0) = \sum_{i=1}^{n}|m_i|\left[u\Big(x + (\text{ sign } m_i)\,\zeta_i\Big) - u(0)\right] + \text{ second order differences.} \qquad (2.31)
$$

By the balance condition,

$$
\sum_{y\in Y} a(y)\, m_i(y) = 0\,,
$$

that is

$$
\sum_{m_i>0} a(y)|m_i| = \sum_{m_i<0} a(y)|m_i|
$$

and hence we obtain, from (??) and (??),

$$
\begin{aligned}
Lu(0) &= \sum_{y\in Y} a(y)\Big(u(y) - u(0)\Big) \qquad (2.32)\\
&= \sum_{y\in\widetilde{Y}} c_{ij}\delta_i\delta_j u(y)
\end{aligned}
$$

as required. The case of general $x \in E$, follows immediately by translation.

3. The General Case

We pass from the frozen to the general case though a method employed by Safonov [?], [?] for fully nonlinear partial differential equations, which was already extended to difference equations by Holtby in [?], [?]. As remarked earlier, our equations are more general than those considered by Holtby and overall our approach in simpler. We begin by considering equation (??) in a mesh ball $E_R = E_R(x_0)$, $R > 0$ and $x_0 \in E$, under the hypotheses (??), (??), (??), (??), (??) and (??). The interior E_R^o and boundary E_R^b of E, with respect to the difference operator F under our hypotheses, are given by

$$\begin{aligned} E_R^o &= \left\{ x \in E_R \,\middle|\, x + Y \subset E_R \right\}, \\ E_R^b &= E_R - E_R^o, \end{aligned}$$

where Y is given by (??). Letting P denote a polynomial of degree two, we consider the "frozen" Dirichlet problem,

$$\begin{aligned} F[v] &:= F(x_0, \tilde{L}v + \tilde{L}P) = 0 \quad \text{in } E_R^o, \\ v &= u - P \quad \text{on } E_R^b, \end{aligned} \tag{3.1}$$

where u is the given solution of equation (??) in E_R. The existence of a unique solution v of (??) may be shown from the method of continuity and the discrete maximum principle [?], [?], [?], as in [?]. Observing that $\tilde{L}P$ is a constant vector, we apply the interior estimate (??), with $\psi \equiv 0$, to obtain, for any $r < R - \overline{h}$,

$$\operatorname*{osc}_{E_r} \delta^2 v \leq C \frac{r^\alpha}{R^{2+\alpha}} \max_{E_R} |\, v \,|, \tag{3.2}$$

where C and α are as in (??). At this point, it is convenient to specify $P = P_0$ so that $\delta^\beta P_0(x_0) = \delta^\beta u(x_0)$ for $\beta \leq 2$. It follows that

$$F(x_0\,, \tilde{L}P_0(x_0)) = 0 \tag{3.3}$$

Hence we can write equation (??) in the form,

$$\sum_{j=1}^{K} \frac{\partial F}{\partial q_j}(\zeta)\, L_j v = 0 \tag{3.4}$$

for some ζ lying between $\tilde{L}v$ and $\tilde{L}(v+P)$. By the maximum principle, we then have

$$\begin{aligned} \max_{E_R} |v| &\leq \max_{E_R^b} |\, v \,| \\ &\leq \max_{E_R^b} |\, u - P_0 \,| \\ &\leq C\, R^{2+\gamma} \left[\delta^2 u \right]_{\gamma; E_R} \end{aligned} \tag{3.5}$$

for any $0 < \gamma \leq 1$ by the discrete Taylor formula (??), and our choice of P_0, (Note that the first order difference in (??) for negative m_i can be controlled through second order differences). Taking $\gamma < \alpha$ and substituting (??) into (??) we then obtain

$$r^{-\gamma} \operatorname*{osc}_{E_r} \delta^2 v \le C \left(\frac{r}{R}\right)^{\alpha-\gamma} \left[\delta^2 u\right]_{\gamma;E_R} \tag{3.6}$$

Next by combining equations (??) and (??), we have

$$\begin{aligned} \left|\sum_{j=1}^{K} \frac{\partial F}{\partial q_j}(\zeta)\, L_j(u - v - P_0)\right| &\le \left|F(x, \tilde{L}u) - F(x_0, \tilde{L}u)\right| \\ &\le \mu\left(1 + \left|\tilde{L}u\right|\right)R^\gamma \end{aligned} \tag{3.7}$$

in E_R, by (??), where ζ lies between $\tilde{L}u$ and $\tilde{L}(v + P_0)$. Applying the discrete maximum principle (see [?], [?], [?]), we thus obtain

$$\left|u - v - P_0\right| \le C R^{2+\gamma}\left(1 + \max\left|\tilde{L}u\right|\right) \tag{3.8}$$

in the mesh ball E_R. Consequently, letting $\wp$ denote the set of second degree polynomials, we obtain from (??), (??) and (??) with appropriate choice of polynomial p,

