STOCHASTIC ASPECTS OF LANCHESTER'S THEORY OF WARFARE

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Dedicated to M.S. Bartlett (1910–2002)

Abstract

A Markov chain model for a battle between two opposing forces is formulated, which is a stochastic version of one studied by F.W. Lanchester. Solutions of the backward equations for the final state yield martingales and stopping identities, but a more powerful technique is a time-reversal analogue of a known method for studying urn models. A general version of a remarkable result of Williams and McIlroy is proved.

1. Deterministic theory

Frederick William Lanchester (1868–1946), pioneering engineer and amateur mathematician, made many contributions to the early years of the motor car and later to aviation. His biography by Ricardo (1948) gives an account of a long and varied career, and a recent article by Cummings (2001) gives an appreciation from an operational research viewpoint.

Cummings emphasises his work on theories of warfare and battles, and in particular his fascination with the battle of Trafalgar, fought in 1805 between a Franco-Spanish fleet commanded by Villeneuve and a British fleet commanded by Nelson. Lanchester suggested that simple deterministic models could provide useful insight into such events, but such models tend to fail when opponents are evenly matched and random effects may be decisive.

For the sake of vividness we shall use the language of a naval battle although Lanchester applied his methods to other modes of conflict. Suppose that two hostile fleets, of x_0 and y_0 ships respectively, approach one another and start firing. The gunfire results from time to time in ships being sunk or disabled; suppose that at time t there are x_t and y_t ships remaining active. Lanchester models these as continuous variables, and postulates the differential equations

$$\frac{dx}{dt} = -by\ ,\ \frac{dy}{dt} = -ax\,. \tag{1.1}$$

The strictly positive constants a and b measure the effectiveness of the firing of the two fleets, so that the rate of attrition of each fleet is proportional to the gunpower of the other.

Equation (1.1) is very easy to resolve. Note first that

$$\frac{d}{dt}(ax^2 - by^2) = 0\,,$$

so that

$$ax_t^2 - by_t^2 = ax_0^2 - by_0^2 = \Delta\,, \tag{1.2}$$

say. The constant Δ measures the advantage or disadvantage of one fleet relative to the other. The point (x_t, y_t) travels down the hyperbolic trajectory (1.2) until it hits one axis or the other, when the battle is won and lost.

If $\Delta > 0$, it is easy to check that this occurs at time

$$\tau = (ab)^{1/2} \cosh^{-1}\left(a^{1/2}\Delta^{-1/2}x_0\right)\,, \tag{1.3}$$

when

$$x_\tau = a^{-1/2}\Delta^{1/2} = \left\{a^{-1}\left(ax_0^2 - by_0^2\right)\right\}^{1/2}\,,\; y_\tau = 0\,. \tag{1.4}$$

Thus the first fleet wins, and (1.4) gives the number of survivors.

Similarly, if $\Delta < 0$, the final result is

$$\tau = (ab)^{-1/2} \cosh^{-1}\left(b^{1/2}|\Delta|^{-1/2}y_0\right)\,,\; y_\tau = \left\{b^{-1}\left(by_0^2 - ax_0^2\right)\right\}^{1/2}\,. \tag{1.5}$$

Thus not only does (1.1) yield a clear criterion to predict the winner, but it also predicts the number of ships surviving after the battle.

The critical case $\Delta = 0$ is much less satisfactory. Equation (1.2) implies that $ax_t^2 = by_t^2$ for all t, and substituting back into (1.1) gives

$$x_t = x_0 e^{-\gamma t}\,,\; y_t = y_0 e^{-\gamma t}\,,\; \gamma = (ab)^{1/2}\,. \tag{1.6}$$

Both fleets decay exponentially, the battle apparently takes an infinite time, and there are not survivors.

This model is defective because it ignores random effects. A few lucky hits early in the battle will move Δ away from zero, to give one fleet or other a decisive advantage. Thus it is important to have stochastic models to refine (1.1).

