TORIC RESIDUES AND MIRROR SYMMETRY

VICTOR V. BATYREV AND EVGENY N. MATEROV

To Yuri Ivanovich Manin on his 65-th birthday

ABSTRACT. We develop some ideas of Morrison and Plesser and formulate a precise mathematical conjecture which has close relations to toric mirror symmetry. Our conjecture, we call it Toric Residue Mirror Conjecture, claims that the generating functions of intersection numbers of divisors on a special sequence of simplicial toric varieties are power series expansions of some rational functions obtained as toric residues. We expect that this conjecture holds true for all Gorenstein toric Fano varieties associated with reflexive polytopes and give some evidences for that. The proposed conjecture suggests a simple method for computing Yukawa couplings for toric mirror Calabi-Yau hypersurfaces without solving systems of differential equations. We make several explicit computations for Calabi-Yau hypersurfaces in weighted projective spaces and in products of projective spaces.

CONTENTS

1. Introduction

The mirror symmetry attracts interest of mathematicians because it allows to identify mathematical objects of a very different nature: generating functions for Gromov-Witten invariants of rational curves on Calabi-Yau manifolds X and power series expansions of special functions on periods of the mirror family of Calabi-Yau manifolds X^*. Many examples of this identification can be computed explicitly for Calabi-Yau hypersurfaces in toric Fano varieties. The general toric mirror construction [Bat2] suggests a duality between Calabi-Yau varieties with Gorenstein canonical singularities. These singularities in general can not be resolved without changing the canonical class. Therefore, a verification of the toric mirror symmetry in full generality requires orbifold versions of quantum cohomology and Gromov-Witten invariants for singular varieties (cf. [CR1, CR2]). The Mirror Theorem states that power series obtained by these two very different methods are actually the same. It is rather nontrivial to prove the Mirror Theorem even in the very special case of Calabi-Yau quintic 3-folds [G2, G4].

In this paper we propose a more elementary mirror symmetry test which is expected to hold for *all* families of Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties associated with dual pairs of reflexive polytopes. We formulate this test as a mathematical conjecture and call it *Toric Residue Mirror Conjecture*. The idea of this conjecture is due to Morrison and Plesser [MP] who have checked it for some toric Calabi-Yau hypersurfaces (including the case of Calabi-Yau quintic 3-folds). Unfortunately, Morrison and Plesser didn't formulate their ideas in the form of a precise mathematical conjecture. The most important contribution of Morrison and Plesser [MP] is the construction of a sequence of toric manifolds $\{\mathbb{P}_\beta\}$ associated with pairs $(\mathbb{P}, \beta)$, where $\mathbb{P}$ is a smooth projective d-dimensional toric manifold $\mathbb{P}$ and β is an element in the monoid of integral points in the Mori cone $K_{\text{eff}}(\mathbb{P})$ of $\mathbb{P}$. Using ideas of Morrison and Plesser, we define a special cohomology class $\Phi_\beta \in H^{2(\dim \mathbb{P}_\beta - d)}(\mathbb{P}_\beta, \mathbb{Q})$ which plays the role of a "virtual fundamental class" in our mirror symmetry test. The manifold $\mathbb{P}_\beta$ and the cohomology class Φ_β will be called *Morrison-Plesser moduli space* and *Morrison-Plesser class* respectively. For our Toric Residue Mirror Conjecture, we will need a slight generalization of the Morrison-Plesser construction for projective simplicial toric varieties $\mathbb{P}$.

Let us make some remarks concerning the relation of our conjecture to the traditional point of view on the mirror symmetry for Calabi-Yau manifolds (see, e.g., [CK]). First of all, it is important to emphasize that Toric Residue Mirror Conjecture can not be obtained as a consequence of the mirror theorems due to Givental and Lian-Liu-Yau [G4, LLY2, LLY3]. On the other hand, it seems that all known methods for proving Mirror Theorem for toric Calabi-Yau hypersurfaces and complete intersections use some versions of the Morrison-Plesser moduli spaces

$\mathbb{P}_\beta$ as naive toric approximations of the Kontsevich moduli spaces of stable maps [G1, G2, G3, G4, G5, LLY1, LLY2, LLY3].

This makes impression that our conjecture could provide a helpful intermediate step for future formulations and verifications of the toric mirror symmetry predictions in case of Calabi-Yau varieties with orbifold singularities. We remark that the Toric Residue Mirror Conjecture also suggests a simple method for computing Yukawa $(d-1)$-point functions for $(d-1)$-dimensional toric Calabi-Yau hypersurfaces without using systems of differential equations.

The main advantage of the Toric Residue Mirror Conjecture is its simplicity. One does not need to know much (e.g., the Hodge theory, Gromov-Witten invariants and quantum cohomology) in order to understand its statement. Our conjecture uses only the intersection theory on simplicial toric varieties $\mathbb{P}_\beta$ and the notion of toric residues which are standard operations in computational commutative algebra. Unfortunately, we were not able to prove our conjecture in general. In this paper, we check it for some classes of reflexive polytopes using direct computations. We hope that a conceptual proof of the Toric Residue Mirror Conjecture for arbitrary reflexive polytopes would be an important contribution to understanding of the mirror symmetry phenomenon.

The paper is organized as follows. In Section 2, we give a review of toric residues and their properties. In Section 3, we discuss Morrison-Plesser moduli spaces $\mathbb{P}_\beta$ associated with lattice points β in the Mori cone $K_{\text{eff}}(\mathbb{P})$ of a simplicial toric variety $\mathbb{P}$. In Section 4, we formulate the Toric Residue Mirror Conjecture and give some evidences for it. In Sections 5 and 6, we illustrate our conjecture for some simplest interesting cases. In Sections 7 and 8, we prove the Toric Residue Mirror Conjecture for Calabi-Yau hypersurfaces in weighted projective spaces and in product of projective spaces. Section 9 explains applications of our conjecture to the toric mirror symmetry and to the computations of Yukawa couplings for toric Calabi-Yau hypersurfaces. Some examples of these computations are given in Sections 10 and 11.

We remark that the Toric Residue Mirror Conjecture can be formulated in the same generality also for toric Calabi-Yau complete intersections associated with nef-partitions of reflexive polytopes. The details of this generalization will be published in the forthcoming paper [BM].

2. TORIC RESIDUES

The notion of toric residue was introduced by D. Cox using homogeneous coordinates on toric varieties [Cox2]. In this paper, we prefer another approach which uses the affine coordinates $t_1, \ldots, t_d$ on the algebraic torus $\mathbb{T} = \operatorname{Spec} \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

Denote by M the group of characters of $\mathbb{T}$ which is isomorphic to $\mathbb{Z}^d$. Let Δ be a d-dimensional convex polytope in $M_{\mathbb{R}} = M \otimes \mathbb{R}$ with vertices in M.

Definition 2.1. We denote by

$$S_\Delta = \bigoplus_{k=0}^{\infty} S_\Delta^k$$

the graded subring in $\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ whose k-th homogeneous component S_Δ^k is spanned as $\mathbb{C}$-vector space by all monomials $t_0^k t_1^{m_1} \cdots t_d^{m_d}$ such that the lattice point $(m_1, \ldots, m_d)$ is contained in $k\Delta$.

Definition 2.2. The variety $\mathbb{P}_\Delta := \operatorname{Proj} S_\Delta$ is called *the projective toric variety associated with the polytope* Δ. We denote by $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ the ample invertible sheaf on $\mathbb{P}_\Delta$ corresponding to the graded S_Δ-module $S_\Delta(-1)$.

Definition 2.3. Denote by $I_\Delta = \bigoplus_{k=0}^{\infty} I_\Delta^k$ the graded ideal in S_Δ whose k-th homogeneous component I_Δ^k is spanned over $\mathbb{C}$ by all monomials $t_0^k t_1^{m_1} \cdots t_d^{m_d}$ such that $(m_1, \ldots, m_d)$ is contained in the interior of $k\Delta$.

It is known that S_Δ is a Cohen-Macaulay ring and I_Δ is its dualizing module [Dan]. Let $g_0(t), g_1(t), \ldots, g_d(t)$ be generic Laurent polynomials in $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ with supports in Δ such that

$$G = \{G_0, G_1, \ldots, G_d\} := \{t_0 g_0(t), t_0 g_1(t), \ldots, t_0 g_d(t)\} \subset S_\Delta^1$$

is a regular sequence in S_Δ. We can consider the polynomials $G_0, G_1, \ldots, G_d$ also as global sections of $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ on $\mathbb{P}_\Delta$ having no common zeros.

Since G is also a regular sequence for the dualizing module I_Δ, we obtain two finite-dimensional graded $\mathbb{C}$-vector spaces

$$S_G := S_\Delta / \langle G_0, G_1, \ldots, G_d \rangle S_\Delta, \quad I_G := I_\Delta / \langle G_0, G_1, \ldots, G_d \rangle I_\Delta,$$

where I_G is a dualizing module of the Artinian graded ring S_G (see [Bat1, Proposition 9.4]). One has a duality

$$S_G^k \times I_G^{d+1-k} \to I_G^{d+1}, \quad k = 0, \ldots, d,$$

where I_G^{d+1} is a 1-dimensional $\mathbb{C}$-vector space. The Jacobian

$$(1) \qquad J_G := \det\left(t_i \frac{\partial G_j}{\partial t_i}\right)_{0 \leq i,j \leq d} = t_0^{d+1} \det \begin{pmatrix} g_0 & g_1 & \cdots & g_d \\ t_1 \frac{\partial g_0}{\partial t_1} & t_1 \frac{\partial g_1}{\partial t_1} & \cdots & t_1 \frac{\partial g_d}{\partial t_1} \\ \vdots & \vdots & \ddots & \vdots \\ t_d \frac{\partial g_0}{\partial t_d} & t_d \frac{\partial g_1}{\partial t_d} & \cdots & t_d \frac{\partial g_d}{\partial t_d} \end{pmatrix}$$

is an element of S_Δ^{d+1}. One can easily check that J_G is contained in $I_\Delta^{d+1} \subset S_\Delta^{d+1}$ (see [CDS, Proposition 1.2]) and the class of J_G in I_G spans I_G^{d+1}.

We define the toric residue in the following algebraic way:

Definition 2.4. Let $\mathrm{Vol}(\Delta)$ be the *normalized volume* of Δ, i.e., $d! \cdot(\text{volume of } \Delta)$. The $\mathbb{C}$-linear map

$$(2) \qquad \mathrm{Res}_G : I_\Delta^{d+1} \to \mathbb{C},$$

vanishing on the $\mathbb{C}$-subspace $\langle G_0, G_1, \ldots, G_d \rangle I_\Delta^d \subset I_\Delta^{d+1}$ and sending J_G to $\mathrm{Vol}(\Delta)$ is called the **toric residue**. This map establishes a canonical isomorphism

$$I_G^{d+1} = I_\Delta^{d+1}/\langle G_0, G_1, \ldots, G_d \rangle I_\Delta^d \cong \mathbb{C}.$$

Remark 2.5. This algebraic definition of the toric residue works as well for any algebraically closed field k of characteristic 0. We compare it with the definition given by D. Cox in Section 9.

There exists a more geometric point of view on the toric residue. For this, one remarks that the coherent sheaf $\widetilde{I_\Delta}$ associated with the graded S_Δ-module I_Δ is exactly the sheaf $\Omega_{\mathbb{P}_\Delta}^d$ of Zariski differential d-forms (or dualizing sheaf) on $\mathbb{P}_\Delta$ [Dan]. Comparing the pairing cohomologies

$$H^d(\mathbb{P}_\Delta, \Omega_{\mathbb{P}_\Delta}^d) \times H^0(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}) \to H^d(\mathbb{P}_\Delta, \Omega_{\mathbb{P}_\Delta}^d)$$

with the pairing

$$I_G^{d+1} \times S_G^0 \to I_G^{d+1},$$

we can canonically identify the 1-dimensional $\mathbb{C}$-space I_G^{d+1} with $H^d(\mathbb{P}_\Delta, \Omega_{\mathbb{P}_\Delta}^d)$. The regular sequence G defines a finite morphism of degree $\mathrm{Vol}(\Delta)$

$$\Psi : \mathbb{P}_\Delta \to \mathbb{P}^d, \quad p \mapsto (G_0(p) : G_1(p) : \cdots : G_d(p)).$$

Consider the Čech cocycle

$$\alpha = \frac{z_0^d}{z_1 \cdots z_d} \, d\left(\frac{z_1}{z_0}\right) \wedge \cdots \wedge d\left(\frac{z_d}{z_0}\right)$$

in the standard open covering $\mathcal{U} = \{z_i \neq 0\}_{i=0,\ldots,d}$ of $\mathbb{P}^d$ with the homogeneous coordinates $z_0, z_1, \ldots, z_d$. One can show that α determines a generator of $H^d(\mathbb{P}^d, \Omega_{\mathbb{P}^d}^d)$

[Ha1, Chapter III, § 7]. It is easy to check that the map

$$\Psi^* : H^d(\mathbb{P}^d, \Omega^d_{\mathbb{P}^d}) \to H^d(\mathbb{P}_\Delta, \Omega^d_{\mathbb{P}_\Delta})$$

sends α to the Čech cocycle

$$\Psi^*(\alpha) = \frac{J_G}{G_0 \cdots G_d} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d}$$

in the covering $\mathcal{U}' = \{G_i \neq 0\}_{i=0,\dots,d}$ of $\mathbb{P}_\Delta$. Since the sheaf $\Omega^d_{\mathbb{P}_\Delta}$ is dualizing, there exists a canonical *trace map* $T_{\mathbb{P}_\Delta} : H^d(\mathbb{P}_\Delta, \Omega^d_{\mathbb{P}_\Delta}) \to \mathbb{C}$ (see [Ha1, Chapter III, § 7]). By the property of the trace map

$$T_{\mathbb{P}_\Delta}([\Psi^*(\alpha)]) = \deg(\Psi) T_{\mathbb{P}^d}([\alpha]) = \mathrm{Vol}(\Delta) T_{\mathbb{P}^d}([\alpha])$$

(see [Ha2, Chapter III]). This naturally explains the normalization $\mathrm{Res}_G(J_G) = \mathrm{Vol}(\Delta)$ in Definition 2.4.

Let us review some properties of the toric residue.

Choose a regular sequence $G = \{G_0, G_1, \dots, G_d\}$ in S^1_Δ defined by Laurent polynomials $g_0(t), g_1(t), \dots, g_d(t)$ supported in Δ as above. It follows from the regularity of G that the set V_g of common zeros of $G_1, \dots, G_d$ in $\mathbb{P}_\Delta$ is finite. Next statement follows immediately from [CCD, Theorem 0.4] and [CDS, Proposition 1.3] and claims that toric residue can be expressed as a sum of local Grothendieck residues.

Theorem 2.6. *Let $p(t_1, \dots, t_d)$ be a Laurent polynomial with support in the interior of $(d+1)\Delta$. We set $P := t_0^{d+1} p(t_1, \dots, t_d) \in I^{d+1}_\Delta$ and choose a sufficiently generic regular sequence $(G_0, G_1, \dots, G_d) = (t_0 g_0, t_0 g_1, \dots, t_0 g_d) \subset S^1_\Delta$ such the set V_g of common zeros of $G_1, \dots, G_d$ in $\mathbb{P}_\Delta$ is contained in $\mathbb{T} \cong (\mathbb{C}^*)^d$. Then*

$$\mathrm{Res}_G(P) = \sum_{\xi \in V_g} \mathrm{res}_{g,\xi}(\omega_P),$$

where

$$\omega_P = \frac{p/g_0}{g_1 \cdots g_n} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d}$$

and $\mathrm{res}_{g,\xi}(\omega_P)$ is the local Grothendieck residue of the form ω_P at the point $\xi \in V_g$.
In particular, if all the common roots of $g_1, \dots, g_d$ in $\mathbb{T}$ are simple, then

$$\mathrm{Res}_G(P) = \sum_{\xi \in V_g} \frac{p(\xi)}{g_0(\xi) J^0_g(\xi)},$$

where

$$J^0_g := \det \left(t_i \frac{\partial g_j}{\partial t_i} \right)_{1 \leq i,j \leq d}.$$

Let A be a finite subset in $\Delta \cap M$ which contains all vertices of Δ. Write each of the polynomials $g_0(t), g_1(t), \ldots, g_d(t)$ as

$$g_j(t) = \sum_{m \in A} a_m^{(j)} t^m, \quad j = 0, 1, \ldots, d.$$

Let $\mathbb{Q}[a]$ be the polynomial ring in the variables $a_m^{(j)}$ ($m \in A, 0 \le j \le d$). Denote by $\mathcal{R}_{g_0,\ldots,g_d}(a) \in \mathbb{Q}[a]$ *the (unmixed) sparse A-resultant of* $g_0, \ldots, g_d$ defined in [GKZ, §8.2]. The following statement, which is a reformulation of [CDS, Theorem 1.4] and [Di, Proposition 3.5], claims that toric residue is a rational function in coefficients of Laurent polynomials $g_0, g_1, \ldots, g_d$.

Theorem 2.7. *For any interior lattice point m in $(d+1)\Delta$, there exists a polynomial $Q_m(a) \in \mathbb{Q}[a]$ such that*

$$\text{Res}_G(t_0^{d+1} t^m) = \frac{Q_m(a)}{\mathcal{R}_{g_0,\ldots,g_d}(a)}.$$

We will be mostly interested in toric residues in the special case when the regular sequence $F = (F_0, F_1, \ldots, F_d)$ of elements in S_Δ^1 is constructed as follows. Take a generic Laurent polynomial

$$f(t) = \sum_{m \in A} a_m t^m \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$$

and define

$$F_0 := t_0 f(t), \ F_1 := t_0 f_1(t), \ldots, F_d := t_0 f_d(t),$$

where $f_i(t) := t_i \partial f / \partial t_i$ ($1 \le i \le d$). In this case, the Jacobians J_F, J_F^0 become Hessians

$$H_f := \det\left(\left(t_i \frac{\partial}{\partial t_i}\right)\left(t_j \frac{\partial}{\partial t_j}\right) t_0 f\right)_{0 \le i,j \le d}, \quad H_f^0 := \det\left(t_j \frac{\partial f_i}{\partial t_j}\right)_{1 \le i,j \le d}.$$

We write for simplicity S_f, I_f, Res_f instead of S_F, I_F, Res_F respectively.