$$\begin{aligned} &r^{-2-\gamma} \inf_{p\in\wp} \max_{E_r} |u - p| \\ &\quad\le r^{-2-\gamma}\left\{\inf_{p\in\wp} \max_{E_r} |v - p| + \max_{E_r}\left|u - P_0 - v\right|\right\} \\ &\quad\le C\left\{\left(\frac{r}{R}\right)^{\alpha-\gamma}\left[\delta^2 u\right]_{\gamma;E_R} + \left(\frac{R}{r}\right)^{2+\gamma}\left(1 + \max_{E_R}\left|\delta^2 u\right|\right)\right\} \end{aligned} \tag{3.9}$$

with constant C depending on $n, \Lambda/\lambda, \mu/\lambda, \overline{h}/h, a_0/\rho h, \tilde{a}/\lambda_0, \ell_0$ and K. To get an interior Schauder estimate from (??), we let Ω be a subset of E, $x_0 \in \Omega^o$ and choose $r = \epsilon R$ with $R < R_0 = \frac{1}{2}\mathrm{dist}(x_0, \Omega^b)$. We thus obtain

$$r^{-2-\gamma} \inf_{p\in\wp} \max_{E_r} |u - p| \le C\left\{\epsilon^{\alpha-\gamma}[\delta^2 u]_{\gamma;E_R} + \epsilon^{-(2+\gamma)}\left(1 + \max_{E_R}|\delta^2 u|\right)\right\}. \tag{3.10}$$

Defining the *interior* Hölder semi-norms,

$$\begin{aligned} [u]^*_{\gamma;\Omega} &= \max_{\Omega'\subset\Omega} (d')^\gamma [u]_{\gamma,\Omega'} \\ [u]^*_{k;\Omega} &= \max_{\substack{\Omega'\subset\Omega \\ |\beta|=k}} (d')^k \left|\delta^\beta u\right|_{0;\Omega'} \\ [u]^*_{k,\gamma;\Omega} &= \max_{\substack{\Omega'\subset\Omega \\ |\beta|=k}} (d')^{k+\gamma} \left[\delta^\beta u\right]_{\gamma,\Omega'} \end{aligned} \tag{3.11}$$

where $d' = \mathrm{dist}(\Omega', \Omega^b)$, we may rewrite (??) as,

$$\left(\frac{r}{R_0}\right)^{-2-\gamma} \inf_{p\in\wp} \max_{E_r} |u - p| \le C\left\{\epsilon^{\alpha-\gamma}[u]^*_{2,\gamma;\Omega} + \epsilon^{-(2+\gamma)}\left(R_0^{2+\gamma} + R_0^{\gamma}[u]^*_{2;\Omega}\right)\right\}. \quad (3.12)$$

Note that (??) will also hold for $n \ge \epsilon R_0$ As in the continuous case, ([?], [?], [?]) the interior k, γ semi-norms in (??) are equivalent to the corresponding L^∞-Campanato semi-norms, whence we infer from (??),

$$[u]^*_{2,\gamma;\Omega} \le C\left\{\epsilon^{\alpha-\gamma}[u]^*_{2,\gamma;\Omega} + \epsilon^{-(2+\gamma)}\left(d^{2+\gamma} + d^{\gamma}[u]^*_{2;\Omega}\right)\right\}. \quad (3.13)$$

where $d = \mathrm{diam}\,\Omega$. By choosing ϵ sufficiently small and using the interpolation inequalities [?], [?], (as in the continuous case [?], [?]), we finally arrive at the interior Schauder estimates,

$$\begin{aligned} \| u\|^*_{2,\gamma;\Omega} &= \sum_{k=0}^{2}[u]^*_{k;\Omega} + [u]^*_{2,\gamma;\Omega} \quad (3.14)\\ &\le C\left(| u |_{0;\Omega} + \left(\mathrm{diam}\ \Omega\right)^{2+\gamma}\right), \end{aligned}$$

where C depends on n, Λ/λ, μ/λ, $\overline{h}/h$, $a_0/\rho h$, $\tilde{a}/\lambda_0$, ℓ_0, γ and K.

Accordingly we have the following theorem.

Theorem 3.1. *Let E be a lattice mesh in $\mathbb{R}^n$ and suppose u is a solution of the difference equation (??) in a bounded subset Ω of E. Assume that F satisfies the structural conditions, (??), (??), (??), (??), (??), (??). Then there exists a constant $\alpha > 0$, depending on n, Λ/λ, $\overline{h}/h, a_0/\rho h, \ell_0, \tilde{a}/\lambda_0$ such that if $\gamma < \alpha$ then*

$$\| u \|^*_{2,\gamma;\Omega} \le C, \quad (3.15)$$

where C depends additionally on μ/λ, γ, $|u|_0$ and $\mathrm{diam}\,\Omega$.

In a sequel paper, we will consider the extension of Theorem ?? to general meshes. The main difficulties here are that we cannot have constant operators $\widetilde{L}$ nor define the iterated difference operators δ^β on mesh functions

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