Lanchester's analysis can be greatly generalised without losing much of the simplicity of his results. Thus (1.1) could be replaced by

$$\frac{dx}{dt} = -b(y)c(x,y,t)\,, \ \frac{dy}{dt} = -a(x)c(x,y,t)\,, \tag{1.7}$$

for strictly positive functions a, b and c. This allows the firepower of each fleet to depend non-linearly on the number of ships (perhaps because they get in the way of each other), and for weather and sea conditions to be modelled in the function c.

Writing

$$A(x) = \int_0^x a(\xi)d\xi\,, \ B(y) = \int_0^y b(\eta)d\eta\,, \tag{1.8}$$

(1.7) shows that

$$\frac{d}{dt}\{A(x) - B(y)\} = 0\,,$$

so that

$$A(x_t) - B(y_t) = A(x_0) - B(y_0) = \Delta\,, \tag{1.9}$$

say. If $\Delta > 0$, (1.9) implies that $A(x_t) \geqslant \Delta$, so that the first fleet never falls below a minimum size. This is not quite the same as saying that the first fleet wins the battle, since a, b and c could be such that the battle might go on for ever. It is however easy to impose conditions that imply that there is a finite τ with $y_\tau = 0$, and then the number of survivors is given by the equation

$$A(x_\tau) = \Delta\,. \tag{1.10}$$

Similarly, if $\Delta < 0$ the second fleet never loses, and if it wins at time τ the number of survivors satisfies

$$B(y_\tau) = -\Delta\,. \tag{1.11}$$

The case $\Delta = 0$ is again critical, and demands a stochastic treatment.

2. A Markov model

We consider a stochastic analogue of (1.7) in the form of a continuous time Markov chain on the quadrant

$$\mathcal{Q} = \{(x,y)\,;\ x,y = 0,1,2,\ldots\,,\ x+y \geqslant 1\} \tag{2.1}$$

with (0,0) removed. The state at time t is (X_t, Y_t), where X_t represents the number of ships in the first fleet still firing at t, and Y_t the number in the second.

If

$$X_0 = x_0\,,\ Y_0 = y_0\,, \tag{2.2}$$

the chain has effective state space

$$\mathcal{Q}(x_0, y_0) = \{(x,y) \in \mathcal{Q};\ x \leqslant x_0\,,\ y \leqslant y_0\} \tag{2.3}$$

which is finite, so that the complications associated with infinite state spaces do not arise.

By analogy with (1.7), it is assumed that the transition rates of the Markov chain are non-zero only for transitions of the form $(x,y) \longmapsto (x, y-1)$ and $(x,y) \longmapsto (x-1, y)$, and that they are of the form $a(x)c(x,y,t)$ and $b(y)c(x,y,t)$. Here the functions a, b and c are strictly positive, except that $a(0) = b(0) = 0$. The states $(x, 0)$ and $(0, y)$, which form the boundary $\partial\mathcal{Q}$ of $\mathcal{Q}$, are absorbing.

The dependence of c on t, if non-trivial, renders the chain one with non-stationary transition probabilities, and this can give complications which do not seem profitable to pursue. We shall therefore take c to be a function $c(x,y)$ on $\mathcal{Q}$, though it would not be hard to remove this restriction if necessary.

Thus we assume that

$$\begin{aligned} \mathbb{P}\{X_{t+h} &= x\,,\ Y_{t+h} = y-1 | X_t = x\,, Y_t = y\} = a(x)c(x,y)h + o(h) \\ \mathbb{P}\{X_{t+h} &= x-1\,,\ Y_{t+h} = y | X_t = x\,,\ Y_t = y\} = b(y)c(x,y)h + o(h) \end{aligned} \tag{2.4}$$

as $h \to 0$ from above for $x, y \geqslant 1$, all other transitions having probability $o(h)$. This determines the transition probabilities of the Markov chain (X_t, Y_t) on the finite state space $\mathcal{Q}(x_0, y_0)$. With probability one the chain reaches, and remains in, one of the states of $\partial\mathcal{Q}$ after a finite time. This final state (X_∞, Y_∞) is either $(S, 0)$ or $(0, S)$ where S is the number of survivors.

