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T-Duality, and the K-Theoretic Partition Function of TypeIIA Superstring Theory

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We study the partition function of type IIA string theory on 10-manifolds of the form $T^2 \times X$ where X is 8-dimensional, compact, and spin. We pay particular attention to the effects of the topological phases in the supergravity action implied by the K-theoretic formulation of RR fields, and we use these to check the T-duality invariance of the partition function. We find that the partition function is only T-duality invariant when we take into account the T-duality anomalies in the RR sector, the fermionic path integral (including 4-fermi interaction terms), and 1-loop corrections including worldsheet instantons. We comment on applications of our computation to speculations about the role of the Romans mass in M-theory. We also discuss some issues which arise when one attempts to extend these considerations to checking the full U-duality invariance of the theory.

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1. Introduction & Summary

Duality symmetries, such as the U-duality symmetry of toroidally compactified M-theory, have been of central importance in the definition of string theory and M-theory. Topologically nontrivial effects associated with the RR sector have also played a crucial role in defining the theory. It is currently believed that RR fieldstrengths (and their D-brane charge sources) are classified topologically using K-theory [1,2,3,4,5,6,7,8]. Unfortunately, this classification is not U-duality invariant. Finding a U-duality invariant formulation of M-theory which at the same time naturally incorporates the K-theoretic formulation of RR fields remains an outstanding open problem.

With this problem as motivation, the present paper investigates the interplay between the K-theoretic formulation of RR fields and the T-duality group, an important subgroup of the full U-duality group. While T-duality invariance of the theory was one of the guiding principles in the definition of the K-theoretic theta function [4][7] we will see that the full implementation of T-duality invariance of the low energy effective action of type II string theory is in fact surprisingly subtle, even on backgrounds as simple as $T^2 \times X$, where T^2 is a two-dimensional torus, and X is an 8-dimensional compact spin manifold. We will show that, in fact, in the RR sector there is a T-duality anomaly. This anomaly is cancelled by a compensating anomaly from fermion determinants together with quantum corrections to the 8D effective action. A by-product of our computation is a complete analysis of the 1-loop determinants of IIA supergravity on $X \times T^2$.

As an application of our discussion, we re-examine a proposal of C. Hull [9] for interpreting the Romans mass of IIA supergravity in the framework of M-theory. We will show that, while the interpretation cannot hold at the level of classical field theory, it might well hold as a quantum-mechanical equivalence. In section 10 we comment on some of the issues which arise in extending our computation to a fully U-duality invariant partition function. This includes the role of twisted K-theory in formulating the partition sum.

This paper is long and technical. Therefore we have attempted to write a readable summary of our results in the remainder of the introduction.

1.1. The effective eight-dimensional supergravity, and its partition function

Previous studies of the partition function in type II string theory [4][7] considered the limit of a large 10-manifold. One chose a family of Riemannian metrics $g = t^2 g_0$ with $t \to \infty$ and g_0 fixed. Simultaneously, one took the string coupling to zero. The focus of

these works was on the sum over classical field configurations of the RR fields. In this paper we consider the limit where only 8 of the dimensions are large. The metric has the form

$$ds^2 = ds^2_{T^2} + t^2 ds^2_X \tag{1.1}$$

where $ds^2_{T^2}$ is flat when pulled back to T^2. The background dilaton $g^2_{\rm string} = e^{2\xi}$ is constant. We will work in the limit

$$\begin{aligned} t &\to \infty \\ e^{-2\xi} &:= e^{-2\phi} V \to \infty \end{aligned} \tag{1.2}$$

where V is the volume of T^2 and ϕ is the 10-dimensional dilaton. Finally - and this is important -until section 10 we assume the background NSNS 3-form flux, $\hat{H}$, is identically zero. In particular, the 2-form potential, $\hat{B}$, is a globally well-defined harmonic form on $X \times T^2$.

As is well-known the background data for the toroidal compactification (1.1) include a pair of points $(\tau, \rho) \in \mathcal{H} \times \mathcal{H}$ where $\mathcal{H}$ is the upper half complex plane. τ is the Teichmuller parameter of the torus and $\rho := B_0 + iV$, where $B_0 d\sigma^8 \wedge d\sigma^9$ is an harmonic 2-form on T^2. While we work in the limit (1.2), within this approximation we work with exact expressions in the geometrical data (τ, ρ). In this way we go beyond [7].

It is extremely well-known that the low energy effective 8D supergravity theory obtained by Kaluza-Klein reduction of type II supergravity on T^2 has a "U-duality symmetry" which is classically $SL(3,R) \times SL(2,R)$, and is broken to $\mathcal{D} := SL(3,Z) \times SL(2,Z)$ by quantum effects [10,11,12,13,14]. These are symmetries of the equations of motion and not of the action. (The implementation of these symmetries at the level of the action involves a Legendre transformation of the fields.) What is perhaps less well-known is that the K-theoretic formulation of RR fields leads to an extra term in the supergravity action which is nonvanishing in the presence of nontrivial flux configurations. Indeed, the proper formulation of this term is unknown for arbitrary flux configurations with $[\hat{H}_3] \neq 0$, but for topologically trivial NSNS flux the extra term is known [7] and is recalled in equations (1.14) and (1.15) below. This term breaks naive duality invariance of the classical supergravity theory already for the T-duality subgroup of the U-duality group, and makes the discussion of T-duality nontrivial.

Let us now summarize the fields and T-duality transformation laws in the conventional description of the eight-dimensional effective supergravity theory on X. The T-duality group is $\mathcal{D}_T = SL(2,Z)_\tau \times SL(2,Z)_\rho$. The theory has the following bosonic fields. From

the NSNS sector there is a scalar t, characterizing the size of X, a unit volume metric g_{MN}, a 2-form potential [1] $B_{(2)}$, with fieldstrength $H_{(3)}$, and a dilaton ξ, all of which are invariant under $\mathcal{D}_T$. In addition, there is a multiplet of 1-form potentials $\mathbf{A}^{m\alpha}_{(1)}$ transforming in the $(\mathbf{2},\mathbf{2})$ of $\mathcal{D}_T$. Finally, the pair of scalars (τ,ρ), transform under $(\gamma_1,\gamma_2)\in\mathcal{D}_T$ as $(\tau,\rho)\to(\gamma_1\cdot\tau,\gamma_2\cdot\rho)$ where $\gamma\cdot$ is the action by a fractional linear transformation. We therefore call the factors $SL(2,Z)_\tau, SL(2,Z)_\rho$, respectively.

The fieldstrengths from the RR sector include a 0-form and a 2-form, $g^\alpha_{(p)}$, $p=0,2,\alpha=1,2$ transforming in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}_T$, and a 1-form and 3-form $g_{(p)m}$, $p=1,3,m=8,9$ transforming in the $(\mathbf{2'},\mathbf{1})$ of $\mathcal{D}_T$. Finally there is a 4-form fieldstrength $g_{(4)}$ on X. This field does not transform locally under T-duality, rather its equation of motion mixes with its Bianchi identity [14]. The fermionic partners are described in section 7 below.

The real part of the standard bosonic supergravity action takes the form

$$Re\left(S^{(8D)}_{\text{boson}}\right)=S_{NSNS}+\sum_{p=0}^{3}S_p\left(g_{(p)}\right)+S_4\left(g_{(4)}\right) \tag{1.3}$$

In the action (1.3) all of the terms except for the last term are manifestly T-duality invariant. The detailed forms of the actions are:

$$\begin{aligned}S_{NSNS}=\frac{1}{2\pi}\int_X e^{-2\xi}\Big\{&t^6\left(\mathcal{R}(g)+4d\xi\wedge *d\xi+28t^{-2}dt\wedge *dt\right)+\frac{1}{2}t^2H_{(3)}\wedge *H_{(3)}\\&+\frac{1}{2}t^6\frac{d\tau\wedge *d\tau}{(\mathrm{Im}\tau)^2}+\frac{1}{2}t^6\frac{d\rho\wedge *d\rho}{(\mathrm{Im}\rho)^2}+\frac{1}{2}t^4g_{mn}\mathcal{G}_{\alpha\beta}\mathbf{F}^{m\alpha}_{(2)}\wedge *\mathbf{F}^{n\beta}_{(2)}\Big\}\end{aligned} \tag{1.4}$$

where $*$ stands for the Hodge dual with the metric g_{MN}, we also denote

$$\mathbf{F}^{m\alpha}_{(2)}=d\mathbf{A}^{m\alpha}_{(1)},\quad H_{(3)}=dB_{(2)}-\frac{1}{2}\epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}^{m\alpha}_{(1)}\mathbf{F}^{n\beta}_{(2)}, \tag{1.5}$$

ϵ_{mn} and $\mathcal{E}_{\alpha\beta}$ are invariant antisymmetric tensors for $SL(2,\mathbf{Z})_\tau$ and $SL(2,\mathbf{Z})_\rho$ respectively,

$$g_{mn}=\mathcal{M}(\tau),\quad g^{mn}=\mathcal{M}(\tau)^{-1},\quad \mathcal{G}_{\alpha\beta}=\mathcal{M}(\rho) \tag{1.6}$$

and finally

$$\mathcal{M}(z):=\frac{1}{\mathrm{Im}z}\begin{pmatrix}1 & \mathrm{Re}z\\ \mathrm{Re}z & |z|^2\end{pmatrix}. \tag{1.7}$$

[1] We will always indicate by the subscript (p) the degree p of a differential p-form on X

The real part of the RR sector action is given by

$$\sum_{p=0}^{3} S_p\left(g_{(p)}\right) = \pi \int_X \Big\{ t^8 \mathcal{G}_{\alpha\beta} g^{\alpha}_{(0)} \wedge * g^{\beta}_{(0)} + t^6 g^{mn} g_{(1)m} \wedge * g_{(1)n} + \tag{1.8}$$

$$t^4 \mathcal{G}_{\alpha\beta} g^{\alpha}_{(2)} \wedge * g^{\beta}_{(2)} + t^2 g^{mn} g_{(3)m} \wedge * g_{(3)n} \Big\}$$

together with

$$S_4\left(g_{(4)}\right) = \pi \int_X \mathrm{Im}(\rho) g_{(4)} \wedge * g_{(4)}. \tag{1.9}$$

1.2. The semiclassical expansion

The vevs of the two fields t and $e^{-2\xi}$ (the 8-dimensional length scale of X and the inverse-square 8D string coupling) define semiclassical expansions when they become large. We will expand around a solution of the equations of motion on X. To leading order in our expansion this means X admits a Ricci flat metric[2] g_{MN}. We also have constant scalars t, ξ, τ, ρ, and $\mathbf{F}^{m\alpha}_{(2)} = 0, \quad H_{(3)} = 0$, so the background action S_{NSNS} is zero. Finally, we expand around a classical field configuration for the RR fluxes, and to leading order these fluxes $g_{(p)}$ are harmonic forms. Nonzero fluxes contribute terms to the partition function going like $\mathcal{O}(e^{-t^{8-2p}})$.

Let us consider the leading order contribution to the partition function. There are several sources of contributions even at leading order, but, since we are interested in questions of T-duality, most of these can be neglected. [3] The volume of X suppresses the contribution of fluxes $g_{(p)}, p = 0, 1, 2, 3$, and, to leading order in the $t \to \infty$ expansion these can be set to their classical values. Note, however, that neither the string coupling, nor the volume of X, suppress the action for $g_{(4)}$, and thus we must work in a fully quantum mechanical way with this field. This is just as well, since (not coincidentally)

[2] Almost nothing in what follows relies on the Ricci flatness of the metric. We avoid using this condition since a T-duality anomaly on non-Ricci flat manifolds would signal an important inconsistency in formulating string theory on manifolds of topology $X \times T^2$.

[3] In particular we are negelecting determinants of KK and string modes, and perturbative corrections $\mathcal{O}(g^2_{\mathrm{string}})$. These are all T-duality invariant. The backreaction of nonzero RR fluxes on the NSNS action simply renormalizes V to V_{eff}, where $\rho = B_0 + iV_{eff}$ is the variable on which $SL(2,\mathbf{Z})_\rho$ acts by fractional linear transformations.

this is the term in the action which is not manifestly T-duality invariant. Fortunately, in our approximation, $g_{(4)}$ is a free, nonchiral field and hence quantization is straightforward (after the K-theory subtleties are taken into account). Including subleading terms in the expansion parameter t involves (among other things) summing over the RR fluxes $g_{(p)}$, $p = 0, 1, 2, 3$.

Finally, in order to be consistent with our approximation scheme we must allow the possibility of *flat* potentials in the background. [4] These contribute nontrivially to the partition function through important phases and accordingly, we will generalize our background to include these. The real part of the action for the flat configurations vanishes, of course, and hence in the physical partition function one must integrate over these flat configurations. In the RR sector the flat potentials are thought to be classified by $K^1(X_{10}; U(1))$ [5]. These contribute no phase to the action and we will henceforth ignore them. [5] The space of flat NSNS potentials is

$$H^2(X; U(1)) \times (H^1(X; U(1)))^4.$$

In this paper we will work only with the identity component of this torus. Accordingly, we will identify the space of flat NSNS potentials with the torus

$$\frac{\mathcal{H}^2(X)}{\mathcal{H}^2_{\mathbf{Z}}(X)} \times \left(\frac{\mathcal{H}^1(X)}{\mathcal{H}^1_{\mathbf{Z}}(X)}\right)^4 \tag{1.10}$$

where $\mathcal{H}^p(X)$ is a space of harmonic p-forms on X and $\mathcal{H}^p_{\mathbf{Z}}(X)$ is the lattice of integrally normalized harmonic p-forms on X. The first factor is for $B_{(2)}$ and the second factor for the fields $\mathbf{A}^{m\alpha}_{(1)}$ transforming in the $(\mathbf{2}, \mathbf{2})$ of $\mathcal{D}_T$.

Putting all these ingredients together the partition function we wish to study can be schematically written as

$$Z(t, g_{MN}, \xi, \tau, \rho) = \int_{\text{flat potentials}} d\mu_{\text{flat}} \sum_{\text{RR fluxes}} \text{Det} \cdot e^{-S_{cl}} + \cdots \tag{1.11}$$

where $d\mu_{\text{flat}}$ is a T-duality invariant measure on the flat potentials, Det is a product of 1-loop determinants and S_{cl} is the classical action. Now, to investigate T-duality it is

[4] By "flat" we mean the DeRham representative of the relevant fieldstrength is zero.

[5] If treated as differential forms, RR zero modes do contribute to the overall dependence of the partition sum on $\tilde{t} = te^{-\xi/3}$. See eq.(7.39) below.) In the K-theoretic treatment they also give a factor of $|K^0_{tors}(X \times T^2)|$.

convenient to denote by $\mathcal{F}$ the collection of all fields occuring in (1.11) which transform locally and linearly under $\mathcal{D}_T$. These include the flat NSNS potentials above as well as the classical fluxes $g_{(p)}, p = 0, \ldots, 3$. We introduce a measure $[d\mathcal{F}]$ on $\mathcal{F}$ which includes integration over the flat potentials and summation over the fluxes for $p = 0, 1, 2, 3$. This measure is T-duality invariant, and we can write

$$Z(t, g_{MN}, \xi, \tau, \rho) = \int [d\mathcal{F}] Z(\mathcal{F}; t, g_{MN}, \xi, \tau, \rho). \tag{1.12}$$

The invariance of (1.12) under the subgroup $SL(2,Z)_\tau$ of the T-duality group is essentially trivial. The relevant actions and determinants are all based on $SL(2,Z)_\tau$-invariant differential operators. The invariance of the theory under $SL(2,Z)_\rho$ is, however, much more nontrivial. Therefore we simplify notation and just write $Z(\mathcal{F}, \rho)$ for the integrand of (1.12). Now, checking T-duality invariance is reduced to checking the invariance of $Z(\mathcal{F}, \rho)$. This function is constructed from

a. The K-theoretic sum over RR fluxes of $g_{(4)}$ in the presence of $\mathcal{F}$.

b. The integration over the Fermi zeromodes in the presence of $g_{(4)}$ and $\mathcal{F}$.

c. The inclusion of 1-loop determinants, including determinants of the 8D supergravity fields and the quantum corrections due to worldsheet instantons.

In the following subsections we sketch how each of these elements enters $Z(\mathcal{F}, \rho)$. Briefly, the K-theoretic sum over RR fluxes $g_{(4)}$ leads to a theta function $\Theta(\mathcal{F}, \rho)$. This function turns out to transform anomalously under T-duality. The integration over the fermion zeromodes corrects this to a function $\widehat{\Theta}(\mathcal{F}, \rho)$. This function still transforms anomalously. The inclusion of 1-loop effects, including the string 1-loop effects finally cancels the anomaly.

1.3. The K-theoretic RR partition function

In order to write explicit formulae for the quantities in (1.12) we must turn to the K-theoretic formulation of RR fields. In practical terms the K-theoretic formulation alters the standard formulation of supergravity in two ways: First it restricts the allowed flux configurations through a "Dirac quantization condition" on the fluxes. Second, it changes the supergravity action by the addition of important topological terms in the action. [6]

[6] It also alters the overall normalization of the bosonic determinants by changing the nature of the gauge group for RR potentials, but we will not discuss this in the present paper.

In more detail, the K-theoretic Dirac quantization condition states that the DeRham class of the total RR fieldstrength $[G/(2\pi)]$ is related to a K-theory class $x \in K^0(X_{10})$ via

$$[\frac{G}{2\pi}] = \mathrm{ch}(x)\sqrt{\hat{A}} \tag{1.13}$$

The topological terms in the action can be described as follows. On a general 10-manifold this term involves the mod-two index of a Dirac operator and cannot even be written as a traditional local term in the supergravity action [4,5,7]. In the case of zero NS-NS fluxes, the general expression for the phase in the supergravity theory is:

$$Im(S_{10D}) = -2\pi\Phi, \quad \Phi = \Phi_1 + \Phi_2 \tag{1.14}$$

where $e^{2\pi i\Phi_2}$ is the mod-two index and Φ_1 is given by the explicit expression

$$\begin{aligned}\Phi_1 = \int_{X_{10}} \Bigg\{ &-\frac{1}{15}\left(\frac{G_2}{2\pi}\right)^5 + \frac{1}{6}\left(\frac{G_2}{2\pi}\right)^3 \left[\left(\frac{G_4}{2\pi}\right) + \frac{p_1}{12}\left(1+\frac{G_0}{8\pi}\right)\right] \\ &- \left(\frac{G_2}{2\pi}\right)\left[\frac{p_1}{48}\left(\frac{G_4}{2\pi}\right) + \frac{\hat{A}_8}{2}\left(1+\frac{G_0}{2\pi}\right) + \frac{G_0}{4\pi}\left(\frac{p_1}{48}\right)^2\right]\Bigg\}\end{aligned} \tag{1.15}$$

where $G_{2j}, j = 0,1,2$ are RR fluxes on X_{10}, $p_i = p_i(TX_{10})$ and $\hat{A}$ is expressed in terms of the Pontryagin classes of X_{10} as

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right). \tag{1.16}$$

In the case that we reduce to 8 dimensions, taking our manifold to be of the form $X \times T^2$ with the choice of supersymmetric spin structure on T^2 the above considerations simplify and can be made much more concrete.

Consider first the Dirac quantization condition. We reduce RR fieldstrengths as: [7]

$$\begin{aligned}\frac{G_0}{2\pi} &= g^2_{(0)} \\ \frac{G_2}{2\pi} &= g^1_{(0)} d\sigma^8 \wedge d\sigma^9 + g_{(1)m} \wedge d\sigma^m + g^2_{(2)} \\ \frac{G_4}{2\pi} &= g_{(4)} + g_{(3)m} \wedge d\sigma^m + g^1_{(2)} \wedge d\sigma^8 \wedge d\sigma^9\end{aligned} \tag{1.17}$$

[7] Beware of notation! The subscript (p) indicates form degree, while the other sub- and superscripts on $g_{(p)}$ indicate $\mathcal{D}_T$ transformation properties. Thus, for example, $g^2_{(0)}$ is the second component of a doublet $g^\alpha_{(0)}$ of 0-forms.

where $\sigma^m, m = 8, 9$ are coordinates on T^2. In the K-theoretic formulation of flux quantization the fieldstrengths $g_{(4)}, g_{(3)m}, g^\alpha_{(2)}, g_{(1)m}, g^\alpha_{(0)}$ are related to certain integral cohomology classes which we denote as

$$a \in H^4(X, \mathbf{Z}), \quad f_m \in H^3(X, \mathbf{Z}) \otimes \mathbf{Z}^2, \quad e^\alpha = \begin{pmatrix} e'' \\ e \end{pmatrix} \in H^2(X, \mathbf{Z}) \otimes \mathbf{Z}^2, \tag{1.18}$$

$$\gamma_m \in H^1(X, \mathbf{Z}) \otimes \mathbf{Z}^2, \quad n^\alpha = \begin{pmatrix} n_1 \\ n_0 \end{pmatrix} \in H^0(X, \mathbf{Z}) \otimes \mathbf{Z}^2$$

The explicit relation between these classes and the $g_{(p)}$ is somewhat complicated and given in equation (4.3) below. The K-theoretic Dirac quantization condition leaves all integral classes in (1.18) unconstrained except for f_m. One finds that $Sq^3(f_m) = 0$. As explained in section 3.3 and 5.2 "turning on" flat NSNS potentials corresponds to acting on the K-theory torus by an automorphism changing the holonomies of the flat connection on the torus. In concrete terms, turning on flat potentials modifies the reduction formulae (1.17) according to equations (5.15) to (5.18) below.

Now let us consider the phase. It turns out that on 10-folds of the form $X \times T^2$ the phase $e^{2\pi i \Phi_2}$ arising from the mod 2 index may be expressed in concrete terms as

$$\exp[2\pi i \Phi_2] = \exp\left[i\pi \int_X \left\{ g_{(3)8} \cup Sq^2(g_{(3)9}) + g_{(3)8} \cup Sq^2(g_{(3)8}) + g_{(3)9} \cup Sq^2(g_{(3)9}) \right\} \right. \tag{1.19}$$

$$+ i\pi \int_X \left\{ g^2_{(0)} \hat{A}_8 + \left(g_{(4)} + \frac{g^2_{(0)}}{48} p_1 - \frac{1}{2} \left(g^2_{(2)} \right)^2 \right) \left(\left[g^1_{(2)} - g^1_{(0)} g^2_{(2)} + g_{(1)8} g_{(1)9} \right]^2 + \frac{p_1}{2} \right) \right.$$

$$\left. \left. + \frac{p_1^2}{8} + g_{(1)8} g_{(1)9} \left(g^2_{(2)} \right)^3 - \left(g^2_{(2)} \right)^2 \epsilon^{mn} g_{(1)m} g_{(3)n} \right\} \right]$$

This expression is cohomological although it is still unconventional in supergravity theory since it involves the mod-two valued Steenrod squares, denoted $Sq^2(g_{(3)})$, in the first line.

The above topological term (1.14) is deduced from the K-theory theta function Θ_K defined in [4,5,7], and reviewed below. As explained above, it is convenient to fix the fields $\mathcal{F}$. We can define a function $\Theta(\mathcal{F}, \rho)$ by writing Θ_K as a sum

$$\Theta_K = \sum e^{-S_B(\mathcal{F})} \Theta(\mathcal{F}, \rho) \tag{1.20}$$

The sum is over all integral classes except a. That is, we sum over $n^\alpha, \gamma_m, e^\alpha, f_m$ subject to the constraint on $Sq^3 f_m$. The action $S_B(\mathcal{F})$ is the manifestly T-duality invariant action

for the fluxes given in (1.8). Θ_K is a function of g_{MN}, ρ, τ and the flat background NSNS fluxes. As we have mentioned, turning on flat potentials corresponds, in the K-theoretic interpretation, to acting by automorphisms of the K-theory group $K^0(X) \otimes R$. These automorphisms act naturally on the theta function. We give concrete formulae for this action by showing how the inclusion of nonzero flat NSNS fields $B_0, B_{(2)}, \mathbf{A}_{(1)}^{m\alpha}$ modifies the phase Φ. The explicit formula is in equations (5.20)-(5.24) below.

Since the K-theoretic constraint $Sq^3 a = 0, a \in H^4(X, \mathbf{Z})$ is automatically satisfied on spin 8-folds X it turns out that $\Theta(\mathcal{F})$ is, essentially, a Siegel-Narain theta function for the lattice $H^4(X; \mathbf{Z})$. More precisely, there is a quadratic form on $H^4(X; \mathbf{R})$ given by $Q = \mathrm{Im}(\rho) HI - i\mathrm{Re}(\rho) I$ where H is the action of Hodge $*$ and I is the integral intersection pairing on $H^4(X, \mathbf{Z})$. Then

$$\Theta(\mathcal{F}, \rho) = e^{i2\pi \Delta\widetilde{\Phi}(\mathcal{F})} \Theta \left[\begin{matrix} \vec{\alpha} \\ \vec{\beta} \end{matrix} \right] (Q) \tag{1.21}$$

Here $\Theta \left[\begin{matrix} \vec{\alpha} \\ \vec{\beta} \end{matrix} \right] (Q)$ is the Siegel-Narain theta function with characteristics. The characteristics are written explicitly in equations (5.10), (5.20), and (5.21) below. Finally, the prefactor $\Delta\widetilde{\Phi}(\mathcal{F})$ in (1.21) is defined in (5.23) and (5.24) below.

1.4. T-duality transformations

One of the more subtle aspects of the K-theoretic formulation of RR fluxes, is that the very formulation of the action depends crucially on a choice of polarization of the K-theory lattice $K(X_{10})$ with respect to the quadratic form defined by the index. In the above discussion we have chosen the "standard polarization" for IIA theory, i.e Γ_2 is the sublattice of $K(X_{10})$ with vanishing G_4, G_2, G_0. Γ_1 is then a complementary Lagrangian sublattice such that $K(X_{10}) = \Gamma_1 + \Gamma_2$. The standard polarization is distinguished for any large 10-manifold in the following sense. When the metric of X_{10} is scaled up $\hat{g}_{\hat{M}\hat{N}} \to t^2 \hat{g}_{\hat{M}\hat{N}}$ the action $\int_{X_{10}} \sqrt{\hat{g}} |G_{2p}|^2$ of the Type IIA RR 2p-form scales as t^{10-4p}. This allows the sensible approximation of first summing only over G_4, with $G_2 = G_0 = 0$, then including G_2 with $G_0 = 0$, and finally summing over all classical fluxes G_4, G_2, G_0.

In the case of $X_{10} = T^2 \times X$ with the metric (1.1) the standard polarization is no longer distinguished. Various equally good choices are related by the action of the T-duality group $\mathcal{D}_T$ on $\Gamma_K := K(X \times T^2)$. [8] In section 4 we explain how the duality group

[8] There is also a polarization on manifolds of the type $S^1 \times X_9$, (in our case $X_9 = S^1 \times X$) where the measure is purely real and the imaginary part of the action is an integral multiple of $i\pi$ (without flat NSNS potentials). However, this polarization does not lead to a good long-distance approximation scheme.

$\mathcal{D}_T$ acts as a subgroup of symplectic transformations on the K-theory lattice and we give an explicit embeding $\mathcal{D}_T \subset Sp(2N, \mathbf{Z})$, where $2N = rank(\Gamma_K)$. As explained in section 4.2, since $\mathcal{D}_T$ acts symplectically, the function $\Theta(\mathcal{F}, \rho)$ must transform under T-duality as $\Theta(\gamma \cdot \mathcal{F}, \gamma \cdot \rho) = j(\gamma, \rho)\Theta(\mathcal{F}, \rho)$ where $j(\gamma, \rho)$ is a standard transformation factor for modular forms. Nevertheless, this transformation law leaves open the possibility of a T-duality anomaly through a multiplier system in $j(\gamma, \rho)$. In order to investigate this potential anomaly more closely we must choose an explicit duality frame and perform the relevant modular transformations.