The *principal A-determinant* $E_A(f)$ of polynomial $f(t)$ is defined to be the sparse resultant $\mathcal{R}_{f,f_1,\ldots,f_d}(a)$ of polynomials $f(t), f_1(t), \ldots, f_d(t)$ [GKZ, §10.1]. It follows from [Bat1, §4] that principal A-determinant $E_A(f)$ is nonzero if and only if $F_0, F_1, \ldots, F_d$ form a regular sequence in S_Δ^1. In the latter case the polynomial $f(t)$ will be called Δ-*regular*. Note that Δ-regularity of f guarantees that all critical points of f are isolated.

Remark 2.8. There exists another definition of Δ-regularity of a Laurent polynomial $f = \sum_{m \in A} a_m t^m$. For any face Γ of Δ, we define the Laurent polynomial

$$f_\Gamma := \sum_{m \in A \cap \Gamma} a_m t^m.$$

The polynomial f is called Δ-*regular* if

$$(3) \qquad \left\{ t \in \mathbb{T} \; : \; f_\Gamma(t) = \frac{\partial f_\Gamma}{\partial t_1}(t) = \cdots = \frac{\partial f_\Gamma}{\partial t_d}(t) = 0 \right\} = \emptyset$$

for all faces $\Gamma \subseteq \Delta$. This definition is equivalent to the previous one, because $E_A(f)$ is equal to the product of discriminants [GKZ, Chapter 10, Theorem 1.2]

$$E_A(f) = \pm \prod_{\Gamma \subseteq \Delta} D_{A \cap \Gamma}(f_\Gamma)^{\mu_\Gamma}, \quad \mu_\Gamma \in \mathbb{N}$$

and for each face $\Gamma \subseteq \Delta$ the condition (3) holds if and only if the discriminant $D_{A \cap \Gamma}(f_\Gamma)$ is nonzero.

The statements of theorems 2.6 and 2.7 for the toric residue Res_f can be summarized as follows:

Theorem 2.9. *Let* $P = t_0^{d+1} p(t_1, \ldots, t_d)$ *be an arbitrary element in* I_A^{d+1}. *Then*

(i) for any sufficiently generic Δ-regular Laurent polynomial $f(t)$ such that the set V_f of all common zeros of $F_1, \ldots, F_d$ in $\mathbb{P}_\Delta$ is contained in $\mathbb{T} \cong (\mathbb{C}^*)^d$ and any critical point $\xi \in V_f$ of f is nondegenerate (i.e., $H_f^0(\xi) \neq 0$), one has

$$\mathrm{Res}_f(P) = \sum_{\xi \in V_f} \frac{p(\xi)}{f(\xi) H_f^0(\xi)}.$$

(ii) the toric residue $\mathrm{Res}_f(P)$ is a rational function in the coefficients of the polynomials $p(t)$ and $f(t)$. In particular, for any interior lattice point m in $(d+1)\Delta$, there exists a polynomial $Q_m(a) \in \mathbb{Q}[a]$ such that

$$\mathrm{Res}_f(t_0^{d+1} t^m) = \frac{Q_m(a)}{E_A(f)}.$$

Let $N := Hom(M, \mathbb{Z})$ be the dual to M lattice and $N_\mathbb{R} := N \otimes \mathbb{R}$. We denote by $\langle *, * \rangle$ the natural pairing $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$.

Definition 2.10 ([Bat2]). A polytope $\Delta \subset M_\mathbb{R}$ with vertices in M is called *reflexive* if it contains 0 as interior point and its polar polytope

$$\Delta^* = \{ y \in N_\mathbb{R} : \langle x, y \rangle \geq -1, \quad \forall x \in \Delta \} \subset N_\mathbb{R}$$

has vertices in N. We remark that Δ is reflexive if and only if $\mathbb{P}_\Delta$ is a Gorenstein toric Fano variety and $\mathcal{O}_{\mathbb{P}_\Delta}(1)$ is the anticanonical sheaf on $\mathbb{P}_\Delta$.

The reflexivity of Δ implies that Δ^* is also reflexive and $(\Delta^*)^* = \Delta$. We will demand that the finite subset $A \subset M \cap \Delta$ contains not only all the vertices of Δ, but also the lattice point 0 as interior.

Now consider the toric residue Res_f in the special case when the support polytope Δ of f is reflexive. In this case, I_Δ is the principal ideal in S_Δ generated by t_0. This

implies that $H_f \in I_\Delta^{d+1} \subset S_\Delta^{d+1}$ can be uniquely written as product $t_0 H_f'$, where

$$H_f' = t_0^d \det \begin{pmatrix} f & f_1 & \cdots & f_d \\ f_1 & t_1\frac{\partial f_1}{\partial t_1} & \cdots & t_1\frac{\partial f_d}{\partial t_1} \\ \vdots & \vdots & \ddots & \vdots \\ f_d & t_d\frac{\partial f_1}{\partial t_d} & \cdots & t_d\frac{\partial f_d}{\partial t_d} \end{pmatrix} \in S_\Delta^d.$$

The ring S_Δ is Gorenstein and the multiplication in Artinian graded ring S_f defines the perfect pairings

$$S_f^k \times S_f^{d-k} \to S_f^d, \quad k = 0, \ldots, d,$$

where S_f^d is a 1-dimensional $\mathbb{C}$-vector space generated by the class of H_f'. By abuse of notations, we denote by Res_f also the $\mathbb{C}$-linear map

$$\mathrm{Res}_f : S_\Delta^d \to \mathbb{C},$$

vanishing on the subspace $\langle F_0, F_1, \ldots, F_d \rangle S_\Delta^{d-1} \subset S_\Delta^d$ such that $\mathrm{Res}_f(H_f') = \mathrm{Vol}(\Delta)$. This map induces a canonical isomorphism

(4) $$S_f^d = S_\Delta^d / \langle F_0, F_1, \ldots, F_d \rangle S_\Delta^{d-1} \cong \mathbb{C}.$$

3. MORRISON-PLESSER MODULI SPACES

Let $\mathbb{P} = \mathbb{P}_\Sigma$ be a d-dimensional projective simplicial toric variety over $\mathbb{C}$ defined by a simplicial fan Σ whose 1-dimensional cones are generated by integral vectors $e_1, \ldots, e_n \in N$. There exists a canonical embedding $M = Hom(N, \mathbb{Z}) \hookrightarrow \mathbb{Z}^n$ defined by

$$m \mapsto (\langle m, e_1 \rangle, \ldots, \langle m, e_n \rangle)$$

which gives rise to the short exact sequence

$$0 \to M \to \mathbb{Z}^n \to \mathrm{Cl}(\mathbb{P}) \to 0,$$

where $\mathrm{Cl}(\mathbb{P})$ is canonically isomorphic to the group of Weil divisor classes on $\mathbb{P}$ modulo linear equivalence.

Let $G \subset (\mathbb{C}^*)^n$ be the diagonalizable algebraic group over $\mathbb{C}$ with the character group $\mathrm{Cl}(\mathbb{P})$. Then $\mathbb{P}$ can be described as a quotient $U(\Sigma)/G$ (space of orbits), where $U(\Sigma)$ is an open dense subset in $\mathbb{C}^n$ and the action of G on $U(\Sigma)$ is induced by the embedding $G \hookrightarrow (\mathbb{C}^*)^n$ defined by the epimorphism $\mathbb{Z}^n \to \mathrm{Cl}(\mathbb{P})$. The standard affine coordinates $z_1, \ldots, z_n$ on $\mathbb{C}^n$ determine *homogeneous coordinates* on $\mathbb{P}$ [Cox1]. Every equation $z_j = 0$ $(1 \le j \le n)$ defines a Weil ($\mathbb{Q}$-Cartier) divisor D_j on $\mathbb{P}$. We denote by $\chi_1, \ldots, \chi_n$ the characters of G defining the representation of G in $GL(n, \mathbb{C})$. These characters correspond to the Weil divisor classes $[D_1], \ldots, [D_n] \in \mathrm{Cl}(\mathbb{P})$. Let $R(\Sigma)$ be the subgroup in $\mathbb{Z}^n$ consisting of all lattice

vectors $\lambda = (l_1, \ldots, l_n)$ such that $l_1 e_1 + \cdots + l_n e_n = 0$ and N' the sublattice of finite index in N generated by $e_1, \ldots, e_n$. We have the exact sequence

$$0 \to R(\Sigma) \to \mathbb{Z}^n \to N \to N/N' \to 0,$$

where the middle map is $(l_1, \ldots, l_n) \mapsto \sum_{j=1}^n l_j e_j$. Tensoring this sequence by $\mathbb{Q}$, one gets

$$0 \to R(\Sigma) \otimes \mathbb{Q} \to \mathbb{Q}^n \to N \otimes \mathbb{Q} \to 0.$$

Comparing the latter with the exact sequence

$$0 \to M \otimes \mathbb{Q} \to \mathbb{Q}^n \to \mathrm{Cl}(\mathbb{P}) \otimes \mathbb{Q} \to 0,$$

we obtain the canonical isomorphisms

$$Hom(R(\Sigma), \mathbb{Q}) \cong \mathrm{Cl}(\mathbb{P}) \otimes \mathbb{Q} \cong \mathrm{Pic}(\mathbb{P}) \otimes \mathbb{Q} \cong H^2(\mathbb{P}, \mathbb{Q}).$$

Therefore, $R(\Sigma)_{\mathbb{Q}} := R(\Sigma) \otimes \mathbb{Q}$ can be identified with the rational homology group $H_2(\mathbb{P}, \mathbb{Q})$. For any $\mathbb{Q}$-divisor $D = \sum_{j=1}^n c_j D_j$ ($c_j \in \mathbb{Q}$) and for any $\lambda = (l_1, \ldots, l_n) \in R(\Sigma)_{\mathbb{Q}}$, one has the intersection number

$$(D, \lambda) = \sum_{i=1}^n c_i l_i \in \mathbb{Q}.$$

If H is an ample Cartier divisor on $\mathbb{P}$, then the fan Σ can be obtained as a normal fan for the simple d-dimensional polytope

$$\Delta_H := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \ : \ \sum_{j=1}^n (D_j, \lambda) x_j = (H, \lambda), \quad \forall \lambda \in R(\Sigma)_{\mathbb{Q}} \right\}.$$

All vertices of Δ_H belong to the sublattice $M_H \subset \mathbb{R}^n$, where M_H is the set of all integral vectors $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that H is linearly equivalent to $\sum_{j=1}^n x_j D_j$.

Let $K_{\mathrm{amp}}(\mathbb{P})$ be the closed *ample (or Kähler) cone* in $\mathrm{Cl}(\mathbb{P}) \otimes \mathbb{R} \cong H^2(\mathbb{P}, \mathbb{R})$ and $K_{\mathrm{eff}}(\mathbb{P})$ the dual to $K_{\mathrm{amp}}(\mathbb{P})$ *Mori cone* of effective curves in $R(\Sigma)_{\mathbb{R}} \cong H_2(\mathbb{P}, \mathbb{R})$. The cone of vectors $\beta = (b_1, \ldots, b_n) \in R(\Sigma)_{\mathbb{R}} = R(\Sigma) \otimes \mathbb{R}$ such that $b_1, \ldots, b_n \geq 0$ will be denoted by $K^+(\mathbb{P})$. It is easy to see that $K^+(\mathbb{P})$ is always a subcone of the Mori cone $K_{\mathrm{eff}}(\mathbb{P})$.

For any lattice point $\beta = (b_1, \ldots, b_n) \in K_{\mathrm{eff}}(\mathbb{P})$, we will construct a simplicial toric variety $\mathbb{P}_\beta$ which can be considered as a "naive compactification" of the moduli space of rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}$ such that the class $[\phi(\mathbb{P}^1)] \in H_2(\mathbb{P}, \mathbb{R})$ is equal to β. In the case $\beta = 0$, the toric variety $\mathbb{P}_\beta$ coincides with $\mathbb{P}$.

First we consider the case $\beta = (b_1, \ldots, b_n) \in K^+(\mathbb{P})$, i.e., all the $b_1, \ldots, b_n$ are supposed to be nonnegative. Since $(D_j, \beta) = b_j$ ($1 \leq j \leq n$), we can construct a map $\phi : \mathbb{P}^1 \to \mathbb{P}$ with $[\phi(\mathbb{P}^1)] = \beta$ by choosing n homogeneous binary forms $\phi_1(u, v), \ldots, \phi_n(u, v) \in \mathbb{C}[u, v]$ such that $\deg \phi_j = b_j$ ($1 \leq j \leq n$). Let $\mathbb{C}_j(\beta)$ be the space of all homogeneous binary forms of degree b_j ($\dim \mathbb{C}_j(\beta) = b_j + 1$). We set $\mathbb{C}(\beta) := \bigoplus_{j=1}^n \mathbb{C}_j(\beta)$ and denote by $z_0^{(j)}, \ldots, z_{b_j}^{(j)}$ the coordinates on $\mathbb{C}_j(\beta)$

corresponding to the standard monomial basis of $\mathbb{C}_j(\beta)$. We define the action of G on $\mathbb{C}_j(\beta)$ as the scalar multiplication by the character χ_j $(1 \leq j \leq n)$. This defines an effective action of G on $\mathbb{C}(\beta)$. If $\chi_H = \chi_1^{c_1} \cdots \chi_n^{c_n}$ is the character of G corresponding to the class of the ample Cartier divisor $H = \sum_{j=1}^n c_j D_j$ on $\mathbb{P}$, then we define the **Morrison-Plesser moduli space** $\mathbb{P}_\beta$ as the GIT-quotient $\mathbb{C}(\beta)//G$ with respect to the linearization by χ_H of the structure sheaf on $\mathbb{C}(\beta)$. We define the dense open subset $U(\beta) \subset \mathbb{C}(\beta)$ as the union of all open subsets

$$U_{i_1,\ldots,i_{n-d}}(\sigma) = \left\{ z \in \mathbb{C}(\beta) \; : \; z_{i_1}^{(j_1)} z_{i_2}^{(j_2)} \cdots z_{i_{n-d}}^{(j_{n-d})} \neq 0 \right\},$$

where σ runs over all d-dimensional cones of Σ, $\{e_{j_1}, e_{j_2}, \ldots, e_{j_{n-d}}\}$ is the set of all vectors from $\{e_1, \ldots, e_n\}$ which do not belong to σ, and each index i_k $(1 \leq k \leq n-d)$ runs independently over all elements of $\{0, 1, \ldots, b_k\}$. It is easy to show that the Morrison-Plesser moduli space $\mathbb{P}_\beta$ is also the space of orbits $U(\beta)/G$. Moreover, $\mathbb{P}_\beta$ is a projective simplicial toric variety of dimension $d + \sum_{j=1}^n b_j$.

In general, the cone $K^+(\mathbb{P})$ is smaller than $K_{\mathrm{eff}}(\mathbb{P})$. In Section 5 we consider such a situation for $\mathbb{P} = \mathbb{F}_1$ and show that there exist infinitely many classes $\beta \in K_{\mathrm{eff}}(\mathbb{F}_1) \setminus K^+(\mathbb{F}_1)$ which can not be represented by irreducible curves $C \subset \mathbb{F}_1$ (see Remark 5.3). Therefore, if one of the coordinates b_j of $\beta = (b_1, \ldots, b_n)$ is negative, it may happen that there is no rational map $\phi : \mathbb{P}^1 \to \mathbb{P}$ such that $[\phi(\mathbb{P}^1)] = \beta$, but the corresponding Morrison-Plesser moduli space $\mathbb{P}_\beta$ (see Definition 3.4 below) is not empty.

Now let $\beta = (b_1, \ldots, b_n)$ be an arbitrary lattice point in $K_{\mathrm{eff}}(\mathbb{P})$. For any $j \in \{1, \ldots, n\}$, we define the free abelian group $\mathbb{Z}_j(\beta)$ as

$$\mathbb{Z}_j(\beta) := \begin{cases} \mathbb{Z}^{b_j+1}, & \text{if } b_j \geq 0, \\ 0, & \text{if } b_j < 0. \end{cases}$$

Using the standard basis of $\mathbb{Z}_j(\beta)$, we write each element of $\mathbb{Z}_j(\beta)$ as the integral vector $(x_0^{(j)}, x_1^{(j)}, \ldots, x_{b_j}^{(j)})$. We set $\mathbb{Z}(\beta) := \bigoplus_{j=1}^n \mathbb{Z}_j(\beta)$, $\mathbb{R}(\beta) := \mathbb{Z}(\beta) \otimes \mathbb{R}$ and denote by $\mathbb{R}_{\geq 0}(\beta)$ the set of all vectors in $\mathbb{R}(\beta)$ having nonnegative coordinates.

Definition 3.1. Let be r the number of negative coordinates of a lattice point $\beta = (b_1, \ldots, b_n) \in K_{\mathrm{eff}}(\mathbb{P})$. Without loss of generality we may assume that $b_1, \ldots, b_{n-r} \geq 0$ and $b_{n-r+1}, \ldots, b_n < 0$. We define the convex set

$$\Delta_H^\beta := \left\{ x \in \mathbb{R}_{\geq 0}(\beta) \; : \; \sum_{j=1}^{n-r} (D_j, \lambda) \left(\sum_{i=0}^{b_j} x_i^{(j)} \right) = (H, \lambda), \quad \forall \lambda \in R(\Sigma)_\mathbb{Q} \right\}.$$

Denote by $M_H^\beta \subset \mathbb{Z}(\beta)$ the set of all lattice points $x \in \mathbb{Z}(\beta)$ whose coordinates $x_i^{(j)}$ satisfy the condition: the divisor

$$\sum_{j=1}^{n-r} \left(\sum_{i=0}^{b_j} x_i^{(j)} \right) D_j$$

is linearly equivalent to H up to a linear combination of $D_{n-r+1}, \ldots, D_n$.