It is convenient to define an indicator variable χ which is 1 in the first case and 0 in the second, and which therefore shows which fleet wins. The pair (χ, S) is a coding of the final state

$$X_\infty = \chi S\,,\ Y_\infty = (1-\chi)S\,, \tag{2.5}$$

and our aim will be to calculate the joint distribution of χ and S.

This process was first studied by Williams and McIlroy (1998) in the very special case

$$a(x) = x\,,\ \ b(y) = y \tag{2.6}$$

which corresponds to Lanchester's model (1.1) with $a = b$. They use the language of the gunfight at the OK Corral, and we shall therefore refer to this special case as the OK Corral. They arrived at the remarkable result that, if $x_0 = y_0$ and $x_0 \to \infty$, then

$$\mathbb{E}(S) \sim 2(3)^{-1/4}\pi^{-1/2}\Gamma(3/4)x_0^{3/4}\,. \tag{2.7}$$

They did not in fact prove (2.7), but observed the 3/4 power from numerical evidence, and then described a diffusion approximation which yields the coefficient. It seems difficult to make this rigorous, but a different approach by Kingman (1999) does succeed in proving (2.7), and in finding the limiting distribution of S. (By symmetry χ is independent of S, taking values 0 and 1 with equal probability.) It is proved that, as $x_0 \to \infty$, the random variable

$$(2\chi - 1)(2x_0)^{-3/2}S^2 \tag{2.8}$$

has a limiting normal distribution $\mathcal{N}(0, \frac{1}{3})$, from which (2.7) follows at once. More generally, if x_0 and y_0 are nearly but not exactly equal, in the sense that $x_0, y_0 \to \infty$ with

$$x_0 - y_0 \sim \mu(x_0 + y_0)^{1/2} \tag{2.9}$$

for some constant μ, then (2.8) has limiting distribution $\mathcal{N}(\mu, \frac{1}{3})$.

The 1999 analysis does not generalise to our chain (2.4), but there is a simpler approach which does. As in the earlier work, this relies on the identification of functions $f(X_t, Y_t)$ which are martingales, or what is the same thing on solutions of the Kolmogorov backward equations. The imaginative use of these equations was one of the hallmarks of the work of the great statistician Maurice Bartlett, and I dedicate this paper with respect, affection and gratitude to his memory.

3. The backward equation

In this section (X_t, Y_t) is a continuous time Markov chain with state space Q and initial state (x_0, y_0) whose transition rates are given by (2.4). The final state (X_∞, Y_∞) is either $(S, 0)$ or $(0, S)$ for some $S \geqslant 1$, and is reached after a finite time with probability one. Together with the indicator variable χ, S labels the final state as in (2.5).

Let g by any function on $\{0,1\} \times \{1,2,3,\ldots\}$ and let

$$f(x_0, y_0) = \mathbb{E}\{g(\chi, S)\}\,. \tag{3.1}$$

Then f, as a function of the initial state, satisfies the backward Kolmogorov equation (see for instance Chung (1967) section II.17):

$$\begin{aligned}\{a(x)c(x,y) + b(y)c(x,y)\}f(x,y)\\ = a(x)c(x,y)f(x,y-1) + b(y)c(x,y)f(x-1,y)\end{aligned}$$

or

$$\{a(x) + b(y)\}f(x,y) = a(x)f(x,y-1) + b(y)f(x-1,y)\,. \tag{3.2}$$

The reason $c(x,y)$ disappears is that the 'jump chain', which determines the successive states of the chain without regard for time, has transition probabilities depending only on a and b.