We find that, in fact, the function $\Theta(\mathcal{F}, \rho)$ does transform as a modular form with a nontrivial "multiplier system" under $SL(2, Z)_\rho$. That is, using the standard generators T, S of $SL(2, Z)_\rho$ we have:

$$\begin{aligned} \Theta(T \cdot \mathcal{F}, \rho + 1) &= \mu(T)\Theta(\mathcal{F}, \rho) \\ \Theta(S \cdot \mathcal{F}, -1/\rho) &= \mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\Theta(\mathcal{F}, \rho) \end{aligned} \tag{1.22}$$

where $T \cdot \mathcal{F}, S \cdot \mathcal{F}$ denotes the linear action of $\mathcal{D}_T$ on the fluxes. Here b_4^+, b_4^- is the dimension of the space of self-dual and anti-self-dual harmonic forms on X and the multiplier system is

$$\begin{aligned} \mu(T) &= \exp\left[\frac{i\pi}{4}\int_X \lambda^2\right] \\ \mu(S) &= \exp\left[\frac{i\pi}{2}\int_X \lambda^2\right] \end{aligned} \tag{1.23}$$

where λ is the integral characteristic class of the spin bundle on X. (So, $2\lambda = p_1$). The multiplier system is indeed nontrivial on certain 8-manifolds. As an example, on all Calabi-Yau 4-folds we have the relation

$$\frac{1}{4}\int_X \lambda^2 = 62\int_X \hat{A}_8 - 4 + \frac{1}{12}\chi \tag{1.24}$$

and hence μ is nontrivial if χ is not divisible by 12. In particular, a homogeneous polynomial of degree 6 in P^5, has $\chi = 2610$. See, e.g. [15].

In more physical language, the "multiplier system" signals a potential T-duality anomaly. Such an anomaly would spell disaster for the theory since the T-duality group should be regarded as a *gauge symmetry* of M-theory. Accordingly, we turn to the remaining functional integrals in the supergravity theory. We will find that the anomalies cancel, of course, but this cancellation is surprisingly intricate.

1.5. Inclusion of 1-loop effects

We first turn to the 1-loop functional determinants of the quantum fluctuations of the bosonic fields. We show that these are all manifestly T-duality invariant functions of $\mathcal{F}$ except for the quantum fluctuations of $g_{(4)}$. The full bosonic 1-loop determinant Det_B is given in equation (6.20) below. The net effect of inlcuding the bosonic determinants is thus to replace

$$e^{-S_B(\mathcal{F})}\Theta(\mathcal{F},\rho) \to Z_B(\mathcal{F},\rho) := \mathrm{Det}_B e^{-S_B(\mathcal{F})}\Theta(\mathcal{F},\rho) \tag{1.25}$$

Inclusion of this determinant alters the modular weight so that $Z_B(\mathcal{F},\rho)$ transforms with weight $(\frac{1}{4}(\chi+\sigma),\frac{1}{4}(\chi-\sigma))$, in close analogy to the theory of abelian gauge potentials on a 4-manifold [16,17]. Here χ,σ are the Euler character and signature of the 8-fold X. The multiplier system (1.23) is left unchanged.

Now let us consider modifications from the fermionic path integral. Recall that we may always regard a modular form as a section of a line bundle over the modular curve $\mathcal{H}/SL(2,Z)_\rho$. On general grounds, we expect the fermionic path integral to provide a trivializing line bundle. The gravitino and dilatino in the 8d theory transform as modular forms under the T-duality group $\mathcal{D}_T$ with half-integral weights and consequently they too are subject to potential T-duality anomalies.

The inclusion of the fermions modifies the bosonic partition function in two ways: through zeromodes and through determinants. The fermion action in the 8D supergravity has the form

$$S^{(8)}_{\mathrm{Fermi}} = S_{\mathrm{kinetic}} + S_{\mathrm{fermi-flux}} + S_{4-\mathrm{fermi}} \tag{1.26}$$

where kinetic terms S_{kinetic} as well as fermion-flux couplings $S_{\mathrm{fermi-flux}}$ are quadratic in fermions and $S_{4-\mathrm{fermi}}$ denotes the four-fermion coupling. S_{kinetic} is T-duality invariant but $S_{\mathrm{fermi-flux}}$ and $S_{4-\mathrm{fermi}}$ contain some non-invariant terms. The non-invariant fermion zeromode couplings are collected together in the form

$$S^{(zm)ninv} = \int_X \left\{ 4\pi \mathrm{Im}\rho \; g_{(4)} \wedge * Y_{(4)} + 2\pi \mathrm{Im}\rho \; Y_{(4)} \wedge * Y_{(4)} \right\} \tag{1.27}$$

where the harmonic 4-form $Y_{(4)}$ is bilinear in the fermion zeromodes. The explicit expression for $Y_{(4)}$ can be found in equations (7.21) and (7.41) below.

The inclusion of the integral over the fermionic zeromodes of S_{kinetic} modifies the partition function by replacing the expression $\Theta(\mathcal{F},\rho)$ in (1.21) by

$$\widehat{\Theta}(\mathcal{F},\rho) = \int d\mu_F^{(zm)}\ e^{i2\pi\widehat{\Delta\Phi}(\mathcal{F})}\Theta\left[\begin{matrix}\vec{\hat{\alpha}}\\ \vec{\hat{\beta}}\end{matrix}\right](Q) \tag{1.28}$$

Here

$$\Theta\left[\begin{matrix}\vec{\hat{\alpha}}\\ \vec{\hat{\beta}}\end{matrix}\right](Q)$$

is a supertheta function for a superabelian variety based on the K-theory theta function. (This is explained in Appendix F.) In particular, the characteristics $\vec{\hat{\alpha}},\vec{\hat{\beta}}$ differ from $\vec{\tilde{\alpha}},\vec{\tilde{\beta}}$ by expressions bilinear in the fermion zeromodes. Similarly, the prefactor $\widehat{\Delta\Phi}$ differs from $\Delta\widetilde{\Phi}$ by an expression quartic in the fermion zeromodes. Finally, $d\mu_F^{(zm)}$ is a T-duality invariant measure for the finite dimensional integral over fermion and ghost zeromodes. It includes the T-duality invariant term $e^{-S^{(zm)inv}}$ from the action.

Including the one-loop fermionic determinants of the non-zero modes we finally arrive at

$$Z_{B+F}(\mathcal{F},\rho) := \text{Det}'_B\text{Det}'_F e^{-S_B(\mathcal{F})}\widehat{\Theta}(\mathcal{F},\rho) \tag{1.29}$$

The formula we derive for (1.29) allows a relatively straightforward check of the T-duality transformation laws and we find:

$$\begin{aligned} &Z_{B+F}(T\cdot\mathcal{F},\rho+1) = \mu(T)Z_{B+F}(\mathcal{F},\rho)\\ &Z_{B+F}(S\cdot\mathcal{F},-1/\rho) = (-i\rho)^{\frac{1}{4}\chi+\frac{1}{8}\int_X(p_2-\lambda^2)}(i\bar{\rho})^{\frac{1}{4}\chi-\frac{1}{8}\int_X(p_2-\lambda^2)}Z_{B+F}(\mathcal{F},\rho)\end{aligned} \tag{1.30}$$

Perhaps surprisingly, the fermion determinants have *not* completely trivialized the RR contribution to the path integral measure. However, there is one final ingredient we must take into account: In the low energy supergravity there are quantum corrections which contribute to leading order in the $t\to\infty$ and $\xi\to-\infty$ limit. From the string worldsheet viewpoint these consist of a 1-loop term in the α' expansion together with worldsheet instanton corrections. From the M-theory viewpoint we must include the one-loop correction $\int C_3X_8$ in M-theory together with the effect [18] of membrane instantons. The net effect is to modify the action by the quantum correction

$$S_{\text{quant}} = \left[\frac{1}{2}\chi+\frac{1}{4}\int_X(p_2-\lambda^2)\right]log\left[\eta(\rho)\right] + \left[\frac{1}{2}\chi-\frac{1}{4}\int_X(p_2-\lambda^2)\right]log\left[\eta(-\bar{\rho})\right] \tag{1.31}$$

Where $\eta(\rho)$ is the Dedekind function. The final combination

$$Z(\mathcal{F},\rho) = e^{-S_{\text{quant}}}Z_{B+F}(\mathcal{F},\rho) \tag{1.32}$$

is the fully T-duality invariant low energy partition function.

1.6. Applictions

As a by-product of the above results we will make some comments on the open problem of the relation of M-theory to massive IIA string theory. In [9] C. Hull made an interesting suggestion for an 11-dimensional interpretation of certain backgrounds in the Romans theory. One version of Hull's proposal states that massive IIA string theory on $T^2 \times X$ is equivalent to M-theory on a certain 3-manifold, the nilmanifold.

In section 9 we review Hull's proposal. For reasons explained there we are motivated to introduce a modification of Hull's proposal, in which one does not try to set up a 1-1 correspondence between M-theory geometries and massive IIA geometries, but nevertheless, the physical partition function $Z(\mathcal{F}, \rho)$ of the massive IIA theory can be identified with a certain sum over M-theory geometries involving the nilmanifold. The detailed proposal can be found in section 9.3.

1.7. U-duality and M-theory

In the final section of the paper we comment on some of the issues which arise in trying to extend these considerations to writing the fully U-duality-invariant partition function. We summarize briefly the M-theory partition function on $X \times T^3$, we comment on the $SL(2, Z)_\rho$ duality invariance, and we make some preliminary remarks on how one can see K-theory theta functions for twisted K-theory from the M theory formulation.

2. Review of T-duality invariance in the standard formulation of type IIA supergravity

We start by reviewing bosonic part of the standard 10D IIA supergravity action [19]. Fermions will be incorporated into the discussion in section 7.

2.1. Bosonic action of the standard 10D IIA supergravity

The 10D NSNS fields are the dilaton ϕ, 2-form potential $\hat{B}_2$ and string frame metric $\hat{g}_{\hat{M}\hat{N}}$, where $\hat{M}, \hat{N} = 0, \ldots 9$. The 10D RR fieldstrenghts are the 4-form G_4, 2-form G_2 and 0-form G_0.

We measure all dimensionful fields in units of 11D Planck length l_p and set $k_{11} = \pi$, so

$$S_{bos}^{(10)} = \frac{1}{2\pi}\int_{X_{10}} e^{-2\phi}\left(\sqrt{g_{10}}\mathcal{R}(\hat{g}) + 4d\phi\wedge\hat{*}d\phi + \frac{1}{2}\hat{H}_3\wedge\hat{*}\hat{H}_3\right)$$
$$+\frac{1}{4\pi}\int_{X_{10}}\left(\tilde{G}_4\wedge\hat{*}\tilde{G}_4 + i\hat{B}\wedge\tilde{G}_4\wedge\tilde{G}_4 + \tilde{G}_2\wedge\hat{*}\tilde{G}_2 + \sqrt{g_{10}}G_0^2\right) \tag{2.1}$$

where $\hat{*}$ stands for the 10D Hodge duality operator. The fields in (2.1) are defined as

$$\tilde{G}_2 = G_2 + \hat{B}_2 G_0, \quad \tilde{G}_4 = G_4 + \hat{B}_2 G_2 + \frac{1}{2}\hat{B}_2\hat{B}_2 G_0, \quad \hat{H}_3 = d\hat{B}_2.$$

We explain the relation between our fields and those of [19] in Appendix(B).

2.2. Reduction of IIA supergravity on a torus

We now recall some basic facts about the reduction of the bosonic part of the 10D action on T^2. Let us consider $X_{10} = T^2 \times X$ and split coordinates as $X^{\hat{M}} = (x^M, \sigma^m)$, where $M = 0, \ldots, 7, \quad m = 8, 9$.

The standard ansatz for the reduction of the 10d metric has the form:

$$ds_{10}^2 = t^2 g_{MN} dx^M dx^N + V g_{mn}\omega^m \otimes \omega^n \tag{2.2}$$

where g_{mn} is defined in (1.6), $t^2 g_{MN}$ is 8D metric, $detg_{MN} = 1$. V is the volume of T^2 and $\omega^m = d\sigma^m + \mathcal{A}_{(1)}^m$. The other bosonic fields of the $8D$ theory are listed below.

1 . $g_{(0)}^\alpha, g_{(2)}^\alpha, \quad \alpha = 1, 2 \quad g_{(1)m}, g_{(3)m} \quad m = 8, 9 \quad$ and $g_{(4)}$ are defined from[9]

$$\begin{aligned}
\frac{G_0}{2\pi} &= g_{(0)}^2 \\
\frac{\tilde{G}_2}{2\pi} &= \left(g_{(0)}^1 + g_{(0)}^2 B_0\right)\frac{1}{2}\epsilon_{mn}\omega^m\omega^n + g_{(1)m}\omega^m + g_{(2)}^2 \\
\frac{\tilde{G}_4}{2\pi} &= g_{(4)} + g_{(3)m}\omega^m + \left(B_0 g_{(2)}^2 + g_{(2)}^1\right)\frac{1}{2}\epsilon_{mn}\omega^m\omega^n
\end{aligned} \tag{2.3}$$

2 . The $8D$ dilaton ξ is defined by

$$e^{-2\xi} = e^{-2\phi}V \tag{2.4}$$

3 . $B_{(2)}, B_{(1)m}, B_0$ are obtained from the KK reduction of the NSNS 2-form potential in the following way

[9] $\epsilon^{89} = 1, \quad \epsilon_{89} = 1$

$$\hat{B}_2 = \frac{1}{2}B_0\epsilon_{mn}\omega^m\omega^n + B_{(1)m}\omega^m + B_{(2)} + \frac{1}{2}\mathcal{A}^m_{(1)}B_{(1)m} \tag{2.5}$$

Now, the real part of the $8D$ bosonic action obtained by the above reduction is

$$Re\left(S^{(8D)}_{\text{boson}}\right) = S_{NS} + \sum_{p=0}^{3} S_p\left(g_{(p)}\right) + S_4\left(g_{(4)}\right) \tag{2.6}$$

where

$$\begin{aligned} S_{NS} = \frac{1}{2\pi}\int e^{-2\xi}\Big\{ & t^6\left(\mathcal{R}(g) + 4d\xi\wedge *d\xi + 28t^{-2}dt\wedge *dt\right) + \frac{1}{2}t^2 H_{(3)}\wedge *H_{(3)} \\ & + \frac{1}{2}t^6\frac{d\tau\wedge *d\tau}{(\mathrm{Im}\tau)^2} + \frac{1}{2}t^6\frac{d\rho\wedge *d\rho}{(\mathrm{Im}\rho)^2} + \frac{1}{2}t^4 g_{mn}\mathcal{G}_{\alpha\beta}\mathbf{F}^{m\alpha}\wedge *\mathbf{F}^{n\beta}\Big\} \end{aligned} \tag{2.7}$$

where $\mathcal{G}_{\alpha\beta}$ is defined in (1.6) and $\mathcal{A}^m_{(1)}$ and $B_{(1)m}$ are combined into 1-form as a collection of

$$\mathbf{A}^{m\alpha}_{(1)} = \begin{pmatrix} \epsilon^{mn}B_{(1)n} \\ \mathcal{A}^m_{(1)} \end{pmatrix} \tag{2.8}$$

Also, we denote[10]

$$H_{(3)} = dB_{(2)} - \frac{1}{2}\epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}_{(1)m\alpha}\mathbf{F}^{n\beta}_{(2)} \tag{2.9}$$

$$\begin{aligned} \sum_{p=0}^{3} S_p\left(g_{(p)}\right) = \pi\int_X \Big\{ & t^8\mathcal{G}_{\alpha\beta}g^\alpha_{(0)}\wedge *g^\beta_{(0)} + t^6 g^{mn}g_{(1)m}\wedge *g_{(1)n} + \\ & t^4\mathcal{G}_{\alpha\beta}g^\alpha_{(2)}\wedge *g^\beta_{(2)} + t^2 g^{mn}g_{(3)m}\wedge *g_{(3)n}\Big\} \end{aligned} \tag{2.10}$$

Finally we have

$$S_4\left(g_{(4)}\right) = \pi\int_X \mathrm{Im}(\rho)g_{(4)}\wedge *g_{(4)} \tag{2.11}$$

It is convenient to introduce the notation $S_B(\mathcal{F}) = \sum_{p=0}^{3} S_p\left(g_{(p)}\right)$ for the value of the actions evaluated on a background flux field configuration. $S_B(\mathcal{F})$ will enter the partition sum $Z_{B+F}(\mathcal{F},\tau,\rho)$ in equation (8.1) below.

[10] $\mathcal{E}_{12} = 1, \quad \mathcal{E}_{21} = -1$

2.3. T-duality action on 8D bosonic fields

The T-duality group of the $8D$ effective theory obtained by reduction on T^2 is known to be $\mathcal{D}_T = SL(2,\mathbf{Z})_\tau \times SL(2,\mathbf{Z})_\rho$, where the first factor is mapping class group of T^2 which acts on τ

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.12}$$

and the second factor acts on $\rho = B_0 + iV$

$$\rho \to \frac{\alpha\rho + \beta}{\gamma\rho + \delta} \tag{2.13}$$

Let us denote generators of $SL(2,\mathbf{Z})_\rho$ by

$$S : \rho \to -1/\rho, \quad T : \rho \to \rho + 1$$

and generators of $SL(2,\mathbf{Z})_\tau$ by

$$\tilde{S} : \tau \to -1/\tau, \quad \tilde{T} : \tau \to \tau + 1$$

We now recall how T-duality acts on the remaining bosonic fields of the 8D theory [14]. First, ξ, t, g_{MN} are T-duality invariant. Next, there is the collection of fields $\mathcal{F}$ mentioned in the introduction. These transform linearly under T-duality. They include the NS potential $B_{(2)}$, which is T-duality invariant, as well as $\mathbf{A}^{m\alpha}_{(1)}$, which transform in the $(\mathbf{2},\mathbf{2})$. The other components of $\mathcal{F}$ are the RR fieldstrengths $g^\alpha_{(0)}, g^\alpha_{(2)}, \alpha = 1,2$ which transform in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}_T$ and $g_{(1)m}, g_{(3)m}, m = 8,9$ which transform in the $(\mathbf{2}',\mathbf{1})$ of $\mathcal{D}_T$.

Finally, the field $g_{(4)}$ is singled out among all the other fields since according to the conventional supergravity [14] $SL(2,\mathbf{Z})_\rho$ mixes $g_{(4)}$ with its Hodge dual $*g_{(4)}$ and hence $g_{(4)}$ does not have a local transformation. More concretely,

$$\begin{pmatrix} -\mathrm{Re}\rho g_{(4)} + i\mathrm{Im}\rho * g_{(4)} \\ g_{(4)} \end{pmatrix} \tag{2.14}$$

transforms in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}_T$. Due to this non-trivial transformation the classical bosonic 8D action $S^{(8D)}_{\text{boson}}$ is not manifestly invariant under $SL(2,\mathbf{Z})_\rho$.

3. Review of the K-theory theta function

In this section we review the basic flux quantization law of RR fields and the definition of the K-theory theta function. We follow closely the treatment in [4,5,7].

3.1. K-theoretic formulation of RR fluxes

As found in [1]-[4] RR fields in IIA superstring theory are classified topologically by an element $x \in K^0(X_{10})$. The relation for $\hat{B}_2 = 0$ is

$$\left[\frac{G}{2\pi}\right] = \sqrt{\hat{A}}chx, \quad G = \sum_{j=0}^{10} G_j \tag{3.1}$$

where ch is the total Chern character and $\hat{A}$ is expressed in terms of the Pontryagin classes as

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right) \tag{3.2}$$

In (3.1), the right hand side refers to the harmonic differential form in the specified real cohomology class. The quantization of the RR background fluxes is understood in the sense that they are derived from an element of $K^0(X_{10})$.

3.2. Definition of the K-theory theta function

Let us recall the general construction of a K-theory theta function, which serves as the RR partition function in Type IIA. One starts with the lattice $\Gamma_K = K^0(X_{10})/K^0(X_{10})_{tors}$. This lattice is endowed with an integer-valued unimodular antisymmetric form by the formula

$$\omega(x, y) = I(x \otimes \bar{y}), \tag{3.3}$$

where for any $z \in K^0(X_{10})$, $I(z)$ is the index of the Dirac operator with values in z.

Given a metric on X_{10}, one can define a metric on Γ_K

$$g(x, y) = \int_{X_{10}} \frac{G(x)}{2\pi} \wedge \frac{\hat{*}G(y)}{2\pi} \tag{3.4}$$

where $\hat{*}$ is the 10D Hodge duality operator.

Let us consider the torus $\mathbf{T} = \left(\Gamma_K \otimes_{\mathbf{Z}} \mathbf{R}\right)/\Gamma_K$. The quantities ω and g can be interpreted as a symplectic form and a metric, respectively, on $\mathbf{T}$. To turn $\mathbf{T}$ into a Kahler manifold one defines the complex structure J on $\mathbf{T}$ as

$$g(x, y) = \omega(Jx, y) \tag{3.5}$$

Now, if it is possible to find a complex line bundle $\mathcal{L}$ over $\mathbf{T}$ with $c_1(\mathcal{L}) = \omega$, then $\mathbf{T}$ becomes a "principally polarized abelian variety." $\mathcal{L}$ has, up to a constant multiple, a

unique[11] holomorphic section which is the contribution of the sum over fluxes to the RR partition function.

As explained in detail in [20], holomorphic line bundles $\mathcal{L}$ over $\mathbf{T}$ with constant curvature ω are in one-one correspondence with U(1)-valued functions Ω on Γ_K such that

$$\Omega(x+y) = \Omega(x)\Omega(y)(-1)^{\omega(x,y)}. \tag{3.6}$$

For weakly coupled Type II superstrings one can take Ω to be valued in $\mathbf{Z}_2$. Motivated by T-duality, and the requirements of anomaly cancellation on D-branes [5], Witten proposed that the natural $\mathbf{Z}_2-$ valued function Ω for the RR partition function is given by a mod two index [4]. For any $x \in K^0(X_{10})$, $x \otimes \bar{x} \in KO(X_{10})$ lies in the real K-theory group on X_{10}, and for any $v \in KO(X_{10})$, there is a well-defined mod 2 index $q(v)$ [21]. We take

$$\Omega(x) = (-1)^{j(x)} \tag{3.7}$$

where $j(x) = q(x \otimes \bar{x})$.

As explained in [4,5,7] there is an anomaly in the theory unless $\Omega(x)$ is identically 1 on the torsion subgroup of $K(X_{10})$. In the absence of this anomaly it descends to a function on $\Gamma_K = K^0(X_{10})/K^0(X_{10})_{tors}$ and can be used to define a line bundle $\mathcal{L}$ and hence the RR partition function.

To define the theta function one must choose a decomposition of Γ_K as a sum $\Gamma_1 \oplus \Gamma_2$, where Γ_1 and Γ_2 are "maximal Lagrangian" sublattices. ω establishes a duality between Γ_1 and Γ_2, and therefore there exists $\theta_K \in \Gamma_1/2\Gamma_1$ such that

$$\Omega(y) = (-1)^{\omega(\theta_K, y)}, \quad \forall y \in \Gamma_2 \tag{3.8}$$

Following [7] we choose the standard polarization:the sublattice Γ_2^{std} is defined as the set of x with vanishing G_0, G_2, G_4. This choice implies that G_0, G_2, G_4 are considered as independent variables. This is a distinguished choice for every large 10-manifold in the sense that it allows for a good large volume semiclassical approximation scheme on any 10-manifold (see sec.5).

[11] The uniqueness follows from the index theorem on $\mathbf{T}$ using unimodularity of ω and the fact that for any complex line bundle M over $\mathbf{T}$ with positive curvature we have $H^i(\mathbf{T}; M) = 0, \quad i > 0$.

It was demonstrated in [7] that Γ_1^{std} in the standard polarization consists of K-theory classes of the form $x = n_0\mathbf{1} + x(c_1, c_2)$. $\mathbf{1}$ is a trivial complex line bundle and $x(c_1, c_2)$ is defined for $c_1 \in H^2(X_{10}, \mathbf{Z})$ and $c_2 \in H^4(X_{10}, \mathbf{Z})$ with $Sq^3 c_2 = 0$, as

$$ch(x(c_1, c_2)) = c_1 + (-c_2 + \frac{1}{2}c_1^2) + \dots. \tag{3.9}$$

The higher Chern classes indicated by ... are such that $x(c_1, c_2)$ is in a maximal Lagrangian sublattice Γ_1^{std} complementary to Γ_2^{std}. Then, θ_K for the standard polarization can be chosen to satisfy

$$ch_0(\theta_K) = 0, \quad ch_1(\theta_K) = 0, \quad ch_2(\theta_K) = -\lambda + 2\hat{a}_0, \quad I(\theta_K) = 0 \tag{3.10}$$

where $\lambda = \frac{1}{2}p_1$ and $\hat{a}_0$ is a fixed element of $H^4(X_{10}, \mathbf{Z})$ such that

$$\forall \hat{c} \in L' \quad f(\hat{c}) = \int_{X_{10}} \hat{c} \cup Sq^2 \hat{a}_0 \tag{3.11}$$

where $L' = \left\{ \hat{c} \in H^4_{tors}(X_{10}, \mathbf{Z})/2H^4_{tors}(X_{10}, \mathbf{Z}), \quad Sq^3(\hat{c}) = 0 \right\}$ and $f(\hat{a})$ stands for the mod 2 index of the Dirac operator coupled to an E_8 bundle on the 11D manifold $X_{10} \times S^1$ with the characteristic class $\hat{a} \in H^4(X_{10}, \mathbf{Z})$ and supersymmetric spin structure on the S^1. (We will show in section 5.1 below that for $X_{10} = X \times T^2$ in fact $\hat{a}_0 = 0$.)

The K-theory theta function in the standard polarization is

$$\Theta_K = e^{iu} \sum_{x \in \Gamma_1} e^{i\pi\tau_K(x + \frac{1}{2}\theta_K)} \Omega(x) \tag{3.12}$$

where $u = -\frac{\pi}{4} \int_{X_{10}} ch_2(\theta_K) ch_3(\theta_K)$ and the explicit form of the period matrix τ_K is given by

$$Re\tau_K(x + \frac{1}{2}\theta_K) = \frac{1}{(2\pi)^2} \int_{X_{10}} (G_0 G_{10} - G_2 G_8 + G_4 G_6) \tag{3.13}$$

$$Im\tau_K(x + \frac{1}{2}\theta_K) = \sum_{p=0}^{2} \frac{1}{(2\pi)^2} \int_{X_{10}} G_{2p} \wedge \hat{*} G_{2p} \tag{3.14}$$

The RR fields which enter (3.13),(3.14) are:

$$\begin{aligned} &\frac{1}{2\pi} G_0(x + \frac{1}{2}\theta_K) = n_0 \\ &\frac{1}{2\pi} G_2(x + \frac{1}{2}\theta_K) = \hat{e} \\ &\frac{1}{2\pi} G_4(x + \frac{1}{2}\theta_K) = \hat{a} + \frac{1}{2}\hat{e}^2 - \frac{1}{2}(1 + n_0/12)\lambda \end{aligned} \tag{3.15}$$

where we denote $\hat{e} = c_1(x), \quad \hat{a} = -c_2(x) + \hat{a}_0$.