Proposition 3.2. Δ_H^β *is a compact convex simple polytope having vertices in M_H^β. The normal fan Σ_β of the polytope Δ_H^β does not depend on the choice of the ample divisor H.*

Proof. In order to see the compactness of Δ_H^β, we remark that there exists a canonical affine linear mapping $\pi_\beta : \Delta_H^\beta \to \Delta_H^0 = \Delta_H$ which replaces every $b_j + 1$ coordinates $x_0^{(j)}, x_1^{(j)}, \ldots, x_{b_j}^{(j)}$ by their sum $x_j = \sum_{i=0}^{b_j} x_i^{(j)}$. It is clear that $\pi_\beta(\Delta_H^\beta) \subset \mathbb{R}^n$ is a face of Δ_H^0 defined by the equations $x_j = 0$ $(n - r \le j \le n)$. Therefore $\dim \pi_\beta(\Delta_H^\beta) = d - r$ if $\pi_\beta(\Delta_H^\beta)$ is not empty (the latter holds if and only if $e_{n-r}, \ldots, e_n$ generate a r-dimensional cone in Σ). We observe that the preimage $\pi_\beta^{-1}(x)$ of a point in $x \in \pi_\beta(\Delta_H^\beta)$ is the product of $n - r$ simplices of dimensions $b_1, \ldots, b_{n-r}$. Therefore, Δ_H^β is compact and

$$\dim \Delta_H^\beta = d - r + \sum_{j=1}^{n-r} b_j.$$

Now we want to describe all faces of codimension 1 (i.e., facets) of Δ_H^β. It is clear that each facet must be defined by an equation $x_i^{(j)} = 0$ for some $0 \le i \le b_j$, $1 \le j \le n - r$. However, it is not true in general that every equation $x_i^{(j)} = 0$ defines a facet. It is easy to show that the equation $x_i^{(j)} = 0$ defines a facet of Δ_H^β if and only if either $b_j > 0$, or if $b_j = 0$ and $e_j, e_{n-r}, \ldots, e_n$ generate a $(r + 1)$-dimensional cone in Σ. The π_β-image of a vertex $y \in \Delta_H^\beta$ is a vertex of Δ_H. Take an arbitrary vertex $x \in \pi_\beta(\Delta_H^\beta)$. Since x is a vertex of Δ_H, there exists a subset $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ which contains $\{n - r + 1, \ldots, n\}$ such that $e_{i_1}, \ldots, e_{i_d}$ are generators of a d-dimensional cone in Σ and $x \in \Delta_H$ is defined by the conditions $x_{i_1} = \cdots = x_{i_d} = 0$. Since Δ_H is a simple polytope we have $x_j > 0$ for all $j \notin \{i_1, \ldots, i_d\}$ and the equation $x_j = \sum_{i=0}^{b_j} x_i^{(j)}$ defines a b_j-dimensional simplex with $(b_j + 1)$-vertices. In this way, we obtain $\prod_j (b_j + 1)$ (j runs over $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_d\}$) vertices of Δ_H^β as π_β-preimages of x. By this method, we get all vertices of Δ_H^β as π_β-preimages of vertices of $\pi_\beta(\Delta_H^\beta)$. Moreover, one sees that each vertex of Δ_H^β is contained in exactly $\dim \Delta_H^\beta$ facets: for each $j \in \{i_1, \ldots, i_d\} \setminus \{n - r + 1, \ldots, n\}$ we get $b_j + 1$ facets $x_j^{(i)} = 0$ $(0 \le i \le b_j)$ containing a

chosen π_β-preimage y of x and for each $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_d\}$ we get b_j facets $x_j^{(i)} = 0$ containing y. Therefore Δ_H^β is a simple polytope. Since the combinatorial structure of Δ_H^β is completely determined by β and Δ_H, it does not depend on the choice of H and the same is true for the normal fan Σ_β. $\qquad\square$

Definition 3.3. Let $\beta = (b_1, \ldots, b_n)$ be an arbitrary lattice point in $K_{\mathrm{eff}}(\mathbb{P})$. The projective simplicial toric variety $\mathbb{P}_\beta$ associated with the normal fan Σ_β of the polytope Δ_H^β is called the **Morrison-Plesser moduli space** corresponding to $\beta \in K_{\mathrm{eff}}(\mathbb{P})$.

It follows from the proof of the last proposition that

$$\dim \mathbb{P}_\beta = \dim \Delta_H^\beta = d - n + \sum_{j=1}^{n-r} (b_j + 1)$$

if $\mathbb{P}_\beta$ is not empty. It is easy to see that the last definition of the Morrison-Plesser moduli space coincides with the previous one in the case $\beta \in K^+(\mathbb{P})$. We also remark that $\mathbb{P}_\beta$ is nonsingular for all $\beta \in K_{\mathrm{eff}}(\mathbb{P})$ if $\mathbb{P}$ is nonsingular. We will need the following property of the Morrison-Plesser moduli spaces $\mathbb{P}_\beta$:

Proposition 3.4. *There exists a canonical surjective homomorphism*

$$\psi_\beta : H^2(\mathbb{P}, \mathbb{Q}) \to H^2(\mathbb{P}_\beta, \mathbb{Q})$$

which is always bijective if $\beta \in K^+(\mathbb{P})$.

Proof. Let $\{e_{j_1}, \ldots, e_{j_k}\}$ be the set of all generators $e_j \in \{e_1, \ldots, e_{n-r}\}$ such that $b_j = 0$ and $e_j, e_{n-r+1}, \ldots, e_n$ do not generate a $(r+1)$-dimensional cone in Σ. We define the subgroup $G_\beta \subset G$ to be the common kernel of the characters $\chi_{j_1}, \ldots, \chi_{j_k}$, i.e.,

$$G_\beta := \{g \in G : \chi_{j_1}(g) = \cdots = \chi_{j_k}(g) = 1\}.$$

Then the simplicial toric variety $\mathbb{P}_\beta$ can be obtained as a geometric quotient of an affine space of dimension $(b_1 + 1) + \cdots + (b_{n-r} + 1) - k$ modulo the linear action of G_β. The embedding $G_\beta \hookrightarrow G$ induces the surjective homomorphism of the character groups $\mathrm{Cl}(\mathbb{P}) \to \mathrm{Cl}(\mathbb{P}_\beta)$. This homomorphism is bijective if $\beta \in K^+(\mathbb{P})$, because in the latter case $G = G_\beta$. Tensoring by $\mathbb{Q}$, we obtain the canonical surjective homomorphism $\psi_\beta : H^2(\mathbb{P}, \mathbb{Q}) \to H^2(\mathbb{P}_\beta, \mathbb{Q})$. $\qquad\square$

Definition 3.5. Assume that the anticanonical class $-K_\mathbb{P}$ of $\mathbb{P}$ is nef, i.e., $(-K_\mathbb{P}, \beta) = \sum_{j=1}^n b_j \geq 0$ for all $\beta \in K_{\mathrm{eff}}(\mathbb{P})$ and r is the number of negative coordinates of β, i.e., $b_1, \ldots, b_{n-r} \geq 0$ and $b_{n-r+1}, \ldots, b_n < 0$. By abuse of notations, let us denote by $[D_j] \in H^2(\mathbb{P}_\beta, \mathbb{Q})$ $(1 \leq j \leq n)$ also the image of $[D_j] \in H^2(\mathbb{P}, \mathbb{Q})$ under ψ_β. Using the multiplication in the cohomology ring $H^*(\mathbb{P}_\beta, \mathbb{Q})$, we define the intersection

product

$$\Phi_\beta := ([D_1] + \cdots + [D_n])^{b_1 + \cdots + b_n} \prod_{j=n-r+1}^{n} [D_j]^{-b_j - 1},$$

considered as a cohomology class in $H^{2(\dim \mathbb{P}_\beta - d)}(\mathbb{P}_\beta, \mathbb{Q})$ and call Φ_β the **Morrison-Plesser class** of $\mathbb{P}_\beta$.

4. Toric Residue Mirror Conjecture

In order to formulate our conjecture, we need some results about Newton polytopes of principal A-determinants due to Gelfand, Kapranov and Zelevinsky [GKZ].

Let $\Delta \subset M_\mathbb{R}$ be a d-dimensional polytope with vertices in M. Denote by A a finite subset in $\Delta \cap M$ which includes all vertices of Δ.

Definition 4.1. By a *triangulation* $\mathcal{T} = \{\tau_1, \ldots, \tau_k\}$ *of* Δ *associated with* A, we mean a decomposition of Δ into a union of d-dimensional simplices $\tau_1, \ldots, \tau_k$ having vertices in A such that any nonempty intersection $\tau_i \cap \tau_j$ is a common face of τ_i and τ_j. A triangulation $\mathcal{T}$ associated with A is called *coherent* if there exists a convex piecewise-linear function $\phi : \Delta \to \mathbb{R}$ whose domains of linearity are precisely the simplices of $\mathcal{T}$.

Definition 4.2. Denote by $\mathbb{R}^A$ the space of all real-valued functions on A. Let $\mathcal{T}$ be a triangulation of Δ associated with A. The function $\chi_\mathcal{T} : A \to \mathbb{R}$ defined as

$$m \mapsto \sum_{i \,:\, m \in \mathrm{Vert}(\tau_i)} \mathrm{Vol}(\tau_i),$$

where the sum of the normalized volumes $\mathrm{Vol}(\tau_i)$ runs over all simplices of $\tau_i \in \mathcal{T}$ containing $m \in A$ as vertex, is called the *characteristic function* of $\mathcal{T}$. The **secondary polytope** $\mathrm{Sec}(A)$ is defined as the convex hull of the vectors $\chi_\mathcal{T} \in \mathbb{R}^A$, where $\mathcal{T}$ runs over all triangulations of Δ associated with A.

Theorem 4.3 ([GKZ], Chapter 7.1). *The secondary polytope* $\mathrm{Sec}(A) \subset \mathbb{R}^A$ *is a* $(|A| - d - 1)$-*dimensional polytope whose vertices are exactly the characteristic functions* $\chi_\mathcal{T}$ *corresponding to all coherent triangulations* $\mathcal{T}$ *of* Δ.

Consider a generic Laurent polynomial

$$f(t) = \sum_{m \in A} a_m t^m \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}].$$

The principal A-determinant $E_A(f)$ is a certain polynomial in $|A|$ independent variables $\{a_m\}_{m \in A}$ with integral coefficients. The following theorem will be very important in the sequel.

Theorem 4.4 ([GKZ], Chapter 10.1, Theorem 1.4). *The Newton polytope of $E_A(f)$ coincides with the secondary polytope $\mathrm{Sec}(A)$. If $\mathcal{T} = \{\tau_1, \ldots, \tau_k\}$ is a coherent triangulation corresponding to some vertex of $\mathrm{Sec}(A)$, then the coefficient at the monomial $\prod_{m \in A} a_m^{\chi_{\mathcal{T}}(m)}$ in $E_A(f)$ is equal (up to sign) to the product*

$$\prod_{i=1}^{k} \mathrm{Vol}(\tau_i)^{\mathrm{Vol}(\tau_i)}.$$

Recall the notion of a Laurent series of a rational function at a vertex of the Newton polytope of its denominator (see, e.g., [GKh] or [GKZ, p. 195]).

Definition 4.5. Let $P(a), Q(a) \in \mathbb{C}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$ be two arbitrary Laurent polynomials and let $v \in \mathbb{Z}^n$ be a vertex of the Newton polytope of $Q = \sum_w c_w a^w$. We write $Q(a) = c_v a^v (1 + \tilde{Q}(a))$, where

$$\tilde{Q}(a) := \sum_{w \neq v} \frac{c_w}{c_v} a^{w-v}.$$

It is easy to see that each Laurent monomial in $a_1, \ldots, a_n$ appears with nonzero coefficient in $(\tilde{Q}(a))^i$ only for finitely many values of i. So the expression

$$\frac{1}{1 + \tilde{Q}(a)} = \sum_{i=0}^{\infty} (-1)^i (\tilde{Q}(a))^i$$

is well-defined as a Laurent power series in the variables $a_1, \ldots, a_n$. The product

$$P(a) \cdot c_v^{-1} a^{-v} \cdot (1 - \tilde{Q}(a) + (\tilde{Q}(a))^2 - \cdots)$$

is called the **Laurent series of the rational function $P(a)/Q(a)$ at the vertex** v of the Newton polytope of Q.

Now we are able to formulate our Toric Residue Mirror Conjecture:

Conjecture 4.6. *Let $\Delta \subset M_{\mathbb{R}}$ be an arbitrary reflexive d-dimensional polytope and A a finite subset in $\Delta \cap M$ containing 0 and all vertices of Δ. Choose any coherent triangulation $\mathcal{T} = \{\tau_1, \ldots, \tau_k\}$ of Δ associated with A such that 0 is a vertex of all the simplices $\tau_1, \ldots, \tau_k$. Denote by $\mathbb{P} = \mathbb{P}_{\Sigma(\mathcal{T})}$ the simplicial toric variety defined by the fan $\Sigma = \Sigma(\mathcal{T}) \subset M_{\mathbb{R}}$ whose d-dimensional cones are exactly $\sigma_i := \mathbb{R}_{\geq 0} \tau_i$ $(1 \leq i \leq k)$. If $A = \{v_0 = 0, v_1, \ldots, v_n\}$ and*

$$f(t) := 1 - \sum_{i=1}^{n} a_i t^{v_i},$$

then for any homogeneous polynomial $P(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ of degree d the Laurent expansion of the toric residue

$$R_P(a) := (-1)^d \mathrm{Res}_f(t_0^d P(a_1 t^{v_1}, \ldots, a_n t^{v_n}))$$

at the vertex $v_{\mathcal{T}} \in \mathrm{Sec}(A)$ *corresponding to the coherent triangulations* $\mathcal{T}$ *coincides with the generating function of intersection numbers*

$$I_P(a) := \sum_{\beta \in K_{\mathrm{eff}}(\mathbb{P})} I(P, \beta) a^\beta,$$

where the sum runs over all integral points $\beta = (b_1, \ldots, b_n)$ *of the Mori cone* $K_{\mathrm{eff}}(\mathbb{P})$, $a^\beta := a_1^{b_1} \cdots a_n^{b_n}$,

$$I(P, \beta) = \int_{\mathbb{P}_\beta} P([D_1], \ldots, [D_n]) \Phi_\beta = \langle P([D_1], \ldots, [D_n]) \Phi_\beta \rangle_\beta,$$

and $\Phi_\beta \in H^{2(\dim \mathbb{P}_\beta - d)}(\mathbb{P}_\beta, \mathbb{Q})$ *is the Morrison-Plesser class of* $\mathbb{P}_\beta$. *We assume* $I(P, \beta)$ *to be zero if* $\mathbb{P}_\beta$ *is empty.*

Remark 4.7. Consider any coherent triangulation $\mathcal{T} = \{\tau_1, \ldots, \tau_k\}$ of Δ associated with A such that 0 is a vertex of all simplices and let $v_{\mathcal{T}}$ be the corresponding vertex of the $(n - d)$-dimensional polytope $\mathrm{Sec}(A)$ as above. It is easy to show that the cone $\mathbb{R}_{\geq 0}(\mathrm{Sec}(A) - v_{\mathcal{T}})$ can be canonically identified with the Mori cone $K_{\mathrm{eff}}(\mathbb{P}) \subset H_2(\mathbb{P}, \mathbb{R})$.

We want to check the statement of our conjecture for the coefficient $I(P, 0)$ of the power series $I_P(a)$. For this purpose, we choose a Σ-piecewise linear function φ on $M_{\mathbb{R}}$ corresponding to an ample Cartier divisor H on $\mathbb{P}_\Sigma$ and make the substitution $a_i = u^{\varphi(v_i)}$ $(1 \leq i \leq n)$, where u is a variable. This substitution is equivalent to the consideration of the 1-parameter family of Laurent polynomials

$$f(t) = 1 - \sum_{i=1}^n u^{\varphi(v_i)} t^{v_i}$$

depending on u. We remark that our substitution transforms $I_P(a)$ into a formal power series

$$I_P(u) = \sum_{\beta \in K_{\mathrm{eff}}(\mathbb{P}_\Sigma)} I(P, \beta) u^{(H, \beta)} \in \mathbb{Q}[[u]].$$

Since $(H, \beta) > 0$ for all nonzero $\beta \in K_{\mathrm{eff}}(\mathbb{P}_\Sigma) \cap R(\Sigma)$, we have

$$\lim_{u \to 0} I_P(u) = I(P, 0) = \int_{\mathbb{P}} P([D_1], \ldots, [D_n]).$$

Conjecture 4.6 for $\beta = 0$ is equivalent to the following:

Theorem 4.8. *For any homogeneous polynomial* $P(x_1, \ldots, x_n)$ *of degree* d, *one has*

$$(-1)^d \lim_{u \to 0} \mathrm{Res}_f(t_0^d P(u^{\varphi(v_1)} t^{v_1}, \ldots, u^{\varphi(v_n)} t^{v_n}) = \int_{\mathbb{P}} P([D_1], \ldots, [D_n]).$$

Proof. Let φ be a Σ-piecewise linear function on $M_{\mathbb{R}}$ corresponding to an ample Cartier divisor on $\mathbb{P}_{\Sigma}$ as above. Without loss of generality we may assume that φ is positively defined, i.e., $\varphi \geq 0$ on $M_{\mathbb{R}}$ and $\varphi(x) = 0$ for some $x \in M_{\mathbb{R}}$ if and only if $x = 0$. Let $S_{\Delta}[u]$ be $\mathbb{C}[u] \otimes_{\mathbb{C}} S_{\Delta}$, which is considered as a graded algebra over the polynomial ring $\mathbb{C}[u]$. We denote by

$$S_{\varphi}[u] = \bigoplus_{l=0}^{\infty} S_{\varphi}^l[u]$$

the graded $\mathbb{C}[u]$-subalgebra in $S_{\Delta}[u]$ whose l-th homogeneous component $S_{\varphi}^l[u]$ is spanned as $\mathbb{C}$-vector space by all monomials $u^r t_0^l t^m$ such that the lattice point m is contained in $l\Delta$ and $r \geq \varphi(m)$. It is easy to see that the set $S_{\varphi}[u]$ is closed under the multiplication: if $u^r t_0^l t^m, u^{r'} t_0^{l'} t^{m'} \in S_{\varphi}[u]$, then $u^{r+r'} t_0^{l+l'} t^{m+m'} \in S_{\varphi}[u]$, because $r + r' \geq \varphi(m) + \varphi(m') \geq \varphi(m + m')$.