Equation (3.2) is fundamental, and was used as the starting point by Williams and McIlroy in the OK Corral. They noted that (by induction on $x+y$), f is uniquely determined by (3.2) and the boundary conditions

$$f(x,0) = g(1,x)\,,\ f(0,y) = g(0,y)\,. \tag{3.3}$$

This viewpoint was reversed in (Kingman, 1999). If f is *any* solution of (3.2) and g is *defined* by (3.3), then (3.1) holds.

Equation (3.2) is just the condition for $f(X_t, Y_t)$ to be a martingale. Thus (3.1) is the familiar stopping time identity

$$f(x_0, y_0) = \mathbb{E}\{f(X_\infty, Y_\infty)\}\,. \tag{3.4}$$

By finding a sufficiently rich family of solutions of (3.2), we can gain enough information to deduce the distribution of (χ, S).

One such solution is easily found. By direct substitution,

$$f(x,y) = A(x) - B(y) \tag{3.5}$$

satisfies (3.2) if

$$A(x) = \sum_{i=1}^{x} a(i)\,,\ B(y) = \sum_{j=1}^{y} b(j)\,. \tag{3.6}$$

This choice of f leads to the *basic martingale*

$$Z_t = A(X_t) - B(Y_t)\,, \tag{3.7}$$

for which (3.1) becomes

$$A(x_0) - B(y_0) = \mathbb{E}\{\chi A(S) - (1-\chi)B(S)\}\,. \tag{3.8}$$

Specialising to the OK Corral,

$$f(x,y) = \tfrac{1}{2}x(x+1) - \tfrac{1}{2}y(y+1) = \tfrac{1}{2}(x-y)(x+y+1)\,, \tag{3.9}$$

and (3.8) becomes

$$(x_0 - y_0)\,(x_0 + y_0 + 1) = \mathbb{E}\{(2\chi - 1)S(S+1)\}\,. \tag{3.10}$$

This is trivial if $x_0 = y_0$, since by symmetry $2\chi - 1$ is ± 1 with equal probabilities independent of S. It is however shown in (Kingman, 1999) that (3.10) is the first of an infinite sequence of identities of which the next is

$$\begin{aligned}(x_0 + y_0 + 1)\,(x_0\ + y_0 + 2)\left\{x_0 + y_0 + 3\,(x_0 - y_0)^2\right\}\\ = \mathbb{E}\{S(S+1)(S+2)(3S+1)\}\,.\end{aligned} \tag{3.11}$$

This is informative even when $x_0 = y_0$, and the whole sequence determines the distribution of (χ, S) and is enough to prove the limit theorem cited in the last section.

Unfortunately, the algebraic methods of 1999 do not easily generalise, even to the case

$$a(x) = ax\,,\ b(y) = by \tag{3.12}$$

with $a \neq b$. We therefore use a different approach to find solutions of (3.2).

4. A martingale generating function

Seek a solution of (3.2) of product form:

$$f(x,y) = \phi(x)\psi(y)\,. \tag{4.1}$$

Substituting and rearranging, we have

$$\frac{\phi(x)-\phi(x-1)}{a(x)\phi(x)} = -\frac{\psi(y)-\psi(y-1)}{b(y)\psi(y)} . \tag{4.2}$$

Since the left hand side is a function only of x, and the right only of y, they must equal a constant λ (say), and then

$$\phi(x) = \phi(0) \prod_{i=1}^{x} \{1-\lambda a(i)\}^{-1}$$

and

$$\psi(y) = \psi(0) \prod_{j=1}^{y} \{1+\lambda b(j)\}^{-1} .$$

The algebra reverses, showing that

$$f(x,y) = \prod_{i=1}^{x} \{1-\lambda a(i)\}^{-1} \prod_{j=1}^{y} \{1+\lambda b(j)\}^{-1} \tag{4.3}$$

satisfies (3.2) for any λ except $a(i)^{-1}$ or $-b(j)^{-1}$ $(i,j=1,2,\ldots)$.

The interested reader will easily verify that, in the OK Corral, the 1999 solutions are (apart from numerical factors) the coefficients of powers of λ in the expansion of (4.3). In the general case, (3.5) is the coefficient of λ, and the other coefficients give further solutions.