From (3.12) and (3.13),(3.14) the following topological term was found in [7] to be the K-theoretic corrections to the 10D IIA supergravity action.

$$e^{2\pi i\Phi(n_0,\hat{e},\hat{a})} = exp\Big[-2\pi i n_0 \int_{X_{10}} \hat{e}\left(\sqrt{\hat{A}}\right)_8\Big]\,(\Omega(\mathbf{1}))^{n_0}\, e^{2\pi i\Phi(\hat{e},\hat{a})} \tag{3.16}$$

$$e^{2\pi i\Phi(\hat{e},\hat{a})} = (-1)^{f(\hat{a}_0)}(-1)^{f(\hat{a})} exp\left[2\pi i \int_{X_{10}} \left(\frac{\hat{e}^5}{60} + \frac{\hat{e}^3\hat{a}}{6} - \frac{11\hat{e}^3\lambda}{144} - \frac{\hat{e}\hat{a}\lambda}{24} + \frac{\hat{e}\lambda^2}{48} - \frac{1}{2}\hat{e}\hat{A}_8\right)\right] \tag{3.17}$$

3.3. Turning on the NSNS 2-form flux with $[\hat{H}_3] = 0$

In the presence of an H-flux we expect K-theory to be replaced by twisted K-theory K_H classifying bundles of algebras with nontrivial Dixmier-Douady class. The Morita equivalence class of the relevant algebras only depends on the cohomology class of H, but this does not mean that the choice of "connection" that is, the choice of B field is irrelevant to formulating the K-theory theta function. Indeed, when $[\hat{H}] = 0$, the choice of trivialization $\hat{B}$ in $\hat{H} = d\hat{B}$ changes the action in supergravity and "turning on" this field in supergravity corresponds to acting with an automorphism on the K-theory torus. In this section we describe this change explicitly. See [22][23] for recent mathematical results relevant to this issue.

Let us turn on $\hat{B}_2 \in H^2(X_{10}, R)$. We normalize $\hat{B}_2$ so that it is defined mod $H^2(X_{10}, \mathbf{Z})$ under global tensorfield gauge transformation. By Morita equivalence, the RR fields are still classified topologically by $x \in K^0(X_{10})$. The standard coupling to the D-branes implies that the cohomology class of the RR field is

$$\frac{\tilde{G}(x)}{2\pi} = e^{\hat{B}_2} ch(x)\sqrt{\hat{A}} \tag{3.18}$$

Let us define

$$\overline{\frac{\tilde{G}(x)}{2\pi}} := e^{-\hat{B}_2} ch(\bar{x})\sqrt{\hat{A}} \tag{3.19}$$

The bilinear form on $\Gamma_K = K^0(X_{10})/K^0(X_{10})_{tors}$ is still given by the index:

$$\omega(x,y) = \frac{1}{(2\pi)^2}\int_{X_{10}} \tilde{G}(x)\wedge\overline{\tilde{G}(y)} = I(x\otimes\bar{y}) \tag{3.20}$$

while the metric on Γ_K is modified to be

$$\tilde{g}(x,y) = \frac{1}{(2\pi)^2}\int_{X_{10}} \tilde{G}(x)\wedge\hat{*}\tilde{G}(y) \tag{3.21}$$

and the $\mathbf{Z_2}$ valued function $\Omega(x)$ is unchanged. If we continue to use the standard polarization then $\theta_K \in \Gamma_1/2\Gamma_1$ is unchanged as well.

The net effect to modify (3.12) is that the period matrix τ_K should be substituted for $\widetilde{\tau_K} = \tau_K(G \to \tilde{G})$.

$$\Theta_K\left(\hat{B}_2\right) = e^{iu}\sum_{x\in\Gamma_1} e^{i\pi\widetilde{\tau_K}(x+\frac{1}{2}\theta_K)}\Omega(x) \tag{3.22}$$

Note, that the constant phase e^{iu} in front of the sum remains the same as in (3.12)

The imaginary part of the 10D Type IIA supergravity action now becomes $Im(S_{10D}) = -2\pi\widetilde{\Phi}$, where

$$\widetilde{\Phi} = \Phi + \frac{1}{8\pi^2}\Big[\hat{B}_2G_4^2 + \hat{B}_2^2G_2G_4 + \frac{1}{3}\hat{B}_2^3\left(G_2^2 + G_0G_4\right) + \frac{1}{4}\hat{B}_2^4G_0G_2 + \frac{1}{20}\hat{B}_2^5G_0^2\Big], \tag{3.23}$$

Φ is defined in (3.16),(3.17) and $G_{2p}(x+\frac{1}{2}\theta_K), \quad p=0,1,2$ are given in (3.15).

From (3.23) we find that corrections to Φ depending on $\hat{B}_2$ coincide with the imaginary part of the standard supergravity action (see, for example [12].)

Note, that $\tilde{G}$ defined in (3.19) is a gauge invariant field if the global tensorfield gauge transformation

$$\hat{B}_2 \to \hat{B}_2 + f_2, \quad f_2 \in H^2(X_{10},\mathbf{Z}) \tag{3.24}$$

also acts on $K^0(X_{10})$ as:

$$x \to L(-f_2)\otimes x, \quad x \in K^0(X_{10}) \tag{3.25}$$

where the line bundle $L(-f_2)$ has $c_1\left(L(-f_2)\right) = -f_2$.

Thus, according to (3.25) a tensorfield gauge transformation acts as an automorphism of Γ_K, preserving the symplectic form ω. (3.25) acts on theta function (3.22) by multiplication by a constant phase:

$$\Theta_K\left(\hat{B}_2 + f_2\right) = e^{i\frac{\pi}{4}\int_{X_{10}} f_2(\lambda-2\hat{a}_0)^2}\Theta_K\left(\hat{B}_2\right) \tag{3.26}$$

4. Action of T-duality in K-theory

In this section we consider $X_{10} = T^2 \times X$ and describe the action of T-duality on the K-theory variables.

As we have mentioned, the standard polarization is distinguished for any large 10-manifold in the following sense. When the metric of X_{10} is scaled up $\hat{g}_{\hat{M}\hat{N}} \to t^2 \hat{g}_{\hat{M}\hat{N}}$ the action $\int_{X_{10}} \sqrt{\hat{g}} |G_{2p}|^2$ of the Type IIA RR 2p-form scales as t^{10-4p}. This allows the successive approximation of keeping only G_4 whose periods have the smallest action, then including G_2 and finally keeping all G_4, G_2, G_0.

In the case of $X_{10} = T^2 \times X$ with the metric (1.1), the standard polarization is no longer distinguished. Various equally good choices are related by the action of the T-duality group $\mathcal{D}_T$ on $\Gamma_K = K^0(T^2 \times X)/K^0_{tors}(T^2 \times X)$.

We argue below that $\mathcal{D}_T$ can be considered as a subgroup of $Sp(2N, \mathbf{Z})$, where N denotes the complex dimension of the K-theory torus $\mathbf{T} = K^0(T^2 \times X) \otimes_{\mathbf{Z}} \mathbf{R}/\Gamma_K$ and $Sp(2N, \mathbf{Z})$ stands for the group of symplectic transformations of the lattice Γ_K.

4.1. Background RR fluxes in terms of integral classes on X.

To describe the action of $\mathcal{D}_T$ on K-theory variables, we will write RR fields in terms of integral classes on X. Let us start from the standard polarization [12] and write a general element of Γ_1^{std} as

$$x = n_0 \mathbf{1} + \Big(L(n_1 e_0 + e + \gamma_m d\sigma^m) - \mathbf{1} \Big) + x(e_0 e' + a + h_m d\sigma^m) + \Delta \qquad (4.1)$$

where $e_0 = d\sigma^8 \wedge d\sigma^9$, so that $\int_{T^2} e_0 = 1$. $L(\hat{e})$ is a line bundle with $c_1(L) = \hat{e} \in H^2(X_{10}; \mathbf{Z})$, $\mathbf{1}$ is a trivial line bundle, and for any $\hat{a} \in H^4(X_{10}; Z)$, $x(\hat{a})$ is a K-theory lift (if it exists). In (4.1) Δ puts x into the Lagrangian lattice Γ_1^{std} and we also introduce the notations:

$$a \in H^4(X; \mathbf{Z}), \quad e, e' \in H^2(X; \mathbf{Z}), \quad h_m \in H^3(X, \mathbf{Z}), \quad \gamma_m \in H^1(X; \mathbf{Z}) \quad m = 8, 9 \quad (4.2)$$

The RR fields entering (3.13),(3.14) are given by

$$\begin{aligned} &\frac{1}{2\pi} G_0(x + \frac{1}{2}\theta_K) = n_0, \\ &\frac{1}{2\pi} G_2(x + \frac{1}{2}\theta_K) = n_1 e_0 + e + \gamma_m d\sigma^m, \\ &\frac{1}{2\pi} G_4(x + \frac{1}{2}\theta_K) = a + \frac{1}{2} e^2 + e_0 e'' + f_m d\sigma^m - \frac{1}{2}(1 + n_0/12)\lambda \end{aligned} \qquad (4.3)$$

where

$$e'' = n_1 e + e' - \gamma_1 \gamma_2, \quad f_m = h_m + a_m + e\gamma_m \qquad (4.4)$$

Note that (3.13) is in fact only a function of these variables, by the Lagrangian property.

From the 10D constraint $Sq^3 \hat{a} = Sq^3 \hat{a}_0$, valid in the case $[\hat{H}_3] = 0$, we find the constraints on the integral cohomology classes: $Sq^3 f_m = Sq^3 a_m, \quad m = 8, 9$. We will show that actually $Sq^3 f_m = 0, \quad m = 8, 9$ (see comment below 5.8).

[12] Γ_1^{std} and Γ_2^{std} are defined on page 19.

4.2. The embedding $\mathcal{D}_T \subset Sp(2N, \mathbf{Z})$

From the transformation rules of the RR fields under the T-duality group [24] we find that f_m and γ_m transform in the $(\mathbf{2}', \mathbf{1})$ of $\mathcal{D}_T$ and we can form a representation $(\mathbf{1}, \mathbf{2})$ out of n_0, n_1 and e, e'' in the following way:

$$n^\alpha = \begin{pmatrix} n_1 \\ n_0 \end{pmatrix}, \quad e^\alpha = \begin{pmatrix} e'' \\ e \end{pmatrix} \tag{4.5}$$

We would like to reformulate the transformation rules for RR fields in terms of the action on Γ_K. [13] The action of $SL(2, \mathbf{Z})_\tau$ on Γ_K is via standard pullback under topologically nontrivial diffeomorphisms. The action of $SL(2, \mathbf{Z})_\rho$ is more novel.

We will explain the action of the two generators S, T of $SL(2, \mathbf{Z})_\rho$ separately. To begin, the action of T on Γ_K is a particular case of the global gauge transformation (3.24),(3.25) with $f_2 = e_0$ and for this reason $T \in Sp(2N, \mathbf{Z})$. The action of T preserves the standard polarization since it maps $\Gamma_2^{std} \to \Gamma_2^{std}$:

$$G_{2p}(y \otimes L(-e_0)) = 0, \quad \forall y \in \Gamma_2^{std} \quad p = 0, 1, 2 \tag{4.6}$$

The action of the generator S on Γ_K is more interesting. By the Kunneth theorem we can decompose

$$K^0(X \times T^2) = K^0(X) \otimes K^0(T^2) \oplus K^1(X) \otimes K^1(T^2) \tag{4.7}$$

Both $K^0(T^2) = \mathbf{Z} \oplus \mathbf{Z}$ and $K^1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$ have natural symplectic bases on which S acts as the standard symplectic operator $i\sigma_2$. For $K^0(T^2)$ we choose basis $\mathbf{1}$ and $L(e_0) - \mathbf{1}$, and for $K^1(T^2)$ we denote the basis as $\zeta^m, m = 8, 9$. We now have a Lagrangian decomposition of $\Gamma_K = \Gamma_1 \oplus \Gamma_2$:

$$\Gamma_1 = K^0(X) \otimes \mathbf{1} \oplus K^1(X) \otimes \zeta^8, \quad \Gamma_2 = K^0(X) \otimes (L(e_0) - \mathbf{1}) \oplus K^1(X) \otimes \zeta^9 \tag{4.8}$$

on which the T-duality generator S acts simply. However, the decomposition (4.8) is not compatible with the standard polarization, and hence the action of S in the standard polarization appears complicated. We now give an explicit description of the action of S in the standard polarization.

Let us write a generic element $y \in \Gamma_2^{std}$ as

$$y = x(\tilde{a}) \otimes \Big(L(e_0) - \mathbf{1}\Big) + z_1 + z_2 + z_3 \otimes \Big(L(e_0) - \mathbf{1}\Big), \quad \tilde{a} \in H^4(X, \mathbf{Z}) \tag{4.9}$$

[13] Some discussion of T-duality in the K-theoretic context can be found in [25].

In (4.9) z_1, z_2, z_3 are such that

$$\frac{G}{2\pi}(z_1) = j_m d\sigma^m, \quad \frac{G}{2\pi}(z_2) = k, \quad \frac{G}{2\pi}(z_3) = k' \tag{4.10}$$

where $j_m \in H^5(X,\mathbf{R}) \oplus H^7(X,\mathbf{R}), \quad k,k' \in H^6(X,\mathbf{R}) \oplus H^8(X,\mathbf{R})$ According to the transformation rules of RR fields [24] S acts on y as

$$S : y \to y', \quad y' = x(\tilde{a}) + z_1 + z_3 - z_2 \otimes \Big(L(e_0) - \mathbf{1}\Big) \tag{4.11}$$

From (4.11) we find that the image $\Gamma_2' := S(\Gamma_2^{std})$ differs from Γ_2^{std}. [14]

Since we have an embedding $\mathcal{D}_T \subset Sp(2N,\mathbf{Z})$, we can deduce the existence of well-defined transformation laws under $\mathcal{D}_T$ of the function $\Theta(\mathcal{F},\rho)$, related by (1.20) to the K-theory theta function Θ_K. This follows from the fact that Θ_K is an holomorphic section of the the line bundle $\mathcal{L}$ over the K-theory torus with $c_1(\mathcal{L}) = \omega$. Since $\mathcal{L}$ is not affected by symplectic transformations, and has a one-dimensional space of holomorphic sections, it follows that under T-duality transformatons Θ_K can at most be multiplied by a constant. Nevertheless, this leaves open the possibility of a T-duality anomaly, as indeed takes place.

To conclude this section we show how the multiplier system of (1.22)(1.23) is related to the standard 8^{th} roots of unity appearing in theta function transformation laws. Let us recall the general transformation rule under $Sp(2N,\mathbf{Z})$ for the theta function $\theta[m](\tau)$ of a principally polarized lattice $\Lambda = \Lambda_1 + \Lambda_2$ of rank $2N$. Here $m = \begin{pmatrix} m' \\ m'' \end{pmatrix} \in R^{2N}$ are the characterstics and the period matrix $\tau \in M_N(\mathbf{C}), \quad \tau^{\mathbf{T}} = \tau$ is a quadratic form on Λ_1.

It was found in [26] that under symplectic transformations

$$\sigma \cdot \tau = \frac{A\tau + B}{C\tau + D}, \quad \sigma \in Sp(2N,\mathbf{Z}) \tag{4.12}$$

the general $\theta[m](\tau)$ transforms as

$$\vartheta[\sigma \cdot m](\sigma \cdot \tau) = \kappa(\sigma) e^{2\pi i \phi(m,\sigma)} det(C\tau + D)^{1/2} \vartheta[m](\tau) \tag{4.13}$$

where

$$\sigma \cdot m = m\sigma^{-1} + \frac{1}{2}\begin{pmatrix} (C^T D)_d \\ (A^T B)_d \end{pmatrix}$$

[14] In following [24] we have actually combined the transformation S with the transformation $\tilde{S}$ from $SL(2,\mathbf{Z})_\tau$. This is a more convenient basis for checking the invariance of the theory.

$$\phi(m,\sigma) = -\frac{1}{2}\left(m'^T D B^T m' - 2m'^T B C^T m'' + m''^T C A^T m''\right) + \\ + \frac{1}{2}\left(m'^T D - m''^T C\right)\left(A^T B\right)_d$$

where $(A)_d$ denotes a vector constructed out of diagonal elements of matrix A.

The factor $\kappa(\sigma)$ in (4.13) has quite nontrivial properties [26]. In particular $\kappa^2(\sigma)$ is a character of $\Gamma(1,2) \subset Sp(2N,\mathbf{Z})$, where

$$\sigma \in \Gamma(1,2) \quad iff \quad \left(A^T B\right)_d \in 2\mathbf{Z}, \quad \left(C^T D\right)_d \in 2\mathbf{Z} \tag{4.14}$$

One can easily check that $SL(2,Z)_\rho \subset \Gamma(1,2)$ by writing out explicit representations $\sigma(S)$ and $\sigma(T)$ in $Sp(2N,\mathbf{Z})$. We give $\sigma(S)$ and $\sigma(T)$ in Appendix(A).

Using the explicit expressions for $\sigma(S)$ and $\sigma(T)$ as well as the definition of τ_K (3.13),(3.14) we find that in (4.13)

$$det(C(S)\tau_K + D(S))^{1/2} = e^{i\frac{\pi}{4}b_4}(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}, \quad \phi(m,\sigma(S)) = 0 \tag{4.15}$$

$$det(C(T)\tau_K + D(T))^{1/2} = 1, \quad \phi(m,\sigma(T)) = 0 \tag{4.16}$$

Now comparing (4.13) and the explicit formulae (5.31) for the transformation laws of $\Theta(\mathcal{F},\rho)$ derived in the next section we find the relation between $\kappa(\sigma)$ and the multiplier system $\mu(S),\mu(T)$

$$\kappa(S)e^{i\frac{\pi}{4}b_4} = \mu(S), \quad \kappa(T) = \mu(T) \tag{4.17}$$

5. $\Theta(\mathcal{F},\rho)$ as a modular form

In this section we derive an explicit expression for $\Theta(\mathcal{F},\rho)$ using its relation (1.20) to the K-theory theta function Θ_K and we check that $\Theta(\mathcal{F},\rho)$ transforms under the T-duality group $\mathcal{D}_T$ as a modular form.

5.1. Zero NSNS fields

We first assume that all NSNS background fields are zero. In this case $\Theta(\mathcal{F},\rho)$, defined in (1.20) is given by an expression of the form

$$\Theta(\mathcal{F},\rho) = \sum_{a \in H^4(X,\mathbf{Z})} e^{i2\pi\Phi(a,\mathcal{F})} e^{-\pi\int_X Im(\rho) g_{(4)}\wedge *g_{(4)}} \tag{5.1}$$

where the imaginary part of the 8D effective action $2\pi\Phi(a,\mathcal{F})$ is derived as follows. We substitute

$$\hat{a} = a + e_0 e' + h_m d\sigma^m, \quad \hat{e} = e + n_1 e_0 + \gamma_m d\sigma^m \tag{5.2}$$

into the definition (3.16) of $e^{i2\pi\Phi(n_0,\hat{e},\hat{a})}$.

We need to evaluate $f(a + e_0 e' + h_m d\sigma^m)$. We use the bilinear identity from [7]

$$f(u+v) = f(u) + f(v) + \int_{X_{10}} uSq^2 v, \quad \forall u, v \in H^4(X_{10}; \mathbf{Z}) \tag{5.3}$$

to find

$$f(a + e_0 e' + h_m d\sigma^m) = f(a + e_0 e') + f(h_m d\sigma^m). \tag{5.4}$$

Let us consider $f(h_m d\sigma^m)$ first. Again using the bilinear identity we obtain:

$$f(h_m d\sigma^m) = f(h_8 d\sigma^8) + f(h_9 d\sigma^9) + \int_X h_8 Sq^2(h_9) \tag{5.5}$$

From (5.3) it follows that $f(h d\sigma^m), m = 8, 9$ are linear functions of $h \in H^3(X, \mathbf{Z})$. Moreover, from the diffeomorphism invariance of the mod two index we see that $f(h d\sigma^8) = f(h d\sigma^8 + \ell h d\sigma^9)$, for any integer ℓ and, using the bilinear identity once more we find that $f(h d\sigma^m) = r(h), m = 8, 9$ where

$$r(h) = \int_X hSq^2 h, \quad h \in H^3(X, \mathbf{Z}) \tag{5.6}$$

is a spin-cobordism invariant $\mathbf{Z}_2$-valued function. In fact, $r(h)$ is a nontrivial invariant since for $X = SU(3)$ and $h = x_3$ the generator of $H^3(SU(3), \mathbf{Z})$ we have $r(h) = 1$. In conclusion:

$$f(h_m d\sigma^m) = \int_X \Big[h_8 Sq^2 h_8 + h_9 Sq^2 h_9 + h_8 Sq^2(h_9) \Big] \tag{5.7}$$

Now we consider $f(a + e_0 e')$:

$$\begin{aligned} f(e_0 e' + a) &= f(a) + f(e_0 e') + \int_{X_{10}} e_0 e' Sq^2 a \\ &= \int_X (a)^2 - \frac{1}{2}(e')^2 \lambda + (e')^2 a = \int_X a\lambda + (e')^2 (a - \frac{1}{2}\lambda) \end{aligned} \tag{5.8}$$

This uses the bilinear identity (5.3), the reduction of the mod two index along T^2, and the formula eq.(8.40) for $f(u \cup v)$ from [7].

We can now evaluate $\hat{a}_0$ defined in (3.11). The kernel of Sq^3 is given by those elements $a + e_0 e' + h_m d\sigma^m$ such that $h_8 \cup h_8 = h_9 \cup h_9 = 0$. If we add the condition that the element

is a torsion class then $f(a + e_0 e') = 0$ and we need only evaluate (5.7). Now, since $Sq^3(h_m) = h_m \cup h_m = 0$ it follows that $Sq^2(h_m)$ has an integral lift. Using again the condition that h_m is torsion we find that the right hand side of (5.7) is zero. It follows that $\hat{a}_0 = 0$.

We can now evaluate the phase. Using (4.4) we reexpress (5.5) as

$$f(h_m d\sigma^m) = \int_X \Big(f_8 Sq^2(f_9) + f_8 Sq^2(f_8) + f_9 Sq^2(f_9) + e^2(\gamma_9 f_8 - \gamma_8 f_9) + e^3 \gamma_8 \gamma_9 \Big) \quad (5.9)$$

Taking into account (5.9) and (5.8) we find the total phase $\Phi(a, \mathcal{F})$ in (5.1) is given by:

$$\Phi(a, \mathcal{F}) = \Delta\Phi + \int_X (a + \alpha)\beta, \quad (5.10)$$

where the characteristics are defined as:

$$\begin{aligned} \alpha &= \frac{1}{2}(e)^2 + \frac{1}{2}(1 - n_0/12)\,\lambda + \frac{1}{2}(e'' + e)\,\epsilon^{mn}\gamma_m\gamma_n \\ \beta &= \frac{1}{2}(e'')^2 + \frac{1}{2}(1 - n_1/12)\,\lambda + \frac{1}{2}(e'' - e)\,\epsilon^{mn}\gamma_m\gamma_n \end{aligned} \quad (5.11)$$

and we recall that $e'' = n_1 e + e' - \frac{1}{2}\epsilon^{mn}\gamma_m\gamma_n$. Note that for convenience we have made a shift of the summation variable in (5.1) $a \to a + \lambda + \frac{1}{2}(e + e'')\,\epsilon^{mn}\gamma_m\gamma_n$.

The prefactor $\Delta\Phi$ is given by

$$exp[2\pi i \Delta\Phi] = exp\left[\pi i \int_X \Big(f_8 Sq^2(f_9) + f_8 Sq^2(f_8) + f_9 Sq^2(f_9) \Big)\right] \quad (5.12)$$

$$exp\Bigg[2\pi i \int_X \Big(-\frac{1}{4}(e''e)^2 - \frac{1}{24}e''e\lambda + \frac{1}{6}e^3e'' - \frac{1}{4}e^2\lambda + \frac{1}{48}n_0\lambda(e'')^2 + \frac{1}{4}(1 + n_0/12)\lambda^2 +$$

$$+\frac{1}{2}(n_0 - n_1)\hat{A}_8 - \frac{1}{2}n_0 n_1 \left[\hat{A}_8 + \left(\frac{\lambda}{24}\right)^2\right] + \frac{\lambda}{24}\epsilon^{mn}\gamma_m f_n +$$

$$+\frac{1}{48}\Big[n_0(e'' - e)\lambda - 12e^2e'' - 4e\lambda - 4e^3\Big]\epsilon^{mn}\gamma_m\gamma_n\Big)\Bigg]$$

In deriving $\Delta\Phi$ we have used

$$(\sqrt{\hat{A}})_8 = \frac{1}{2}\left[\hat{A}_8 - \left(\frac{\lambda}{24}\right)^2\right]$$

Also, in bringing $\Delta\Phi$ to the form (5.12) we have used the congruences

$$\int_X \frac{1}{6}\left[(e'')^3 e + e'' e^3\right] + \frac{1}{4}(e'')^2 e^2 - \frac{1}{12}\lambda e'' e \in \mathbf{Z} \tag{5.13}$$

$$\int_X (e''e)^2 \in 2\mathbf{Z}, \quad \int_X e''e\lambda \in 2\mathbf{Z}. \tag{5.14}$$

which follow from the index theorem on X :

$$\int_X \frac{1}{24}e^4 - \frac{1}{24}\lambda e^2 \in \mathbf{Z}, \qquad \forall e \in H^2(X, \mathbf{Z}). \tag{5.15}$$

5.2. Including flat NSNS potentials

Let us now take into account globally defined NSNS fields:

$$\hat{B}_2 = \frac{1}{2}B_0\epsilon_{mn}\omega^m\omega^n + B_{(1)m}\omega^m + B_{(2)} + \frac{1}{2}\mathcal{A}^m_{(1)}B_{(1)m}, \quad \mathcal{A}^m_{(1)}$$

and recall that $\mathcal{A}^m_{(1)}$ and $B_{(1)m}$ are combined into the $(\mathbf{2},\mathbf{2})$ of $\mathcal{D}_T$ as in (2.8).