Let us set $y_0 := t_0$ and $y_i := -u^{\varphi(v_i)} t_0 t^{v_i}$ $(1 \leq i \leq n)$. By definition, the elements $u, y_0, y_1, \ldots, y_n$ are contained in $S_{\varphi}[u]$. Denote by $\langle u \rangle$ the principal ideal in $S_{\varphi}[u]$ generated by u. Using [CDS, Proposition 1.2], we obtain the formula

$$H_f = \sum_{J \subset I} (V(J))^2 \prod_{j \in J} y_j,$$

where the sum runs over all subsets $J = \{j_0, j_1, \ldots, j_d\}$ in $I = \{0, 1, \ldots, n\}$ and $V(J)$ is the normalized d-dimensional volume of the convex hull $conv(\{v_{j_0}, v_{j_1}, \ldots, v_{j_d}\})$ (in particular, $V(J) = 0$ if dimension of $conv(\{v_{j_0}, v_{j_1}, \ldots, v_{j_d}\})$ is less than d). If $v_{i_1}, \ldots, v_{i_l}$ are not vertices of any d-dimensional simplex $\tau_i \in \mathcal{T}$, then

(5)
$$y_{i_1} \cdots y_{i_l} = u^{\varphi(v_{i_1}) + \cdots + \varphi(v_{i_l})} t_0^l t^{v_{i_1} + \cdots + v_{i_l}} =$$

$$= u^{\varphi(v_{i_1}) + \cdots + \varphi(v_{i_l}) - \varphi(v_{i_1} + \cdots + v_{i_l})} \left(u^{\varphi(v_{i_1} + \cdots + v_{i_l})} t_0^l t^{v_{i_1} + \cdots + v_{i_l}} \right) \in \langle u \rangle,$$

because the strict convexity of φ implies that

$$\varphi(v_{i_1}) + \cdots + \varphi(v_{i_l}) > \varphi(v_{i_1} + \cdots + v_{i_l}).$$

Therefore, the Hessian H_f can be written as

$$H_f = \sum_{i=1}^{k} (\text{Vol}(\tau_i))^2 \prod_{j : v_j \in \tau_i} y_j \quad + \quad h$$

for some $h \in \langle u \rangle$. Since every simplex $\tau_i \in \mathcal{T}$ contains $v_0 = 0$ and H_f is divisible by y_0, then

(6)
$$H_f' = \sum_{i=1}^{k} (\text{Vol}(\tau_i))^2 \prod_{j : v_j \in \tau_i, j \neq 0} y_j \quad + \quad h'$$

for some $h' \in \langle u \rangle$.

Let $e_1, \ldots, e_d$ be any basis of the dual lattice $N = Hom(M, \mathbb{Z})$. Then we can write every monomial t^m ($m \in M$) as product $t_1^{m_1} \cdots t_d^{m_d}$, where $m_i := \langle m, e_i \rangle$. Denote by $\mathbb{C}(u)$ the field of rational functions in variable u. The toric residue over $\mathbb{C}(u)$ is uniquely determined by $\mathbb{C}[u]$-linear mapping

$$\mathrm{Res}_f[u] : S_\Delta^d[u] \to \mathbb{C}(u),$$

having following two properties:

(1) $\mathrm{Res}_f[u](H_f') = \mathrm{Vol}(\Delta)$;

(2) $\mathrm{Res}_f[u]$ vanishes on all $\mathbb{C}[u]$-submodules $F_i S_\Delta^{d-1}[u] \subset S_\Delta^d[u]$ $(0 \le i \le d)$, where

$$F_0 := t_0 f(t) = y_0 + \sum_{l=1}^n y_l,$$

$$F_i := t_0 t_i \partial f / \partial t_i = \sum_{l=1}^n \langle v_l, e_i \rangle y_l, \quad i = 1, \ldots, d.$$

Since the toric residue Res_f is the specialization of the $\mathbb{C}[u]$-linear mapping $\mathrm{Res}_f[u]$ at the point $u = 0$, $R_P(u) = \mathrm{Res}_f[u](P(y_1, \ldots, y_n)) \in \mathbb{Q}(u)$ is regular at the point $u = 0$ and

$$R_P(0) = (-1)^d \lim_{u \to 0} \mathrm{Res}_f(t_0^d P(u^{\varphi(v_1)} t^{v_1}, \ldots, u^{\varphi(v_n)} t^{v_n})$$

for any homogeneous polynomial $P(y_1, \ldots, y_n) \in \mathbb{Q}[y_1, \ldots, y_n]$ of degree d.

The cohomology ring $H^*(\mathbb{P}, \mathbb{Q})$ of the projective simplicial toric variety $\mathbb{P}$ can be computed as a quotient of the polynomial ring $\mathbb{Q}[y_1, \ldots, y_n]$ by the sum of two ideals: $\langle F_1, \ldots, F_d \rangle$ and the ideal generated by all monomials $y_{i_1} \cdots y_{i_l}$ such that $v_{i_1}, \ldots, v_{i_l}$ are not vertices of any d-dimensional simplex $\tau_i \in \mathcal{T}$. In this description, the variables $y_1, \ldots, y_n$ represent the classes of Weil divisors $D_1, \ldots, D_n$ in $H^2(\mathbb{P}, \mathbb{Q})$. Moreover, if $v_{i_1}, \ldots, v_{i_d}$ are the vertices of a d-dimensional simplex $\tau_i \in \mathcal{T}$, then the intersection number $[D_{i_1}] \cdots [D_{i_d}]$ equals $1/\mathrm{Vol}(\tau_i)$.

It follows from (5) and from the property (2) of $\mathrm{Res}_f[u]$ that two linear maps

$$P(y_1, \ldots, y_n) \mapsto \int_{\mathbb{P}} P([D_1], \ldots, [D_n])$$

and

$$P(y_1, \ldots, y_n) \mapsto R_P(0)$$

have the same kernel in the space of homogeneous polynomials of degree d. Therefore, in order to identify these linear maps, it is sufficient to compare their values on the special polynomial

$$\widetilde{P}(y_1, \ldots, y_n) := \sum_{i=1}^k (\mathrm{Vol}(\tau_i))^2 \prod_{j: v_j \in \tau_i, j \ne 0} y_j.$$

By the property (1) of $\mathrm{Res}_f[u]$ and (6), we obtain $R_{\widetilde{P}}(0) = \mathrm{Vol}(\Delta)$. On the other hand,

$$\int_{\mathbb{P}} \widetilde{P}([D_1], \ldots, [D_n]) = \sum_{i=1}^{k} (\mathrm{Vol}(\tau_i))^2 \int_{\mathbb{P}} \prod_{j: v_j \in \tau_i, j \neq 0} [D_j]$$

$$= \sum_{i=1}^{k} (\mathrm{Vol}(\tau_i))^2 \frac{1}{\mathrm{Vol}(\tau_i)} = \sum_{i=1}^{k} \mathrm{Vol}(\tau_i) = \mathrm{Vol}(\Delta).$$

This finishes the proof. $\square$

Remark 4.9. We remark that the number $\mathrm{Vol}(\Delta)$ equals the stringy Euler number $e_{\mathrm{st}}(\mathbb{P})$ of the simplicial toric variety $\mathbb{P}$ [Bat3]. It is known that the usual Euler number $e(\mathbb{P})$ equals k (the number of d-dimensional cones in Σ). The top Chern class of $\mathbb{P}$ is represented in the cohomology ring $H^*(\mathbb{P}, \mathbb{Q})$ by the polynomial

$$C_d(y_1, \ldots, y_n) := \sum_{i=1}^{k} \mathrm{Vol}(\tau_i) \prod_{j: v_j \in \tau_i, j \neq 0} y_j.$$

In the proof of the last theorem, we have shown that the Hessian H_f' specializes at $u = 0$ to the polynomial $\widetilde{P}(y_1, \ldots, y_n)$ which represents the stringy top Chern class of $\mathbb{P}$.

5. Hirzebruch surface $\mathbb{F}_1$

We consider below a simplest example which illustrates many interesting ingredients of our conjecture.

Let $M \cong \mathbb{Z}^2$ and $\Delta \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ be a reflexive polytope with the vertices

$$v_1 = (-1, 1), \quad v_2 = (0, -1), \quad v_3 = (1, 0), \quad v_4 = (0, 1).$$

Denote by A the set of points $\{0, v_1, \ldots, v_4\}$. Take a coherent triangulation $\mathcal{T} = \{\tau_1, \ldots, \tau_4\}$ of Δ associated with A such that 0 is a vertex of every τ_i. The surface $\mathbb{F}_1$ is a toric variety $\mathbb{P}_\Sigma$ defined by the fan $\Sigma = \Sigma(\mathcal{T}) \subset M_{\mathbb{R}}$ (see Figure 1) whose 1-dimensional cones are generated by v_i. Take a generic Laurent polynomial

$$f(t) = 1 - a_1 t^{v_1} - a_2 t^{v_2} - a_3 t^{v_3} - a_4 t^{v_4} = 1 - a_1 t_1^{-1} t_2 - a_2 t_2^{-1} - a_3 t_1 - a_4 t_2$$

with support in Δ.

Proposition 5.1. *The principal A-determinant of $f(t)$ is*

$$E_A(f) = \underline{a_1^2 a_2^2 a_3^2 a_4^2} + \underline{a_1^3 a_2^2 a_3^3 a_4} - 8a_1^2 a_2^3 a_3^2 a_4^3 + \underline{16 a_1^2 a_2^4 a_3^2 a_4^4} - 36 a_1^3 a_2^3 a_3^3 a_4^2 - \underline{27 a_1^4 a_2^3 a_3^4 a_4}.$$

The underlined terms are in one-to-one correspondence with the vertices of the secondary polytope $\mathrm{Sec}(A)$. Moreover, the vertex $v_{\mathcal{T}}$ corresponding to the triangulation $\mathcal{T}$ is related with the monomial $a_1^2 a_2^2 a_3^2 a_4^2$.
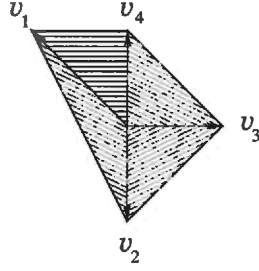
FIGURE 1. Fan Σ for the Hirzebruch surface $\mathbb{P}_\Sigma = \mathbb{F}_1$

Proof. It is easy to find all coherent triangulations of Δ (see Figure 2). By knowing the coherent triangulations, using Theorem 4.4, we can compute (up to sign) the terms of $E_A(f)$ corresponding to the vertices of $\mathrm{Sec}(A)$:

$$a_1^2 a_2^2 a_3^2 a_4^2, \ a_1^3 a_2^2 a_3^3 a_4, \ 16 a_1^2 a_2^4 a_3^2 a_4^4, \ 27 a_1^4 a_2^3 a_3^4 a_4.$$

In order to find the other terms, we compute the discriminant $D_A(f)$, i.e., the set of those coefficients $\{a_i\}$ of $f(t)$ such that the system

$$f(t) = t_1 \frac{\partial f}{\partial t_1}(t) = t_2 \frac{\partial f}{\partial t_2}(t) = 0$$

has a solution in the torus $\mathbb{T} \cong (\mathbb{C}^*)^2$. If we define

$$Z_1 := t_1^{-1} t_2, \ Z_2 := t_2^{-1}, \ Z_3 := t_1, \ Z_4 := t_2,$$

then we obtain $Z_4 = Z_1 Z_3$, $Z_2 Z_4 = 1$ and previous three equations can be rewritten as

$$1 - a_1 Z_1 - \cdots - a_4 Z_4 = 0, \ a_1 Z_1 - a_3 Z_3 = 0,$$
$$a_2 Z_2 - a_1 Z_1 - a_4 Z_4 = 0.$$

Excluding Z_3 and Z_4, we get

$$1 - a_1 Z_1 - 2 a_2 Z_2 = 0, \ a_3(a_2 Z_2 - a_1 Z_1) = a_1 a_4 Z_1^2, \ Z_2(a_2 Z_2 - a_1 Z_1) = a_4.$$

The last system is equivalent to two homogeneous equations

$$a_3(a_1 Z_1 + 2 a_2 Z_2)(a_2 Z_2 - a_1 Z_1) = a_1 a_4 Z_1^2, \ Z_2(a_2 Z_2 - a_1 Z_1) = a_4(a_1 Z_1 + 2 a_2 Z_2)^2.$$

Computing the resultant of two polynomials, we get

$$D_A(f) = a_1^2 a_2^2 a_4^2 + a_1^3 a_2^2 a_3 a_4 - 8 a_1^2 a_2^3 a_4^3 + 16 a_1^2 a_2^4 a_4^4 - 36 a_1^3 a_2^3 a_3 a_4^2 - 27 a_1^4 a_2^3 a_3^2 a_4.$$

We remark that the Newton polytope of $E_A(f)$ (secondary polytope) is obtained from the Newton polytope of $D_A(f)$ via a shift by vector $(0, 0, 2, 0)$. So the multiplication of $D_A(f)$ by a_3^2 yields a polynomial having $\mathrm{Sec}(A)$ as its Newton polytope $\mathrm{Sec}(A)$. Since $D_A(f)$ divides $E_A(f)$ (see Remark 2.8), we conclude that $a_3^2 D_A(f)$ is exactly $E_A(f)$. $\square$

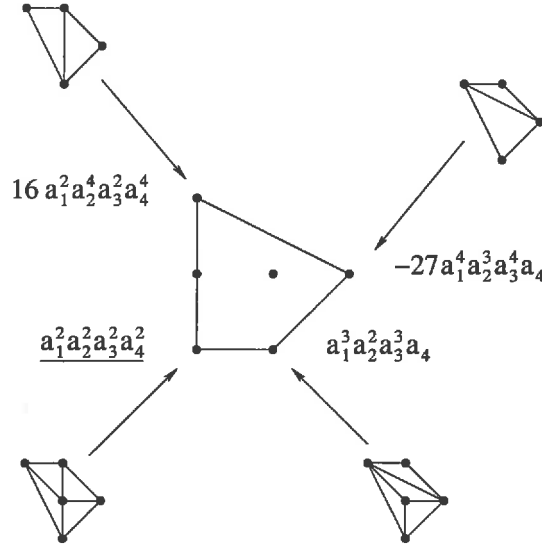For the case $\mathbb{P} = \mathbb{F}_1$, Conjecture 4.6 can be reformulated as follows:

$16\, a_1^2 a_2^4 a_3^2 a_4^4$

$-27 a_1^4 a_2^3 a_3^4 a_4$

$\underline{a_1^2 a_2^2 a_3^2 a_4^2}$

$a_1^3 a_2^2 a_3^3 a_4$

FIGURE 2. Secondary polytope Sec(A)

Conjecture 5.2. *Let $D_1, \ldots, D_4$ be the torus-invariant divisors on $\mathbb{P}_\Sigma$ corresponding to the vectors $v_1, \ldots, v_4$ respectively. Fix a homogeneous polynomial $P(x_1, \ldots, x_4)$ of degree two in $\mathbb{Q}[x_1, \ldots, x_4]$. Then the series expansion of the toric residue*

$$R_P(a_1, \ldots, a_4) = \mathrm{Res}_f(t_0^2\, P(a_1 t^{v_1}, \ldots, a_4 t^{v_1}))$$

at the vertex $v_{\mathcal{T}} \in \mathrm{Sec}(A)$ defined by the triangulation $\mathcal{T}$ coincides with the generating function of the intersection numbers

$$I_P(a_1, \ldots, a_4) = \sum_{\beta \in K_{\mathrm{eff}}(\mathbb{P}_\Sigma)} \langle P([D_1], \ldots, [D_4])\, \Phi_\beta \rangle_\beta\, a^\beta,$$

where $\Phi_\beta = ([D_1] + \cdots + [D_4])^{b_1 + \cdots + b_4} \prod_{i : b_i < 0} [D_i]^{-b_i - 1}$, $a^\beta = a_1^{b_1} \cdots a_4^{b_4}$.

It is possible to compute explicitly both the generating function for intersection numbers and the toric residue. In this way, we check the equality of Conjecture 5.2 by direct calculation. We will omit the details of this calculation and sketch only ideas of how it can be done.

First, we remark that the Mori cone $K_{\mathrm{eff}}(\mathbb{P}_\Sigma) \subset R(\Sigma)_\mathbb{R}$ is spanned by two elements

$$l^{(1)} = (1, 0, 1, -1), \quad l^{(2)} = (0, 1, 0, 1),$$

i.e., β runs over all lattice points

$$(b_1, \ldots, b_4) = \lambda_1(1, 0, 1, -1) + \lambda_2(0, 1, 0, 1) = (\lambda_1, \lambda_2, \lambda_1, \lambda_2 - \lambda_1), \quad \lambda_1, \lambda_2 \geq 0.$$

The dual to $K_{\mathrm{eff}}(\mathbb{P}_\Sigma)$ Kähler cone is generated by the classes of D_1 and D_2. There are two independent linear relations between the classes of torus-invariant divisors

D_1, D_2, D_3, D_4:

$$[D_1] - [D_3] = 0, \quad [D_1] - [D_2] + [D_4] = 0.$$

We can consider the classes of D_1, D_2 as generators of $H^2(\mathbb{P}_\Sigma, \mathbb{Z})$ and put

$$y_1 := a^{l^{(1)}} = \frac{a_1 a_3}{a_4}, \quad y_2 := a^{l^{(2)}} = a_2 a_4$$

so that $a^\beta = a_1^{b_1} \cdots a_4^{b_4}$ is equal to $y_1^{\lambda_1} y_2^{\lambda_2}$. Substituting above linear relations, we get

$$P([D_1], \ldots, [D_4]) = P([D_1], [D_2], [D_1], [D_2] - [D_1]).$$

By linearity, it is sufficient to compute the generating functions

$$I_{x_1^i x_2^{2-i}}(y_1, y_2) = \sum_{\lambda_1, \lambda_2 \geq 0} \langle [D_1]^i [D_2]^{2-i} \Phi_\beta \rangle_\beta \, y_1^{\lambda_1} y_2^{\lambda_2}, \quad i = 0, 1, 2.$$

We divide the Mori cone of $\mathbb{P}_\Sigma$ into two parts (see Figure 3):

(1) $\lambda_1 > \lambda_2 \geq 0$;

(2) $\lambda_2 \geq \lambda_1 \geq 0$.