It is however more productive to apply (3.1) directly to (4.3), giving the very general identity

$$\begin{aligned} &\prod_{i=1}^{x_0} \{1-\lambda a(i)\}^{-1} \prod_{j=1}^{y_0} \{1+\lambda b(j)\}^{-1} \\ &= \mathbb{E}\left[\chi \prod_{i=1}^{S} \{1-\lambda a(i)\}^{-1} + (1-\chi) \prod_{j=1}^{S} \{1+\lambda b(j)\}^{-1}\right] . \end{aligned} \tag{4.4}$$

This holds as an identity between rational functions of λ (since the expectation is a finite sum), and contains enough information to determine the joint distribution of χ and S. To see this, first suppose that the $a(i)$ are distinct, multiply (4.4) by $1-\lambda a(x)$ and let $\lambda \to a(x)^{-1}$ to get

$$\prod_{\substack{i=1 \\ i\neq x}}^{x_0} \frac{a(x)}{a(x)-a(i)} \prod_{j=1}^{y_0} \frac{a(x)}{a(x)+b(j)} = \mathbb{E}\left\{\chi_x \prod_{\substack{i=1 \\ i\neq x}}^{S} \frac{a(x)}{a(x)-a(i)}\right\} , \tag{4.5}$$

where $\chi_x = 1$ if $\chi = 1$ and $S \geqslant x$ and 0 otherwise. Downward recursion on $x = x_0$, $x_0 - 1, \ldots, 1$ determines the distribution of S when $\chi = 1$, and a similar argument deals with $\chi = 0$. Finally the case when the $a(i)$ are not distinct can be handled by a continuity argument.

Equation (4.4) can be used to derive useful inequalities. If λ satisfies

$$0 < \lambda < a(i)^{-1} \quad (i = 1, 2, \ldots, x_0) \;, \tag{4.6}$$

then

$$\begin{aligned} \mathbb{P}\{\chi = 1\} &\leqslant \mathbb{E}\left[\chi \prod_{i=1}^{S} \{1 - \lambda a(i)\}^{-1}\right] \\ &\leqslant \prod_{i=1}^{x_0} \{1 - \lambda a(i)\}^{-1} \prod_{j=1}^{y_0} \{1 + \lambda b(j)\}^{-1} \\ &= 1 + \lambda \{A(x_0) - B(y_0)\} + O(\lambda^2) \end{aligned}$$

as $\lambda \to 0$. This is a non-trivial upper bound for small λ if

$$A(x_0) < B(y_0) \tag{4.7}$$

and can be used for suitable a and b to show that the probability that $\chi = 1$ is small when x_0 and y_0 are large. To illustrate in more detail, suppose that (as in the OK Corral)

$$a(i) = b(i) \tag{4.8}$$

for all i, and that a is non-decreasing. Then (4.7) is equivalent to $x_0 < y_0$, and then

$$\begin{aligned} \mathbb{P}\{\chi = 1\} &\leqslant \prod_{i=1}^{x_0} \{1 - \lambda^2 a(i)^2\}^{-1} \prod_{j=x_0+1}^{y_0} \{1 + \lambda a(j)\}^{-1} \\ &\leqslant \prod_{i=1}^{x_0} \left\{1 - \lambda^2 a(x_0)^2\right\}^{-1} \prod_{j=x_0+1}^{y_0} \{1 + \lambda a(x_0)\}^{-1} \\ &= \{1 - \lambda a(x_0)\}^{-x_0} \{1 + \lambda a(x_0)\}^{-y_0} \; . \end{aligned}$$

The best value of λ in this last upper bound has

$$\lambda a(x_0) = \frac{y_0 - x_0}{y_0 + x_0} = \zeta$$

(say), when the upper bound becomes

$$\mathbb{P}\{\chi = 1\} \leqslant (1-\zeta)^{-x_0}(1+\zeta)^{-y_0} < \exp\left\{-\tfrac{1}{2}(x_0+y_0)\zeta^2\right\}. \tag{4.9}$$

Thus if

$$(y_0 - x_0)^2 / (x_0 + y_0)$$

is large, there is high probability that the larger fleet will win.