We define a gauge invariant fieldstrength $\tilde{G} = e^{\hat{B}_2}G$ as in (3.18) where G are given in (4.3) and we expand $\tilde{G}\left(x + \frac{1}{2}\theta_K\right)$ as

$$\begin{aligned}
\frac{\tilde{G}_0}{2\pi}\left(x + \frac{1}{2}\theta_K\right) &= g^2_{(0)} \\
\frac{\tilde{G}_2}{2\pi}\left(x + \frac{1}{2}\theta_K\right) &= \left(g^1_{(0)} + g^2_{(0)}B_0\right)\frac{1}{2}\epsilon_{mn}\omega^m\omega^n + g_{(1)m}\omega^m + g^2_{(2)} \\
\frac{\tilde{G}_4}{2\pi}\left(x + \frac{1}{2}\theta_K\right) &= g_{(4)} + g_{(3)m}\omega^m + \left(B_0 g^2_{(2)} + g^1_{(2)}\right)\frac{1}{2}\epsilon_{mn}\omega^m\omega^n
\end{aligned} \tag{5.16}$$

The first effect of including flat NSNS fields is to modify the fields which enter $S_B(\mathcal{F})$. These fields $g^\alpha_{(0)}, g_{(1)m}, g^\alpha_{(2)}, g_{(3)m}$ are now linear combinations of the integral classes $\gamma_m, f_m, e^\alpha, n^\alpha$ defined in (4.2),(4.4) with coefficients constructed from $\mathbf{A}^{m\alpha}_{(1)}$ and $B_{(2)}$:

$$g^\alpha_{(0)} = \begin{pmatrix} n_1 \\ n_0 \end{pmatrix}, \quad g_{(1)m} = \gamma_m + \xi_{(1)m}, \quad g^\alpha_{(2)} = e^\alpha + \mathbf{A}^{m\alpha}_{(1)}\left(\gamma_m + \frac{1}{2}\xi_{(1)m}\right) + B_{(2)}g^\alpha_{(0)} \tag{5.17}$$

$$g_{(3)m} = f_m + B_{(2)}g_{(1)m} + \lambda_{(3)m} + \frac{1}{2}k_{(3)m} + \frac{1}{6}\epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}^{p\alpha}_{(1)}\xi_{(1)p}\mathbf{A}^{n\beta}_{(1)} \tag{5.18}$$

where we denote

$$\xi_{(1)m} = \epsilon_{mn}\mathcal{E}_{\alpha\beta}g^\alpha_{(0)}\mathbf{A}^{n\beta}_{(1)}, \quad \lambda_{(3)m} = \epsilon_{mn}\mathcal{E}_{\alpha\beta}e^\alpha\mathbf{A}^{n\beta}_{(1)}, \quad k_{(3)m} = \epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}^{p\alpha}_{(1)}\gamma_p\mathbf{A}^{n\beta}_{(1)} \tag{5.19}$$

The other effect of including flat NSNS fields is to shift the characteristics and the prefactor of $\Theta(\mathcal{F},\rho)$. Now $\Theta(\mathcal{F},\rho)$ has the form:

$$\Theta(\mathcal{F},\rho)=e^{2\pi i\Delta\widetilde{\Phi}}\sum_{a\in H^4(X,\mathbf{Z})}exp\left[\int_X\left(-\pi Im(\rho)g_{(4)}\wedge *\, g_{(4)}+i\pi Re(\rho)g_{(4)}\wedge g_{(4)}+2\pi i g_{(4)}\tilde{\beta}\right)\right] \tag{5.20}$$

where $[g_{(4)}]=a+\tilde{\alpha},\quad a\in H^4(X,\mathbf{Z})$, and the shifted characteristics $\tilde{\alpha},\tilde{\beta}$ are

$$\tilde{\alpha}=\alpha+\varphi^2,\quad \tilde{\beta}=\beta+\varphi^1 \tag{5.21}$$

where α,β are defined in terms of integral classes $n_0,n_1,\gamma_m,e^\alpha$ in (5.11), while φ^α transform in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}_T$. Explicitly,

$$\varphi^\alpha=\mathbf{A}^{m\alpha}_{(1)}\left(f_m+\frac{1}{2}\lambda_{(3)m}+\frac{1}{6}k_{(3)m}\right)+B_{(2)}\left[e^\alpha+\mathbf{A}^{m\alpha}_{(1)}\left(\gamma_m+\frac{1}{2}\xi_{(1)m}\right)\right]+ \tag{5.22}$$

$$+\frac{1}{2}B_{(2)}B_{(2)}g^\alpha_{(0)}-\zeta_{(4)}g^\alpha_{(0)}$$

where $\xi_{(1)m},\lambda_{(3)m},k_{(3)m}$ are given in (5.19) and we also denote

$$\zeta_{(4)}=\frac{1}{64}\mathcal{E}_{\beta_1\beta_2}\mathcal{E}_{\beta_3\beta_4}\mathbf{A}^{n_1\beta_1}_{(1)}\epsilon_{n_1n_2}\mathbf{A}^{n_2\beta_3}_{(1)}\mathbf{A}^{m_1\beta_2}_{(1)}\epsilon_{m_1m_2}\mathbf{A}^{m_2\beta_4}_{(1)} \tag{5.23}$$

The shifted prefactor $\Delta\widetilde{\Phi}$ in (5.20) is given by

$$\Delta\widetilde{\Phi}=\Delta\Phi-\int_X\left[\beta\wedge\varphi^2+\frac{1}{2}\varphi^1\wedge\varphi^2\right]+(\Delta\Phi)_{inv} \tag{5.24}$$

where $\Delta\Phi$ is defined in terms of integral classes $n_0,n_1\gamma_m,e^\alpha,f_m$ in (5.12) and $(\Delta\Phi)_{inv}$ is the part of the phase which is manifestly invariant under the T-duality group $\mathcal{D}_T$. Explicitly,

$$(\Delta\Phi)_{inv}=\int_X B^3_{(2)}\Big[\frac{1}{12}\mathcal{E}_{\alpha\beta}g^\alpha_{(0)}e^\beta-\frac{1}{6}\epsilon^{mn}\gamma_m\gamma_n-\frac{1}{4}\epsilon^{mn}\xi_{(1)m}\gamma_n-\frac{1}{8}\epsilon^{mn}\xi_{(1)m}\xi_{(1)n}\Big]+ \tag{5.25}$$

$$\int_X B^2_{(2)}\Big[-\frac{1}{4}\epsilon^{mn}\xi_{(1)m}f_n-\frac{1}{2}\epsilon^{mn}\lambda_{(3)m}\gamma_n-\frac{3}{8}\epsilon^{mn}\lambda_{(3)m}\xi_{(1)n}-\frac{1}{24}\epsilon^{mn}k_{(3)m}\xi_{(1)n}\Big]+$$

$$\int_X B_{(2)}\Big[-\frac{1}{2}\epsilon^{mn}f_mf_n-\frac{1}{2}\epsilon^{mn}\lambda_{(3)m}f_n-\frac{1}{4}\epsilon^{mn}\lambda_{(3)m}\lambda_{(3)n}-\frac{1}{6}\epsilon^{mn}\lambda_{(3)m}k_{(3)n}+$$

$$+\frac{1}{12}\xi_{(1)m}q^m_{(5)}+\frac{1}{2}\zeta_{(4)}\mathcal{E}_{\alpha\beta}e^\alpha g^\beta_{(0)}+\zeta_{(4)}\epsilon^{mn}\gamma_m\gamma_n\Big]+\int_X\Big[\frac{1}{12}\lambda_{(3)m}q^m_{(5)}+\zeta_{(4)}\epsilon^{mn}\gamma_mf_n\Big]$$

where $q^m_{(5)}=\mathcal{E}_{\alpha\beta}\mathbf{A}^{p\alpha}_{(1)}f_p\mathbf{A}^{m\beta}_{(1)}$

5.3. Derivation of T-duality transformations.

Let us study transformations of $\Theta(\mathcal{F},\rho)$ defined in (5.20) under $\mathcal{D}_T$. First, we note that $\Theta(\mathcal{F},\rho)$ is invariant under $SL(2,Z)_\tau$. Next, we consider the action of the generator S. For any function $h(\mathcal{F})$ of fluxes $\mathcal{F}$, we denote

$$S\Big[h(\mathcal{F})\Big] := h(S\cdot\mathcal{F})$$

and

$$\delta_S[h] := S[h] - h$$

where $S\cdot\mathcal{F}$ denotes the linear action on fluxes. To check the transformation under S we need to do a Poisson resummation on the self-dual lattice $H^4(X,\mathbf{Z})$. The basic transformation law is:

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0|-1/\tau) = (-i\tau)^{1/2}e^{2\pi i\theta\phi}\vartheta\begin{bmatrix}-\phi\\ \theta\end{bmatrix}(0|\tau) \tag{5.26}$$

and its generalization to self-dual lattices (4.13).

After the Poisson resummation and a shift of summation variable $a\to a+e^2+\lambda$ we find that $\Theta(\mathcal{F},\rho)$ transforms under S as

$$\Theta(S\cdot\mathcal{F},-1/\rho) = e^{2\pi i\Big\{\int_X S\left[\tilde\alpha\right]S\left[\tilde\beta\right]+\delta_S\left[\Delta\widetilde{\Phi}\right]\Big\}}(-i\rho)^{\frac{1}{2}b_4^+}(i\bar\rho)^{\frac{1}{2}b_4^-}\Theta(\mathcal{F},\rho) \tag{5.27}$$

Now using the definitions of $\tilde\alpha,\tilde\beta$ (5.21),(5.22) and $\Delta\widetilde{\Phi}$ (5.24) as well as the transformation rules for $\mathcal{F}$, we find after some tedious algebra

$$\delta_S\left[\Delta\widetilde{\Phi}\right] = -\int_X S[\tilde\alpha]S[\tilde\beta] + \int_X\frac{\lambda^2}{4}+\mathbf{Z} \tag{5.28}$$

We conclude that the generator S acts as

$$\Theta(S\cdot\mathcal{F},-1/\rho) = e^{i\pi\int_X\lambda^2/2}(-i\rho)^{\frac{1}{2}b_4^+}(i\bar\rho)^{\frac{1}{2}b_4^-}\Theta(\mathcal{F},\rho) \tag{5.29}$$

To check how $\Theta(\mathcal{F},\rho)$ transforms under the generator T we use its relation (1.20) to the K-theory theta function Θ_K as well as the transformation of Θ_K under global gauge transformation $\hat B_2\to\hat B_2+f_2$ (3.26) where the action of the generator T corresponds to $f_2=e_0$. In this way we find from (3.26) that

$$\Theta(T\cdot\mathcal{F},\rho+1) = e^{i\pi\int_X\lambda^2/4}\Theta(\mathcal{F},\rho) \tag{5.30}$$

5.4. Summary of T-duality transformation laws

Below we summarize the transformation laws of the function $\Theta(\mathcal{F},\rho)$ under the generators of T-duality group $\mathcal{D}_T$.

$\Theta(\mathcal{F},\rho)$ is invariant under $SL(2,Z)_\tau$:

$$\begin{aligned}\Theta(\tilde{T}\cdot\mathcal{F},\rho)&=\Theta(\mathcal{F},\rho)\\ \Theta(\tilde{S}\cdot\mathcal{F},\rho)&=\Theta(\mathcal{F},\rho)\end{aligned}\tag{5.31}$$

$\Theta(\mathcal{F},\rho)$ transforms as a modular form with a nontrivial "multiplier system" under $SL(2,\boldsymbol{Z})_\rho$. That is, using the standard generators T,S of $SL(2,Z)_\rho$ we have:

$$\begin{aligned}\Theta(T\cdot\mathcal{F},\rho+1)&=\mu(T)\Theta(\mathcal{F},\rho)\\ \Theta(S\cdot\mathcal{F},-1/\rho)&=\mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\Theta(\mathcal{F},\rho)\end{aligned}\tag{5.32}$$

where $T\cdot\mathcal{F}, S\cdot\mathcal{F}$ denotes the linear action of $\mathcal{D}_T$ on the fluxes. Here b_4^+, b_4^- is the dimension of the space of self-dual and anti-self-dual harmonic forms on X and the multiplier system is

$$\begin{aligned}\mu(T)&=\exp\Big[\frac{i\pi}{4}\int_X\lambda^2\Big]\\ \mu(S)&=\exp\Big[\frac{i\pi}{2}\int_X\lambda^2\Big]\end{aligned}\tag{5.33}$$

where $p_1=p_1(TX)$. These define the "T-duality anomaly of RR fields."

6. The bosonic determinants

In this section we compute bosonic quantum determinants around the background specified in section 2.

Let us factorize bosonic quantum determinants as: $Det_B = \mathcal{D}_{RR}\mathcal{D}_{NS}$, where $\mathcal{D}_{RR}(\mathcal{D}_{NS})$ denotes the contribution from RR (NSNS) fields.

6.1. Quantum determinants $\mathcal{D}_{RR}$ for RR fields

Quantum determinants $\mathcal{D}_{RR}$ for RR fields have the form

$$\mathcal{D}_{RR}=\prod_{p=1}^{4}Z_{RR,p}\tag{6.1}$$

where $Z_{RR,p}$ is the quantum determinant for $g_{(p)}$. First, we present the contribution $Z_{RR,4}$ arising from the fluctuation $dC_{(3)}$ of $g_{(4)}$. From (2.11) we find the kinetic term for $C_{(3)}$

$$S_{3,cl} = \pi Im(\rho)\big(dC_{(3)}, dC_{(3)}\big) \tag{6.2}$$

where (,) denotes the standard inner product on the space of p-forms on X, constructed with the background metric g_{MN}.

We use the standard procedure [27,28] for path-integration over p-forms, which can be summarized as follows. Starting from the classical action for the p-form $S_{p,cl} = \alpha\big(dC_{(p)}, dC_{(p)}\big)$ one constructs the quantum action as[15]:

$$S_{p,qu} = \alpha\big(C_{(p)}, \Delta_p C_{(p)}\big) + \sum_{m=1}^{p} \alpha^{\frac{1}{m+1}} \sum_{k=1}^{m+1} \Big(u^k_{(p-m)}, \Delta_{p-m} u^k_{(p-m)}\Big) \tag{6.3}$$

where $u^k_{(p-m)}$, $k = 1, \ldots m+1, m = 1, \ldots p$ are ghosts of alternating statistics. For example, $u^k_{(p-1)}$, $k = 1, 2$ are fermions, $u^k_{(p-2)}$, $k = 1, 2, 3$ are bosons, etc. In (6.3) Δ_p is the Laplacian acting on p-forms and constructed with g_{MN}[16].

To compute $Z_{RR,4}$ we apply (6.3) for $p = 3$, $\alpha = \pi \mathrm{Im}(\rho)$ and use the measure $[DC_p]$ normalized as $\int [DC_p] e^{-(C_p, C_p)} = 1$:

$$Z_{RR,4} = (\alpha)^{-\frac{1}{2}(B'_3 - B'_2 + B'_1 - B'_0)} \Big[\frac{det' \Delta_3}{V_3}\Big]^{-\frac{1}{2}} \Big[\frac{det' \Delta_2}{V_2}\Big] \Big[\frac{det' \Delta_1}{V_1}\Big]^{-3/2} \Big[\frac{det' \Delta_0}{V_0}\Big]^2 \tag{6.4}$$

where $det' \Delta_p$ is the determinant of nonzero modes of the Laplacian acting on p-forms. $B'_p = B_p - b_p$, where B_p denotes the (infinite) number of eigen-p-forms and b_p and V_p are the dimension and the determinant of the metric of the harmonic torus $T^p_{harm} = \mathcal{H}^p / \mathcal{H}^p_{\mathbf{Z}}$. The appearance of V_p in (6.4) is due to the appropriate treatment of zeromodes and is explained in Appendix(E).

The determinants $det' \Delta_p$ together with the infinite powers depending on B_p, here and below, require regularization and renormalization, of course. These can be handled using, for example, the techniques of [29]. In particular the expression

$$q(Im\rho) := (\mathrm{Im}\rho)^{-\frac{1}{2}(B_3 - B_2 + B_1 - B_0)} \tag{6.5}$$

---

[15] Factors $\alpha^{\frac{1}{m+1}}$ should be understood as a mnemonic rule to keep track of the dependence on α which follows from the analysis of various cancellations between ghosts and gauge-fixing fields

[16] $\Delta = dd^\dagger + d^\dagger d$

is a local counterterm of the form $e^{-\pi \mathrm{Im}\rho \int_X (u\lambda^2 + vp_2)}$, where the numbers u, v depend on the regularization. From now on we will assume that $\pi Im\rho \int_X (u\lambda^2 + vp_2)$ is included into the 1-loop action:

$$S_{1-loop} = \pi \mathrm{Im}\rho \int_X (u\lambda^2 + vp_2) + \frac{i\pi}{24} \mathrm{Re}\rho \int_X (p_2 - \lambda^2) \tag{6.6}$$

In section 8 we will show that T-duality invariance determines u and v uniquely.

Next, we consider the contributions to $\mathcal{D}_{RR}$ from $dC_{(2)m}$, $dC^{\alpha}_{(1)}$, $d\tilde{C}_{(0)m}$ which are the fluctuations for $g_{(3)m}$, $g^{\alpha}_{(2)}$, $g_{(1)m}$ respectively. Let us also make field redefinition of the quantum fields $\tilde{C}_{(0)m}, m = 8, 9$ to fields $C_{(0)m}, m = 8, 9$ which have well defined transformation properties under the full U-duality group[17]

$$C_{(0)8} = \sqrt{\tau_2} e^{\xi} \tilde{C}_{(0)8}, \quad C_{(0)9} = \frac{1}{\sqrt{\tau_2}} e^{\xi} \tilde{C}_{(0)9} \tag{6.7}$$

From (2.10) we find classical action quadratic in the above fluctuations:

$$S_{0,cl} = \pi \tilde{t}^6 g'^{mn} \left(C_{(0)m}, d^{\dagger} dC_{(0)n} \right), \quad S_{1,cl} = \pi t^4 \mathcal{G}_{\alpha\beta} \left(C^{\alpha}_{(1)}, d^{\dagger} dC^{\beta}_{(1)} \right)$$

$$S_{2,cl} = \pi t^2 g^{mn} \left(C_{(2)m}, d^{\dagger} dC_{(2)n} \right)$$

where $\tilde{t} = te^{-\xi/3}$ is U-duality invariant, and $g'^{88} = \frac{1}{\tau_2} g^{88}$, $g'^{99} = \tau_2 g^{99}$, $g'^{89} = g^{89}$. Now, using (6.3) with $a = \pi \tilde{t}^6 g'^{mn}, \pi t^4 \mathcal{G}_{\alpha\beta}, \pi t^2 g^{mn}$ and $p = 0, 1, 2$ correspondingly we find:

$$Z_{RR,1} = \left(\pi \tilde{t}^6 \right)^{-B'_0} \left[\frac{det' \Delta_0}{V_0} \right]^{-1} \tag{6.8}$$

$$Z_{RR,2} = \left(\pi t^4 \right)^{B'_0 - B'_1} \left[\frac{det' \Delta_1}{V_1} \right]^{-1} \left[\frac{det' \Delta_0}{V_0} \right]^{2} \tag{6.9}$$

$$Z_{RR,3} = \left(\pi t^2 \right)^{-B'_2 + B'_1 - B'_0} \left[\frac{det' \Delta_2}{V_2} \right]^{-1} \left[\frac{det' \Delta_1}{V_1} \right]^{2} \left[\frac{det' \Delta_0}{V_0} \right]^{-3} \tag{6.10}$$

In computing (6.8)-(6.10) we also used that $det_{m,n} g^{mn} = 1$, $det_{m,n} g'^{mn} = 1$ and $det_{\alpha,\beta} \mathcal{G}_{\alpha\beta} = 1$.

Collecting together (6.4) and (6.8)-(6.10) we find that $\mathcal{D}_{RR}$ has the form:

$$\mathcal{D}_{RR} = r_{RR}(t, \rho) \left[\frac{det' \Delta_3}{V_3} \right]^{-\frac{1}{2}} \left[\frac{det' \Delta_1}{V_1} \right]^{-\frac{1}{2}} \tag{6.11}$$

[17] For some discussion of U-duality see sec.10

where

$$r_{RR}(t,\rho) = (e^{\xi})^{2B_0'}\Big(Im\rho\Big)^{\frac{1}{2}(b_3-b_2+b_1-b_0)} t^{-2B_2'-2B_1'-4B_0'}(\pi)^{-\frac{1}{2}(B_0'+B_1'+B_2'+B_3')}$$

and we recall that $q(Im\rho)$ was included into S_{1-loop}.

We have computed the quantum determinants $\mathcal{D}_{RR}$ treating RR fluctuations as differential forms. It would be more natural if these determinants had a K-theoretic formulation. This might be an interesting application to physics of differential K-theory.

6.2. Quantum determinants for NSNS fields

Let us first consider fluctuations $da_{(1)}^{m\alpha}$ and $db_{(2)}$ of the NSNS field $\mathbf{F}_{(2)}^{m\alpha}$ and $H_{(3)}$. From (2.7) we find the quadratic action for fluctuations:

$$S_{cl} = \frac{1}{4\pi}e^{-2\xi}\left\{t^4 g_{mn}\mathcal{G}_{\alpha\beta}\left(\mathbf{a}_{(1)}^{m\alpha}, d^{\dagger}d\mathbf{a}_{(1)}^{n\beta}\right) + t^2\left(b_{(2)}, d^{\dagger}db_{(2)}\right)\right\} \tag{6.12}$$

Now, again using (6.3) we find

$$Z_{NS,2} = \Big(\frac{t^4}{4\pi}e^{-2\xi}\Big)^{2(B_0'-B_1')}\Big[\frac{det'\Delta_1}{V_1}\Big]^{-2}\Big[\frac{det'\Delta_0}{V_0}\Big]^4 \tag{6.13}$$

and

$$Z_{NS,3} = \Big(\frac{t^2}{4\pi}e^{-2\xi}\Big)^{\frac{1}{2}(B_1'-B_2'-B_0')}\Big[\frac{det'\Delta_2}{V_2}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_1}{V_1}\Big]\Big[\frac{det'\Delta_0}{V_0}\Big]^{-3/2} \tag{6.14}$$

Let us now consider fluctuations of scalars: $\delta\xi, \delta\tau, \delta\rho$. From (2.7) we write the action quadratic in these fluctuations:

$$S_{scal} = \beta\int_X\left\{8\partial^M\delta\xi\partial_M\delta\xi + \frac{1}{(\tau_2)^2}\partial^M\delta\tau\partial_M\delta\bar{\tau} + \frac{1}{(\rho_2)^2}\partial^M\delta\rho\partial_M\delta\bar{\rho}\right\} \tag{6.15}$$

where $\beta = \frac{1}{4\pi}e^{-2\xi}t^6$. Now using the scalar measures defined as

$$\int[D\delta\rho][D\delta\bar{\rho}]e^{-\int_X\frac{\delta\rho\wedge*\delta\bar{\rho}}{(\mathrm{Im}\rho)^2}} = 1, \quad \int[D\delta\tau][D\delta\bar{\tau}]e^{-\int_X\frac{\delta\tau\wedge*\delta\bar{\tau}}{(\mathrm{Im}\tau)^2}} = 1 \tag{6.16}$$

$$\int[D\delta\xi]e^{-8\int_X\delta\xi\wedge*\delta\xi} = 1 \tag{6.17}$$

we find the quantum determinants for the NSNS scalars $Z_{NS,0}$:

$$Z_{NS,0} = \beta^{-\frac{5}{2}B_0'}\Big[\frac{det'\Delta_0}{V_0}\Big]^{-\frac{5}{2}} \tag{6.18}$$

Finally, we consider the fluctuation h_{MN} of the metric $t^2 g_{MN}$. Recall that we work in the limit $e^{-\xi} \to \infty$ so that in computing the quantum determinant for the metric we drop couplings to RR background fluxes.

From (2.7) we find the quadratic action:

$$S_{metr} = \beta \int_X \Big\{ (D_N h_{MP}) P^{MPQS} \left(D^N h_{QS}\right) + h^{MP} \mathcal{R}_{MNPQ} h^{NQ} \qquad (6.19)$$

$$- \left(D^M h_{MN} - \frac{1}{2} D_N h \right)^2 \Big\}$$

where $h = g^{MN} h_{MN}$ and

$$P^{MPQS} = \frac{1}{2} g^{MQ} g^{PS} - \frac{1}{4} g^{MP} g^{QS}$$

In (6.19) $\mathcal{R}_{MNPQ}$ is the Riemann tensor of the Ricci-flat[18] background metric g_{MN}. The covariant derivative D_M is performed with the background metric, and indices are raised and lowered with this metric.

Following standard procedure [30,31] we first insert the gauge fixing condition into the path-integral $\delta\left(\kappa_N - (D^M h_{MN} - \frac{1}{2} D_N h)\right)$ Then, we insert the unit

$$1 = \sqrt{det(\beta \mathbf{1}_1)} \int D\kappa_{(1)} e^{-\beta(\kappa_{(1)}, \kappa_{(1)})} \qquad (6.20)$$

and integrate over $\kappa_{(1)}$ in the path-integral. This procedure brings the kinetic term for the fluctuation h_{MN} to the form

$$\beta \int_X h_{MP} P^{MPNR} \mathcal{K}^{QS}_{NR} h_{QS}, \quad \mathcal{K}^{QS}_{NR} = -\delta^Q_N \delta^S_R D_L D^L + 2\mathcal{R}_N{}^Q{}_R{}^S \qquad (6.21)$$

Gauge fixing also introduces fermionic ghosts $k_{(1)}, l_{(1)}$ with the action

$$S_{gh} = \beta^{1/2} \left(l_{(1)}, \Delta_1 k_{(1)}\right) \qquad (6.22)$$

Using the measure $\int [Dh_{MN}] e^{-\int_X h_{MN} P^{MNPQ} h_{PQ}} = 1$ we obtain the result for the quantum determinant Z_{metr} of the metric:

$$Z_{metr} = (\beta)^{-\frac{1}{2}(N'_{\mathcal{K}} - B'_1)} \left[det' \mathcal{K} \right]^{-\frac{1}{2}} \frac{det' \Delta_1}{V_1} \qquad (6.23)$$

[18] If the background metric is not Ricci-flat there are terms involving the Ricci-tensor in (6.19) as well as in (6.22) below.

where $det'\mathcal{K}$ is a regularized determinant of nonzero modes of the operator $\mathcal{K}$ defined in (6.21) and $N'_{\mathcal{K}} = N_{\mathcal{K}} - n_{\mathcal{K}}$, where $N_{\mathcal{K}}$ stands for the dimension (infinite) of the space of the second rank symmetric tensors and $n_{\mathcal{K}}$ is the number of zeromodes of the operator $\mathcal{K}$. We will explain how we regularize $det'\mathcal{K}$ shortly.

Combining all NSNS determinants together we find:

$$\mathcal{D}_{NS} = r_{NS}(t,\xi)\Big[det'\mathcal{K}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_2}{V_2}\Big]^{-\frac{1}{2}} \tag{6.24}$$

where

$$r_{NS}(t,\xi) = (4\pi)^{\frac{1}{2}N'_{\mathcal{K}}+B'_0+B'_1+\frac{1}{2}B'_2}\left(e^{\xi}\right)^{N'_{\mathcal{K}}+B'_2+2B'_1+2B'_0} t^{-3N'_{\mathcal{K}}-B'_2-4B'_1-8B'_0} \tag{6.25}$$

Finally, from (6.11) and (6.24) we find the full expression for bosonic determinants

$$Det_B = Q(t,g_{MN})\Big(Im\rho\Big)^{\frac{1}{2}(b_3-b_2+b_1-b_0)} \tag{6.26}$$

where Q is a function only of the T-duality invariant variables g_{MN}, t and ξ. Explicitly,

$$Q(t,g_{MN}) = r_{tot}\Big[det'\mathcal{K}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_3}{V_3}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_2}{V_2}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_1}{V_1}\Big]^{-\frac{1}{2}} \tag{6.27}$$

where we regularized $det'\mathcal{K}$ in a way that eliminates dependence on ifinite numbers B_p and $N_{\mathcal{K}}$ so that

$$r_{tot} = (\tilde{t})^{3(n_{\mathcal{K}}+b_2+2b_1+4b_0)} \tag{6.28}$$

where we recall $\tilde{t} = te^{-\xi/3}$.

Now, let us check the transformation laws of Det_B under $\mathcal{D}_T$. From (6.26) it is obvious that Det_B is manifestly invariant under all generators of $\mathcal{D}_T$ except generator S.

Using,

$$Im(-1/\rho) = \frac{Im(\rho)}{\rho\bar{\rho}} \tag{6.29}$$

we find that under S, Det_B transforms as

$$Det_B(-1/\rho) = s_B Det_B(\rho), \quad s_B = \left(\rho\bar{\rho}\right)^{\frac{1}{2}(b_0-b_1+b_2-b_3)} \tag{6.30}$$

7. Inclusion of the fermion determinants

In this section we include the effects of the fermionic path integral. We recall the fermion content in the 10-dimensional and 8-dimensional supergravity theories and derive their actions. In the presence of nontrivial fluxes these fermionic path integrals are nonvanishing, even for the supersymmetric spin structure on T^2.

7.1. Fermions in 8D theory and their T-duality transformations.

Let us begin by listing the fermionic content in the 8-dimensional supergravity theory (this content will be derived from the 10-dimensional theory below.)

The fermions in the 8D theory include two gravitinos $\psi^A, \quad \eta^A, \quad A = 0, \ldots, 7$ and spinors $\Sigma, \quad \Lambda, \quad l, \quad \mu, \quad \tilde{l}, \quad \tilde{\mu}$. [19] The relation of these fields to the 10D fields is explained in (7.13),(7.14) below. There are also bosonic spinor ghosts b_1, c_1, Υ_2 and b_2, c_2, Υ_1 which accompany ψ^A and η^A respectively.

