ON A CLASSICAL CORRESPONDENCE BETWEEN K3 SURFACES

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ABSTRACT. Let X be a K3 surface which is intersection of three (a net $\mathbb{P}^2$) of quadrics in $\mathbb{P}^5$. The curve of degenerate quadrics has degree 6 and defines a double covering of $\mathbb{P}^2$ K3 surface Y ramified in this curve. This is the classical example of a correspondence between K3 surfaces which is related with moduli of vector bundles on K3 studied by Mukai. When general (for fixed Picard lattices) X and Y are isomorphic? We give necessary and sufficient conditions in terms of Picard lattices of X and Y.

E.g. for Picard number 2 the Picard lattice of X and Y is defined by its determinant $-d$ where $d > 0$, $d \equiv 1 \mod 8$, and one of equations $a^2 - db^2 = 8$ or $a^2 - db^2 = -8$ should have an integral solution (a, b). Clearly, the set of these d is infinite: $d \in \{(a^2 \mp 8)/b^2\}$ where a and b are odd integers. This describes all possible divisorial conditions on 19-dimensional moduli of intersections of three quadrics in $\mathbb{P}^5$ when $Y \cong X$. One of them when X has a line is classical and corresponds to $d = 17$.

Similar considerations can be applied for a realization of an isomorphism $(T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y))$ of transcendental periods over $\mathbb{Q}$ of two K3 surfaces X and Y by a fixed sequence of types of Mukai vectors.

0. INTRODUCTION

In this paper we study a classical correspondence between algebraic K3 surfaces over $\mathbb{C}$.

Let a K3 surface X be an intersection of three quadrics in $\mathbb{P}^5$ (more generally, X is a K3 surface with a primitive polarization H of degree 8). The three quadrics define the projective plane $\mathbb{P}^2$ (the net) of quadrics up to scaling. Let $C \subset \mathbb{P}^2$ be the curve of degenerate quadrics. The curve C has degree 6 and defines another K3 surface Y which is the minimal resolution of singularities of the double covering of $\mathbb{P}^2$ ramified in C. It has the natural linear system $|h|$ with $h^2 = 2$ which is preimage of lines in $\mathbb{P}^2$. This is a classical and a very beautiful example of a correspondence between K3 surfaces. It is defined by a 2-dimensional algebraic cycle $Z \subset X \times Y$.

This example is related with the moduli of vector bundles on K3 surfaces studied by Mukai [Mu1], [Mu2]. It is well-known that the K3 surface Y is the moduli of vector bundles $\mathcal{E}$ on X with the rank $r = 2$, the first Chern class $c_1(\mathcal{E}) = H$ and

[1]Supported by Grant of Russian Fund of Fundamental Research.

the Euler characteristic $\chi = \chi(\mathcal{E}) = 4$. We apply this construction to study the following questions.

Question 1. *When Y is isomorphic to X?*

We want to answer this question in terms of the Picard lattices $N(X)$ and $N(Y)$ of X and Y. Thus, our question is as follows:

Question 2. *Assume that N is a hyperbolic lattice, $\widetilde{H} \in N$ a primitive element with square 8. What are conditions on N and $\widetilde{H}$ such that for any K3 surface X with the Picard lattice $N(X)$ and a primitive polarization $H \in N(X)$ of degree 8 the corresponding K3 surface Y is isomorphic to X if the the pairs of lattices $(N(X), H)$ and $(N, \widetilde{H})$ are isomorphic as abstract lattices with fixed elements?*

Other speaking, what are conditions on $(N(X), H)$ as an abstract lattice with a primitive vector H with $H^2 = 8$ which guarantee that Y is isomorphic to X, and these conditions are necessary if X is a general K3 surfaces with this Picard lattice?

We give the answer to the questions in terms of Picard lattices of X and Y. See Theorems 2.2.3 and 2.2.4. See also Propositions 2.2.1 and 2.2.2. In particular, if the Picard number $\rho(X) = \rho(Y) \geq 12$, the result is very simple: $X \cong Y$, if and only if there exists $x \in N(X)$ such that $x \cdot H = 1$.