For each of the part of the Mori cone we find the coefficients of the generating function $I_{x_1^i x_2^{2-i}}(y_1, y_2)$.
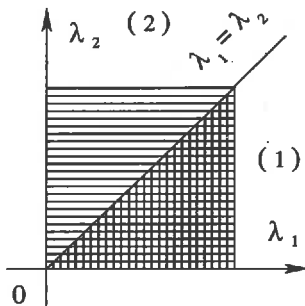

FIGURE 3. Two parts of the Mori cone of $\mathbb{P}_\Sigma$

CASE 1. If $\lambda_1 > \lambda_2 \geq 0$, then the last coordinate b_4 of

$$\beta = (b_1, \ldots, b_4) = (\lambda_1, \lambda_2, \lambda_1, \lambda_2 - \lambda_1)$$

is negative. By definition of the Morrison-Plesser moduli spaces $\mathbb{P}_\beta$ and the Morrison-Plesser classes Φ_β, we obtain $\mathbb{P}_\beta = \mathbb{P}^{2\lambda_1 + 1} \times \mathbb{P}^{\lambda_2}$ and

$$\Phi_\beta = ([D_1] + 2[D_2])^{\lambda_1 + 2\lambda_2} ([D_2] - [D_1])^{\lambda_1 - \lambda_2 - 1}.$$

The intersection theory on $\mathbb{P}_\beta$ shows immediately that

$$\langle [D_1]^{l_1} [D_2]^{l_2} \rangle_\beta = \begin{cases} 1, & l_1 = 2\lambda_1 + 1, \quad l_2 = \lambda_2, \\ 0, & \text{otherwise,} \end{cases}$$

So we obtain

$$\langle [D_1]^i [D_2]^{2-i} \Phi_\beta \rangle_\beta = \langle [D_1]^i [D_2]^{2-i} ([D_1] + 2[D_2])^{\lambda_1+2\lambda_2} ([D_2] - [D_1])^{\lambda_1-\lambda_2-1} \rangle_\beta$$

$$= \sum_{k=0}^{2\lambda_1-1} (-1)^k 2^{2\lambda_2-\lambda_1+k-1+i} \binom{\lambda_1 - \lambda_2 - 1}{k} \binom{\lambda_1 + 2\lambda_2}{2\lambda_1 - k + 1 - i}.$$

Remark 5.3. We remark that there is no irreducible curve $C \subset \mathbb{F}_1$ such that $[C] = \beta = (b_1, \ldots, b_4)$, if $b_4 < 0$, and $b_2 > 0$. Indeed, if $C \subset \mathbb{F}_1$ were such a curve, then $b_4 = (C, D_4) < 0$ would imply that $[C]$ is proportional to $[D_4]$ (the latter contradicts $b_2 = (C, D_2) > 0$ because of $(D_4, D_2) = 0$). On the other hand, the corresponding Morrison-Plesser moduli spaces $\mathbb{P}_\beta = \mathbb{P}^{2b_1+1} \times \mathbb{P}^{b_2}$ are always nonempty.

CASE 2. Let $\lambda_2 \geq \lambda_1 \geq 0$. Then the Morrison-Plesser moduli spaces $\mathbb{P}_\beta$ are toric varieties of dimension $\lambda_1 + 2\lambda_2 + 2$. The Morrison-Plesser class is equal to $\Phi_\beta = ([D_1] + 2[D_2])^{\lambda_1+2\lambda_2}$ and the coefficients of the series $I_{x_1^i x_2^{2-i}}(y_1, y_2)$ are the intersection numbers

$$\langle [D_1]^i [D_2]^{2-i} \Phi_\beta \rangle_\beta = \langle [D_1]^i [D_2]^{2-i} ([D_1] + 2[D_2])^{\lambda_1+2\lambda_2} \rangle_\beta.$$

These numbers can be computed directly using the intersection theory on the Morrison-Plesser moduli spaces $\mathbb{P}_\beta$. We omit the details of the proof. It is remarkable that the obtained formula

$$\langle [D_1]^i [D_2]^{2-i} \Phi_\beta \rangle_\beta = \sum_{k=0}^{2\lambda_1-1} 2^{2\lambda_2-\lambda_1+k-1+i} \binom{\lambda_2 - \lambda_1 + k}{k} \binom{\lambda_1 + 2\lambda_2}{2\lambda_1 - k + 1 - i}$$

can be found from the formula obtained in Case 1 by the analytic continuation of the binomial coefficients:

$$\binom{-m}{n} := (-1)^n \binom{m + n - 1}{n}, \quad n = 0, \pm 1, \pm 2, \ldots$$

for any positive integer m.

Summarizing all the considered cases, we obtain the generating function of the intersection numbers in the form

$$I_{x_1^i x_2^{2-i}}(y_1, y_2) =$$

$$= \sum_{\lambda_1, \lambda_2 \geq 0} \left(\sum_{k=0}^{2\lambda_1-1} (-1)^k 2^{2\lambda_2-\lambda_1+k-1+i} \binom{\lambda_1 - \lambda_2 - 1}{k} \binom{\lambda_1 + 2\lambda_2}{2\lambda_1 - k + 1 - i} \right) y_1^{\lambda_1} y_2^{\lambda_2}.$$

In order to compare this series with toric residues, we use the following integral representation:

$$I_{x_1^i x_2^{2-i}}(y_1, y_2) = \sum_{\lambda_1, \lambda_2 \geq 0} \left(\frac{1}{2\pi i} \int_\gamma \frac{(1 + 2u)^{\lambda_1 + 2\lambda_2}(u - 1)^{\lambda_1 - \lambda_2 - 1} du}{u^{\lambda_2 + i - 1}} \right) y_1^{\lambda_1} y_2^{\lambda_2},$$

where $\gamma = \{u \in \mathbb{C} : |u| = \varepsilon\}$, $\varepsilon > 0$ is sufficiently small. Changing the order of summation and integration, we get

$$I_{x_1^i x_2^{2-i}}(y_1, y_2) = \frac{1}{2\pi i} \int_\gamma \frac{u^{2-i} du}{(1 - (1 + 2u)(u - 1)y_1)(u(u - 1) - (1 + 2u)^2 y_2)}.$$

This integral can be computed directly. We write down the obtained rational functions $I_Q(y_1, y_2)$ together with some first terms of their expansions:

- $Q(x_1, x_2) = x_2^2$:

$$\frac{1 + y_1 + 4y_2 + 3y_1 y_2}{1 + y_1 - 8y_2 + 16y_2^2 - 36y_1 y_2 - 27y_1^2 y_2} = 1 + 12y_2 + 27y_1 y_2 + 80y_2^2$$
$$+ 568y_1 y_2^2 + 448y_2^3 + 728y_1^2 y_2^2 + 6544y_1 y_2^3 + 2304y_2^4 + y_1^3 y_2^2 + 21888y_1^2 y_2^3 + \cdots$$

- $Q(x_1, x_2) = x_1 x_2$:

$$\frac{1 + y_1 - 4y_2 - 6y_1 y_2}{1 + y_1 - 8y_2 + 16y_2^2 - 36y_1 y_2 - 27y_1^2 y_2} = 1 + 4y_2 + 26y_1 y_2 + 16y_2^2 + y_1^2 y_2$$
$$+ 336y_1 y_2^2 + 64y_2^3 - y_1^3 y_2 + 716y_1^2 y_2^2 + 2784y_1 y_2^3 + 256y_2^4 + y_1^4 y_2 + 14y_1^3 y_2^2 + \cdots$$

- $Q(x_1, x_2) = x_1^2$:

$$\frac{y_1(1 + 12y_2)}{1 + y_1 - 8y_2 + 16y_2^2 - 36y_1 y_2 - 27y_1^2 y_2} = y_1 - y_1^2 + 20y_1 y_2 + y_1^3 + 8y_1^2 y_2$$
$$+ 144y_1 y_2^2 - y_1^4 - 9y_1^3 y_2 + 656y_1^2 y_2^2 + 832y_1 y_2^3 + y_1^5 + 10y_1^4 y_2 + 84y_1^3 y_2^2 + \cdots$$

These rational functions can be identified with the toric residues $R_{x_1^i x_2^{2-i}}(y_1, y_2)$ $(i = 0, 1, 2)$ which one computes, for example, by the method from Section 9. This verifies Conjecture 5.2 directly.

6. TORIC RESIDUE AND FLOP

In this section, we consider a simplest reflexive polytope Δ which has two different coherent triangulations such that 0 is a vertex of all simplices. These triangulations correspond to two different vertices of the secondary polytope. We compute the corresponding expansions of the toric residues at each of these vertices.

Denote by A the union of the origin $v_0 = (0, 0, 0)$ in $M \cong \mathbb{Z}^3$ together with the points

$$v_1 = (1, 0, 0), \ v_2 = (0, 1, 0), \ v_3 = (-1, -1, 0), \ v_4 = (0, 0, 1), \ v_5 = (1, 1, -1).$$

Then $\Delta := conv(A) \subset M_{\mathbb{R}} \cong \mathbb{R}^3$ is the reflexive polytope. Let

$$f(t) = 1 - \sum_{i=1}^{5} a_i t^{v_i} = 1 - a_1 t_1 - a_2 t_2 - a_3 t_1^{-1} t_2^{-1} - a_4 t_3 - a_5 t_1 t_2 t_3^{-1}$$

be a generic Laurent polynomial.

Proposition 6.1. *The principal A-determinant of $f(t)$ equals*

$$E_A(f) = \underline{a_1^4 a_2^4 a_3^4 a_4^3 a_5^3} - 54 a_1^5 a_2^5 a_3^5 a_4^3 a_5^3 - \underline{a_1^3 a_2^3 a_3^4 a_4^4 a_5^4} + \underline{729 a_1^6 a_2^6 a_3^6 a_4^3 a_5^3}$$

$$+ 54 a_1^3 a_2^3 a_3^5 a_4^5 a_5^5 - 2187 a_1^5 a_2^5 a_3^6 a_4^4 a_5^4 + 2187 a_1^4 a_2^4 a_3^6 a_4^5 a_5^5 - \underline{729 a_1^3 a_2^3 a_3^6 a_4^6 a_5^6},$$

where the terms corresponding to the vertices of the secondary polytope $Sec(A)$ *are underlined.*

Proof. The idea of the proof is the same as in Proposition 5.1. The terms corresponding to the vertices of the polytope $Sec(A)$ can be easily found (up to sign) from the coherent triangulations of the polytope Δ:

$$a_1^4 a_2^4 a_3^4 a_4^3 a_5^3, \; a_1^3 a_2^3 a_3^4 a_4^4 a_5^4, \; 729 a_1^6 a_2^6 a_3^6 a_4^3 a_5^3, \; 729 a_1^3 a_2^3 a_3^6 a_4^6 a_5^6.$$

We find the discriminant $D_A(f)$, i.e., those values of $\{a_i\}$ for which the equations

$$f(t) = t_1 \frac{\partial f}{\partial t_1}(t) = t_2 \frac{\partial f}{\partial t_2}(t) = t_3 \frac{\partial f}{\partial t_3}(t) = 0$$

have a solution in the torus $\mathbb{T} \cong (\mathbb{C}^*)^3$, by putting

$$Z_1 := t_1, \; Z_2 := t_2, \; Z_3 := t_1^{-1} t_2^{-1}, \; Z_4 := t_3, \; Z_5 := t_1 t_2 t_3^{-1}.$$

Then the last system is equivalent to another one

$$1 - a_1 Z_1 - \cdots - a_5 Z_5 = a_4 Z_4 - a_5 Z_5 = a_1 Z_1 - a_3 Z_3 + a_5 Z_5$$
$$= a_2 Z_2 - a_3 Z_3 + a_5 Z_5 = 0; \; Z_1 Z_2 Z_3 = 1, \; Z_1 Z_2 = Z_4 Z_5.$$

By excluding Z_1, Z_2, Z_4, we obtain

$$1 - 3 a_3 Z_3 = (a_3 Z_3 - a_5 Z_5)^2 Z_3 - a_1 a_2 = a_4 (a_3 Z_3 - a_5 Z_5)^2 - a_1 a_2 a_5 Z_5^2 = 0.$$

Hence, the discriminant $D_A(f)$ is equal to the resultant of two homogeneous forms

$$(a_3 Z_3 - a_5 Z_5)^2 Z_3 - 27 a_1 a_2 a_3^3 Z_3^3 = a_4 (a_3 Z_3 - a_5 Z_5)^2 - a_1 a_2 a_5 Z_5^2 = 0.$$

One can find the resultant by hands. It is straightforward to see that the difference between the Newton polytopes of $D_A(f)$ and $E_A(f)$ is the vector $(1, 1, 0, 3, 0)$. Now since $D_A(f)$ divides $E_A(f)$ (see Remark 2.8), we see that the multiplication of $D_A(f)$ by $-a_1 a_2 a_4^3$ yields $E_A(f)$. $\qquad\square$

The toric residue in the following statement can be computed using, for example, the method from Section 9.

Proposition 6.2. *Let* $P(x_1, x_2, x_3, x_4) = x_1 x_2 x_4 \in \mathbb{Q}[x_1, \ldots, x_4]$ *be the input polynomial. Then the corresponding toric residue*

$$R_{x_1 x_2 x_4}(a_1, \ldots, a_5) = -\mathrm{Res}_f(t_0^3 (a_1 t_1)(a_2 t_2)(a_4 t_3))$$

can be expressed as a quotient

$$R_{x_1 x_2 x_4}(a_1, \ldots, a_5) = (a_1^4 a_2^4 a_3^4 a_4^3 a_5^3 - 27 a_1^5 a_2^5 a_3^5 a_4^3 a_5^3 - 81 a_1^4 a_2^4 a_3^5 a_4^4 a_5^4)/E_A(f).$$
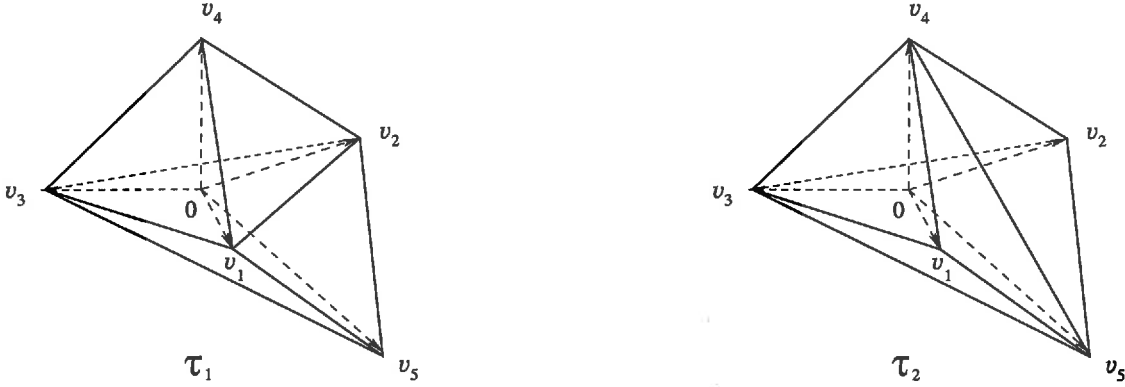

FIGURE 4. Triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$

There are two triangulations related with the vectors $v_1, \ldots, v_5$ with the property that their maximal-dimensional simplices contain 0 (see Figure 4). Namely,

$$\mathcal{T}_1 = \{\tau_{124}, \tau_{125}, \tau_{134}, \tau_{135}, \tau_{235}, \tau_{234}\}$$

and

$$\mathcal{T}_2 = \{\tau_{145}, \tau_{245}, \tau_{134}, \tau_{135}, \tau_{235}, \tau_{234}\},$$

here τ_{124} means that this simplex is generated by 0, v_1, v_2, v_4, etc. Let $\Sigma_1 = \Sigma(\mathcal{T}_1) \subset M_{\mathbb{R}}$ ($\Sigma_2 = \Sigma(\mathcal{T}_2) \subset M_{\mathbb{R}}$) be the fan whose d-dimensional simplices are defined as $\sigma := \mathbb{R}_{\geq 0}\tau$, $\tau \in \mathcal{T}_1$ ($\tau \in \mathcal{T}_2$). Toric varieties $\mathbb{P}_{\Sigma_1}$ and $\mathbb{P}_{\Sigma_2}$ corresponding to the fans Σ_1 and Σ_1 are related by a *flop*.
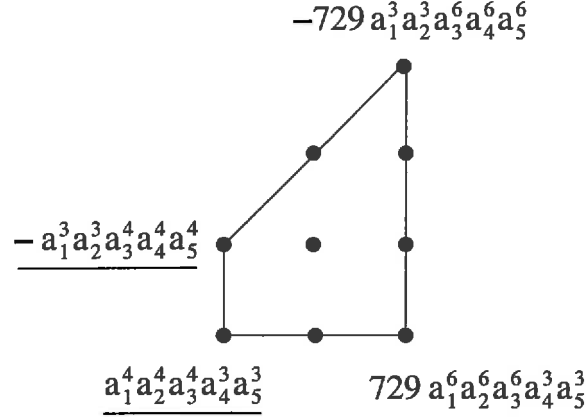
The secondary polytope $\mathrm{Sec}(A)$ is depicted in Figure 5, where the monomials $a_1^4 a_2^4 a_3^4 a_4^3 a_5^3$ and $a_1^3 a_2^3 a_3^4 a_4^4 a_5^4$ of $E_A(f)$ corresponding to the vertices $v_{\mathcal{T}_1}$ and $v_{\mathcal{T}_2}$ of $\mathrm{Sec}(A)$ respectively are underlined.

In oder to find the power series expansion of the toric residue $R_{x_1 x_2 x_4}(a_1, \ldots, a_5)$ at the vertex $v_{\mathcal{T}_1}$ of $\mathrm{Sec}(A)$, we rewrite it as follows

$$R_{x_1 x_2 x_4}(u_1, u_2) = \frac{1 - 27u_1 - 81u_1 u_2}{(1 - u_2)(1 - 54u_1 + 729u_1^2 - 54u_1 u_2 - 1458u_1^2 u_2 + 729u_1^2 u_2^2)},$$

where the variables

$$u_1 := a_1 a_2 a_3, \; u_2 := \frac{a_4 a_5}{a_1 a_2}$$

$$-729\,a_1^3 a_2^3 a_3^6 a_4^6 a_5^6$$

$$-\,a_1^3 a_2^3 a_3^4 a_4^4 a_5^4$$

$$a_1^4 a_2^4 a_3^4 a_4^3 a_5^3 \qquad 729\,a_1^6 a_2^6 a_3^6 a_4^3 a_5^3$$

FIGURE 5. Secondary polytope $\mathrm{Sec}(A)$

correspond to the generators

$$l_1 = (1,1,1,0,0), \quad l_2 = (-1,-1,0,1,1)$$

of the Mori cone of $\mathbb{P}_{\Sigma_1}$. Here are the first terms of the Taylor expansion

$$R_{x_1 x_2 x_4}(u_1, u_2) = 1 + 27u_1 + u_2 + 729u_1^2 + u_2^2 + 19683u_1^3 + 2187u_1^2 u_2 + u_2^3 + \cdots$$

The power series expansion of the toric residue $R_{x_1 x_2 x_4}(a_1, \ldots, a_5)$ at the vertex $v_{\mathcal{T}_2} \in \mathrm{Sec}(A)$ (corresponding to the second triangulation $\mathcal{T}_2$) needs another variables

$$w_1 := u_1 u_2 = a_3 a_4 a_5, \quad w_2 := u_2^{-1} = \frac{a_1 a_2}{a_4 a_5},$$

corresponding to the generators

$$l_1' = (0,0,1,1,1), \quad l_2' = (1,1,0,-1,-1)$$

of the Mori cone of $\mathbb{P}_{\Sigma_2}$. We rewrite $R_{x_1 x_2 x_4}(a_1, \ldots, a_5)$ in the form

$$R_{x_1 x_2 x_4}(w_1, w_2) =$$
$$= \frac{w_2 - 27w_1 w_2^2 - 81w_1 w_2}{(w_2 - 1)(1 - 54w_1 - 54w_1 w_2 + 729w_1^2 - 1458w_1^2 w_2 + 729w_1^2 w_2^2)}.$$

We note that the power series expansion of $R_{x_1 x_2 x_4}(u_1, u_2)$ begins with 1, whereas the series expansion of $R_{x_1 x_2 x_4}(w_1, w_2)$ does not contain a constant term. This fact agrees with different intersection numbers of three divisors D_1, D_2, D_4 on $\mathbb{P}_{\Sigma_1}$ and $\mathbb{P}_{\Sigma_2}$ corresponding to the vectors v_1, v_2, v_4: $\int_{\mathbb{P}_{\Sigma_1}} [D_1][D_2][D_4] = 1$ and $\int_{\mathbb{P}_{\Sigma_2}} [D_1][D_2][D_4] = 0$.