The fermions and ghosts transform under T-duality generators as follows. The generators $T, \tilde{T}, \tilde{S}$ act trivially on fermions and ghosts while the under the generator S they transform as

$$\psi^A \to e^{i\alpha\bar{\Gamma}}\psi^A, \quad \eta^A \to \eta^A, \quad \Lambda \to e^{-i\alpha\bar{\Gamma}}\Lambda, \quad \Sigma \to \Sigma \tag{7.1}$$

$$l \to e^{2i\alpha\bar{\Gamma}}l, \quad \tilde{l} \to e^{-2i\alpha\bar{\Gamma}}\tilde{l}, \quad \mu \to e^{i\alpha\bar{\Gamma}}\mu, \quad \tilde{\mu} \to e^{i\alpha\bar{\Gamma}}\tilde{\mu} \tag{7.2}$$

and ghosts transform as

$$\Upsilon_1 \to \Upsilon_1, \quad \Upsilon_2 \to e^{-i\alpha\bar{\Gamma}}\Upsilon_2 \tag{7.3}$$

$$\{c_1, b_1\} \to e^{i\alpha\bar{\Gamma}}\{c_1, b_1\} \quad \{c_2, b_2\} \to \{c_2, b_2\} \tag{7.4}$$

where α is defined by

$$\alpha = \nu + \frac{1}{2}\pi, \quad i\bar{\rho} = e^{i\nu}|\rho| \tag{7.5}$$

and $\bar{\Gamma}$ is the $8D$ chirality matrix.

The above transformation rules for space-time fermions follow from the transformation rules for the appropriate vertex operators on the world-sheet (as discussed for example in [11]). The only generator of $\mathcal{D}_T$ acting non-trivially on fermions is S. The components $\mathcal{V}^a_{NS}, a = 8, 9$ of the right-moving NS vertex are rotated by 2α, while the components $\mathcal{V}^A_{NS}$ are invariant. This follows since S does not act on the left-moving components of vertex operators. In this way we find the transformation rules for $\eta^A, b_2, c_2, \Sigma, \Upsilon_1, l, \tilde{l}$, which originate from $R \otimes NS$ sector. To account for the transformation rules for $\psi^A, b_1, c_1, \Lambda, \Upsilon_2, \mu, \tilde{\mu}$ we recall that these fields originate from $NS \otimes R$ sector and that the right-moving R vertex $\mathcal{V}_R$ transforms under S as

$$S : \mathcal{V}_R \to e^{i\alpha\bar{\Gamma}}\mathcal{V}_R. \tag{7.6}$$

[19] These fields are MW in Lorentzian signature. We supress 16 component spinor indices below

7.2. 10D fermion action

We start from the part of the $10D$ IIA supergravity action quadratic in fermions[19]. We work in the string frame. [20]

$$S^{(10)}_{ferm} = \int \sqrt{-g_{10}} e^{-2\phi} \left[\frac{1}{2} \bar{\hat{\psi}}_{\hat{A}} \hat{\Gamma}^{\hat{A}\hat{N}\hat{B}} D_{\hat{N}} \hat{\psi}_{\hat{B}} + \frac{1}{2} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{N}} D_{\hat{N}} \hat{\Lambda} - \frac{1}{\sqrt{2}} (\partial_{\hat{N}} \phi) \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{A}} \hat{\Gamma}^{\hat{N}} \hat{\psi}_{\hat{A}} \right]$$

$$+ \frac{1}{16} \int \sqrt{-g_{10}} e^{-\phi} \tilde{G}_{\hat{A}\hat{C}} \left[\bar{\hat{\psi}}^{\hat{E}} \hat{\Gamma}_{[\hat{E}} \hat{\Gamma}^{\hat{A}\hat{C}} \hat{\Gamma}_{\hat{F}]} \hat{\Gamma}^{11} \hat{\psi}^{\hat{F}} + \frac{3}{\sqrt{2}} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{E}} \hat{\Gamma}^{\hat{A}\hat{C}} \hat{\Gamma}^{11} \hat{\psi}_{\hat{E}} + \frac{5}{4} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{A}\hat{C}} \hat{\Gamma}^{11} \hat{\Lambda} \right]$$

$$+ \int \sqrt{-g_{10}} e^{-\phi} G_0 \left[\frac{1}{8} \bar{\hat{\psi}}_{\hat{A}} \hat{\Gamma}^{\hat{A}\hat{B}} \hat{\psi}_{\hat{B}} + \frac{5}{8\sqrt{2}} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{A}} \hat{\psi}_{\hat{A}} - \frac{21}{32} \bar{\hat{\Lambda}} \hat{\Lambda} \right] + \tag{7.7}$$

$$+ \frac{1}{192} \int \sqrt{-g_{10}} e^{-\phi} \tilde{G}_{\hat{A}\hat{B}\hat{C}\hat{D}} \left[\bar{\hat{\psi}}^{\hat{E}} \hat{\Gamma}_{[\hat{E}} \hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}} \hat{\Gamma}_{\hat{F}]} \hat{\psi}^{\hat{F}} + \frac{1}{\sqrt{2}} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{E}} \hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}} \hat{\psi}_{\hat{E}} + \frac{3}{4} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}} \hat{\Lambda} \right]$$

$$+ \frac{1}{48} \int \sqrt{-g_{10}} e^{-2\phi} H_{\hat{A}\hat{B}\hat{C}} \left[\bar{\hat{\psi}}^{\hat{E}} \hat{\Gamma}_{[\hat{E}} \hat{\Gamma}^{\hat{A}\hat{B}\hat{C}} \hat{\Gamma}_{\hat{F}]} \hat{\Gamma}^{11} \hat{\psi}^{\hat{F}} + \sqrt{2} \bar{\hat{\Lambda}} \hat{\Gamma}^{\hat{E}} \hat{\Gamma}^{\hat{A}\hat{B}\hat{C}} \hat{\Gamma}^{11} \hat{\psi}_{\hat{E}} \right]$$

where $\hat{\Lambda}$ and $\hat{\psi}^{\hat{A}}$ are the Majorana dilatino and gravitino and covariant derivatives act on them as

$$D_{\hat{N}} \hat{\psi}^{\hat{A}} = \partial_{\hat{N}} \hat{\psi}^{\hat{A}} + \omega_{\hat{N}}{}^{\hat{A}}{}_{\hat{B}} \hat{\psi}^{\hat{B}} + \frac{1}{4} \omega_{\hat{N}\hat{B}\hat{C}} \Gamma^{\hat{B}\hat{C}} \hat{\psi}^{\hat{A}}$$

$$D_{\hat{N}} \hat{\Lambda} = \partial_{\hat{N}} \hat{\Lambda} + \frac{1}{4} \omega_{\hat{N}\hat{B}\hat{C}} \Gamma^{\hat{B}\hat{C}} \hat{\Lambda}$$

There are also terms quartic in fermions in the action. It turns out that it is important to take them into account to check the T-duality invariance of partition sum. We recall the 4-fermionic terms in Appendix(C).

7.3. Reduction on T^2.

To carry out the reduction of the fermionic action to 8D we choose the gauge for the 10D veilbein as

$$\hat{E}^{\hat{A}}_{\hat{M}} = \begin{pmatrix} tE^A_M & \mathcal{A}^m_M V^{\frac{1}{2}} e^a_m \\ 0 & V^{\frac{1}{2}} e^a_m \end{pmatrix}, \tag{7.8}$$

(recall $a = 8, 9$ and $A = 0, ..., 7$) and use the following basis of 10D 32×32 matrices $\hat{\Gamma}^{\hat{A}}$,

$$\hat{\Gamma}^A = \sigma_2 \otimes \Gamma^A \quad A = 0, \ldots 7, \quad \hat{\Gamma}^8 = \sigma_1 \otimes 1_{16}, \quad \hat{\Gamma}^9 = \sigma_2 \otimes \bar{\Gamma}, \quad \bar{\Gamma} = \Gamma^0 \ldots \Gamma^7 \tag{7.9}$$

[20] We explain the relation between our conventions and those of [19] in Appendix(B).

Here Γ^A are symmetric 8D Dirac matrices, which in Euclidean signature can be all chosen to be real, and $\sigma_{1,2,3}$ are Pauli matrices. In this basis the 10D chirality $\hat{\Gamma}^{11}$ and charge conjugation matrices $C^{(10)}$ have the form

$$\hat{\Gamma}^{11} = \sigma_3 \otimes \mathbf{1}_{16}, \qquad C^{(10)} = i\sigma_2 \otimes \mathbf{1}_{16}. \tag{7.10}$$

The 8D fermions listed in section 7.1 are related to 10D fields $\hat{\psi}^{\hat{A}}$ and $\hat{\Lambda}$ in the following way [21]:

$$\begin{pmatrix} \psi^A \\ \eta^A \end{pmatrix} = \hat{\psi}^A + \frac{1}{6}\hat{\Gamma}^A\hat{\Gamma}_a\hat{\psi}^a, \quad \begin{pmatrix} \Sigma \\ \Lambda \end{pmatrix} = \frac{3}{4}\hat{\Lambda} + \frac{\sqrt{2}}{4}\hat{\Gamma}_a\hat{\psi}^a, \tag{7.11}$$

$$\begin{pmatrix} l \\ \mu \end{pmatrix} = \hat{\Gamma}_a\hat{\psi}^a - \frac{\sqrt{2}}{2}\hat{\Lambda}, \quad \begin{pmatrix} \tilde{\mu} \\ \tilde{l} \end{pmatrix} = \hat{\psi}_8 - \hat{\Gamma}^{89}\hat{\psi}_9 \tag{7.12}$$

7.4. 8D fermion action

Now we present the 8D action $S^{(8)}_{quad} = S_{kin} + S_{fermi-flux}$ quadratic in fermionic fluctuations [22] over the 8D background specified in section 2.2. The kinetic term is standard

$$S_{kin} = \int_X e^{-2\xi}t^7\Big\{\frac{1}{2}\bar{\psi}_A\Gamma^{AMB}D_M\psi_B + \frac{1}{2}\bar{\eta}_A\Gamma^{AMB}D_M\eta_B + \frac{2}{3}\overline{\Sigma}\Gamma^M D_M\Sigma + \frac{2}{3}\overline{\Lambda}\Gamma^M D_M\Lambda$$
$$+\frac{1}{4}\bar{l}\Gamma^M D_M l + \frac{1}{4}\bar{\mu}\Gamma^M D_M\mu + \frac{1}{4}\bar{\tilde{l}}\Gamma^M D_M\tilde{l} + \frac{1}{4}\bar{\tilde{\mu}}\Gamma^M D_M\tilde{\mu}\Big\} \tag{7.13}$$

The coupling of fluxes to fermion bilinears is:

$$S_{fermi-flux} = \frac{\pi}{4}\int_X e^{-\xi}\Big\{t^8\Big[\frac{n_0\rho + n_1}{\sqrt{\mathrm{Im}\rho}}X_{(0)} - \frac{n_0\bar{\rho} + n_1}{\sqrt{\mathrm{Im}\rho}}\tilde{X}_{(0)}\Big] + t^7 g_{(1)m}\wedge * X^m_{(1)} + \tag{7.14}$$
$$+t^6\Big[\frac{g^2_{(2)}\rho + g^1_{(2)}}{\sqrt{\mathrm{Im}\rho}}\wedge * X_{(2)} - \frac{g^2_{(2)}\bar{\rho} + g^1_{(2)}}{\sqrt{\mathrm{Im}\rho}}\wedge * \tilde{X}_{(2)}\Big] + t^5 g_{(3)m}\wedge * X^m_{(3)}$$
$$+t^4\sqrt{\mathrm{Im}\rho}g_{(4)}\wedge * \Big[X_{(4)} + \tilde{X}_{(4)}\Big]\Big\}$$

[21] $\hat{\Lambda}$ and $\hat{\Gamma}_a\hat{\psi}^a$ are mixed to give the 8D "dilatino", the superpartner of $e^{-2\xi} = e^{-2\phi}V$.

[22] In Minkowski signature $\bar{\psi}_A = \psi^\dagger_A\Gamma^0$. In Euclidean signature $\bar{\psi}_A$ and ψ_A are treated as independent fields.

where the harmonic fluxes $g_{(p)}, p = 0, \ldots, 4$ were defined in (5.16). These harmonic fields couple to differential p-forms $X_{(p)}, \tilde{X}_{(p)}$ constructed out of fermi bilinears. We now give explicit formulae for $X_{(p)}$:

$$\begin{aligned} X_{(0)} = &-\overline{\Psi}_A^{(-)}\Gamma^{AB}\mathbf{W}_B^{(-)} - \overline{\mathbf{W}}_B^{(-)}\Gamma^{AB}\Psi_A^{(-)} + i\sqrt{2}\overline{\Lambda}^{(+)}\Gamma^A\mathbf{W}_A^{(+)} \\ &-i\sqrt{2}\overline{\mathbf{W}}_A^{(-)}\Gamma^A\Lambda^{(+)} + i\sqrt{2}\overline{\Sigma}^{(+)}\Gamma^A\Psi_A^{(-)} - i\sqrt{2}\overline{\Psi}_A^{(-)}\Gamma^A\Sigma^{(+)} \\ &+\frac{i}{2}\bar{l}^{(-)}\Gamma^A\Psi_A^{(+)} - \frac{i}{2}\overline{\Psi}_A^{(+)}\Gamma^A l^{(-)} + \frac{i}{2}\bar{\mu}^{(-)}\Gamma^A\mathbf{W}_A^{(+)} - \frac{i}{2}\overline{\mathbf{W}}_A^{(+)}\Gamma^A\mu^{(-)} \\ &+4\overline{\Sigma}^{(+)}\Lambda^{(+)} - 4\overline{\Lambda}^{(+)}\Sigma^{(+)} - \frac{1}{2}\bar{\tilde{l}}^{(+)}\tilde{\mu}^{(+)} + \frac{1}{2}\bar{\tilde{\mu}}^{(+)}\tilde{l}^{(+)} \end{aligned} \tag{7.15}$$

$$\begin{aligned} \left(X_{(2)}\right)_{MN} = &\overline{\Psi}_A^{(-)}\Gamma^{[A}\Gamma_{MN}\Gamma^{B]}\mathbf{W}_B^{(-)} + \overline{\mathbf{W}}_A^{(-)}\Gamma^{[A}\Gamma_{MN}\Gamma^{B]}\Psi_B^{(-)} + \\ &i\sqrt{2}\overline{\Lambda}^{(+)}\Gamma_{MN}\Gamma^A\mathbf{W}_A^{(-)} - i\sqrt{2}\overline{\mathbf{W}}_A^{(-)}\Gamma^A\Gamma_{MN}\Lambda^{(+)} + i\sqrt{2}\overline{\Sigma}^{(+)}\Gamma_{MN}\Gamma^A\Psi_A^{(-)} \\ &+i\sqrt{2}\overline{\Psi}_A^{(-)}\Gamma^A\Gamma_{MN}\Sigma^{(+)} + \frac{i}{2}\bar{l}^{(-)}\Gamma^A\Gamma_{MN}\Psi_A^{(+)} + \frac{i}{2}\overline{\Psi}_A^{(+)}\Gamma_{MN}\Gamma^A l^{(-)} \\ &-\frac{i}{2}\bar{\mu}^{(-)}\Gamma^A\Gamma_{MN}\mathbf{W}_A^{(+)} - \frac{i}{2}\overline{\mathbf{W}}_A^{(+)}\Gamma_{MN}\Gamma^A\mu^{(-)} + 4\overline{\Sigma}^{(+)}\Gamma_{MN}\Lambda^{(+)} \\ &+4\overline{\Lambda}^{(+)}\Gamma_{MN}\Sigma^{(+)} - \frac{1}{2}\bar{\tilde{l}}^{(+)}\Gamma_{MN}\tilde{\mu}^{(+)} - \frac{1}{2}\bar{\tilde{\mu}}^{(+)}\Gamma_{MN}\tilde{l}^{(+)} \end{aligned} \tag{7.16}$$

where $\psi_A^{(\pm)} = \frac{1}{2}\left(\mathbf{1}_{16} \pm \bar{\Gamma}\right)\psi_A$,etc. and we use the combinations of $8D$ fields

$$\Psi_A = \psi_A + i\frac{\sqrt{2}}{3}\Gamma_A\Lambda, \quad \mathbf{W}_A = \eta_A - i\frac{\sqrt{2}}{3}\Gamma_A\Sigma$$

to make the expressions for $X_{(0)}, X_{(2)}$ have nicer coefficients.

The forms $\tilde{X}_{(0)}, \tilde{X}_{(2)}$ can be obtained from $X_{(0)}, X_{(2)}$ by exchange of 8D chiralities $(-) \leftrightarrow (+)$.

Under the T-duality generator S the above forms transform as

$$\left\{X_{(0)}, X_{(2)}\right\} \to e^{-i\alpha}\left\{X_{(0)}, X_{(2)}\right\}, \quad \left\{\tilde{X}_{(0)}, \tilde{X}_{(2)}\right\} \to e^{i\alpha}\left\{\tilde{X}_{(0)}, \tilde{X}_{(2)}\right\} \tag{7.17}$$

so that the combinations $\frac{1}{\sqrt{\mathrm{Im}\rho}}(n_0\rho + n_1)X_{(p)}$, $\frac{1}{\sqrt{\mathrm{Im}\rho}}(n_0\bar{\rho} + n_1)\tilde{X}_{(p)}$ for $p = 0, 2$ which appear in the action (7.13) are invariant under S.

Also we have defined the 1-form

$$\left(X_{(1)}^m\right)_M = e_+^m\Big[\overline{\Psi}_A^{(-)}\Gamma^{[A}\Gamma_M\Gamma^{B]}\mathbf{W}_B^{(+)} - \overline{\mathbf{W}}_A^{(+)}\Gamma^{[A}\Gamma_M\Gamma^{B]}\Psi_B^{(-)} \tag{7.18}$$

$$-i\sqrt{2}\overline{\Lambda}^{(+)}\Gamma_M\Gamma^A\mathbf{W}_A^{(+)}+i\sqrt{2}\overline{\mathbf{W}}_A^{(+)}\Gamma^A\Gamma_M\Lambda^{(+)}-i\sqrt{2}\overline{\Sigma}^{(-)}\Gamma_M\Gamma^A\Psi_A^{(-)}$$
$$+i\sqrt{2}\overline{\Psi}_A^{(-)}\Gamma^A\Gamma_M\Sigma^{(-)}-\frac{i}{2}\bar{l}^{(+)}\Gamma^A\Gamma_M\Psi_A^{(+)}+\frac{i}{2}\overline{\Psi}_A^{(+)}\Gamma_M\Gamma^A\tilde{l}^{(+)}$$
$$-\frac{i}{2}\bar{\tilde{\mu}}^{(-)}\Gamma^A\Gamma_M\mathbf{W}_A^{(-)}+\frac{i}{2}\overline{\mathbf{W}}_A^{(-)}\Gamma_M\Gamma^A\tilde{\mu}^{(-)}-4\overline{\Sigma}^{(-)}\Gamma_M\Lambda^{(+)}$$
$$+4\overline{\Lambda}^{(+)}\Gamma_M\Sigma^{(-)}-\frac{1}{2}\bar{\mu}^{(+)}\Gamma_M l^{(-)}+\frac{1}{2}\bar{l}^{(-)}\Gamma_M\mu^{(+)}\Big]+e_-^m\Big[(+)\leftrightarrow(-)\Big]$$

and the 3-form

$$\left(X_{(3)}^m\right)_{MNP}=e_+^m\Big[-\overline{\Psi}_A^{(-)}\Gamma^{[A}\Gamma_{MNP}\Gamma^{B]}\mathbf{W}_B^{(+)}-\overline{\mathbf{W}}_A^{(+)}\Gamma^{[A}\Gamma_{MNP}\Gamma^{B]}\Psi_B^{(-)}\tag{7.19}$$
$$+i\sqrt{2}\overline{\Lambda}^{(+)}\Gamma_{MNP}\Gamma^A\mathbf{W}_A^{(+)}+i\sqrt{2}\overline{\mathbf{W}}_A^{(+)}\Gamma^A\Gamma_{MNP}\Lambda^{(+)}-i\sqrt{2}\overline{\Sigma}^{(-)}\Gamma_{MNP}\Gamma^A\Psi_A^{(-)}$$
$$-i\sqrt{2}\overline{\Psi}_A^{(-)}\Gamma^A\Gamma_{MNP}\Sigma^{(-)}-\frac{i}{2}\bar{l}^{(+)}\Gamma^A\Gamma_{MNP}\Psi_A^{(+)}-\frac{i}{2}\overline{\Psi}_A^{(+)}\Gamma_{MNP}\Gamma^A\tilde{l}^{(+)}$$
$$+\frac{i}{2}\bar{\tilde{\mu}}^{(-)}\Gamma^A\Gamma_{MNP}\mathbf{W}_A^{(-)}+\frac{i}{2}\overline{\mathbf{W}}_A^{(-)}\Gamma_{MNP}\Gamma^A\tilde{\mu}^{(-)}-4\overline{\Sigma}^{(-)}\Gamma_{MNP}\Lambda^{(+)}$$
$$-4\overline{\Lambda}^{(+)}\Gamma_{MNP}\Sigma^{(-)}+\frac{1}{2}\bar{\mu}^{(+)}\Gamma_{MNP}l^{(-)}+\frac{1}{2}\bar{l}^{(-)}\Gamma_{MNP}\mu^{(+)}\Big]+e_-^m\Big[(+)\leftrightarrow(-)\Big]$$

where we denote $e_\pm^m=e_8^m\mp ie_9^m$.

The forms $X_{(1)}^m$ and $X_{(3)}^m$ transform in the **2** of $SL(2,\mathbf{Z})_\tau$. Also from (7.18),(7.19) we find that $X_{(1)}^m$ and $X_{(3)}^m$ are invariant under $SL(2,\mathbf{Z})_\rho$ if we accompany the action of the generator S by the U(1) rotation of e_a^m

$$e_\pm^m\to e^{\pm i\alpha}e_\pm^m\tag{7.20}$$

Since there are no local Lorentz anomalies, we can make this transformation.

The most important objects in (7.14) are the self-dual[23] form $X_{(4)}$ and the anti-self-dual form $\tilde{X}_{(4)}$ which couple to the flux $g_{(4)}$. $X_{(4)}$ is defined by

$$\left(X_{(4)}\right)_{MNPQ}=-i\overline{\Psi}_A^{(+)}\Gamma^{[A}\Gamma_{MNPQ}\Gamma^{B]}\mathbf{W}_B^{(+)}+i\overline{\mathbf{W}}_A^{(+)}\Gamma^{[A}\Gamma_{MNPQ}\Gamma^{B]}\Psi_B^{(+)}\tag{7.21}$$
$$-\sqrt{2}\overline{\Lambda}^{(-)}\Gamma_{MNPQ}\Gamma^A\mathbf{W}_A^{(+)}+\sqrt{2}\overline{\mathbf{W}}_A^{(+)}\Gamma^A\Gamma_{MNPQ}\Lambda^{(-)}$$
$$-\sqrt{2}\overline{\Sigma}^{(-)}\Gamma_{MNPQ}\Gamma^A\Psi_A^{(+)}+\sqrt{2}\overline{\Psi}_A^{(+)}\Gamma^A\Gamma_{MNPQ}\Sigma^{(-)}-\frac{1}{2}\bar{l}^{(+)}\Gamma^A\Gamma_{MNPQ}\Psi_A^{(-)}$$
$$+\frac{1}{2}\overline{\Psi}_A^{(-)}\Gamma_{MNPQ}\Gamma^A l^{(+)}-\frac{1}{2}\bar{\mu}^{(+)}\Gamma^A\Gamma_{MNPQ}\mathbf{W}_A^{(-)}+\frac{1}{2}\overline{\mathbf{W}}_A^{(-)}\Gamma_{MNPQ}\Gamma^A\mu^{(+)}$$

[23] In our conventions $\Gamma_{A_1A_2A_3A_4}=-\frac{1}{4!}\epsilon_{A_1A_2A_3A_4B_1B_2B_3B_4}\Gamma^{B_1B_2B_3B_4}\bar{\Gamma}$

$$+4i\overline{\Sigma}^{(-)}\Gamma_{MNPQ}\Lambda^{(-)} - 4i\overline{\Lambda}^{(-)}\Gamma_{MNPQ}\Sigma^{(-)} - \frac{i}{2}\bar{\tilde{l}}^{(-)}\Gamma_{MNPQ}\tilde{\mu}^{(-)} + \frac{i}{2}\bar{\tilde{\mu}}^{(-)}\Gamma_{MNPQ}\tilde{l}^{(-)}$$

and $\tilde{X}_{(4)}$ can be obtained from $X_{(4)}$ by the exchange of 8D chiralities $(+) \leftrightarrow (-)$.

Under the T-duality generator S these forms transform as

$$X_{(4)} \to e^{i\alpha}X_{(4)}, \quad \tilde{X}_4 \to e^{-i\alpha}\tilde{X}_{(4)} \tag{7.22}$$

We have also checked using Appendix(C) that the 4-fermion terms in the 8D action can be written as

$$S^{(8D)}_{4-ferm} = S'_{4-ferm} + S''_{4-ferm}, \quad S'_{4-ferm} = \frac{\pi}{128}\int_X e^{-2\xi}t^8\Big[X_{(4)}\wedge * X_{(4)} + \tilde{X}_{(4)}\wedge * \tilde{X}_{(4)}\Big] \tag{7.23}$$

While S''_{4-ferm} is manifestly invariant under T-duality, we will see that the non-invariant term S'_{4-ferm} is required for T-duality invariance of the total partition sum $Z(\mathcal{F},\rho)$ of (1.32).

7.5. T-duality invariance of the ghost interactions

The classical 8D action obtained from the reduction of 10D IIA supergravity on T^2 is invariant under local supersymmetry (all 32 components survive the reduction). To construct the quantum action we have to impose a gauge fixing condition on the gravitino $\hat{\psi}_{(8D)} := \begin{pmatrix}\psi_A \\ \eta_A\end{pmatrix}$ and include ghosts. Since the susy transformation laws involve fluxes, there is a potential T-duality anomaly from the ghost sector. In fact no such anomaly will occur as we now demonstrate. There are two generic properties of supergravity theories:

1 .) In addition to a pair of Faddeev-Popov ghosts associated to the local susy gauge transformation $\hat{\psi}_{(8D)} \to \hat{\psi}^A_{(8D)} + \delta_{\hat{\epsilon}}\hat{\psi}^A_{(8D)}$ a "third ghost," the Nielsen-Kallosh ghost, appears [32].

2 .) Terms quartic in Faddeev-Popov ghosts are required [33].

Let us recall first how the "third ghost" appears. Following the standard procedure we fix the local susy gauge by inserting $\delta\left(f - \hat{\Gamma}_A\hat{\psi}^A_{(8D)}\right)$ into the path integral. Then we also insert the unit[24]

$$\mathbf{1} = \frac{1}{\sqrt{det\left(\frac{1}{2}e^{-2\xi}t^7\hat{D}\right)}}\int[df]e^{\frac{i}{2}\int_X e^{-2\xi}t^7\bar{f}\hat{D}f}, \quad \hat{D} = i\hat{\Gamma}^N D_N \tag{7.24}$$

[24] We use the measure $\int[df]e^{i\int_X \bar{f}f} = 1$.

and integrate over $[df]$. (If $\hat{D}$ has zeromodes this expression is formally 0/0, but (7.27) below still makes sense.)