We are especially interesting in K3 surfaces X and Y with $\rho(X) = \rho(Y) = 2$. Really, it is well-known that the moduli space of the K3 surfaces X, which are intersections of three quadrics, is 19-dimensional. If X is general, i.e. $\rho(X) = 1$, then the surface Y cannot be isomorphic to X because $N(X) = \mathbb{Z}H$ where $H^2 = 8$, and $N(X)$ does not have elements with square 2 which is necessary if $Y \cong X$. Thus, if $Y \cong X$, then $\rho(X) \geq 2$, and X belongs to codimension 1 submoduli space of K3 surfaces which is a *divisor in 19-dimensional moduli space of intersections of three quadrics* (up to codimension 2). To describe connected components of this divisor, it is equivalent to describe Picard lattices $N(X) \cong N(Y)$ of the surfaces $X \cong Y$ above with $\rho(X) = \rho(Y) = 2$ such that general K3 surfaces X and Y with these Picard lattices have $X \cong Y$. We show that the Picard lattice $N(X) \cong N(Y)$ of these $X \cong Y$ is defined by its determinant which is equal to $-d$ where $d > 0$ and $d \equiv 1 \mod 8$. We show that the set $\mathcal{D}$ of these numbers d is exactly the set of $d \in \mathbb{N}$ such that one of equations

$$a^2 - db^2 = 8 \tag{0.1}$$

or

$$a^2 - db^2 = -8 \tag{0.2}$$

has an integral solution. It is easy to see that solutions (a, b) of these equations are odd. It follows that $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$ where

$$\mathcal{D}_+ = \{\frac{a^2 - 8}{b^2} \in \mathbb{N} \mid \text{where } a,\ b \in \mathbb{N} \text{ are odd}\} \tag{0.3}$$

and

$$\mathcal{D}_- = \{\frac{a^2+8}{b^2} \in \mathbb{N} \mid \text{where } a,\ b \in \mathbb{N} \text{ are odd}\}. \tag{0.4}$$

Clearly, both sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are infinite. In Theorem 3.2.1 we give the list of first numbers from $\mathcal{D}$: 1, 9, 17, 33, 41, 57,...

The set of numbers $\mathcal{D}$ labels connected components of the divisor, where $Y \cong X$, in 19-dimensional moduli of intersections of three quadrics X. Each $d \in \mathcal{D}$, gives a connected 18-dimensional moduli space of K3 surfaces with the Picard lattice $N(X) = N(Y)$ of the rank 2 and of the determinant $-d$ where $X \cong Y$. E.g. it is well-known that $Y \cong X$ if X has a line. This is a divisorial condition on moduli of X. This component is labeled by $d = 17 \in \mathcal{D}$.

Our results show that *There exists an infinite number of similar divisorial conditions on moduli of intersections of three quadrics in $\mathbb{P}^5$ where $Y \cong X$. They are labeled by elements of the infinite set $\mathcal{D}$ which was described above. The $d = 17$ corresponds to the classical example above of the line in X.*

We mentions that solutions (a, b) of the equations (0.1) and (0.2) can be interpreted as elements of Picard lattices of X and Y. E. g. we have that for general X and Y with $\rho(X) = \rho(Y) = 2$, we have $X \cong Y$ if and only if $\det N(X)$ is odd and there exists $h_1 \in N(X)$ such that $(h_1)^2 = \pm 4$ (for one of sings) and $h_1 \cdot H \equiv 0 \mod 2$. Similar condition in terms of Y is: $\det N(X)$ is odd and there exists $h_1 \in N(Y)$ such that $h_1^2 = \pm 4$ (for one of signs).

Similar method can be developed for general case: Let X and Y are K3 surfaces, which are general for their Picard lattices, and

$$\phi : (T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y)) \tag{0.5}$$

an isomorphism of their transcendental periods over $\mathbb{Q}$, and

$$(a_1, H_1, b_1)^{\pm}, \dots (a_k, H_k, b_k)^{\pm}, \tag{0.6}$$

a sequence of types of isotropic Mukai vectors of moduli of vector bundles on K3, and $\pm$ shows the direction of the correspondence.

Similar methods and calculations can be applied to study the following question:

When there exists a correspondence between X and Y which is given by the sequence (0.6) *of Mukai vectors and which gives the isomorphism* (0.5) *between their transcendental periods?*

In [N4], some necessary and sufficient conditions for (0.6) are given when there exists at least one such a sequence (0.5) with comprime Mukai vectors (a_i, H_i, b_i).