7. Weighted projective spaces

Consider the d-dimensional weighted projective space $\mathbb{P}(w_1, \ldots, w_n)$, $n = d + 1$, where $\gcd(w_1, \ldots, w_n) = 1$ and

$$w_i | (w_1 + \cdots + w_n), \quad i = 1, \ldots, n.$$

Let $\{v_1, \ldots, v_n\} \subset M \cong \mathbb{Z}^d$ be the vectors generating M and satisfying the relation

$$w_1 v_1 + \cdots + w_n v_n = 0.$$

If we set $A := \{v_0 = 0, v_1, \ldots, v_n\}$, then the polytope $\Delta := conv(A) \subset M_\mathbb{R}$ is a reflexive simplex. There are exactly two coherent triangulations of Δ: the triangulation $\mathcal{T}_1$ coinciding with the whole polytope Δ, and the triangulation $\mathcal{T} = \mathcal{T}_2$ consisting of the union of the d-dimensional simplices

$$\tau_i = conv\{0, v_1, \ldots, \widehat{v_i}, \ldots v_n\}, \quad i = 1, \ldots, n,$$

where $\widehat{v_i}$ means that v_i is omitted. Note that $\mathrm{Vol}(\tau_i) = w_i$ and $\mathrm{Vol}(\Delta) = \sum_{i=1}^n w_i$. The fan $\Sigma = \Sigma(\mathcal{T}) \subset M_\mathbb{R}$ defining $\mathbb{P}(w_1, \ldots, w_n)$ has the generators $\{v_1, \ldots, v_n\}$. Let

$$f(t) := 1 - \sum_{i=1}^n a_i t^{v_i}$$

be a generic Laurent polynomial. It is easy to find its principal A-determinant from Theorem 4.4.

Proposition 7.1. *The secondary polytope $\mathrm{Sec}(A)$ is an interval and the principal A-determinant of $f(t)$ is equal (up to sign) to*

$$E_A(f) = \prod_{i=1}^n w_i^{w_i} a_i^{w_1 + \cdots + \widehat{w_i} + \cdots + w_n} - \left(\sum_{i=1}^n w_i \right)^{\sum_{i=1}^n w_i} (a_1 \cdots a_n)^{\sum_{i=1}^n w_i},$$

where $\widehat{w_i}$ means that w_i is omitted. The first summand in $E_A(f)$ corresponds to the triangulation $\mathcal{T}$.

Theorem 7.2. *Let $P(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ be an arbitrary homogeneous polynomial of degree d. Denote $y := a_1^{w_1} \cdots a_n^{w_n}$. Then the toric residue*

$$R_P(a) = (-1)^d \operatorname{Res}_f(t_0^d P(a_1 t^{v_1}, \ldots, a_n t^{v_n}))$$

is equal to the rational function

$$R_P(a_1^{w_1} \cdots a_n^{w_n}) = R_P(y) = \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y},$$

where

$$\nu := \frac{1}{w_1 \cdots w_n}, \quad \mu := \left(\frac{\left(\sum_{i=1}^n w_i \right)^{\sum_{i=1}^n w_i}}{\prod_{i=1}^n w_i^{w_i}} \right).$$

Proof. By Theorem 2.9(i), the toric residue R_P can be computed as a sum over the critical points ξ of f:

$$R_P = (-1)^d \sum_{\xi \in V_f} \frac{P(a_1 \xi^{v_1}, \ldots, a_n \xi^{v_n})}{f(\xi) H_f^0(\xi)}.$$

If ξ is a critical point of f, i.e., a solution of the system of equations

$$t_1 \frac{\partial f}{\partial t_1}(t) = \cdots = t_d \frac{\partial f}{\partial t_d}(t) = 0, \quad t \in (\mathbb{C}^*)^d,$$

then

$$a_1 \frac{\xi^{v_1}}{w_1} = \cdots = a_n \frac{\xi^{v_n}}{w_n} = z$$

and

(7)
$$z^{w_1 + \cdots + w_n} = \left(\frac{a_1}{w_1} \right)^{w_1} \cdots \left(\frac{a_n}{w_n} \right)^{w_n}.$$

These relations simplify our computations. For example, we may write at the critical points of f:

$$f(\xi) = 1 - \left(\sum_{i=1}^n w_i \right) z,$$

and

$$P(a_1 \xi^{v_1}, \ldots, a_n \xi^{v_n}) = P(w_1, \ldots, w_n) z^d.$$

The value of the polynomial

$$H_f^0(t) = \det \left(\left(t_i \frac{\partial}{\partial t_i} \right) \left(t_j \frac{\partial}{\partial t_j} \right) f \right)_{1 \leq i,j \leq d}$$

at a critical point ξ of f equals

$$H_f^0(\xi) = (-1)^d w_1 \cdots w_n \left(\sum_{i=1}^n w_i \right) z^d.$$

Since the summation over the critical points is (**??**) is equivalent to the summation over the roots of the equation (7), we get

$$
\begin{aligned}
R_P(y) &= \sum_{z^{w_1 + \cdots + w_n} = \left(\frac{a_1}{w_1}\right)^{w_1} \cdots \left(\frac{a_n}{w_n}\right)^{w_n}} \frac{P(w_1, \ldots, w_n)}{w_1 \cdots w_n (\sum_{i=1}^n w_i)(1 - (\sum_{i=1}^n w_i)z)} \\
&= \frac{P(w_1, \ldots, w_n)}{w_1 \ldots w_n} \sum_{b \geq 0} \left(\sum_{i=1}^n w_i \right)^{(\sum_{i=1}^n w_i)b} \left(\frac{a_1^{w_1} \cdots a_n^{w_n}}{w_1^{w_1} \cdots w_n^{w_n}} \right)^b \\
&= \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y},
\end{aligned}
$$

as required.   □

Now we show that the expansion of $R_P(y)$ at the vertex $v_{\mathcal{T}} \in \mathrm{Sec}(A)$ corresponding to the triangulation $\mathcal{T}$ coincides with the generating function of intersection numbers:

Theorem 7.3. *Let $P(x_1, \ldots, x_n)$ be any homogeneous polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$ of degree d. The generating function of intersection numbers on the Morrison-Plesser moduli spaces has the form*

$$
I_P(y) = \nu \cdot P(w_1, \ldots, w_n) \sum_{b \geq 0} \mu^b y^b = \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y}.
$$

Proof. The Morrison-Plesser moduli spaces $\mathbb{P}_\beta$ for the weighted projective space $\mathbb{P} = \mathbb{P}(w_1, \ldots, w_n)$ are also weighted projective spaces of dimension $(\sum_{i=1}^n w_i)b + d$. Let $D_1, \ldots, D_n$ be the divisors corresponding to the vectors $v_1, \ldots, v_n$. We have the following relations between the torus-invariant divisors on $\mathbb{P}_\beta$ modulo rational equivalence:

$$
\frac{[D_1]}{w_1} = \cdots = \frac{[D_n]}{w_n} = [D_0].
$$

Then the Mori cone $K_{\mathrm{eff}}(\mathbb{P})$ consists of $\beta = (b)$, $b \geq 0$. The Morrison-Plesser class is exactly

$$
\Phi_\beta = \left(\left(\sum_{i=1}^n w_i \right) [D_0] \right)^{(\sum_{i=1}^n w_i)b},
$$

and the generating function of intersection numbers on $\mathbb{P}_\beta$ can be written as

$$
I_P(y) = \sum_{b \geq 0} \langle P([D_1], \ldots, [D_n]) \left(\left(\sum_{i=1}^n w_i \right) [D_0] \right)^{(\sum_{i=1}^n w_i)b} \rangle_\beta \, y^b.
$$

Using $\langle [D_0]^{(\sum_{i=1}^n w_i)b+d} \rangle_\beta = 1/(w_1^{w_1 b+1} \cdots w_n^{w_n b+1})$, we obtain

$$
\begin{aligned}
I_P(y) &= P(w_1, \ldots, w_n) \sum_{b \geq 0} \left(\sum_{i=1}^n w_i \right)^{(\sum_{i=1}^n w_i)b} \langle [D_0]^{(\sum_{i=1}^n w_i)b+d} \rangle_\beta \, y^b \\
&= P(w_1, \ldots, w_n) \sum_{b \geq 0} \left(\sum_{i=1}^n w_i \right)^{(\sum_{i=1}^n w_i)b} \frac{1}{w_1^{w_1 b+1} \cdots w_n^{w_n b+1}} \, y^b \\
&= \nu \cdot P(w_1, \ldots, w_n) \sum_{b \geq 0} \mu^b y^b = \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y}.
\end{aligned}
$$

$\square$

8. Product of projective spaces

In this section we check the Toric Residue Mirror Conjecture in the case $\mathbb{P} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}$.

For all $j = 1, \ldots, r$, we set $n_j := d_j + 1$ and denote by M_j the free abelian group of rank d_j generated by the elements $v_{j1}, \ldots, v_{jn_j}$ satisfying the linear relation:

$$
v_{j1} + \cdots + v_{jn_j} = 0.
$$

Let $M := M_1 \oplus \cdots \oplus M_r$. Consider a Laurent polynomial

$$
f(t) = 1 - \sum_{i_1=1}^{n_1} a_{1i_1} t^{v_{1i_1}} - \cdots - \sum_{i_r=1}^{n_r} a_{ri_r} t^{v_{ri_r}}
$$

with support in the reflexive polytope

$$
\Delta = conv(\{v_{11}, \ldots, v_{1n_1}, \ldots, v_{r1}, \ldots, v_{rn_r}\}) \subset M_{\mathbb{R}}.
$$

The fan $\Sigma \subset M_{\mathbb{R}}$ consisting of cones over all faces of Δ determines the toric variety $\mathbb{P} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}$ of dimension $d := d_1 + \cdots + d_r$. Consider $n := n_1 + \cdots + n_r$ variables x_{ji} $(1 \leq i \leq n_j, 1 \leq j \leq r)$. We will use the following notations:

$$
x^k := x_{11}^{k_{11}} \cdots x_{1n_1}^{k_{1n_1}} \cdots x_{r1}^{k_{r1}} \cdots x_{rn_r}^{k_{rn_r}},
$$

$$
k_j := k_{j1} + \cdots + k_{jn_j}, \quad u_j := n_j^{n_j} a_{j1} \cdots a_{jn_j}, \quad j = 1, \ldots, r.
$$

Let $K = \mathbb{C}(u_1, \ldots, u_r)$ be the field of rational functions in $u_1, \ldots, u_r$. Let z_j be a root of the equation $z_j^{n_j} = u_j$ $(1 \leq j \leq r)$. We obtain a finite Galois extension $L = \mathbb{C}(z_1, \ldots, z_r)$ of K of degree $[L : K] = n_1 \cdots n_r$. One has the algebraic trace map

$$
tr_{L/K} : L \to K,
$$

which can be defined by the formula

$$\mathrm{tr}_{L/K}(g) = \sum_{\substack{z_1^{n_1}=u_1 \\ \cdots \\ z_r^{n_r}=u_r}} g(z_1, \ldots, z_r),$$

where the sum runs over all roots of the system of equations $z_j^{n_j} = u_j$ $(1 \le j \le r)$.

Theorem 8.1. *Let $P(x) = x^k$ be any monomial of degree d (i.e., $\sum_{j=1}^r k_j = d$). Then the toric residue R_P corresponding to the monomial x^k is the rational function in $u_1, \ldots, u_r$:*

$$R_{x^k}(u) = n_1^{d_1-k_1-1} \cdots n_r^{d_r-k_r-1} \mathrm{tr}_{L/K}\left(\frac{1}{z_1^{d_1-k_1} \cdots z_r^{d_r-k_r}(1 - z_1 - \cdots - z_r)} \right).$$

Proof. The proof is based on the formula from Theorem 2.9(i) which express the toric residue R_P as the following sum over the critical points ξ of f:

$$R_P = (-1)^d \sum_{\xi \in V_f} \frac{P(a_{11}\xi^{v_{11}}, \ldots, a_{rn_r}\xi^{v_{rn_r}})}{f(\xi)H_f^0(\xi)}.$$

If ξ is a critical point of f, then the vanishing of partial derivatives of f implies:

$$a_{j1}\xi^{v_{j1}} = \cdots = a_{jn_j}\xi^{v_{jn_j}}, \quad j = 1, \ldots, r.$$

We set $z_j := a_{j1}\xi^{v_{j1}} = \cdots = a_{jn_j}\xi^{v_{jn_j}}$. It follows from $v_{j1} + \cdots + v_{jn_j} = 0$ that

$$z_j^{n_j} = a_{j1} \cdots a_{jn_j}, \quad j = 1, \ldots, r.$$

Denote the product $a_{j1} \cdots a_{jn_j}$ by y_j. It is easy to compute the values of the polynomials f and $H_f^0(t)$ at a critical point ξ:

$$f(\xi) = 1 - n_1 z_1 - \cdots - n_r z_r,$$

$$H_f^0(\xi) = (-1)^d n_1 \cdots n_r z_1^{d_1} \cdots z_r^{d_r}.$$

On the other hand, we have

$$P(a_{11}\xi^{v_{11}}, \ldots, a_{rn_r}\xi^{v_{rn_r}}) = z_1^{k_1} \cdots z_r^{k_r}.$$

Thus, we obtain

$$R_{x^k}(u) = \frac{1}{n_1 \cdots n_r} \sum_{\substack{z_1^{n_1}=y_1 \\ \cdots \\ z_r^{n_r}=y_r}} \frac{z_1^{k_1} \cdots z_r^{k_r}}{(1 - n_1 z_1 - \cdots - n_r z_r)z_1^{d_1} \cdots z_r^{d_r}}.$$

Setting $u_j := n_j^{n_j} y_j$ $(1 \le j \le r)$, we get the required formula. $\qquad \square$

In order to compute the generating functions of intersection numbers $I(P, \beta)$ on the Morrison-Plesser moduli spaces associated with $\mathbb{P}$, we remark that the Mori cone $K_{\text{eff}}(\mathbb{P})$ is a simplicial r-dimensional cone generated by the canonical basis of $H_2(\mathbb{P}, \mathbb{Z}) \cong \mathbb{Z}^r$. Moreover, all Morrison-Plesser moduli spaces $\mathbb{P}_\beta$ are projective spaces $\mathbb{P}^{n_1 b_1 + d_1} \times \cdots \times \mathbb{P}^{n_r b_r + d_r}$, where β can be identified with a lattice point $(b_1, \ldots, b_r) \in K_{\text{eff}}(\mathbb{P}) = \mathbb{R}_{\geq 0}^r$. Therefore, the generating function

$$I_P(y) = \sum_{\beta \in K_{\text{eff}}(\mathbb{P})} I(P, \beta) \, a^\beta,$$

can be rewritten as

$$I_P(y) = \sum_{b_1, \ldots, b_r \geq 0} I(P, \beta) \, y_1^{b_1} \cdots y_r^{b_r},$$

where $y_j = a_{j1} \cdots a_{jn_j}$ $(1 \leq j \leq r)$.