As a result we first find that the gravitino kinetic term gets modified to

$$-\frac{i}{2}\int_X e^{-2\xi}t^7\left\{\bar{\psi}^A\mathcal{M}_{AB}\psi^B+\bar{\eta}^A\mathcal{M}_{AB}\eta^B\right\} \tag{7.25}$$

where the operator $\mathcal{M}_{AB}$ acts on sections of the bundle [25] $Spin(X)\otimes TX$ as

$$\mathcal{M}_{AB}=\delta_{AB}i\Gamma^M D_M-2i\Gamma_A D_B \tag{7.26}$$

where $D_A=E_A^M D_M$. The determinant in (7.24) is expressed as the partition function for the "third ghost" $\hat{\Upsilon}$ with action

$$S_{\hat{\Upsilon}}=-\frac{i}{2}\int_X e^{-2\xi}t^7\bar{\hat{\Upsilon}}\hat{D}\hat{\Upsilon} \tag{7.27}$$

$\hat{\Upsilon}$ is a bosonic 32 component spinor, which we decompose into 16 component spinors as

$$\hat{\Upsilon}=\begin{pmatrix}\Upsilon_1\\ \Upsilon_2\end{pmatrix}$$

Now we come to the most interesting part of quantum action which involves Faddeev-Popov ghosts $\hat{b},\hat{c}$.

$$S_{bc}=S_{bc}^{(2)}+S_{bc}^{(4)} \tag{7.28}$$

where $S_{bc}^{(2)}\left(S_{bc}^{(4)}\right)$ denotes the parts of the action quadratic (quartic) in FP ghosts. Let us discuss the quadratic part first. According to the standard FP procedure we have

$$S_{bc}^{(2)}=\int_{X_8} t^7 e^{-2\xi}\bar{\hat{b}}\hat{\Gamma}_A\delta_{\hat{c}}\hat{\psi}^A_{(8D)} \tag{7.29}$$

We decompose bosonic 32 component spinors $\hat{b},\hat{c}$ as

$$\hat{c}=\begin{pmatrix}c_1\\ c_2\end{pmatrix},\quad \hat{b}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}.$$

We can write the action as a sum of two pieces

$$S_{bc}^{(2)}=S_{bc}^{(2)0}+S_{bc}^{(2)2}$$

[25] $Spin(X)$ and TX are spinor and tangent bundles on X

Here $S_{bc}^{(2)0}$ does not contain fermionic matter fields while $S_{bc}^{(2)2}$ is quadratic in fermions. We now present $S_{bc}^{(2)0}$ and put $S_{bc}^{(2)2}$ in Appendix(D).

$$S_{bc}^{(2)0} = \int_{X_8} t^7 e^{-\xi} \bar{\hat{b}}(-i\hat{D})\hat{c} - \pi e^{-\xi}\Big\{ \frac{2}{3}t^8\Big[\frac{n_0\rho+n_1}{\sqrt{\mathrm{Im}\rho}}X_{(0)}^{gh} - \frac{n_0\bar{\rho}+n_1}{\sqrt{\mathrm{Im}\rho}}\tilde{X}_{(0)}^{gh}\Big] \tag{7.30}$$

$$+\frac{1}{2}t^7 g_{(1)m}\wedge * X_{(1)}^{gh\ m} + \frac{1}{3}t^6\mathrm{Im}\Big[\frac{g_{(2)}^2\rho+g_{(2)}^1}{\sqrt{\mathrm{Im}\rho}}\wedge * X_{(2)}^{gh} - \frac{g_{(2)}^2\bar{\rho}+g_{(2)}^1}{\sqrt{\mathrm{Im}\rho}}\wedge * \tilde{X}_{(2)}^{gh}\Big]$$

$$+\frac{1}{8}t^5 g_{(3)m}\wedge * X_{(3)}^{gh\ m}\Big\}$$

where we define forms bilinear in FP ghosts as

$$X_{(0)}^{gh} = \frac{1}{2}\Big\{\overline{b_2}^{(-)}c_1^{(-)} - \overline{c_1}^{(-)}b_2^{(-)} - \overline{b_1}^{(-)}c_2^{(-)} + \overline{c_2}^{(-)}b_1^{(-)}\Big\} \tag{7.31}$$

$$\left(X_{(2)}^{gh}\right)_{MN} = \frac{1}{2}\Big\{\overline{b_2}^{(-)}\Gamma_{MN}c_1^{(-)} + \overline{c_1}^{(-)}\Gamma_{MN}b_2^{(-)} + \overline{b_1}^{(-)}\Gamma_{MN}c_2^{(-)} + \overline{c_2}^{(-)}\Gamma_{MN}b_1^{(-)}\Big\} \tag{7.32}$$

$$\left(X_{(1)}^{gh\ m}\right)_M = \frac{1}{2}e_+^m\Big[\overline{b_2}^{(+)}\Gamma_M c_1^{(-)} - \overline{c_1}^{(-)}\Gamma_M b_2^{(+)} - \overline{b_1}^{(-)}\Gamma_M c_2^{(+)} + \overline{c_2}^{(+)}\Gamma_M b_1^{(-)}\Big] \tag{7.33}$$

$$+\frac{1}{2}e_-^m\Big[(+)\leftrightarrow(-)\Big]$$

$$\left(X_{(3)}^{gh\ m}\right)_{MNP} = \frac{1}{2}e_+^m\Big[\overline{b_2}^{(+)}\Gamma_{MNP}c_1^{(-)} + \overline{c_1}^{(-)}\Gamma_{MNP}b_2^{(+)} + \overline{b_1}^{(-)}\Gamma_{MNP}c_2^{(+)} \tag{7.34}$$

$$+\overline{c_2}^{(+)}\Gamma_{MNP}b_1^{(-)}\Big] + \frac{1}{2}e_-^m\Big[(+)\leftrightarrow(-)\Big]$$

The forms $\tilde{X}_{(0)}^{gh}, \tilde{X}_{(2)}^{gh}$ can be obtained from $X_{(0)}^{gh}, X_{(2)}^{gh}$ by exchange of 8D chiralities $(-)\leftrightarrow(+)$. Note, that $\hat{b}, \hat{c}$ do not couple to the flux $g_{(4)}$.

Let us now present the part of the quantum 8D action which is quartic in ghosts (as obtained by following the procedure of [33]):

$$S_{bc}^{(4)} = e^{-2\xi}t^8\Big\{\frac{1}{84}\left(\bar{\hat{b}}\hat{\Gamma}^{ABC}\hat{c}\right)\left(\bar{\hat{b}}\hat{\Gamma}_{ABC}\hat{c}\right) + \frac{1}{3}\left(\bar{\hat{b}}\hat{\Gamma}^{A}\hat{c}\right)\left(\bar{\hat{b}}\hat{\Gamma}_{A}\hat{c}\right)\Big\} \tag{7.35}$$

The presence of this quartic action is due to the fact that gauge symmetry algebra is open in supergravity: $[\delta_{\hat{\epsilon}_1}, \delta_{\hat{\epsilon}_1}]\,\hat{\psi}_{(8D)}^A$ contains a term proportional to the equation of motion of $\hat{\psi}_{(8D)}^A$.

The T-duality invariance of $S_{bc}^{(4)}, S_{bc}^{(2)0}$ and $S_{\hat{\Upsilon}}$ is manifest and we have also checked that $S_{bc}^{(2)2}$ is T-duality invariant, so we conclude that the part of the 8D quantum action which contains ghosts is T-duality invariant.

7.6. Computation of the determinants

We can now compute the fermionic quantum determinants including ghosts. Let us expand the fields $\Lambda, \Sigma, l, \tilde{l}, \mu, \tilde{\mu}, b_1, b_2, c_1, c_2, \Upsilon_1, \Upsilon_2$ and ψ_A, η_A in the full orthonormal basis of the operators $\not{D} = i\Gamma^N D_N$ and $\mathcal{M}$ respectively, where the operator $\mathcal{M}$ was defined in (7.26). Note that since we are assuming that background fluxes are harmonic, fermionic non-zero modes do not couple to them. Moreover,we can rescale non-zero modes by a factor of $e^{-\xi}t^{7/2}$ so that kinetic terms appear without any dependence on ξ and t, but four-fermionic terms are supressed as $e^{2\xi}t^{-6}$ with respect to the kinetic terms. Since kinetic terms are manifestly T-duality invariant the integration over nonzero modes will just give a factor Det'_F depending only on the Ricci flat metric g_{MN} and the constants t and ξ, all of which are T-duality invariant. Det'_F has the form

$$\mathrm{Det}'_F = r_F(\xi, t) det' \mathcal{M} \tag{7.36}$$

where $det'\mathcal{M}$ is determinant of the operator $\mathcal{M}$ defined in (7.26) regularized in a way that

$$r_F(\xi, t) = const \left(e^{-2\xi}t^7\right)^{-n_{\mathcal{M}}} \tag{7.37}$$

where $n_{\mathcal{M}}$ denotes the number of zero modes of $\mathcal{M}$.

Note, that determinants of nonzero modes of the fermions $\Sigma, \Lambda, l, \mu, \tilde{l}, \tilde{\mu}$ and bosons $\Upsilon_1, \Upsilon_2, b_1, b_2, c_1, c_2$ cancel each other and do not contribute to Det'_F.

The situation is quite different for zero-modes: the kinetic terms are zero but there is nonzero coupling to harmonic fluxes, so that if we rescale fermion zeromodes by $e^{-\frac{1}{2}\xi}t^2$ we make both the fermion coupling to $g_{(4)}$ and the fermion quartic terms independent of ξ and t. We will also rescale ghost zeromodes by $e^{-\frac{1}{2}\xi}t^2$ and include the factor $\left(e^{-\xi}t^4\right)^{n_{\mathcal{M}}}$ which comes from the rescaling of fermion and ghost zeromodes into the definition of Det'_F, i.e. we define new r_F:

$$r_F^{new}(\xi, t) := r_F(\xi, t) \left(e^{-\xi}t^4\right)^{n_{\mathcal{M}}} = const(t)^{-3n_{\mathcal{M}}} (e^{\xi})^{n_{\mathcal{M}}} \tag{7.38}$$

From (6.28) and (7.38) we find that the full quantum determinants depend on t and ξ in the following way

$$(\tilde{t}^{-3})^{n_{\mathcal{M}} - n_{\mathcal{K}} - b_2 - 2b_1 - 4b_0} \tag{7.39}$$

where we recall that $\tilde{t} = te^{-\xi/3}$ is the U-duality invariant combination.[26] Note that the dependence on $\tilde{t}$ in (7.39) comes entirely from the volume of the space of zero modes. The volume of bosonic zero modes is blowing up in the limit $\tilde{t} \to \infty$, but the volume of fermion zero modes is shrinking. Since (7.39) is an overall factor in the partition sum, it is a question of a net balance between fermion and boson zero modes whether the partition sum blows up or vanishes in the limit $\tilde{t} \to \infty$.

7.7. Integration over the space of fermion zeromodes

We can split the action of the rescaled fermion and ghost zeromodes as

$$S^{(zm)} = S^{(zm)inv} + S^{(zm)ninv}.$$

Here the part $S^{(zm)inv}$ is invariant under T-duality and includes all the ghost zeromode interactions, the coupling of the fermion zeromodes to all RR fluxes except for $g_{(4)}$ and the invariant part of the 4-fermion zeromode couplings, denoted $S^{(zm)''}_{4-ferm}$.

$S^{(zm)ninv}$ transforms non-trivially under the generator S of T-duality and can be recast in the following way:

$$S^{(zm)ninv} = \int_X \left\{ 4\pi \mathrm{Im}\rho g_{(4)} \wedge * Y_{(4)} + 2\pi \mathrm{Im}\rho Y_{(4)} \wedge * Y_{(4)} \right\} \tag{7.40}$$

where we define the harmonic 4-form $Y_{(4)}$ as

$$Y_{(4)} = \frac{1}{16} \frac{1}{\sqrt{\mathrm{Im}\rho}} \left[X^{(zm)}_{(4)} + \tilde{X}^{(zm)}_{(4)} \right]. \tag{7.41}$$

This object transforms under S as

$$S \cdot Y_{(4)} = -\mathrm{Re}\rho Y_{(4)} + i\mathrm{Im}\rho * Y_{(4)}. \tag{7.42}$$

We now expand the harmonic 4-forms in the basis ω_i of $H^4(X, \mathbf{Z})$

$$g_{(4)} = (n^i + \tilde{\alpha}^i)\omega_i, \quad Y_{(4)} = y^i \omega_i, \quad \tilde{\beta} = \tilde{\beta}^i \omega_i$$

where the chracteristics $\tilde{\alpha}, \tilde{\beta}$ are given in (5.21). Next, we define

$$\widehat{\Theta}(\mathcal{F}, \rho) = \int d\mu_F^{(zm)} \hat{h} e^{i2\pi\widehat{\Delta\Phi}} \Theta \left[\begin{matrix} \widehat{\alpha} \\ \widehat{\beta} \end{matrix} \right] (Q) \tag{7.43}$$

[26] For any Ricci-flat spin 8-manifold the numbers $n_{\mathcal{M}}$ and $n_{\mathcal{K}}$ can be expressed in terms of topological invariants.

where the shifted characterstics are defined as $\widehat{\alpha}^i = \tilde{\alpha}^i + y^i$, $\widehat{\beta}^i = \tilde{\beta}^i + S \cdot y^i$, and $d\mu_F^{(zm)}$ denotes the measure of the rescaled fermion and ghost zeromodes. Recall that $Q(\rho) = [H\mathrm{Im}\rho - i\mathrm{Re}\rho]I$. In (7.43) $\hat{h} = e^{-S^{(zm)inv}}$ is a T-duality invariant expression which depends on τ, ρ, t, g_{MN} as well as fermion and ghost zeromodes. The dependence on τ, ρ, t, g_{MN} comes entirely from the coupling of the rescaled zeromodes (of fermions and ghosts) to the fluxes $g_{(p)}, p = 0, 1, 2, 3$. Finally, we have also defined

$$\widehat{\Delta\Phi}(\mathcal{F}, \rho, \vec{y}) := \Delta\widetilde{\Phi} - \frac{1}{2}\vec{y}IS \cdot \vec{y} - \vec{y}I\vec{\beta} \tag{7.44}$$

where $\Delta\widetilde{\Phi}$ was defined in (5.24).

$\widehat{\Theta}(\mathcal{F}, \rho)$ is invariant under $SL(2, \mathbf{Z})_\tau$ and transforms under $SL(2, \mathbf{Z})_\rho$ as

$$\widehat{\Theta}(S \cdot \mathcal{F}, -1/\rho) = s_F \mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\widehat{\Theta}(\mathcal{F}, \rho) \tag{7.45}$$

$$\widehat{\Theta}(T \cdot \mathcal{F}, \rho + 1) = \mu(T)\widehat{\Theta}(\mathcal{F}, \rho) \tag{7.46}$$

We do Poisson ressumation to find (7.45) and the extra phase s_F is due to the transformation[27] of $d\mu_F^{zm}$

$$s_F = \left(e^{i\alpha}\right)^{I(\mathcal{M})} = (i)^{I(\mathcal{M})}(-i\rho)^{-\frac{1}{2}I(\mathcal{M})}(i\bar{\rho})^{\frac{1}{2}I(\mathcal{M})} \tag{7.47}$$

where $I(\mathcal{M})$ is the index of the operator $\mathcal{M}$ defined in (7.26). As in the standard computation of the chiral anomaly [34], only the zeromodes contribute to the transformation of fermionic measure. Indeed, the contribution of the bosonic ghosts c_1, b_1, Υ_2 to the transformation of the measure cancels that of the contribution of the fermions $\mu, \tilde{\mu}, \Lambda, l, \tilde{l}$.

8. T-duality invariance

8.1. Transformation laws for $Z_{B+F}(\mathcal{F}, \tau, \rho)$

Now we study the transformation laws for

$$Z_{B+F}(\mathcal{F}, \tau, \rho) = Det_B Det'_F e^{-S_B(\mathcal{F})}\widehat{\Theta}(\mathcal{F}, \rho) \tag{8.1}$$

where $\widehat{\Theta}(\mathcal{F}, \rho)$ is defined in (7.43), while Det_B and Det'_F are defined in (6.26) and (7.36),(7.38) respectively. We also recall that $S_B(\mathcal{F})$ is the real part of the classical action evaluated on the background field configuration.

[27] Here we use the fact that the 10D fermions are Majorana fermions in Minkowski signature.

First, we note that $Z_{B+F}(\mathcal{F},\tau,\rho)$ is invariant under $SL(2,Z)_\tau$. Second, we learn how $Z_{B+F}(\mathcal{F},\tau,\rho)$ transforms under $SL(2,Z)_\rho$ by using the transformation rules of Det_B (6.30) and $\widehat{\Theta}(\mathcal{F},\rho)$ (7.45),(7.46). We find:

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=s_B s_F\mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}Z_{B+F}(\mathcal{F},\tau,\rho) \tag{8.2}$$

$$Z_{B+F}(T\cdot\mathcal{F},\tau,\rho+1)=\mu(T)Z_{B+F}(\mathcal{F},\tau,\rho) \tag{8.3}$$

where s_B is taken from the transformation of D_B.

Now, using the definition of χ and σ

$$\frac{1}{2}(b_0-b_1+b_2-b_3+b_4^\pm)=\frac{1}{4}(\chi\pm\sigma),\quad \sigma=b_4^+-b_4^- \tag{8.4}$$

as well as the index theorem:

$$I(\mathcal{M})+\int_X\lambda^2=\int_X 248\hat{A}_8$$

we obtain the final result for the transformation under the generator S

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=(-i\rho)^{\frac{1}{4}\chi+\frac{1}{8}\int_X(p_2-\lambda^2)}(i\bar{\rho})^{\frac{1}{4}\chi-\frac{1}{8}\int_X(p_2-\lambda^2)}Z_{B+F}(\mathcal{F},\tau,\rho) \tag{8.5}$$

From (8.3) and (8.5) we find that there is a T-duality anomaly.

Let us note in passing that the transformations (8.3),(8.5) are consistent for any 8-dimensional spin manifold. This can be seen by computing [28]

$$Z_{B+F}\big((ST)^6\cdot\mathcal{F},\tau,\rho\big)=e^{i\frac{\pi}{4}\int_X(7\lambda^2-p_2)}Z_{B+F}\big(\mathcal{F},\tau,\rho\big) \tag{8.6}$$

$$Z_{B+F}\big(S^4\cdot\mathcal{F},\tau,\rho\big)=Z_{B+F}\big(\mathcal{F},\tau,\rho\big)$$

and then noting that the index theorem for 8-dimensional spin manifolds implies

$$\int_X(7\lambda^2-p_2)\in 1440\mathbf{Z}. \tag{8.7}$$

Incidentally, when X admits a nowhere-vanishing Majorana spinor of $\pm$ chirality the Euler characteristic is given by [35]:

$$\chi=\pm\frac{1}{2}\int_X(p_2-\lambda^2) \tag{8.8}$$

and the transformation rule (8.5) simplifies to:

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=(-i\rho)^{\frac{1}{2}\chi}Z_{B+F}(\mathcal{F},\tau,\rho) \tag{8.9}$$

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=(i\bar{\rho})^{\frac{1}{2}\chi}Z_{B+F}(\mathcal{F},\tau,\rho) \tag{8.10}$$

for positive and negative chirality, respectively.

[28] The branches for the $8-th$ roots of unity are chosen in such a way that $S^2=(-)^{F_R}$, where F_R is a space-time fermion number in right-moving sector of type IIA string

8.2. Including quantum corrections

Now we recall that there is a 1-loop correction to the effective $8D$ action:

$$S_{1-loop} = \pi \mathrm{Im}\rho \int_X (u\lambda^2 + vp_2) + \frac{i\pi}{24}\mathrm{Re}\rho \int_X (p_2 - \lambda^2) \tag{8.11}$$

where we recall that $\pi \mathrm{Im}\rho \int_X (u\lambda^2 + vp_2)$ comes from the regularization of $q(\mathrm{Im}\rho)$ in(6.5) and the numbers u and v depend on the regularization.

We now demonstrate that to construct a T-duality invariant partition function this term should be replaced with

$$S_{quant} = \left[\frac{1}{2}\chi + \frac{1}{4}\int_X (p_2 - \lambda^2)\right] log\left[\eta(\rho)\right] + \left[\frac{1}{2}\chi - \frac{1}{4}\int_X (p_2 - \lambda^2)\right] log\left[\eta(-\bar{\rho})\right] \tag{8.12}$$

where $\eta(\rho)$ is Dedekind function. Taking the limit $Im\rho \to \infty$ one can uniquely determine $u = -\frac{1}{24}$ and $v = \frac{1}{24}$ in (8.11).

η has the following transformation laws:

$$\eta\left(-1/\rho\right) = (-i\rho)^{\frac{1}{2}}\eta(\rho), \quad \eta(\rho + 1) = e^{\frac{\pi i}{12}}\eta(\rho) \tag{8.13}$$

so that $e^{-S_{quant}}$ transforms as

$$e^{-S_{quant}}\left(-1/\rho\right) = (-i\rho)^{-\frac{1}{4}\chi - \frac{1}{8}\int_X (p_2 - \lambda^2)} (i\bar{\rho})^{-\frac{1}{4}\chi + \frac{1}{8}\int_X (p_2 - \lambda^2)} e^{-S_{quant}}\left(\rho\right) \tag{8.14}$$

$$e^{-S_{quant}}\left(\rho + 1\right) = e^{-i\frac{\pi}{24}\int_X (p_2 - \lambda^2)} e^{-S_{quant}}\left(\rho\right) \tag{8.15}$$

Finally, we find that the total partition function

$$Z(\mathcal{F}, \rho) := e^{-S_{quant}} Z_{B+F}(\mathcal{F}, \rho) \tag{8.16}$$

is invariant:

$$Z\big(T \cdot \mathcal{F}, \rho + 1\big) = Z\big(\mathcal{F}, \rho\big), \tag{8.17}$$

$$Z\big(S \cdot \mathcal{F}, -1/\rho\big) = Z\big(\mathcal{F}, \tau, \rho\big). \tag{8.18}$$

This is our main result.

As a consistency check consider(for simplicity) the case when X admits a nowehere-vanishing spinor of positive chirality and take the limit $Im\rho = V \to \infty$

$$S_{quant} \to \left(\frac{i\pi}{12}\rho + \sum_{n\geq 1}\sum_{m\geq 1}\frac{1}{m}e^{2\pi i n m \rho}\right)\chi. \tag{8.19}$$

We recognize the multiple cover formula for world-sheet instantons on T^2 from [18].

9. Application: Hull's proposal for interpreting the Romans mass in the framework of M-theory

As a by-product of the above results we will make some comments on an interesting open problem concerning the relation of M-theory to IIA string theory.

It is well known that IIA supergravity admits a massive deformation, leading to the Romans theory. The proper interpretation of this massive deformation in 11-dimensional terms is an intriguing open problem. In [9] C. Hull suggested an 11-dimensional interpretation of certain backgrounds in the Romans theory. His interpretation involved T-duality in an essential way, and in the light of the above discussion we will make some comments on his proposal. (For a quite different proposal for interpreting this massive deformation see [36].)

9.1. Review of the relation of M-theory to IIA supergravity

Naive Kaluza-Klein reduction says that for an appropriate transformation of fields $\{g_{M-theory}, C_{M-theory}\} \to \{g_{IIA}, H_{IIA}, \phi_{IIA}, C_{IIA}\}$ we have

$$S_{M-theory} = S_{IIA} \tag{9.1}$$

One of the main points of [7] was that, in the presence of topologically nontrivial fluxes equation (9.1) is not true! Indeed, given our current understanding of these fields, there is not even a 1-1 correspondence between classical M-theory field configurations and classical IIA field configurations. Rather, certain *sums* of IIA-theoretic field configurations were asserted to be equal to certain sums of M-theoretic field configurations. In this sense, the equivalence of type IIA string theory to M-theory on a circle fibration is a quantum equivalence.

To be more precise, in [7] it was shown that for product manifolds $Y = X_{10} \times S^1$, the sum over K-theory lifts $x(\hat{a})$ of a class $\hat{a} \in H^4(X_{10}; \mathbf{Z})$ is proportional to the sum over torsion shifts of the M-theory 4-form of Y. We have:

$$\frac{N(-)^{\mathrm{Arf}(q)+f(\hat{a}_0)}}{\sqrt{N_2 N_K}} \sum_{x(\hat{a})} e^{-S_{IIA}} = exp\left(-||G_{M-theory}(\hat{a})||^2\right) \sum_{\hat{c} \in H^4_{tors}(X_{10}, \mathbf{Z})} (-1)^{f(\hat{a}+\hat{c})} \tag{9.2}$$

The above formula is the main technical result of [7]. We recall that $[G_{M-theory}(\hat{a})] = 2\pi\left(\hat{a} - \frac{1}{2}\lambda\right)$ and the equivalence class of $\hat{a}$ is defined to contain M-theory field configurations with fixed kinetic energy

$$||G_{M-theory}(\hat{a})||^2 = \frac{1}{4\pi} \int_{X_{10}} G_{M-theory}(\hat{a}) \wedge \hat{*} G_{M-theory}(\hat{a}),$$

from which follows that these fields are characterized by $\hat{a}' = \hat{a} + \hat{c}, \quad \hat{c} \in H^4_{tors}(X_{10}, \mathbf{Z})$. Also, in (9.2) N_K and N is the order of $K^0_{tors}(X_{10})$ and $H^4_{tors}(X_{10}; \mathbf{Z})$ respectively, N_2 stands for the number of elements in the quotient $L'' = L/L'$, where $L = H^4_{tors}(X_{10}; \mathbf{Z})/2H^4_{tors}(X_{10}; \mathbf{Z})$ and $L' = \left\{\hat{c} \in L, \quad Sq^3\hat{c} = 0\right\}$. Finally, $\mathrm{Arf}(q)$ is the Arf invariant of the quadratic form $q(\hat{c}) = f(\hat{c}) + \int_{X_{10}} \hat{c} \cup Sq^2\hat{a}_0$ on L''. The identity (9.2) extends to the case where Y is a nontrivial circle bundle over X_{10} [7].

As we have mentioned, we interpret the fact that we must sum over field configurations in (9.2) as a statement that IIA-theory on X_{10} and M-theory on $Y = X_{10} \times S^1$ are really only quantum-equivalent. This point might seem somewhat tenuous, relying, as it does, on the fact that the torsion groups in cohomology and K-theory are generally different. Nevertheless, as we will now show, a precise version of Hull's proposal again requires equating sums over IIA and M-theory field configurations. In this case, however, the sums are over non-torsion cohomology classes, and in this sense the claim that IIA-theory and M-theory are only quantum equivalent becomes somewhat more dramatic.

9.2. Review of Hull's proposal

One version of Hull's proposal states that massive IIA string theory on $T^2 \times X$ is equivalent to M-theory on a certain 3-manifold which is a nontrivial circle bundle over a torus. The proposal is based on T-duality invariance, which allows one to transform away G_0 at the expense of introducing G_2 along the torus, combined with the interpretation of G_2 flux as the first Chern class of a nontrivial M-theory circle bundle [7]. We now describe this in more detail.

Hull's proposal is based on the result [12] that dimensional reduction of massive IIA supergravity with mass m on a circle of radius R, (denoted S^1_R), gives the same theory as Scherk-Schwarz reduction of IIB supergravity on $S^1_{1/R}$. The IIB fields are twisted by

$$g(\theta) = \begin{pmatrix} 1 & m\theta \\ 0 & 1 \end{pmatrix} \tag{9.3}$$

where the coordinate on $S^1_{1/R}$ is $z = \frac{2\pi}{R}\theta, \quad \theta \in [0, 1]$ and the monodromy is

$$g(1)g(0)^{-1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbf{Z}) \tag{9.4}$$

Schematically:

$$\frac{IIA_m}{S^1_R \times X_9} = \left(\frac{IIB}{S^1_{1/R} \times X_9}\right)_{g(\theta)} \tag{9.5}$$

where X_9 is an arbitrary 9-manifold. Note, in particular, that the twist acts on the IIB axiodil $\tau_B = C_0 + ie^{-\phi_B}$ as

$$\tau_B(\theta) = \tau_B(0) + m\theta \tag{9.6}$$

which implies that the IIB RR field G_1 has a nonzero period.