We are grateful to A.Tyurin and A.Verra for very useful and stimulating discussions.

1. Preliminary notations and results about lattices and K3 surfaces

1.1. Some notations about lattices. We use notations and terminology from [N2] about lattices, their discriminant groups and forms. A *lattice* L is a non-degenerate integral symmetric bilinear form. I. e. L is a free $\mathbb{Z}$-module equipped with a symmetric pairing $x \cdot y \in \mathbb{Z}$ for $x, y \in L$, and this pairing should be non-degenerate. We denote $x^2 = x \cdot x$. The *signature* of L is the signature of the corresponding real form $L \otimes \mathbb{R}$. The lattice L is called *even* if x^2 is even for any $x \in L$. Otherwise, L is called *odd*. The *discriminant* of L is defined to be $\det L = \det(e_i \cdot e_j)$ where $\{e_i\}$ is some basis of L. The lattice L is *unimodular* if $\det L = \pm 1$.

The *dual lattice* of L is $L^* = Hom(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$. The *discriminant group* of L is $A_L = L^*/L$. It has the order $|\det L|$. The group A_L is equipped with the *discriminant bilinear form* $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and the *discriminant quadratic form* $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ if L is even. To get this forms, one should extend the form of L to the dual lattice L^*.

For $x \in L$, we shall consider the invariant $\gamma(x) \geq 0$ where

$$x \cdot L = \gamma(x)\mathbb{Z}. \tag{1.1.1}$$

Clearly, $\gamma(x) | x^2$ if $x^2 \neq 0$.

We denote by $L(k)$ the lattice obtained from a lattice L by multiplication of the form of L by $k \in \mathbb{Q}$.

The orthogonal sum of lattices L_1 and L_2 is denoted by $L_1 \oplus L_2$.

For a symmetric integral matrix A, we denote by $\langle A \rangle$ a lattice which is given by the matrix A in some bases. We denote

$$U = \left\langle \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\rangle. \tag{1.1.2}$$

Any even unimodular lattice of the signature $(1,1)$ is isomorphic to U.

An embedding $L_1 \subset L_2$ of lattices is called *primitive* if L_2/L_1 has no torsion.

We denote by $O(L)$, $O(b_L)$ and $O(q_L)$ the automorphism groups of the corresponding forms. Any $\delta \in L$ with $\delta^2 = -2$ defines a reflection $s_\delta \in O(L)$ which is given by the formula $x \to x + (x \cdot \delta)\delta$, $x \in L$. All such reflections generate the *2-reflection group* $W^{(-2)}(L) \subset O(L)$.

1.2. Some notations about K3 surfaces. Here we remind some basic results and notions about K3 surfaces, e. g. see [PS-Sh], [S-D], [Sh]. A K3 surface S is a projective algebraic surface over $\mathbb{C}$ such that its canonical class K_S is zero and the irregularity $q_S = 0$. We denote by $N(S)$ the *Picard lattice* of S which is a hyperbolic lattice with the intersection pairing $x \cdot y$ for $x, y \in N(S)$. Since the canonical class $K_S = 0$, the space $H^{2,0}(S)$ of 2-dimensional holomorphic differential forms on S has dimension one over $\mathbb{C}$, and

$$N(S) = \{x \in H^2(S, \mathbb{Z}) \mid x \cdot H^{2,0}(S) = 0\}$$

where $H^2(S, \mathbb{Z})$ is a 22-dimensional even unimodular lattice of signature $(3, 19)$. The orthogonal lattice $T(S)$ to $N(S)$ in $H^2(S, \mathbb{Z})$ is called the *transcendental lattice of* S. We have $H^{2,0}(S) \subset T(S) \otimes \mathbb{C}$. The pair $(T(S), H^{2,0}(S))$ is called the *transcendental periods of* S. The *Picard number* of S is $\rho(S) = \text{rk}\,(N(S))$. An element $x \in N(S) \otimes \mathbb{R}$

is called *nef* if $x \cdot C \geq 0$ for any effective curve $C \subset S$. It is known that an element $x \in N(S)$ is ample if $x^2 > 0$, x is nef, and the orthogonal complement $x^\perp$ to x in $N(S)$ has no elements with square -2. For any element $x \in N(S)$ with $x^2 \geq 0$, there exists a reflection $w \in W^{(-2)}(N(S))$ such that the element $\pm w(x)$ is nef; it then is ample, if $x^2 > 0$ and $x^\perp$ had no elements with square -2.