Theorem 8.2. *Let* $u_j = n_j^{n_j} y_j$ $(1 \leq j \leq r)$. *Then the generating function of intersection numbers* $I(x^k, \beta)$ *can be written as*

$$I_{x^k}(u) = n_1^{d_1 - k_1} \cdots n_r^{d_r - k_r} \sum_{b_1, \ldots, b_r \geq 0} \frac{(n_1 b_1 + \cdots + n_r b_r)!}{(n_1 b_1 + d_1 - k_1)! \cdots (n_r b_r + d_r - k_r)!} u_1^{b_1} \cdots u_r^{b_r}.$$

Proof. Let $[H_j]$ be the hyperplane class on $\mathbb{P}^{d_j}$. Since $\mathbb{P}$ contains exactly n_j torus-invariant divisors having the class $[H_j]$, the intersection number $I(x^k, \beta)$ equals

$$\langle [H_1]^{k_1} \cdots [H_r]^{k_r} (n_1 [H_1] + \cdots + n_r [H_r])^{n_1 b_1 + \cdots + n_r b_r} \rangle_\beta.$$

We have

$$I(x^k, \beta) = \sum_{m_1 + \cdots + m_r = m} \frac{m!}{m_1! \cdots m_r!} n_1^{m_1} \cdots n_r^{m_r} \langle [H_1]^{m_1 + k_1} \cdots [H_r]^{m_r + k_r} \rangle_\beta,$$

where $m = \sum_{j=1}^r n_j b_j$. The intersection theory on $\mathbb{P}_\beta$ implies

$$\langle [H_1]^{l_1} \cdots [H_r]^{l_r} \rangle_\beta = \begin{cases} 1, & l_j = n_j b_j + d_j, \quad j = 1, \ldots, r, \\ 0, & \text{otherwise}, \end{cases}$$

So we obtain

$$I_{x^k}(u) = n_1^{d_1 - k_1} \cdots n_r^{d_r - k_r} \sum_{b_1, \ldots, b_r \geq 0} \frac{(n_1 b_1 + \cdots + n_r b_r)!}{(n_1 b_1 + d_1 - k_1)! \cdots (n_r b_r + d_r - k_r)!} u_1^{b_1} \cdots u_r^{b_r},$$

where $u_j = n_j^{n_j} y_j$ $(1 \leq j \leq r)$. $\qquad \square$

It is sufficient to verify the Toric Residue Mirror Conjecture (Conjecture 4.6) for any monomial $P = x^k$ of degree d. By Theorem 8.1 and Theorem 8.2, the equality $R_{x^k}(u) = I_{x^k}(u)$ follows from

Proposition 8.3. *Let*

$$T_k(u) := \operatorname{tr}_{L/K}\left(\frac{1}{z_1^{d_1-k_1}\cdots z_r^{d_r-k_r}(1-z_1-\cdots-z_r)}\right),$$

then T_k has the following power series expansion

$$T_k(u) = n_1\cdots n_r \sum_{b_1,\ldots,b_r\geq 0}\frac{(n_1b_1+\cdots+n_rb_r)!}{(n_1b_1+d_1-k_1)!\cdots(n_rb_r+d_r-k_r)!}u_1^{b_1}\cdots u_r^{b_r},$$

Proof. We have the chain of equalities:

$$T_k(u) = \sum_{\substack{z_1^{n_1}=u_1\\ \cdots\\ z_r^{n_r}=u_r}}\frac{1}{z_1^{d_1-k_1}\cdots z_r^{d_r-k_r}}\sum_{b_1,\ldots,b_r\geq 0}\frac{(b_1+\cdots+b_r)!}{b_1!\cdots b_r!}z_1^{b_1}\cdots z_r^{b_r}$$

$$= \sum_{\substack{z_1^{n_1}=u_1\\ \cdots\\ z_r^{n_r}=u_r}}\frac{1}{z_1^{d_1-k_1}\cdots z_r^{d_r-k_r}}\sum_{b}\frac{(b_1+\cdots+b_r-\sum_{i=1}^{r}(d_i-k_i))!}{b_1!\cdots b_r!}z_1^{b_1}\cdots z_r^{b_r}$$

$$= \sum_{\substack{z_1^{n_1}=u_1\\ \cdots\\ z_r^{n_r}=u_r}}\frac{1}{z_1^{d_1-k_1}\cdots z_r^{d_r-k_r}}\sum_{b}\frac{(b_1+\cdots+b_r)!}{(b_1+d_1-k_1)!\cdots(b_r+d_r-k_r)!}z_1^{b_1+d_1-k_1}\cdots z_r^{b_r+d_r-k_r}$$

$$= \sum_{\substack{z_1^{n_1}=u_1\\ \cdots\\ z_r^{n_r}=u_r}}\sum_{b}\frac{(b_1+\cdots+b_r)!}{(b_1+d_1-k_1)!\cdots(b_r+d_r-k_r)!}z_1^{b_1}\cdots z_r^{b_r}$$

$$= n_1\cdots n_r\sum_{b_1,\ldots,b_r\geq 0}\frac{(n_1b_1+\cdots+n_rb_r)!}{(n_1b_1+d_1-k_1)!\cdots(n_rb_r+d_r-k_r)!}u_1^{b_1}\cdots u_r^{b_r},$$

where in the last row we have used the identity

$$\sum_{z_j^{n_j}=u_j}z_j^{b_j} = \begin{cases}n_ju_j^{b_j}, & b_j=kn_j,\\ 0, & b_j\neq kn_j,\end{cases}\quad k=1,2,\ldots$$

for each $j=1,\ldots,r$. $\square$

9. Yukawa $(d-1)$-point functions for Calabi-Yau hypersurfaces

Let $\Delta\subset M_{\mathbb{R}}$ be a reflexive polytope of dimension d, $A=\{0,v_1,\ldots,v_n\}$ a finite subset in $\Delta\cap M$ containing 0 and all vertices of Δ, $f(t)$ a Laurent polynomial in

the variables $t_1, \ldots, t_d$ with support in Δ of the form

$$f(t) := 1 - \sum_{i=1}^{n} a_i t^{v_i}.$$

We denote by Z_f a Calabi-Yau hypersurface defined by the equation $f = 0$ in the torus $\mathbb{T} \cong (\mathbb{C}^*)^d$. Let

$$\Omega := \mathbf{Res} \left(\frac{1}{f} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \right),$$

be the image of the canonical d-form

$$\frac{1}{f} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \in H^d(\mathbb{T} \backslash Z_f)$$

under the *Poincaré residue mapping*

$$\mathbf{Res} : H^d(\mathbb{T} \backslash Z_f) \to H^{d-1}(Z_f).$$

Definition 9.1. Assign to each nonzero lattice point $v_i \in A$ a variable x_i. Let $Q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d - 1$. The *Q-Yukawa $(d-1)$-point function* is defined by the formula

$$(8) \quad Y_Q(a_1, \ldots, a_n) := (-1)^{\frac{(d-1)(d-2)}{2}} \frac{1}{(2\pi i)^{d-1}} \int_{Z_f} \Omega \wedge Q \left(a_1 \frac{\partial}{\partial a_1}, \ldots, a_n \frac{\partial}{\partial a_n} \right) \Omega,$$

where the differential operators $a_1 \partial/\partial a_1, \ldots, a_n \partial/\partial a_n$ are determined by the Gauß-Manin connection.

Remark 9.2. The sign $(-1)^{\frac{(d-1)(d-2)}{2}}$ in the definition of the Yukawa $(d-1)$-point function is inherited by the variation of the Hodge structure (see in [CK, Section 8.6.3]). The 3-point Yukawa functions are also called *Yukawa couplings*.

Example 9.3. Consider the mirror family of Calabi-Yau hypersurfaces in the projective space $\mathbb{P}^d$ defined by the Laurent polynomial

$$f(t) = 1 - \sum_{i=1}^{n} a_i t^{v_i}, \quad n = d + 1,$$

where $v_1, \ldots, v_d$ form a basis of the lattice M and

$$v_{d+1} = -(v_1 + \cdots + v_d).$$

If we set $y := a_1 \cdots a_n$, then the Q-Yukawa $(d-1)$-point function is equal to

$$(9) \qquad Y_Q(y) = \frac{n \, Q(1, \ldots, 1)}{1 - n^n \, y}, \quad \deg Q = d - 1,$$

(see e.g. [JN]). In the particular case $d = 4$ and $Q(x) = x^3$ the formula (9) gives the Yukawa 3-point function for mirrors of Calabi-Yau quintic hypersurfaces in $\mathbb{P}^4$

$$Y_{x^3}(y) = \frac{5}{1 - 5^5 y},$$

which is well-known from [COGP].

In order to establish the relation between Yukawa $(d-1)$-point functions and toric residues, we need the notion of homogeneous coordinate ring of a toric variety [Cox1]. Let Σ be a complete simplicial fan in $N_{\mathbb{R}}$ defining a projective simplicial toric variety $\mathbb{P} = \mathbb{P}_{\Sigma}$, $\Sigma(1) := \{e_1, \ldots, e_r\}$ the set of generators of 1-dimensional cones in Σ, and $z_1, \ldots, z_r$ the corresponding homogeneous coordinates. The polynomial ring

$$S(\mathbb{P}) := \mathbb{C}[z_1, \ldots, z_r]$$

having a natural grading by $\mathrm{Cl}(\mathbb{P})$ is called the *homogeneous coordinate ring* of $\mathbb{P}$.

Let $H = \sum_{i=1}^{r} c_i D_i$ be a big and nef divisor on $\mathbb{P}$. One obtains the convex polytope $\Delta_H \subset M_{\mathbb{R}}$ as intersection of r half-spaces $\langle m, e_i \rangle \geq -c_i$ $(1 \leq i \leq r)$. For any lattice point m in $k\Delta_H$, one has $\langle m, e_i \rangle + kc_i \geq 0$ $(1 \leq i \leq r)$. Thus we can define a mapping $S_{\Delta_H} \to S(\mathbb{P})$ which sends a monomial $t_0^k t^m \in S_{\Delta_H}^k$ to the monomial $\prod_{i=1}^{r} z_i^{\langle m, e_i \rangle + kc_i} \in S(\mathbb{P})$. This mapping yields an isomorphism of graded rings

$$(10) \qquad\qquad S_{\Delta_H} \cong \bigoplus_{k=0}^{\infty} S(\mathbb{P})_{k\alpha},$$

where α is the class of H in $\mathrm{Cl}(\mathbb{P})$. For all k, this isomorphism identifies the subspace $I_{\Delta_H}^k \subset S_{\Delta_H}^k$ with the image of $S(\mathbb{P})_{k\alpha - \omega_0}$ in $S(\mathbb{P})_{k\alpha}$ under the mapping

$$S(\mathbb{P})_{k\alpha - \omega_0} \xrightarrow{\prod_{i=1}^{r} z_i} S(\mathbb{P})_{k\alpha},$$

where $\omega_0 \in \mathrm{Cl}(\mathbb{P})$ is the anticanonical class of $\mathbb{P}$. This bijection allows to compare our notion of toric residue from Section 2 with the definition of toric residue given by D. Cox in [Cox2].

By the isomorphism (10), we identify the regular sequence $G = (G_0, \ldots, G_d)$ of elements in $S_{\Delta_H}^1$ with its image in $S(\mathbb{P})_{\alpha}$. Thus, by [Cox2, Theorem 5.1] (see also [Mav1, Thorem 4.8], where the theorem of Cox was extended to the case when H is big and nef), the toric residue mapping (2) coincides with the residue mapping

$$\mathrm{Res}_G : S(\mathbb{P})_{\rho} \to \mathbb{C}, \qquad \rho = (d+1)\alpha - \omega_0$$

considered by Cox and induces the canonical isomorphism

$$S(\mathbb{P})_{\rho} / \langle G_0, \ldots, G_d \rangle_{\rho} \cong \mathbb{C}.$$

Remark 9.4. In particular, let $\alpha = \omega_0$, i.e., Δ_H be a reflexive polytope (we denote it shortly by Δ), $f(t)$ be a generic Laurent polynomial with support Δ defining a Calabi-Yau hypersurface $Z_f \subset \mathbb{T}$, and $F_0(z), \ldots, F_d(z) \in S(\mathbb{P})_{\omega_0}$ are the images of the following regular sequence in S_Δ^1:

$$t_0 f, \ t_0 t_1 \partial f / \partial t_1, \ldots, t_0 t_d \partial f / \partial t_d.$$

Then the isomorphism (4) from Section 2 coincides with

(11)
$$S(\mathbb{P})_\rho / \langle F_0, \ldots, F_d \rangle_\rho \cong \mathbb{C}, \quad \rho = d\omega_0.$$

Definition 9.5 ([Mav1]). Given a subset $I = \{e_{i_0}, \ldots, e_{i_d}\} \subset \Sigma(1)$ consisting of d elements and an integral basis $m_1, \ldots, m_d$ of the lattice M, denote by c_I a determinant of the $(d+1) \times (d+1)$-matrix obtained from the matrix $(\langle m_j, e_{i_k} \rangle_{1 \le j \le d, i_k \in I})$ by adding the first row $(1, \ldots, 1)$.

The following statement shows how Yukawa $(d-1)$-point functions can be computed by means of toric residues Res_{F_I} with respect to some sequence F_I of polynomials in the homogeneous coordinate ring of a toric variety (it is a reformulation of the result in [Mav1, p. 104]):

Theorem 9.6. *Let Δ be a d-dimensional reflexive polytope and $F_0(z) \in S(\mathbb{P})_{\omega_0}$ a generic homogeneous polynomial as in Remark 9.4. Choose a subset $I = \{e_{i_0}, \ldots, e_{i_d}\} \subset \Sigma(1)$ is such that $c_I \ne 0$ (see Definition 9.5) and define the sequence F_I of homogeneous polynomials as $F_I = (z_{i_0} \partial F_0 / \partial z_{i_0}, \ldots, z_{i_d} \partial F_0 / \partial z_{i_d})$. Let $Q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d-1$. We set*

$$q(t) := (-1)^{d-1} t_0^{d-1} Q(a_1 t^{v_1}, \ldots, a_n t^{v_n})$$

and denote by $\tilde{q}(z) \in S(\mathbb{P})_{(d-1)\omega_0}$ the image of the polynomial $q(t)$ under the isomorphism (10). Then the Q-Yukawa $(d-1)$-point function is equal to the toric residue

$$Y_Q(a) = -c_I \, \mathrm{Res}_{F_I} \left(\tilde{q}(z) \cdot \prod_{i=1}^r z_i \right).$$

There exists another formula for Yukawa $(d-1)$-point function $Y_Q(a)$ which does not depend on the choice of a subset $I = \{e_{i_0}, \ldots, e_{i_d}\} \subset \Sigma(1)$:

Theorem 9.7. *Let $Q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $(d-1)$ and*

$$P(x_1, \ldots, x_n) := (x_1 + \cdots + x_n) Q(x_1, \ldots, x_n).$$

Then the Q-Yukawa $(d-1)$-point function is equal to the toric residue

$$Y_Q(a_1, \ldots, a_n) = (-1)^d \, \mathrm{Res}_f(t_0^d \, P(a_1 t^{v_1}, \ldots, a_n t^{v_n})).$$

Proof. It follows from the definition of toric residue that

$$
\begin{aligned}
\operatorname{Res}_f(t_0^d\, P(a_1 t^{v_1}, \ldots, a_n t^{v_n})) &= \operatorname{Res}_f(t_0^d\, (1 - f(t))\, Q(a_1 t^{v_1}, \ldots, a_n t^{v_n})) \\
&= \operatorname{Res}_f(t_0^d\, Q(a_1 t^{v_1}, \ldots, a_n t^{v_n})).
\end{aligned}
$$

Let $F_0(z), \ldots, F_d(z) \in S(\mathbb{P})_{\omega_0}$ be the homogeneous polynomials as in Remark 9.4. Then the last residue can be written in homogeneous coordinates as

$$
\operatorname{Res}_f(t_0^d\, Q(a_1 t^{v_1}, \ldots, a_n t^{v_n})) = (-1)^{d-1} \operatorname{Res}_F\left(\tilde{q}(z) \cdot \prod_{i=1}^r z_i\right),
$$

where Res_F is the Cox's residue with respect to the sequence $F = (F_0, \ldots, F_d)$. Compare this residue with the residue from Theorem 9.6. To do this, note that the reflexivity of Δ implies that each monomial $t_0\, t^m \in S_\Delta^1$ maps to the monomial $\prod_{i=1}^r z_i^{\langle m, e_i \rangle + 1} \in S(\mathbb{P})_{\omega_0}$ in (10). Hence, it is easy to see that the sequences of homogeneous polynomials $F = (F_0, \ldots, F_d)$ and $F_I = (z_{i_0} \partial F_0 / \partial z_{i_0}, \ldots, z_{i_d} \partial F_0 / \partial z_{i_d})$ are related by the formulae

$$
z_{i_k} \partial F_0 / \partial z_{i_k} = F_0 + \sum_{j=1}^d \langle m_j, e_{i_k} \rangle F_j = \sum_{j=0}^d A_{jk} F_j, \quad k = 0, \ldots, d.
$$

Now, we have

$$
\begin{aligned}
\operatorname{Res}_f(t_0^d\, P(a_1 t^{v_1}, \ldots, a_n t^{v_n})) &= (-1)^{d-1} \operatorname{Res}_F\left(\tilde{q}(z) \cdot \prod_{i=1}^r z_i\right) \\
&= (-1)^{d-1} \operatorname{Res}_{F_I}\left(\det(A_{jk})\, \tilde{q}(z) \cdot \prod_{i=1}^r z_i\right) \\
&= (-1)^{d-1} c_I \operatorname{Res}_{F_I}\left(\tilde{q}(z) \cdot \prod_{i=1}^r z_i\right) = (-1)^d Y_Q(a),
\end{aligned}
$$

where the second row follows from the *Global Transformation Law* for toric residue (see Theorem 9.8 below) and the third row follows from the equality $\det(A_{jk}) = c_I$. $\qquad\square$

Next statement is a particular case of [CCD, Theorem 0.4].

Theorem 9.8. *Let $G = (G_0, \ldots, G_d)$ and $H = (H_0, \ldots, H_d)$ be the regular sequences of elements in $S(\mathbb{P})_{\omega_0}$. If*

$$
H_j = \sum_{i=0}^d A_{ij} G_i,
$$

where A_{ij} are complex numbers, then for each $P \in S(\mathbb{P})_{d\omega_0}$, we have

$$
\operatorname{Res}_G(P) = \operatorname{Res}_H(P \det(A_{ij})).
$$

Toric Residue Mirror Conjecture (Conjecture 4.6) implies the following:

Corollary 9.9. *Let $\Delta \subset M_{\mathbb{R}}$ be an arbitrary reflexive d-dimensional polytope and A a finite subset in $\Delta \cap M$ containing 0 and all vertices of Δ. Choose any coherent triangulation $\mathcal{T} = \{\tau_1, \ldots, \tau_k\}$ of Δ associated with A such that 0 is a vertex of all the simplices $\tau_1, \ldots, \tau_k$. Denote by $\mathbb{P} = \mathbb{P}_{\Sigma(\mathcal{T})}$ the simplicial toric variety defined by the fan $\Sigma = \Sigma(\mathcal{T}) \subset M_{\mathbb{R}}$ whose d-dimensional cones are exactly $\sigma_i := \mathbb{R}_{\geq 0}\tau_i$ $(1 \leq i \leq k)$. If $A = \{0, v_1, \ldots, v_n\}$ and*

$$f(t) := 1 - \sum_{i=1}^{n} a_i t^{v_i},$$

then for any homogeneous polynomial $Q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ of degree $d-1$ the Laurent expansion of the Q-Yukawa $(d-1)$-point function $Y_Q(a)$ at the vertex $v_{\mathcal{T}} \in \mathrm{Sec}(A)$ corresponding to the coherent triangulations $\mathcal{T}$ coincides with the generating function of intersection numbers

$$\sum_{\beta \in K_{\mathrm{eff}}(\mathbb{P})} \langle Q([D_1], \ldots, [D_n])([D_1] + \cdots + [D_n])\, \Phi_\beta \rangle_\beta \, a^\beta,$$

where the sum runs over all integral points $\beta = (b_1, \ldots, b_n)$ of the Mori cone $K_{\mathrm{eff}}(\mathbb{P})$, and $a^\beta := a_1^{b_1} \cdots a_n^{b_n}$.