This has so far only been proved under the symmetry condition (4.8) and for monotone a, but we shall return to the general case in Section 6.

5. Decoupling the fleets

The existence of the large family (4.3) of martingales of product form suggests that in some sense the component processes X_t and Y_t should be independent, but they are not, nor are they even Markov processes in the simplest case of the OK Corral. It was noticed however by S.E Volkov that the OK Corral can be seen as a time-reversed form of a particular urn model, and that some urn models can be decoupled using ideas of S. Karlin and H. Rubin.

The consequences of this idea were worked out for the OK Corral in Kingman and Volkov (2002), but the argument is more transparent in the context of the general Lanchester model. We have seen that the distribution of the final state (though not of the time to reach it) does not depend on the function c, and there is accordingly no loss of generality in taking

$$c(x,y) = a(x)^{-1}b(y)^{-1}. \tag{5.1}$$

Then the transition rate from (x,y) to $(x-1,y)$ is

$$b(y)c(x,y) = a(x)^{-1}$$

and is independent of y. Similarly the transition rate from (x,y) to $(x,y-1)$ is $b(y)^{-1}$, and is independent of x. It follows easily that, under (5.1), X_t and Y_t are independent pure death processes with respective rates $a(x)^{-1}$ and $b(y)^{-1}$, up to the time when one of them reaches zero.

It must be stressed that (5.1) is not a realistic choice in modelling terms. It is simply a mathematical convenience to facilitate calculation of the distribution of (χ, S). To see how this works, let $\xi_r (r = 0, 1, 2, \ldots, x_0 - 1)$ be the first time that X_t enters r, and η_r be a similar quantity for Y_t. Then the event $\{\chi = 1, S = r\}$ is the event that, at $t = \eta_0$, $X_t = r \geqslant 1$. Thus

$$\mathbb{P}\{\chi = 1\,,\ S = r\} = \mathbb{P}\{\xi_r < \eta_0 < \xi_{r-1}\} \tag{5.2}$$

and similarly

$$\mathbb{P}\{\chi = 0\,,\ S = r\} = \mathbb{P}\{\eta_r < \xi_0 < \eta_{r-1}\}\,. \tag{5.3}$$

Since the increments $\xi_{r-1} - \xi_r$ and $\eta_{s-1} - \eta_s$ are independent with known (negative exponential) distributions, these probabilities can be calculated by standard techniques of renewal theory. This is done in detail for the OK Corral in (Kingman and Volkov (2002)) and the generalisation to the present model is entirely straightforward.

Under mild conditions on a and b, the central limit theorem can be applied to ξ_r and η_r to give limit theorems for (χ, S). It is however simpler to proceed directly from the identity (4.4).

6. Asymptotic normality of the basic martingale

The basic martingale (3.7) has a final value, at the end of the battle, given by

$$Z = A(X_\infty) - B(Y_\infty) = \chi A(S) - (1-\chi)B(S)\,. \tag{6.1}$$

No information is lost in passing from (X_∞, Y_∞) or (χ, S) to Z. If $Z > 0$, then $\chi = 1$ and S is given uniquely by $A(S) = Z$. If $Z < 0$, then $\chi = 0$ and $B(S) = -Z$. Thus Z is a coding of the outcome of the battle, and it is also very convenient for formulating limiting results for large x_0, y_0.

To express this precisely, consider a sequence of initial values, replacing (x_0, y_0) by (x_n, y_n) and imposing conditions on the way in which

$$x_n \to \infty\,,\ y_n \to \infty\ (n \to \infty)\,. \tag{6.2}$$

The limiting result cited in Section 2 shows what sort of result can be expected. In the OK Corral,

$$Z = \tfrac{1}{2}(2\chi - 1)S(S+1)\,,$$

which can be normalised so as to have a limiting normal distribution if (2.9) is satisfied.