Let us also recall the duality between IIB on a circle and M-theory on T^2:

$$\frac{IIB}{S^1_{R'} \times S^1_{1/R} \times X} = \frac{M}{T^2(\tau_M, A_M) \times S^1_{1/R} \times X} \tag{9.7}$$

where the $T^2(\tau_M, A_M)$ on the M-theory side has complex structure $\tau_M = \tau_B(0)$ and area $A_M = e^{\frac{\phi_B}{3}}(R')^{-\frac{4}{3}}$.

Now, invoking the adiabatic argument we have:

$$\left(\frac{IIB}{S^1_{1/R} \times S^1_{R'} \times X}\right)_{g(\theta)} = \frac{M}{B(m; R', R) \times X} \tag{9.8}$$

where $B(m; R', R)$ is a 3-manifold with metric:

$$ds^2 = \left(\frac{2\pi}{R}\right)^2 (d\theta)^2 + A_M \left[\frac{1}{\mathrm{Im}\tau_M}\left(dx + (\mathrm{Re}\tau_M + m\theta)dy\right)^2 + \mathrm{Im}\tau_M dy^2\right] \tag{9.9}$$

where x, y are periodic $x \sim x + 1$ and $y \sim y + 1$. [29]

Combining (9.5) with (9.8) we get the basic statement of Hull's proposal:

$$\frac{IIA_m}{S^1_R \times S^1_{R'} \times X} = \frac{M}{B(m; R', R) \times X} \tag{9.10}$$

We can now see the connection between Hull's proposal and T-duality. A duality transformation exchanges G_0 for a flux of G_2 through the torus. Then we can interpret the nontrivial flux G_2 as the first chern class of a line bundle in the M-theory setting.

9.3. A modified proposal

In view of what we have discussed in the present paper, the equivalence of classical actions - when proper account is taken of the various phases of the supergravity action - cannot be true. This is reflected, for example, in the asymmetry of the phase (5.12) in

[29] It is not entirely obvious that the invocation is justified, since for a large M-theory torus the twist is carried out over a small radius on the IIB side.

exchanging n_0 for n_1. However, we follow the lead of (9.2) and therefore modify Hull's proposal by identifying sums over certain geometries on the IIA and M-theory side. [30]

A modified proposal identifies $Z(\mathcal{F},\rho,\tau)$ defined in (8.1),(8.16) with a sum over M-theory geometries as follows. Recall first that in the 8D theory there is a doublet of zeroforms $g^{\alpha}_{(0)}$, arising from G_0 and G_2. Next, let us factor $g_{(0)} = \ell \begin{pmatrix} p \\ q \end{pmatrix}$ where p, q are relatively prime integers and ℓ is an integer. Then we take a matrix $\mathcal{N} \in SL(2,\mathbf{Z})_\rho$

$$\mathcal{N} = \begin{pmatrix} r & -s \\ -q & p \end{pmatrix} \qquad rp - sq = 1 \tag{9.11}$$

such that

$$\mathcal{N} g_{(0)} = \begin{pmatrix} \ell \\ 0 \end{pmatrix} \tag{9.12}$$

This is the T-duality transformation that eliminates Romans flux.

Now, thanks to the invariance of $Z(\mathcal{F},\tau,\rho)$ under T-duality transformations (see (8.17)(8.18) above) we find:

$$Z(\mathcal{F},\rho) = Z\left(\mathcal{N}\cdot\mathcal{F}, \frac{p\rho+s}{q\rho+r}\right) \tag{9.13}$$

By the results of [7] the right hand side of (9.13), having $G_0 = 0$, *does* have an interpretation as a sum over M-theory geometries. The M-theory geometry is indeed a circle bundle over $T^2 \times X$ defined by $c_1 = \ell e_0 + pe - qe'' + \gamma_m d\sigma^m$ (as in Hull's proposal), but in addition it is necessary to sum over E_8 bundles on the 11-manifold $B \times X$. While it is essential to sum over $g_{(4)}$, all other fluxes $\mathcal{F}$ may be treated as classical - that is, they may be fixed and it is not necessary to sum over them.

Both sides of (9.13) should be regarded as wavefunctions in the quantization of self-dual fields. For this reason we propose that there is only an intrinsically *quantum mechanical* equivalence between IIA theory and M-theory in the presence of G_0.

[30] In making these statements we are including the K-theoretic phase as part of the "classical" action. Since the phase is formally at 1-loop order it is possible that one could associate it with a 1-loop effect in such a way that classical equivalence does hold.

10. Comments on the U-duality invariant partition function

The present paper has been based on weakly coupled string theory. However, our motivation was understanding the relationship between K-theory and U-duality. In generalizing our considerations to the full U-duality group $\mathcal{D} = SL(3,\mathbf{Z}) \times SL(2,\mathbf{Z})_\rho$ of toroidally compactified IIA theory it is necessary to go beyond the weak coupling expansion. Thus, it is appropriate to start with the M-theory formulation. In the present section we make a few remarks on the U-duality of the M-theory partition function and its relation to the K-theory partition functions of type IIA strings. In particular, we will address the following points:

a.) The invariance of the M-theory partition function under the nongeometrical $SL(2,Z)_\rho$ is not obvious and appears to require surprising properties of η invariants. In section 10.2 we state this open problem in precise terms.

b.) We will sketch how one can recover "twisted K-theory theta functions," at weak coupling cusps when the H-flux is nonzero in section 10.3.

We believe that one can clarify the relation between K-theory and U-duality by studying the behavior of the M-theory partition function at different cusps of the M-theory moduli space. At a given cusp the summation over fluxes is supported on fluxes which can be related to K-theory. (See, for example, (9.2).) A U-duality invariant formulation of the theory must map the equations defining the support at one cusp to those at any other cusp. This should define the U-duality invariant extension of the K-theory constraints.

10.1. The M-theory partition function

Let us consider the contribution to the M-theory partition function from a background Y which is a T^3 fibration over X.

$$ds_{11}^2 = V^{-\frac{1}{3}} \tilde{t}^2 g_{MN} dx^M dx^N + V^{\frac{2}{3}} \tilde{g}_{\mathbf{mn}} \theta^{\mathbf{m}} \theta^{\mathbf{n}} \tag{10.1}$$

where $\theta^{\mathbf{m}} = dx^{\mathbf{m}} + \mathcal{A}^{\mathbf{m}}_{(1)}$ and $x^{\mathbf{m}} \in [0,1]$. $\tilde{t}^2 g_{MN}$ is an 8D Einstein metric with $det g_{MN} = 1$. $\tilde{g}_{\mathbf{mn}}$ and V are the shape and the volume of the T^3 fiber. We denote world indices on T^3 by $\mathbf{m} = (m, 11), m = 8, 9$ and $M = 0, \ldots, 7$ as before.

Topologically, one can specify the T^3 fibration over X by a triplet of line bundles $L^{\mathbf{m}}$ which transform in the representation $\mathbf{3}$ of $SL(3,\mathbf{Z})$ and have first Chern classes $c_1(L^{\mathbf{m}}) = \mathcal{F}^{\mathbf{m}}_{(2)}$, where $\mathcal{F}^{\mathbf{m}}_{(2)} = d\theta^{\mathbf{m}}$. Such a specification is valid up to possible monodromies. These are characterized by a homomorphism $\pi_1(X) \to SL(3,\mathbf{Z})$.

On a manifold Y of the type (10.1) we reduce the M-theory 4-form $G_{M-theory}$ as

$$\frac{G_{M-theory}}{2\pi} = G_{(4)} + G_{(3)\mathbf{m}}\theta^{\mathbf{m}} + \frac{1}{2}\left(F_{(2)\mathbf{mn}} + \varepsilon_{\mathbf{mnk}}B_0\mathcal{F}^{\mathbf{k}}_{(2)}\right)\theta^{\mathbf{m}}\theta^{\mathbf{n}} \tag{10.2}$$

We also include the flat potential

$$c_{(0)} = \frac{1}{6}B_0\varepsilon_{\mathbf{mnk}}\theta^{\mathbf{m}}\theta^{\mathbf{n}}\theta^{\mathbf{k}} \tag{10.3}$$

in the Kaluza-Klein reduction.[31] (We will list the full set of flat potentials in this background below.)

From the Bianchi identity $dG_{M-theory} = 0$ we have

$$dG_{(4)} = \mathcal{F}^{\mathbf{m}}_{(2)}G_{(3)\mathbf{m}}, \quad dG_{(3)\mathbf{m}} = \mathcal{F}^{\mathbf{n}}_{(2)}F_{(2)\mathbf{mn}} \quad dF_{(2)\mathbf{mn}} = 0 \quad d\mathcal{F}^{\mathbf{m}}_{(2)} = 0 \tag{10.4}$$

which implies that fluxes $G_{(4)}$ and $G_{(3)\mathbf{m}}$ are in general not closed forms.[32]

Let us recall how the various fields transform under $\mathcal{D} = SL(3,\mathbf{Z}) \times SL(2,\mathbf{Z})_\rho$ [14].

- $\tilde{t}, g_{MN}$ are U-duality invariant.
- $SL(2,\mathbf{Z})_\rho$ acts on $\rho = B_0 + iV \in \mathcal{H}$ by fractional linear transformations.
- $SL(3,\mathbf{Z})$ acts on the scalars $\tilde{g}_{\mathbf{mn}}$ parametrizing $SL(3,R)/SO(3)$ via the mapping class group of T^3.
- $\mathbf{F}^{m\alpha}_{(2)} = \begin{pmatrix} F^{\mathbf{m}}_{(2)} \\ \mathcal{F}^{\mathbf{m}}_{(2)} \end{pmatrix}$ transform in the $(\mathbf{3},\mathbf{2})$ of $\mathcal{D}$, where $F^{\mathbf{m}}_{(2)} := \frac{1}{2}\varepsilon^{\mathbf{mnk}}F_{(2)\mathbf{nk}}$.
- $G_{(3)\mathbf{m}}$ transform in the $(\mathbf{3'},\mathbf{1})$ of $\mathcal{D}$
- $G_{(4)}$ is singled out among all the other fields since according to conventional supergravity [14] $SL(2,\mathbf{Z})_\rho$ mixes $G_{(4)}$ with its Hodge dual $*G_{(4)}$. More concretely,

$$\begin{pmatrix} -\mathrm{Re}\rho\ G_{(4)} + i\mathrm{Im}\rho\ *G_{(4)} \\ G_{(4)} \end{pmatrix} \tag{10.5}$$

transforms in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}$. Due to this non-trivial transformation the classical bosonic 8D action is not manifestly invariant under $SL(2,\mathbf{Z})_\rho$. In detail, the action has real part:

$$Re(S_{8D}) = \pi\int_X \left\{ \mathrm{Im}\rho G_{(4)}\wedge *G_{(4)} + \tilde{t}^2\tilde{g}^{\mathbf{mn}}G_{(3)\mathbf{m}}\wedge *G_{(3)\mathbf{n}} + \tilde{t}^4\tilde{g}_{\mathbf{mn}}\mathcal{G}_{\alpha\beta}\mathbf{F}^{\mathbf{m}\alpha}_{(2)}\wedge *\mathbf{F}^{\mathbf{n}\beta}_{(2)} \right\} \tag{10.6}$$

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[31] $\varepsilon^{11,8,9} = \epsilon_{11,8,9} = 1$.

[32] In IIA at weak coupling we assumed $G_{(3)11} = 0$ and $\mathcal{F}^n_{(2)} = 0, n = 8,9$, so that all background fluxes are closed forms.

$$+\frac{1}{2\pi}\int_X \tilde{t}^6\left\{\mathcal{R}+28\tilde{t}^{-2}\partial_M\tilde{t}\partial_M\tilde{t}+\frac{1}{2\rho_2^2}\partial_M\rho\partial^M\bar{\rho}+\frac{1}{3}\tilde{g}^{\mathbf{mn}}\tilde{g}^{\mathbf{kl}}\partial_M\tilde{g}_{\mathbf{mk}}\partial^M\tilde{g}_{\mathbf{nl}}\right\}$$

where $\mathcal{G}_{\alpha\beta}$ is defined in (1.6), $\tilde{g}^{\mathbf{kl}}$ is inverse of $\tilde{g}_{\mathbf{mk}}$ and $\mathcal{R}$ is the Ricci-scalar of the metric g_{MN}.

The imaginary part of the 8D bosonic action follows from the reduction of the M-theory phase $\Omega_M(C)$. This phase is subtle to define in topologically nontrivial field configurations of the G-field. It may be formulated in two ways. The first formulation was given in [37]. It uses Stong's result that the spin-cobordism group $\Omega_{11}(K(\mathbf{Z},4))=0$ [38]. That is, given a spin 11-manifold Y and a 4-form flux $\frac{G}{2\pi}$ one can always find a bounding spin 12-manifold Z and an extension $\tilde{G}$ of the the flux to Z. In these terms the M-theory phase $\Omega_M(C)$ is given as:

$$\Omega_M(C)=\epsilon\exp\left[\frac{2\pi i}{6}\int_Z\tilde{G}^3-\frac{2\pi i}{48}\int_Z\tilde{G}(p_2-\lambda^2)\right] \tag{10.7}$$

Here ϵ is the sign of the Rarita-Schwinger determinant. The phase does not depend on the choice of bounding manifold Z, but does depend on the "trivializing" C-field at the boundary Y.

A second formulation [7,39,40] proceeds from the observation of [37] that the integrand of (10.7) may be identified as the index density for a Dirac operator coupled to an E_8 vector bundle. The M-theory 4-form can be formulated in the following terms [7,39,40]. We set:

$$\frac{G_{M-theory}}{2\pi}=\bar{G}+dc \tag{10.8}$$

where $\bar{G}=\frac{1}{60}Tr_{248}\frac{F^2}{8\pi^2}+\frac{1}{32\pi^2}TrR^2$, F is the curvature of a connection A on an E_8 bundle V on Y and R is the curvature of the metric connection on TY. $G_{M-theory}$ is a real differential form, and $c\in\Omega^3(Y,R)/\Omega^3_Z(Y)$, where $\Omega^3_Z(Y)$ are 3-forms with integral periods. The pair (A,c) is subject to an equivalence relation. In these terms the M-theory phase is expressed as:

$$\Omega_M(C)=exp\left[2\pi i\left(\frac{\eta(D_V)+h(D_V)}{4}+\frac{\eta(D_{RS})+h(D_{RS})}{8}\right)\right]\omega(c) \tag{10.9}$$

where D_V is the Dirac operator coupled to the connection A, D_{RS} is the Rarita-Schwinger operator, $h(D)$ is the number of zeromodes of the operator D on Y, and $\eta(D)$ is the η invariant of Atiyah-Patodi-Singer. The phase $\omega(c)$ is given by

$$\omega(c)=exp\left[\pi i\int_Y\left(c(\bar{G}^2+X_8)+cdc\bar{G}+\frac{1}{3}c(dc)^2\right)\right] \tag{10.10}$$

10.2. The semiclassical expansion

For large $\tilde{t}$ there is a well-defined semiclassical expansion of the M-theory partition function, which follows from the appearance of kinetic terms in the action (10.6) scaling as $\tilde{t}^{2k}$ for $k = 0, 1, 2, 3$. In the leading approximation we can fix all the fields except $G_{(4)}$, but this last field must be treated quantum mechanically. Note that this semiclassical expansion can differ from that described in the previous sections because we do not necessarily require weak string coupling. In the second approximation we treat $G_{(4)}$ and $G_{(3)\mathbf{n}}$ as quantum fields, and so on.

In the leading approximation in addition to the sum over fluxes $G_{(4)}$ we must integrate over the flat potentials. These include flat connection $\mathcal{A}^{\mathbf{m}}_{(1)}$ of the T^3 fibration and potentials coming from the KK reduction of c

$$c = C'_{(3)} + C'_{(2)\mathbf{m}}\theta^{\mathbf{m}} + \frac{1}{2}C_{(1)\mathbf{mn}}\theta^{\mathbf{m}}\theta^{\mathbf{n}} + c_{(0)} \tag{10.11}$$

where $C'_{(2)\mathbf{m}} = C_{(2)\mathbf{m}} - \frac{1}{2}C_{(1)\mathbf{pm}}\mathcal{A}^{\mathbf{p}}_{(1)}$ and $C'_{(3)} = C_{(3)} - C'_{(2)\mathbf{m}}\mathcal{A}^{\mathbf{m}}_{(1)}$, and $c_{(0)}$ is defined in (10.3). $C_{(3)}$ is invariant under U-duality, $C_{(2)\mathbf{m}}$ transforms in the $(\mathbf{3}, \mathbf{1})$ of $\mathcal{D}$. We can combine the flat potentials $C_{(1)\mathbf{mn}}$ and $\mathcal{A}^{\mathbf{m}}_{(1)}$ in a U-duality multiplet of $\mathcal{D}$ transforming as $(\mathbf{3}, \mathbf{2})$ by writing

$$\mathbf{A}^{\mathbf{m}\alpha}_{(1)} = \begin{pmatrix} \frac{1}{2}\varepsilon^{\mathbf{mnk}}C_{(1)\mathbf{nk}} \\ \mathcal{A}^{\mathbf{m}}_{(1)} \end{pmatrix}. \tag{10.12}$$

The duality invariance in the leading approximation is straightforward to check. We keep only $G_{(4)}$. The flux is quantized by $[G_{(4)}] = a - \frac{1}{2}\lambda$, where $a \in H^4(X, \mathbf{Z})$ is the characteristic class of the E_8 bundle and λ is the characteristic class of the spin bundle. We sum over $a \in H^4(X, \mathbf{Z})$. The 8D action, including the imaginary part is $SL(3, \mathbf{Z})$ invariant. The imaginary part of the 8D effective action in this case takes a simple form which can be found from (10.9):

$$Im(S_{8D}) = -\pi \int_X \left(a \cup \lambda + B_0 \left(a - \frac{1}{2}\lambda \right)^2 \right) \tag{10.13}$$

The invariance under $SL(2, \mathbf{Z})_\rho$ then follows in the same way as in our discussion in the weak string coupling regime.

Let us now try to go beyond the first approximation. In the second approximation $[G_{(4)}] = a - \frac{1}{2}\lambda + [\mathcal{A}^{\mathbf{m}}_{(1)}G_{(3)\mathbf{m}}]$. We allow nonzero fluxes $G_{(3)\mathbf{m}}$, but still set to zero the fieldstrengths $F_{(2)}$ and $\mathcal{F}_{(2)}$. We thus have a family of tori with flat connections. Already in

the second approximation, when we switch on nonzero fluxes $G_{(3)\mathbf{m}}$ there does not appear to be a simple expression for the M-theory phase.

Nevertheless, one can get some information about the M-theory phase from the requirement of U-duality invariance. We know that $SL(3,\mathbf{Z})$ invariance is again manifest from the definition of $\Omega_M(C)$ and $\mathrm{Re}(S_{8D})$. But the expected $SL(2,\mathbf{Z})_\rho$ invariance is nontrivial. We would simply like to state this precisely. To do that we write M-theory partition function in the second approximation as

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}},\rho) := \int d\mu_{flat} \sum_{G_{(3)\mathbf{m}}} Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, G_{(3)\mathbf{m}}, \rho) \tag{10.14}$$

where $Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, G_{(3)\mathbf{m}}, \rho)$ is the partition function with fixed, but nonzero, flux, $G_{(3)\mathbf{m}}$, $d\mu_{flat}$ stands for the integration over

$$\frac{\mathcal{H}^3(X)}{\mathcal{H}^3_{\mathbf{Z}}(X)} \times \left(\frac{\mathcal{H}^2(X)}{\mathcal{H}^2_{\mathbf{Z}}(X)}\right)^3 \times \left(\frac{\mathcal{H}^1(X)}{\mathcal{H}^1_{\mathbf{Z}}(X)}\right)^6, \tag{10.15}$$

where $\mathcal{H}^p(X)$ is a space of harmonic p-forms on X and $\mathcal{H}^p_{\mathbf{Z}}(X)$ is the lattice of integrally normalized harmonic p-forms on X. The first factor is for $C_{(3)}$, the second factor for $C_{(2)\mathbf{m}}$ and the third factor is for the fields $\mathbf{A}^{\mathbf{m}\alpha}_{(1)}$ transforming in the $(\mathbf{3},\mathbf{2})$ of $\mathcal{D}$. The integration measure $d\mu_{flat}$ is U-duality invariant.

The summand in (10.14) with fixed $G_{(3)\mathbf{m}}$ is given by

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, G_{(3)\mathbf{m}}, \rho) = \sum_{a\in H^4(X,\mathbf{Z})} Det(G_{(4)}, G_{(3)\mathbf{m}}) e^{-S_{quant}} e^{-S_{cl}} \tag{10.16}$$

where

$$e^{-S_{cl}} = \Omega_M\left(G_{(4)}, G_{(3)\mathbf{m}}, B_0\right) e^{-\pi \int_X \left(Im(\rho) G_{(4)}\wedge *G_{(4)} + \tilde{t}^2 \tilde{g}^{\mathbf{mn}} G_{(3)\mathbf{m}}\wedge *G_{(3)\mathbf{n}} \right)}$$

and $Det(G_{(4)}, G_{(3)\mathbf{m}})$ denotes 1-loop determinants. These depend implicitly on the scalars $\rho, \tilde{g}_{\mathbf{mn}}, \tilde{t}$ as well as on the metric g_{MN}. We include 1-loop corrections in S_{quant} (see below).

The M-theory phase Ω_M in (10.16) depends on the fieldstrenths $G_{(4)}, G_{(3)\mathbf{m}}$ and the flat potentials, but it is metric-independent, and hence should be a topological invariant. The dependence of Ω_M on flat potentials is explicit from (10.10) for c as in (10.11). For example dependence of Ω_M on B_0 has the form

$$e^{i\pi \int_X B_0 G_{(4)} G_{(4)}} \tag{10.17}$$

It is conveneient to include 1-loop corrections $\int_X B_0 X_8$ together with effect of membrane instantons in S_{quant}. The nontrivial question is dependence on $G_{(4)}$ and $G_{(3)\mathbf{m}}$ which also comes from $\eta(D_V) + h(D_V)$.

The independence of Ω_M on the metric on $Y = X \times T^3$ (in the second approximation) follows from the standard variation formula for η-invariant. To show this let us fix the connection on the E_8 bundle V with curvature F and consider the family of veilbeins $e(s)$ on $Y = X \times T^3$ parametrized by $s \in [0,1]$ such that the metric on T^3 remains flat and independent of the coordinates on X. The corresponding family of Rieman tensors $\mathcal{R}(s)$ gives an A-roof genus $\hat{A}(s)$ which is a pullback from $X \times [0,1]$. Now we can write the standard formula for the change in η-invariant under the variation of veilbein [41]:

$$\eta(e(1)) - \eta(e(0)) = j + \int_{Y\times[0,1]} ch(V)\hat{A}(s) \tag{10.18}$$

where integer j is a topological invariant of $Y \times [0,1]$ and $ch(V) := \frac{1}{30}[Tr_{248} e^{\frac{iF}{2\pi}}]$. In the second approximation we only switch on $\bar{G} = G_{(4)} + G_{(3)\mathbf{m}} dx^m$ so that neither $ch_2(V) = -2(\bar{G} + \frac{1}{2}\lambda)$ nor $ch_4(V) = \frac{1}{5}(\bar{G} + \frac{1}{2}\lambda)^2$ have a piece $\sim dx^8 dx^9 dx^{11}$ and integral in (10.18) vanishes.

Now we come to the main point. The requirement of the invariance under the standard generators S, T of $SL(2,\mathbf{Z})_\rho$

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, -1/\rho) = Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho) \tag{10.19}$$

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho + 1) = Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho) \tag{10.20}$$

gives a nontrivial statement about the properties of the function $\Omega_M(G_{(4)}, G_{(3)\mathbf{m}}, B_0)$.

The sum over fluxes $G_{(3)\mathbf{m}} \in H^3(X,\mathbf{Z})$ in (10.14) might be entirely supported by classes which satisfy a system of $SL(3,\mathbf{Z})$ invariant constraints. These constraints can in principle be determined by summing over torsion classes once the phase Ω_M is known in sufficently explicit terms. In the simple case when $G_{(3)\mathbf{m}}$ are all 2-torsion classes, one can act by the generators of $SL(3,\mathbf{Z})$ on the constraint

$$Sq^3(G_{(3)9}) + Sq^3(G_{(3)11}) + G_{(3)9} \cup G_{(3)11} = 0 \tag{10.21}$$

which follows from [7]. If we assume that this constraint is part of $SL(3,\mathbf{Z})$ invariant system of constraints then we find

$$G_{(3)\mathbf{m}} \cup G_{(3)\mathbf{n}} = 0, \quad \mathbf{m},\mathbf{n} = 8,9,11 \tag{10.22}$$

10.3. Comment on the connection with twisted K-theory

In this section we discuss the behavior of the partition function near a weak-coupling cusp. There is a twisted version of K-theory which is thought to be related to the classification of D-brane charges in the presence of nonzero NSNS H-flux [2,42,43,44]. It is natural to ask if the contributions to the M-theory partition function $Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho)$ from fluxes with nonzero $H_{(3)} := G_{(3)11} \in H^3(X, \mathbf{Z})$ are related, in the weak string-coupling cusp, to some kind of twisted K-theory theta function.

The weak-coupling cusp may be described by relating the fields in (10.1) to the fields in IIA theory. First, the scale $\tilde{t}$ is related to the expansion parameter used in our previous sections by $\tilde{t}^2 = e^{-\frac{2}{3}\xi} t^2$. Next, we parametrize the shape of T^3 as $\tilde{g}_{\mathbf{mn}} = e^{\mathbf{a}}{}_{\mathbf{m}} e^{\mathbf{b}}{}_{\mathbf{n}} \delta_{\mathbf{ab}}$ where

$$e^{\mathbf{a}}{}_{\mathbf{m}} = \begin{pmatrix} \frac{e^{-\xi/3}}{\sqrt{\tau_2}} & 0 & 0 \\ 0 & e^{-\xi/3}\sqrt{\tau_2} & 0 \\ 0 & 0 & e^{2\xi/3} \end{pmatrix} \begin{pmatrix} 1 & \tau_1 & C_{(0)8} \\ 0 & 1 & C_{(0)9} \\ 0 & 0 & 1 \end{pmatrix} \tag{10.23}$$

We denote frame indices by $\mathbf{a} = (a, 11), a = 8, 9$. The weak coupling cusp may be written as

$$R \times R^2 \times SL(2,R)/SO(2) \tag{10.24}$$

where the first factor is for the dilaton ξ, the second for $C_{(0)8}, C_{(0)9}$,[33] and the third for the modular parameter τ of the IIA torus.