10. ALGORITHMIC CALCULATION OF TORIC RESIDUES

An effective procedure for computing of toric residues in homogeneous coordinates using Gröbner basis calculus was developed in [CCD, CD]. Next we describe how these ideas can be used in concrete calculations of Yukawa $(d-1)$-point functions. We hold the same notations as in Section 9.

Definition 10.1 ([Cox1]). Pick a subset $I = \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ such that $e_{i_1}, \ldots, e_{i_d}$ are linearly independent. Then define a *toric Jacobian* for any $(d+1)$ homogeneous polynomials $G_0, \ldots, G_d \in S(\mathbb{P})_\alpha$ by the formula

$$J_G = \begin{pmatrix} G_0 & \cdots & G_d \\ \partial G_0/\partial z_{i_1} & \cdots & \partial G_d/\partial z_{i_d} \\ \vdots & \ddots & \vdots \\ \partial G_0/\partial z_{i_d} & \cdots & \partial G_d/\partial z_{i_d} \end{pmatrix} / \det(e_I)\widehat{z}_I,$$

where $e_I = \det(\langle m_j, e_{i_k} \rangle_{1 \leq j,k \leq d})$ and $\widehat{z}_I = \prod_{i \notin I} z_i$.

Remark 10.2. Suppose that $G = (G_0, \ldots, G_d)$ is the image in $S(\mathbb{P})_\alpha$ of the regular sequence of elements in $S^1_{\Delta_H}$. Then it is easy to show (see [CDS]) that the Jacobian J_G in Definition 10.1 can be identified with the image of the Jacobian J_G in (1) under the isomorphism (10).

As we have seen in the proof of Theorem 9.7, the function $Y_Q(a)$ is equal to the Cox's residue $-\mathrm{Res}_F(\tilde{q}(z) \cdot \prod_{i=1}^r z_i)$. We can compute this residue using the following method. Choose a Gröbner basis of the ideal generated by $F_0(z), \ldots, F_d(z) \in S(\mathbb{P})_{\omega_0}$. Then compute the normal form $normalf(H)$ for the polynomial $H = -\tilde{q}(z) \cdot \prod_{i=1}^r z_i$ and the normal form $normalf(J_F)$ for the toric Jacobian J_F. Since the quotient in (11) is one-dimensional, both normal forms are the multiples

$$normalf(H) = c\,z^\sigma, \quad normalf(J_F) = c_F\,z^\sigma$$

of some monomial $z^\sigma \in S(\mathbb{P})_{d\omega_0}$ by constants c and $c_F \neq 0$ modulo $\langle F_0, \ldots, F_d \rangle$. Recall that the toric residue $\mathrm{Res}_F(J_F)$ is equal to the normalized volume $\mathrm{Vol}(\Delta)$ of polytope Δ. Now given a polynomial $f(t)$ supported in the reflexive polytope Δ and a homogeneous polynomial Q, we get in output

$$Y_Q(a) = \frac{c}{c_F}\,\mathrm{Vol}(\Delta).$$

Example 10.3. We illustrate this method by the following example (cf. [COFKM, Section 8.1], [MP, Section 4.4], [HKTY, Appendix A.1], [CK, Example 5.6.2.1]). Let $\mathbb{P}(1,1,2,2,2)$ be the weighted projective space defined by the fan $\Sigma' \subset M_\mathbb{R} \cong \mathbb{R}^4$ with one-dimensional generators

$$v_1 = (-1,-2,-2,-2), \qquad v_2 = (1,0,0,0), \qquad v_3 = (0,1,0,0),$$
$$v_4 = (0,0,1,0), \qquad\qquad v_5 = (0,0,0,1),$$

which are the vertices of the reflexive polytope $\Delta \subset M_\mathbb{R}$. The polytope Δ contains only two lattice points except of listed above: $v_6 = (0,-1,-1,-1) = \frac{1}{2}(v_1 + v_2)$ and the origin $v_0 = (0,0,0,0)$. Including the additional lattice point v_6 to the generators of Σ' corresponds to the blowup of $\mathbb{P}(1,1,2,2,2)$. The dual polytope $\Delta^* \subset N_\mathbb{R}$ to Δ is spanned by the vectors

$$e_1 = (-1,-1,-1,-1), \qquad e_2 = (7,-1,-1,-1), \qquad e_3 = (-1,3,-1,-1),$$
$$e_4 = (-1,-1,3,-1), \qquad e_5 = (-1,-1,-1,3),$$

generating the fan $\Sigma \subset N_\mathbb{R}$ of $\mathbb{P} = \mathbb{P}_\Sigma$.

Take the Laurent polynomial

$$f(t) = 1 - \sum_{i=1}^6 a_i t^{v_i} = 1 - a_1 t_1^{-1} t_2^{-2} t_3^{-2} t_4^{-2} - a_2 t_1 - a_3 t_2 - a_4 t_3 - a_5 t_4 - a_6 t_2^{-1} t_3^{-1} t_4^{-1}$$

having Δ as support polytope. After choosing the new variables

$$y_1 := a_3 a_4 a_5 a_6, \quad y_2 := \frac{a_1 a_2}{a_6^2},$$

we can put the equation for $f(t)$ in the form

$$f(t) = 1 - y_2 t_1^{-1} t_2^{-2} t_3^{-2} t_4^{-2} - t_1 - y_1 t_2 - t_3 - t_4 - t_2^{-1} t_3^{-1} t_4^{-1}.$$

Let Z_f be a hypersurface in $\mathbb{T} \cong (\mathbb{C}^*)^4$ defined by $f(t)$ and forming the mirror family of Calabi-Yau hypersurfaces in $\mathbb{P}$. Denote

$$\Omega := \mathbf{Res}\left(\frac{1}{f}\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_4}{t_4}\right) \in H^3(Z_f).$$

We compute the Yukawa couplings

$$Y^{(3-k,k)}(y_1, y_2) = \frac{-1}{(2\pi i)^3}\int_{Z_f}\Omega \wedge (y_1\partial/\partial y_1)^{3-k}(y_2\partial/\partial y_2)^k\Omega, \quad k = 0, 1, 2, 3$$

corresponding to polynomials $Q(x_1, x_2) = x_1^{3-k}x_2^k$. Show how to compute, say, the Yukawa coupling $Y^{(3,0)}(y_1, y_2)$ corresponding to the polynomial $Q(x_1, x_2) = x_1^3$. Let z_i be the homogeneous coordinates on $\mathbb{P}$ related with the vectors e_i ($1 \le i \le 5$). Then the homogenization of $f(t)$ defining the anticanonical hypersurface on $\mathbb{P}$ is

$$F(z) = z_1z_2z_3z_4z_5 - y_2z_1^8 - z_2^8 - y_1z_3^4 - z_4^4 - z_5^4 - z_1^4z_2^4.$$

Denote by $F_1(z), \ldots, F_4(z)$ the images of $t_0t_1\partial f/\partial t_1, \ldots, t_0t_4\partial f/\partial t_4$ under the isomorphism (10). Fix a Gröbner basis of the ideal $\langle F_0, \ldots, F_4\rangle$ with respect to the reverse lex order. Next, note that the homogenization of

$$q(t) = -t_0^3 Q(y_1t^{v_1}, y_2t^{v_3}) = -t_0^3 Q(y_1t_2, y_2t_1^{-1}t_2^{-2}t_3^{-2}t_4^{-2}) = -t_0^3(y_1t_2)^3$$

is $\tilde{q}(z) = -y_1^3z_3^{12}$. Applying the Gröbner basis calculation, we have found (using MAPLE) the following normal forms:

$$normal f(-\tilde{q}(z) \cdot z_1 \cdots z_5) = 4y_1^4z_3^{16},$$
$$normal f(J_F) = 4y_1^4((1 - 2^8y_1)^2 - 2^{18}y_1^2y_2)z_3^{16},$$

where J_F is the toric Jacobian. This easily yields that the Yukawa coupling is given by the formula

$$Y^{(3,0)}(y_1, y_2) = \frac{8}{(1 - 2^8y_1)^2 - 2^{18}y_1^2y_2},$$

since $\mathrm{Vol}(\Delta) = 8$. Using the same procedure, we obtain

$$Y^{(2,1)}(y_1, y_2) = \frac{4(1 - 2^8y_1)}{(1 - 2^8y_1)^2 - 2^{18}y_1^2y_2},$$

$$Y^{(1,2)}(y_1, y_2) = \frac{8y_2(-1 + 2^9y_1)}{(1 - 4y_2)((1 - 2^8y_1)^2 - 2^{18}y_1^2y_2)},$$

$$Y^{(0,3)}(y_1, y_2) = \frac{4y_2(1 - 2^8y_1 + 2^2y_2 - 2^{10}3y_1y_2)}{(1 - 4y_2)^2((1 - 2^8y_1)^2 - 2^{18}y_1^2y_2)}.$$

The series expansions for $Y^{(3-k,k)}(y_1, y_2)$ were computed in [MP, Section 4.4]. The result of these computations is

$$Y^{(3-k,k)}(y_1, y_2) = \sum_{\lambda_1,\lambda_2\ge 0} 2^{8\lambda_1+2\lambda_2+3-k}\binom{\lambda_1 + 1 - k}{2\lambda_2 + 1 - k}y_1^{\lambda_1}y_2^{\lambda_2}, \quad k = 0, 1, 2, 3.$$

Now suppose that we do not blowup $\mathbb{P}(1,1,2,2,2)$. It means that we do not add v_6 to the generators of Σ'. Then the polynomial

$$f(t) = 1 - \sum_{i=1}^{5} a_i t^{v_i} = 1 - a_1 t_1^{-1} t_2^{-2} t_3^{-2} t_4^{-2} - a_2 t_1 - a_3 t_2 - a_4 t_3 - a_5 t_4$$

can be transformed to

$$f(t) = 1 - t_1^{-1} t_2^{-2} t_3^{-2} t_4^{-2} - y t_1 - t_2 - t_3 - t_4$$

after introducing the new variable $y := a_1 a_2 a_3^2 a_4^2 a_5^2$. It is nice to observe that the Yukawa coupling

$$Y^3(y) = \frac{8}{1 - 2^{18} y} = 8 + 2097152 y + 549755813888 y^2 + O(y^3)$$

for $Q(x) = x^3$ can be computed either by theorems 7.2 and 9.7 or as a limit $a_6 \to 0$ in $Y^{(3,0)}(y_1, y_2)$ found above.

11. MIRRORS OF CALABI-YAU HYPERSURFACES IN $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$

Let us illustrate our method for computation of Yukawa $(d-1)$-point functions for Calabi-Yau hypersurfaces $Z_f \subset \mathbb{T} \cong (\mathbb{C}^*)^2$ defined by polynomial

$$f(t) = 1 - \sum_{i_1=1}^{n_1} a_{1i_1} t^{v_{1i_1}} - \sum_{i_2=1}^{n_2} a_{2i_2} t^{v_{2i_2}},$$

where the exponents v_{ij} satisfy the relations

$$v_{11} + \cdots + v_{1n_1} = 0, \quad v_{21} + \cdots + v_{2n_2} = 0.$$

The toric variety $\mathbb{P}_\Sigma$ with the fan Σ of dimension $d = d_1 + d_2$ (here, $d_1 = n_1 - 1$, $d_2 = n_2 - 1$) with generators $\{v_{jn_j}\}$ is the product of two projective spaces $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$. According to [Bat2], the hypersurfaces Z_f form the mirror family of Calabi-Yau hypersurfaces in $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$.

Denote by $y_1 = a_{11} \cdots a_{1n_1}$, $y_2 = a_{21} \cdots a_{2n_2}$ the new variables and by $\theta_1 = y_1 \partial / \partial y_1$, $\theta_2 = y_2 \partial / \partial y_2$ the logarithmic partial derivations. Given a form-residue

$$\Omega := \mathbf{Res} \left(\frac{1}{f} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \right) \in H^{d-1}(Z_f),$$

the 2-parameter Yukawa $(d-1)$-point functions are defined as the integrals

$$Y^{(k_1,k_2)}(y_1, y_2) = \frac{(-1)^{\frac{(d-1)(d-2)}{2}}}{(2\pi i)^{d-1}} \int_{Z_f} \Omega \wedge \theta_1^{k_1} \theta_2^{k_2} \Omega, \quad k_1 + k_2 = d - 1.$$

Theorem 9.7, Theorem 8.1 and Proposition 8.3 easily implies the following statement.

Proposition 11.1. *The Yukawa* $(d-1)$-*point function* $Y^{(k_1,k_2)}(y_1, y_2)$ *is equal to the toric residue related with the polynomial* $P(x_1, x_2) = x_1^{k_1} x_2^{k_2}(n_1 x_1 + n_2 x_2)$. *Let* $u_1 = n_1^{n_1} y_1$, $u_2 = n_2^{n_2} y_2$. *Then the function* $Y^{(k_1,k_2)}(y_1, y_2)$ *can be computed*

1) *as a trace of rational function:*

$$Y^{(k_1,k_2)}(u_1, u_2) = n_1^{d_1-k_1-1} n_2^{d_2-k_2-1} \sum_{\substack{z_1^{n_1}=u_1 \\ z_2^{n_2}=u_2}} \frac{1}{z_1^{d_1-k_1} z_2^{d_2-k_2}(1 - z_1 - z_2)};$$

2) *as a series:*

$$Y^{(k_1,k_2)}(u_1, u_2) = n_1^{d_1-k_1} n_2^{d_2-k_2} \sum_{b_1,b_2 \geq 0} \frac{(n_1 b_1 + n_2 b_2 + 1)!}{(n_1 b_1 + d_1 - k_1)!(n_2 b_2 + d_2 - k_2)!} u_1^{b_1} u_2^{b_2}.$$

Some examples of explicit calculation of Yukawa $(d-1)$-point functions as rational functions are given below.

Example 11.2. $\mathbb{P}^1 \times \mathbb{P}^1$; $u_1 = 2^2 y_1$, $u_2 = 2^2 y_2$.

$$
\begin{aligned}
Y^{(1,0)}(u_1, u_2) &= \frac{1}{2} \sum_{\substack{z_1^2=u_1 \\ z_2^2=u_2}} \frac{1}{z_2(1 - z_1 - z_2)} = 2 \sum_{b_1,b_2 \geq 0} \frac{(2b_1 + 2b_2 + 1)!}{(2b_1)!(2b_2 + 1)!} u_1^{b_1} u_2^{b_2} \\
&= \frac{2(1 + u_1 - u_2)}{(1 - u_1 - u_2)^2 - 4u_1 u_2}.
\end{aligned}
$$

By symmetry, $Y^{(0,1)}(u_1, u_2) = Y^{(1,0)}(u_2, u_1)$.

Example 11.3. $\mathbb{P}^1 \times \mathbb{P}^2$; $u_1 = 2^2 y_1$, $u_2 = 3^3 y_2$.

$$
\begin{aligned}
Y^{(2,0)}(u_1, u_2) &= \frac{3}{2} \sum_{\substack{z_1^2=u_1 \\ z_2^3=u_2}} \frac{z_1}{z_2^2(1 - z_1 - z_2)} = \frac{9}{2} \sum_{b_1,b_2 \geq 0} \frac{(2b_1 + 3b_2 + 1)!}{(2b_1 - 1)!(3b_2 + 2)!} u_1^{b_1} u_2^{b_2} \\
&= \frac{9}{2} \frac{u_1(3 + u_1)}{(1 - u_1)^3 - 2u_2(1 - 3u_1)}.
\end{aligned}
$$

$$
\begin{aligned}
Y^{(1,1)}(u_1, u_2) &= \frac{1}{2} \sum_{\substack{z_1^2=u_1 \\ z_2^3=u_2}} \frac{1}{z_2(1 - z_1 - z_2)} = 3 \sum_{b_1,b_2 \geq 0} \frac{(2b_1 + 3b_2 + 1)!}{(2b_1)!(3b_2 + 1)!} u_1^{b_1} u_2^{b_2} \\
&= \frac{3(1 - u_2 - u_1^2)}{(1 - u_1)^3 - 2u_2(1 - 3u_1)}.
\end{aligned}
$$

$$Y^{(0,2)}(u_1, u_2) = \frac{1}{3} \sum_{\substack{z_1^2 = u_1 \\ z_2^3 = u_2}} \frac{1}{z_1(1 - z_1 - z_2)} = 2 \sum_{b_1, b_2 \geq 0} \frac{(2b_1 + 3b_2 + 1)!}{(2b_1 + 1)!(3b_2)!} u_1^{b_1} u_2^{b_2}$$

$$= \frac{2((1 - u_1)^2 + 2u_2)}{(1 - u_1)^3 - 2u_2(1 - 3u_1)}.$$

The next example of hypersurfaces in product of projective spaces was considered in [BvS] in the case of diagonal one-parameter subfamily and in [HKTY].

Example 11.4. $\mathbb{P}^2 \times \mathbb{P}^2$; $u_1 = 3^3 y_1$, $u_2 = 3^3 y_2$.

$$Y^{(3,0)}(u_1, u_2) = \frac{1}{3} \sum_{\substack{z_1^3 = u_1 \\ z_2^3 = u_2}} \frac{z_1}{z_2^2(1 - z_1 - z_2)} = 3 \sum_{b_1, b_2 \geq 0} \frac{(3b_1 + 3b_2 + 1)!}{(3b_1 - 1)!(3b_2 + 2)!} u_1^{b_1} u_2^{b_2}$$

$$= \frac{9u_1(2 + u_1 + u_2)}{(1 - u_1 - u_2)^3 - 27u_1 u_2}.$$

$$Y^{(2,1)}(u_1, u_2) = \frac{1}{3} \sum_{\substack{z_1^3 = u_1 \\ z_2^3 = u_2}} \frac{1}{z_2(1 - z_1 - z_2)} = 3 \sum_{b_1, b_2 \geq 0} \frac{(3b_1 + 3b_2 + 1)!}{(3b_1)!(3b_2 + 1)!} u_1^{b_1} u_2^{b_2}$$

$$= \frac{3((1 - u_2)^2 + u_1(1 - 2u_1 - u_2))}{(1 - u_1 - u_2)^3 - 27u_1 u_2}.$$

By symmetry, $Y^{(k_1, k_2)}(u_1, u_2) = Y^{(k_2, k_1)}(u_2, u_1)$.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, TÜBINGEN D-72076, GERMANY

E-mail address: victor.batyrev@uni-tuebingen.de

MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, TÜBINGEN D-72076, GERMANY

E-mail address: evgeny.materov@uni-tuebingen.de