To extend this result to the general case, define

$$\mathcal{A}(x) = \sum_{i=1}^{x} a_i^2 \ , \ \mathcal{B}(y) = \sum_{j=1}^{y} b_j^2 \tag{6.3}$$

and

$$\sigma_n^2 = \mathcal{A}(x_n) + \mathcal{B}(y_n) \, , \tag{6.4}$$

Suppose that, as $n \to \infty$

$$A(x_n) - B(y_n) \sim \mu\sigma_n \tag{6.5}$$

for some constant μ, and assume also the uniform asymptotic negligibility condition

$$\max\{a(i)\,(1 \leqslant i \leqslant x_n) \ , \ b(j)\,(1 \leqslant j \leqslant y_n)\} = o\,(\sigma_n) \ . \tag{6.6}$$

Then, if

$$\lambda = \theta\sigma_n^{-1} \tag{6.7}$$

for fixed θ, the left hand side of (4.4) has logarithm

$$\begin{aligned}
&-\sum_{i=1}^{x_n} \log\left\{1 - \theta\sigma_n^{-1}a(i)\right\} - \sum_{j=1}^{y_n} \log\left\{1 + \theta\sigma_n^{-1}b(j)\right\} \\
&= \sum_{i=1}^{x_n} \left\{\theta\sigma_n^{-1}a(i) + \tfrac{1}{2}\theta^2\sigma_n^{-2}a(i)^2\right\} - \sum_{j=1}^{y_n} \left\{\theta\sigma_n^{-1}b(j) - \tfrac{1}{2}\theta^2\sigma_n^{-2}b(j)^2\right\} + o(1) \\
&= \theta\sigma_n^{-1}\left\{A\,(x_n) - B\,(y_n)\right\} + \tfrac{1}{2}\theta^2\sigma_n^{-2}\left\{\mathcal{A}\,(x_n) + \mathcal{B}\,(y_n)\right\} + o(1) \\
&= \mu\theta + \tfrac{1}{2}\theta^2 + o(1) \, .
\end{aligned}$$

Hence

$$\mathbb{E}\left[\chi \prod_{i=1}^{S} \left\{1 - \theta\sigma_n^{-1}a(i)\right\}^{-1} + (1-\chi) \prod_{j=1}^{S} \left\{1 + \theta\sigma_n^{-1}b(j)\right\}^{-1}\right] \to e^{\mu\theta + \frac{1}{2}\theta^2} \tag{6.8}$$

as $n \to \infty$, for all real θ. Now use the fact that

$$e^{-u} \leqslant \frac{1}{1+u} \leqslant e^{-u}/(1-\delta)e^{\delta}$$

when $|u| \leqslant \delta < 1$ to deduce that, for large n, the expression in square brackets in (6.8) lies between Ψ and $\Psi/(1-\delta)e^{\delta}$, where

$$\begin{aligned}
\Psi &= \chi \prod_{i=1}^{S} \exp\left\{\theta\sigma_n^{-1}a(i)\right\} + (1-\chi) \prod_{j=1}^{S} \exp\left\{-\theta\sigma_n^{-1}b(j)\right\} \\
&= \chi \exp\left\{\theta\sigma_n^{-1}A(S)\right\} + (1-\chi)\exp\left\{-\theta\sigma_n^{-1}B(S)\right\} \qquad , \\
&= \exp\left\{\theta\sigma_n^{-1}Z\right\}
\end{aligned}$$

and since δ can be arbitrarily small, this implies that

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left\{\theta\sigma_n^{-1}Z\right\}\right] = e^{\mu\theta+\frac{1}{2}\theta^2}\,. \tag{6.9}$$