As far as we know, nobody has precisely defined what should be meant by the " K_H theta function." Since the Chern character has recently been formulated in [22][23], this should be possible. Nevertheless, even without a precise definition we do expect it to be a sum over a "Lagrangian" sublattice of $K_H(X \times T^2)$. At the level of DeRham cohomology, this should be a "maximal Lagrangian" sublattice of $\ker d_3/\mathrm{Im} d_3$ where $d_3 : H^*(X_{10}, \mathbf{Z}) \to H^*(X_{10}, \mathbf{Z})$ is the differential $d_3(\omega) = \omega \wedge [H_{(3)}]$. Using the filtration implied by the semiclassical expansion, and working to the approximation of $e^{-\tilde{t}^2}$ this means that we should first define a sublattice of the cohomology lattice by the set of integral cohomology classes $(a, G_{(3)8}, G_{(3)9})$ such that $(G_{(4)}, G_{(3)8}, G_{(3)9})$ are in the kernel of d_3:

$$H_{(3)} \wedge G_{(4)} = 0, \quad H_{(3)} \wedge G_{(3)m} = 0, \quad m = 8, 9 \tag{10.25}$$

[33] These are related to the RR potentials $\tilde{C}_{(0)m}$ transforming in the $\mathbf{2}'$ of $SL(2,\mathbf{Z})_\tau$ as $C_{(0)8} = e^{\xi}\sqrt{\tau_2}\tilde{C}_{(0)8}, \quad C_{(0)9} = e^{\xi}\frac{1}{\sqrt{\tau_2}}\tilde{C}_{(0)9}$

Then the theta function should be a sum over the quotient lattice obtained by modding out by the image of d_3

$$G_{(3)8} \sim G_{(3)8} - pH_{(3)}, \quad G_{(3)9} \sim G_{(3)9} - sH_{(3)}, \quad G_{(4)} \sim G_{(4)} - \omega_{(1)}H_{(3)}. \tag{10.26}$$

Here $p, s \in \mathbf{Z}$ and $\omega_{(1)} \in H^1(X, \mathbf{Z})$. Thus, our exercise is to describe how a sum over this quotient lattice emerges from (10.14).

Let us consider the couplings of flat potentials $C_{(1)89}$ and $C_{(2)m}$ to the fluxes which follow from (10.10):

$$e^{2\pi i \int_X C_{(1)89} H_{(3)} G_{(4)}} e^{2\pi i \int_X \epsilon^{mn} C_{(2)m} G_{(3)n} H_{(3)}} \tag{10.27}$$

Integrating over $C_{(1)89}$ and $C_{(2)m}$ gives $H_{(3)} \wedge G_{(4)} = 0$ and $\epsilon^{mn} H_{(3)} \wedge G_{(3)n} = 0$ respectively.

Next, we note that, due to the $SL(3, \mathbf{Z})$ invariance of the M-theory action we have (suppressing many irrelevant variables)

$$Z_{M-theory}(C_{(0)m}, G_{(3)m} - p_m H_{(3)}, \mathcal{A}^{11}_{(1)}, G_{(4)} - \omega_{(1)} H_{(3)}) = \tag{10.28}$$

$$Z_{M-theory}(C_{(0)m} + p_m, G_{(3)m}, \mathcal{A}^{11}_{(1)} + \omega_{(1)}, G_{(4)})$$

Now we use (10.28) to write the sum over all fluxes $G_{(4)}, G_{(3)m}, \quad m = 8, 9$ in the kernel of d_3 as

$$Z_H = \sum_{d_3 - kernel} Z_{M-theory}(C_{(0)m}, G_{(3)m}, \mathcal{A}^{11}_{(1)}, G_{(4)}) = \sum_{\mathcal{M}_{fund}} W \tag{10.29}$$

where $\mathcal{M}_{fund}$ stands for the fluxes in the fundamental domain for the image of d_3 within the kernel of d_3 and

$$W = \sum_{p_m \in \mathbf{Z}^2} \sum_{\omega_{(1)} \in H^1(X, \mathbf{Z})} Z_{M-theory}(C_{(0)m} + p_m, G_{(3)m}, \mathcal{A}^{11}_{(1)} + \omega_{(1)}, G_{(4)}) \tag{10.30}$$

Now, we can recognize that Z_H descends naturally to the quotient of the weak-coupling cusp.

$$\Gamma'_\infty \backslash \Big[R \times R^2 \times SL(2, R)/SO(2) \Big] \tag{10.31}$$

where $\Gamma'_\infty \cong \mathbf{Z}^2$ is the subgroup of the parabolic group Γ_∞ consisting of elements of the form

$$L_{\mathbf{m}}^{\mathbf{n}} = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad p, s \in Z \tag{10.32}$$

Written this way, Z_H is clearly a sum over a Lagrangian sublattice of the $K_H(X \times T^2)$ lattice. (Recall that we are working in the DeRham theory, with the filtration appropriate to the second approximation.)

The interesting point that we learn from this exercise is that in formulating the K_H theta function, the weighting factor for the contribution of a class in K_H should be given by (10.30). The dependence of the action on the integers p_m and $\omega_{(1)} \in H^1(X, \mathbf{Z})$ behaves like $\exp[-Q(p_m, \omega_{(1)})]$ where Q is quadratic form. Therefore W is itself already a theta function. This follows because the dependence on $C_{(0)m}$ and $\mathcal{A}^{11}_{(1)}$ comes entirely from the real part of the classical action (10.6), since, as we have shown, the phase is independent of the metric on $X \times T^3$. The dependence on $C_{(0)m}$ comes from $\int_X \tilde{t}^2 \tilde{g}^{\mathbf{mn}} G_{(3)\mathbf{m}} \wedge * G_{(3)\mathbf{n}}$ and the dependence on $\mathcal{A}^{11}_{(1)}$ from $\int_X Im\rho G_{(4)} \wedge * G_{(4)}$, where we recall that $[G_{(4)}] = a - \frac{1}{2}\lambda + [\mathcal{A}^{\mathbf{m}}_{(1)} G_{(3)\mathbf{m}}]$.

It would be very interesting to see if the function Z_H defined in (10.29) is in accord with a mathematically natural definition of a theta function for twisted K-theory. But we will leave this for future work.

As an example, let us consider $X = SU(3)$. Let x_3 generate $H^3(X, \mathbf{Z})$. Then fixing $H_{(3)} = kx_3$ we find that the fundamental domain of the image of d_3 within the kernel of d_3 is given by

$$G_{(3)8} = rx_3, \quad G_{(3)9} = px_3, \quad 0 \leq r, p \leq k-1 \tag{10.33}$$

so that the sum over RR fluxes in (10.29) is finite and it is in this sense that RR fluxes are k-torsion. This example of $X = SU(3)$ is especially interesting since it is well known [45,46,44] that at weak string coupling D-brane charges on $SU(3)$ in the presence of $H_{(3)} = kx_3$ are classified by twisted K-theory groups of $SU(3)$, and these groups are k-torsion. As argued in [5], from Gauss's law it is then natural to expect that RR fluxes are also k-torsion. This is indeed what we find in (10.33). [34] On the other hand, the M-theory sum is indeed a full sum over all fluxes. This is in harmony with the result of [47] for brane charges. Clearly, there is much more to understand here.

Acknowledgements:

[34] In fact, from [44] we know the order of the torsion group is actually k or $k/2$, according to the parity of k. However, given the crude level at which we are working we do not expect to see that distinction. We expect that a more accurate account of the phases in the partition function will reproduce this result.

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Appendix A. Duality transformations as symplectic transformations

Here we give the explicit expressions for representations of S and T in $Sp(2N, \mathbf{Z})$. Let us choose the following basis of the lattice Γ

$$\vec{\mathbf{x}} = \left(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2\right) \tag{A.1}$$

$$\vec{\mathbf{x}}_1 = \Big(y_l \mathbf{1}, y_l \otimes (L(e_0) - \mathbf{1}), (L(e_s) - \mathbf{1}), (L(e_s) - \mathbf{1}) \otimes (L(e_0) - \mathbf{1}), (L(\gamma_r d\sigma^m) - \mathbf{1}), \tag{A.2}$$
$$x\left(f_k d\sigma^m\right), x\left(\omega_i\right)\Big)$$

$$\vec{\mathbf{x}}_2 = \Big(x(\omega_i) \otimes (L(e_0) - \mathbf{1}), x\left(d_k d\sigma^m\right), x\left(w_r d\sigma^m\right), x(u_s), x(u_s) \otimes (L(e_0) - \mathbf{1}), \tag{A.3}$$
$$x(h_l), x(h_l) \otimes (L(e_0) - \mathbf{1})\Big)$$

where we introduce

$$y_l \in H^0(X, \mathbf{Z}), \quad h_l \in H^8(X, \mathbf{Z}), \quad l = 1, \ldots, b_0$$

$$\gamma_r \in H^1(X, \mathbf{Z}), \quad w_r \in H^7(X, \mathbf{Z}), \quad r = 1, \ldots b_1$$

$$e_s \in H^2(X, \mathbf{Z}), \quad u_s \in H^6(X, \mathbf{Z}) \quad s = 1, \ldots, b_2,$$

$$f_k \in H^3(X, \mathbf{Z}), \quad d_k \in H^5(X, \mathbf{Z}), k = 1, \ldots, \mathbf{b_3}, \quad \omega_i \in H^4(X, \mathbf{Z}), i = 1, \ldots, b_4,$$

where b_p is the rank of $H^p(X, \mathbf{Z})$ and $\mathbf{b_3}$ is the rank of the sublattice of $H^3(X, \mathbf{Z})$ which is span by classes f such that $Sq^3 f = 0$.

In the above basis the generators S and T are represented by

$$\sigma(S) = \begin{pmatrix} A(S) & B(S) \\ C(S) & D(S) \end{pmatrix}, \quad \sigma(T) = \begin{pmatrix} A(T) & B(T) \\ C(T) & D(T) \end{pmatrix} \tag{A.4}$$

$$A(S) = \begin{pmatrix} \mathbf{1}_{b_0} & & & & & \\ -\mathbf{1}_{b_0} & & & & & \\ & & \mathbf{1}_{b_2} & & & \\ & & -\mathbf{1}_{b_2} & & & \\ & & & \mathbf{1}_{2b_1} & & \\ & & & & \mathbf{1}_{2\mathbf{b_3}} & \\ & & & & & \mathbf{0}_{b_4} \end{pmatrix} \quad B(S) = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \mathbf{1}_{b_4} \end{pmatrix} \tag{A.5}$$

$$C(S) = \begin{pmatrix} -\mathbf{1}_{b_4} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \quad D(S) = \begin{pmatrix} \mathbf{0}_{b_4} & & & & & & \\ & \mathbf{1}_{2\mathbf{b_3}} & & & & & \\ & & \mathbf{1}_{2b_1} & & & & \\ & & & \mathbf{1}_{b_2} & & & \\ & & & & -\mathbf{1}_{b_2} & & \\ & & & & & \mathbf{1}_{b_0} & \\ & & & & & & -\mathbf{1}_{b_0} \end{pmatrix} \tag{A.6}$$

$$A(T) = \begin{pmatrix} \mathbf{1}_{b_0} & & & & & & \\ -\mathbf{1}_{b_0} & \mathbf{1}_{b_0} & & & & & \\ & & \mathbf{1}_{b_2} & & & & \\ & & -\mathbf{1}_{b_2} & \mathbf{1}_{b_2} & & & \\ & & & & \mathbf{1}_{2b_1} & & \\ & & & & & \mathbf{1}_{2\mathbf{b_3}} & \\ & & & & & & \mathbf{1}_{b_4} \end{pmatrix} \quad B(T) = \mathbf{0}_N \tag{A.7}$$

$$C(T) = \mathbf{0}_N, \quad D(T) = \begin{pmatrix} \mathbf{1}_{b_4} & & & & & & \\ & \mathbf{1}_{2\mathbf{b_3}} & & & & & \\ & & \mathbf{1}_{2b_1} & & & & \\ & & & \mathbf{1}_{b_2} & & & \\ & & & -\mathbf{1}_{b_2} & \mathbf{1}_{b_2} & & \\ & & & & & \mathbf{1}_{b_0} & \\ & & & & & -\mathbf{1}_{b_0} & \mathbf{1}_{b_0} \end{pmatrix} \tag{A.8}$$

Appendix B. Supergravity conventions

The 10D fields that we use are related to the fields in [19]as:

$$\frac{\tilde{G}_4}{\sqrt{2\pi}} = e^{-\frac{3\phi}{4}} F_4^{Rom}, \quad \frac{G_2}{\sqrt{2\pi}} = -e^{-\frac{9\phi}{4}} F_2^{Rom}, \qquad \frac{\hat{B}_2}{\sqrt{2\pi}} = -e^{\frac{3\phi}{2}} B_2^{Rom}, \quad m = G_0 e^{\frac{15\phi}{4}},$$

$$\hat{\psi}_{\hat{A}} = e^{-\frac{\phi}{8}} \psi_{\hat{A}}^{(Rom)}, \quad \hat{\Lambda} = e^{-\frac{\phi}{8}} \lambda^{(Rom)}, g_{\hat{M}\hat{N}} = e^{\frac{1}{2}\phi} g_{\hat{M}\hat{N}}^{(Rom)}$$

We also remind that we set $k_{11} = \pi$ while in [19] $k_{11} = \sqrt{2\pi}$ was assumed.

Appendix C. 4-Fermion terms

Below we collect 4-fermionic terms in D=10 IIA supergravity action which are obtained from circle reduction of the D=11 action of [48].

$$S^{(10)}_{4-ferm} = \frac{\pi}{2}\int \sqrt{-g_{10}}e^{-2\phi}\Big\{-\frac{1}{64}\Big[\bar\chi_{\mathbf{E}}\hat\Gamma^{\mathbf{ABCDEF}}\chi_{\mathbf{F}} + 12\bar\chi^{[\mathbf{A}}\hat\Gamma^{\mathbf{BC}}\chi^{\mathbf{D}]}\Big]\bar\chi_{[\mathbf{A}}\hat\Gamma_{\mathbf{BC}}\chi_{\mathbf{D}]} \quad \text{(C.1)}$$

$$+\frac{1}{32}\left(\bar\chi_{\mathbf{E}}\hat\Gamma^{\mathbf{ABCEF}}\chi_{\mathbf{F}}\right)\left(\bar\chi_{\mathbf{A}}\hat\Gamma_{\mathbf{B}}\chi_{\mathbf{C}}\right) + \frac{1}{4}\left(\bar\chi_{\mathbf{A}}\hat\Gamma^{\mathbf{A}}\chi_{\mathbf{C}}\right)\left(\bar\chi_{\mathbf{B}}\hat\Gamma^{\mathbf{B}}\chi^{\mathbf{C}}\right)$$

$$-\frac{1}{8}\left(\bar\chi_{\mathbf{A}}\hat\Gamma^{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar\chi_{\mathbf{B}}\hat\Gamma^{\mathbf{A}}\chi^{\mathbf{C}}\right) - \frac{1}{16}\left(\bar\chi_{\mathbf{A}}\hat\Gamma_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar\chi^{\mathbf{A}}\hat\Gamma^{\mathbf{B}}\chi^{\mathbf{C}}\right)\Big\}$$

where

$$\chi_{\hat A} = \Big[\hat\psi_{\hat A} + \frac{1}{6\sqrt{2}}\hat\Gamma_{\hat A}\hat\Lambda\Big]$$

$$\chi_{11} = -\frac{2\sqrt{2}}{3}\hat\Gamma^{11}\hat\Lambda$$

and $\mathbf{A} = (\hat A, 11)$.

Recall that the graviton $\mathbf{E}^{\mathbf{A}}_{\mathbf{M}}$ and the gravitino $\psi^{(11)}_{\mathbf{A}}$ of 11D supergravity are related to 10D fields as [48]:

$$\mathbf{E}^{\hat A}_{\hat M} = e^{-\frac{\phi}{3}}\hat E^{\hat A}_{\hat M}, \quad \mathbf{E}^{11}_{11} = e^{\frac{2\phi}{3}}, \quad \mathbf{E}^{11}_{\hat M} = e^{\frac{2\phi}{3}}C_{\hat M}$$

$$\psi^{(11)}_{\mathbf{A}} = \frac{1}{\sqrt{2\pi}}e^{\frac{\phi}{6}}\chi_{\mathbf{A}}$$

Appendix B. Quartic couplings of ghosts and fermions

Below we collect terms in the 8D quantum action which are bilinear in FP ghosts and bilinear in fermions:

$$S^{(2)2}_{bc} = \frac{\pi}{2}\int_X t^8 e^{-2\xi}\Big\{\frac{1}{8}\left(\bar\chi_{\mathbf{B}}\hat\Gamma_A\chi_{\mathbf{C}} + 2\bar\chi_A\hat\Gamma_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\hat b}\hat\Gamma^A\hat\Gamma^{\mathbf{BC}}\hat c\right) + \quad \text{(B.1)}$$

$$\frac{1}{6}\left(\bar\chi_{\mathbf{B}}\hat\Gamma_{\bar a}\chi_{\mathbf{C}} + 2\bar\chi_{\bar a}\hat\Gamma_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\hat b}\hat\Gamma^{\bar a}\hat\Gamma^{\mathbf{BC}}\hat c\right)$$

$$+\frac{1}{6}\left(\bar\chi_A\hat\Gamma_{\mathbf{BC}}\chi_{\mathbf{D}}\right)\left(\bar{\hat b}\hat\Gamma^{A\mathbf{BCD}}\hat c\right) + \frac{2}{9}\left(\bar\chi_{\bar a}\hat\Gamma_{\mathbf{BC}}\chi_{\mathbf{D}}\right)\left(\bar{\hat b}\hat\Gamma^{\bar a\mathbf{BCD}}\hat c\right)$$

$$-\frac{1}{48}\bar{\hat b}\Big[\hat\Gamma_A\hat\Gamma^{A\mathbf{BCDE}} + \frac{4}{3}\hat\Gamma_{\bar a}\hat\Gamma^{\bar a\mathbf{BCDE}}\Big]\hat c\left(\bar\chi_{\mathbf{B}}\hat\Gamma_{\mathbf{CD}}\chi_{\mathbf{E}}\right)$$

$$-\left(\bar{\hat{c}}\hat{\Gamma}^{\mathbf{B}}\chi_A\right)\left(\bar{\hat{b}}\hat{\Gamma}^{A}\chi_{\mathbf{B}}\right)-\frac{4}{3}\left(\bar{\hat{c}}\hat{\Gamma}^{\mathbf{B}}\chi_{\bar{a}}\right)\left(\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\chi_{\mathbf{B}}\right)+\frac{1}{4}\left(\bar{\hat{c}}\hat{\Gamma}^{11}\chi_{11}\right)\left[\bar{\hat{b}}\hat{\Gamma}^{A}\chi_A+\frac{4}{3}\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\chi_{\bar{a}}\right]$$

$$+L_{A\bar{a}}\left(\bar{\hat{b}}\hat{\Gamma}^{A}\chi^{\bar{a}}\right)+\frac{4}{3}L_{\bar{a}\mathbf{D}}\left(\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\chi^{\mathbf{D}}\right)+\frac{1}{4}L_{\mathbf{DE}}\bar{\hat{b}}\left(\hat{\Gamma}^{A}\hat{\Gamma}^{\mathbf{DE}}\chi_A+\frac{4}{3}\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\mathbf{DE}}\chi_{\bar{a}}\right)\Bigg\}$$

where we now split indices as $\mathbf{A}=(A,\bar{a})\,,\quad A=0,\dots 7,\quad \bar{a}=(a,11),\quad a=8,9$. Nonzero components of $L_{\mathbf{DE}}$ are given by:

$$L_{A\bar{d}}=-\bar{\hat{c}}\hat{\Gamma}_A\chi_{\bar{d}},\quad L_{a11}=-\bar{\hat{c}}\hat{\Gamma}_a\chi_{11}$$

$S_{bc}^{(2)2}$ is obtained by relating 8D gauge field $\hat{\psi}^{A}_{(8D)}$(gauge parameter $\hat{\epsilon}$) to 11D gravitino $\psi^{(11)}_{\mathbf{A}}$ (gauge parameter $\epsilon^{(11)}$) as

$$\hat{\psi}^{A}_{(8D)}=\sqrt{2\pi}e^{-\frac{\phi}{6}}\left[\psi^{(11)}_{A}+\frac{1}{6}\hat{\Gamma}_A\hat{\Gamma}^{\bar{a}}\psi^{(11)}_{\bar{a}}\right],\quad \hat{\epsilon}=\sqrt{2\pi}e^{\frac{\phi}{6}}\epsilon^{(11)}$$

Let us also remind a standard fact that to keep the gauge

$$\mathbf{E}^{\hat{A}}_{11}=0,\quad \mathbf{E}^{A}_{m}=0$$

used in reduction from 11D one has to accompany supersymmetry transformations of [48] with field dependent Lorentz transformations.

The last line in the action $S_{bc}^{(2)2}$ originates from such Lorentz transformations.

To write out $S_{bc}^{(2)2}$ in terms of 8D fields

$$\hat{\psi}^{A}_{(8D)}:=\begin{pmatrix}\psi^{A}\\ \eta^{A}\end{pmatrix},\quad \hat{\Lambda}_{(8D)}:=\begin{pmatrix}\Sigma\\ \Lambda\end{pmatrix},\quad \hat{\theta}_{(8D)}:=\begin{pmatrix}l\\ \mu\end{pmatrix},\quad \hat{\nu}_{(8D)}:=\begin{pmatrix}\bar{l}\\ \tilde{\mu}\end{pmatrix}$$

one should substitute

$$\chi^{A}=\hat{\psi}^{A}_{(8D)}+\frac{1}{12}\hat{\Gamma}^{A}\hat{\theta}_{(8D)}+\frac{\sqrt{2}}{6}\hat{\Gamma}^{A}\hat{\Lambda}_{(8D)},\quad A=0,\dots,7$$

$$\chi^{8}=\frac{1}{2}\hat{\nu}_{(8D)}+\frac{1}{3}\hat{\Gamma}^{8}\left(\hat{\theta}_{(8D)}+\sqrt{2}\hat{\Lambda}_{(8D)}\right),\quad \chi^{9}=\frac{1}{2}\hat{\Gamma}^{89}\hat{\nu}_{(8D)}+\frac{1}{3}\hat{\Gamma}^{9}\left(\hat{\theta}_{(8D)}+\sqrt{2}\hat{\Lambda}_{(8D)}\right)$$

$$\chi_{11}=-\frac{2\sqrt{2}}{3}\hat{\Gamma}^{11}\left(\hat{\Lambda}_{(8D)}-\frac{\sqrt{2}}{4}\hat{\theta}_{(8D)}\right)$$

We do not present the final expression but we have checked that $S_{bc}^{(2)2}$ is T-duality invariant.

Appendix E. Measures for path integrals

Here we explain why $det'\Delta_p$ are divided by V_p in (6.4) This is related to the integration over zeromodes.

Introducing a basis $a^i_{(p)}$, $i = 1, \dots, b^p$ in $\mathcal{H}^p_{\mathbf{Z}}$ let us denote

$$V_p^{ij} = \int_X a^i_{(p)} \wedge * a^j_{(p)}, \quad V_p = det_{i,j} V_p^{ij} \tag{E.1}$$

Note, that V_p is invariant under the choice of basis in $\mathcal{H}^p_{\mathbf{Z}}$.

To explain integration over fermionic zero modes let us consider the following path-integral over fermionic p-forms u and v.

$$\int DuDv \left[\prod_{i=1}^{b^p} \int_{\gamma_i} u \prod_{j=1}^{b^p} \int_{\gamma_j} v \right] e^{-\left(v, \Delta_p u\right)} \tag{E.2}$$

where $\gamma_i, i = 1, \dots b^p$ is a basis of $H_p(X, \mathbf{Z})$.

In (E.2) we have inserted $\prod_{i=1}^{b^p} \int_{\gamma_i} u \prod_{j=1}^{b^p} \int_{\gamma_j} v$, to get non-zero answer, i.e. to saturate fermion zero modes.

To perform the integration in (E.2) we expand u and v in an orthonormal basis $\{\psi_n\}$ of eigen p-forms of Δ_p.

$$u = \sum_n u_n \psi_n, \quad v = \sum_n v_n \psi_n, \quad \left(\psi_n, \psi_m\right) = \delta_{n,m} \tag{E.3}$$

Let us choose the basis $a^i_{(p)}$, $i = 1, \dots, b^p$ of the lattice $\mathcal{H}^p_{\mathbf{Z}}$, dual to the basis $\gamma_i \in H_p(X, \mathbf{Z})$, i.e

$$\int_{\gamma_i} a^j_{(p)} = \delta_{ij}$$

Then, orthonormal zero-modes are expressed as

$$\psi^i_{zm} = a^j_{(p)} \left(W_p^{-1}\right)^i_j \tag{E.4}$$

where W_p^{-1} is the inverse of the vielbein for the metric on $\mathcal{H}^p_{\mathbf{Z}}$: $(V_p)^{ij} = \left(W_p^T W_p\right)^{ij}$.

Now, we integrate (E.2) and obtain

$$\left[\frac{det'\Delta_p}{V_p}\right] \tag{E.5}$$

In the case of bosonic p-forms u and v we do not need to insert anything to get a non-zero answer:

$$\int DuDve^{-\left(u,\Delta_p v\right)} = \left[\frac{det'\Delta_p}{V_p}\right]^{-1} \tag{E.6}$$

where (E.6) the integration over bosonic zero-modes was performed

$$\int \prod_{i=1}^{b^p} Du^i_{zm} \prod_{j=1}^{b^p} Dv^j_{zm} = \frac{1}{\left(det_{i,j} \int_{\gamma_i} \psi^j_{zm}\right)^2} = V_p \tag{E.7}$$

Appendix F. Super-K-theory theta function

Here we explain why $\widehat{\Theta}(\mathcal{F}, \rho)$ defined in (1.28) is a supertheta function for a family of principally polarized superabelian varieties. To show this we use the results of [49], where supertheta functions were studied.

A generic complex supertorus is defined as a quotient of the affine superspace with even coordinates $z_i, \quad i = 1, \ldots, N_{even}$ and odd coordinates $\xi_a, \quad a = 1, \ldots, N_{odd}$ by the action of the abelian group generated by $\{\lambda_i, \lambda_{i+N_{even}}\}$

$$\lambda_i : z_j \to z_j + \delta_{ij}, \quad \xi_a \to \xi_a \tag{F.1}$$

$$\lambda_{i+N_{even}} : z_j \to z_j + (\Omega_{even})_{ij}, \quad \xi_a \to \xi_a + (\Omega_{odd})_{ia} \tag{F.2}$$

We will restrict to the special case $(\Omega_{odd})_{ia} = 0$ relevant for our discussion. Let us also assume that the reduced torus (obtained from the supertorus by forgetting all odd coordinates) has a structure of a principally polarized abelian variety and denote its Kahler form by ω.

It follows from the results of [49], that a complex line bundle L on the supertorus with $c_1(L) = \omega$ has a unique section (up to constant multiple) iff $\Omega^T_{even} = \Omega_{even}$ together with the positivity of the imaginary part of the reduced matrix. This section is a supertheta function.

Now we can find a family of principally polarized superabelian varieties relevant to our case simply by setting $N_{even} = N$ and $N_{odd} = N_{ferm.zm}$ and by *defining* symmetric Ω_{even} as

$$Re(\Omega_{even})_{ij} = Re\tau_K(x_i, x_j), \tag{F.3}$$

$$Im(\Omega_{even})_{ij} = Im\tau_K(x_i, x_j) + \tag{F.4}$$

$$\sum_{p=0}^{2}\int_{X_{10}}\Big(G_{2p}(x_i)+G_{2p}(x_j)\Big)\wedge\hat{*}\mathcal{J}_{2p}(zm)+\delta_{ij}F(zm)$$

where $x_i, i=1,\ldots,N$ is a basis of Γ_1. In (F.3) $\mathcal{J}_{2p}(zm)$ is a 2p-form on X_{10} constructed as a bilinear expression in fermion(and ghosts) zeromodes and $F(zm)$ is a functional quartic in fermion(and ghosts) zeromodes, both $\mathcal{J}_{2p}(zm)$ and $F(zm)$ can in principle be found from the 10D fermion action (7.10),(14.1) as well as from the ghost action (7.35),(7.40),(15.1). The modified characteristics $\vec{\tilde{\alpha}}, \vec{\tilde{\beta}}$ and prefactor $\widehat{\Delta\tilde{\Phi}}(\mathcal{F})$ in (1.28) all originate from the shift of the imaginary part of the period matrix described in (F.4). It would be very nice if one could formulate this superabelian variety in a more natural way, without reference to a Lagrangian splitting of Γ_K.

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