This proves that, under conditions (6.5) and (6.6), Z/σ_n has limiting distribution $\mathcal{N}(\mu, 1)$. This is an exact generalisation of the theorem of Kingman (1999), since in the OK Corral

$$Z \sim \tfrac{1}{2}(2\chi - 1)S^2$$

and

$$A(x) \sim \tfrac{1}{2}x^2\,,\ B(y) \sim \tfrac{1}{2}y^2\,,\ \mathcal{A}(x) \sim \tfrac{1}{3}x^3\,,\ \mathcal{B}(y) \sim \tfrac{1}{3}y^3\,.$$

Thus (2.9) is the special case of (6.5), and

$$\sigma_n^2 \sim \tfrac{1}{3}(x_n^3 + y_n^3) \sim \frac{1}{12}(x_n + y_n)^3\,.$$

To say that Z/σ_n has limiting distribution $\mathcal{N}(\mu, 1)$ is to say that, for any fixed ζ,

$$\lim_{n\to\infty} \mathbb{P}\{Z \leqslant \sigma_n\zeta\} = \Phi(\zeta - \mu)\,, \tag{6.10}$$

where

$$\Phi(u) = (2\pi)^{-1/2}\int_{-\infty}^{u} e^{-\frac{1}{2}v^2}\,dv\,.$$

In particular, putting $\zeta = 0$ and recalling that $Z > 0$ if and only if $\chi = 1$, we have

$$\lim_{n\to\infty} \mathbb{P}\{\chi = 1\} = \Phi(\mu)\,. \tag{6.11}$$

Thus μ measures the advantage of the first fleet over the second; if μ is positive and large the first fleet is highly likely to win. Conversely, if μ is strongly negative the second fleet will very probably win. This tallies with the interpretation of Δ in the deterministic model.

More precisely, suppose that (6.5) is replaced by

$$\Delta_n = A(x_n) - B(y_n) > 0\,,\ \sigma_n/\Delta_n \to 0\,. \tag{6.12}$$

Then the same limiting argument from (4.4) now shows that

$$\mathbb{P}\{\chi = 1\} \to 1 \tag{6.13}$$

and that

$$A(S) \sim \Delta_n \tag{6.14}$$

in probability as $n \to \infty$.

7. Conclusions

The analysis of the stochastic model validates Lanchester's deterministic model so long as x_0 and y_0 are large, and

$$\Delta = A(x_0) - B(y_0) \tag{7.1}$$

is large (in absolute value) compared with

$$\sigma = \{\mathcal{A}(x_0) + \mathcal{B}(y_0)\}^{1/2} . \tag{7.2}$$

If $\Delta > 0$ the first fleet will prevail, and will be left with a number of survivors determined by

$$A(S) = \Delta . \tag{7.3}$$

If $\Delta < 0$ the second fleet will win, with S ships still afloat, where

$$B(S) = -\Delta . \tag{7.4}$$

If Δ is not large compared with σ, a stochastic treatment is essential. The probability that the first fleet wins is approximately $\Phi(\Delta/\sigma)$, and the distribution of S is approximated by (6.10).

To see how this works out, suppose for simplicity complete symmetry between the two fleets, so that

$$x_0 = y_0 , \ a(i) = b(i) \ (i = 1, 2, \ldots) . \tag{7.5}$$

Then

$$\Delta = 0 , \ \sigma^2 = 2\mathcal{A}(x_0) , \ Z = (2\chi - 1) A(S) . \tag{7.6}$$

and in the normal approximation

$$\mathbb{P}\left\{A(S) \leqslant \zeta \, [2\mathcal{A}(x_0)]^{1/2}\right\} \doteqdot 2\Phi(\zeta) - 1 . \tag{7.7}$$

The $\frac{3}{4}$ power in the OK Corral can be seen as a consequence of the linearity of $a(i)$. If (2.6) is replaced by

$$a(x) = x^{\alpha}\,,\ b(y) = y^{\alpha} \tag{7.8}$$

for some constant α, (7.7) shows that $\frac{3}{4}$ must be replaced by

$$\frac{2\alpha+1}{2(\alpha+1)}\,.$$

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