1.3. K3 surfaces with polarizations of degree 8 and 2. The following results are well-known, see Mayer [Ma], Saint-Donat [S-D].

Let X be a K3 surface with a primitive polarization $H \in N(X)$ of degree $H^2 = 8$. Here primitive means that the sublattice $\mathbb{Z}H \subset N(X)$ is primitive.

Proposition 1.3.1. *The linear system $|H|$ has dimension* 5, *and there are the following and the only following cases for the linear system $|H|$:*

(a) $|H \cdot E| > 2$ for any elliptic curve E (it has $E^2 = 0$) on X. Then the linear system $|H|$ gives embedding of X to $\mathbb{P}^5$ as intersection of three quadrics.

(b) $|H \cdot E| \geq 2$ for any elliptic curve E on X, and there exists an elliptic curve E on X such that $H \cdot E = 2$. Then the linear system $|H|$ is hyperelliptic and gives a double covering by X of a rational scroll. For this case, H and E generate a sublattice isomorphic to $U(2)$ in $N(X)$.

(c) $|H \cdot E| \geq 1$ for any elliptic curve E on X, and there exists an elliptic curve E on X such that $H \cdot E = 1$. Then the linear system $|H|$ has a fixed component D which is a non-singular rational curve on X (it has $D^2 = -2$), and $|H| = 5|E| + D$ where $|E|$ is an elliptic pencil on X and $E \cdot D = 1$. For this case, H and E generate a sublattice isomorphic to U in $N(X)$.

Since $H^2 = 8$, we then have $\gamma(H) = 1$, 2, 4 or 8.

For the case (c), $\rho(X) \geq 2$ and $\gamma(H) = 1$.

For the case (b), $\rho(X) \geq 2$ and $\gamma(H) = 1$ or 2. If $\rho(X) = 2$, then $\gamma(H) = 2$.

For the case (a), $\rho(X) \geq 1$ and $\gamma(H) = 1$, 2, 4 or 8. If $\rho(X) = 1$, then $\gamma(H) = 8$.

A general K3 surface X has $\rho(X) = 1$, then one has the case (a) and X is intersection of three quadrics.

Now let Y be a K3 surface with a nef element $h \in N(Y)$ of degree $h^2 = 2$. Obviously, h is primitive. One has

Proposition 1.3.2. *The linear system $|h|$ has dimension* 2, *and there are the following and the only following cases:*

(a) $|h \cdot E| \geq 2$ for any elliptic curve E on Y. Then the linear system $|h|$ gives a double covering by Y of $\mathbb{P}^2$ ramified in a curve of degree 6 with double singularities.

(b) $|h \cdot E| \geq 1$ for any elliptic curve E on Y, and there exists an elliptic curve E such that $h \cdot E = 1$. Then the linear system $|h|$ has a fixed component D which is a non-singular rational curve on Y (it has $D^2 = -2$), and $|H| = 2|E| + D$ where $|E|$ is an elliptic pencil on Y and $E \cdot D = 1$. For this case, E and D generate a primitive sublattice isomorphic to U in $N(Y)$.

Since $h^2 = 2$, we have $\gamma(h) = 1$ or 2.

For the case (b), $\rho(Y) \geq 2$ and $\gamma(h) = 1$.

For the case (a), $\rho(Y) \geq 1$ and $\gamma(h) = 1$ or 2. If $\rho(Y) = 1$, then $\gamma(h) = 2$. A general K3 surface Y has $\rho(Y) = 1$, then $\gamma(h) = 2$, and one has the case (a).

2. General results on the classical correspondence between K3 surfaces with primitive polarizations of degree 8 and 2 which gives isomorphic K3

2.1. The correspondence. Let a K3 surface X be an intersection of three quadrics in $\mathbb{P}^5$ (more generally, X is a K3 surface with a primitive polarization H of degree 8). The three quadrics define the projective plane $\mathbb{P}^2$ (the net) of quadrics up to scaling. Let $C \subset \mathbb{P}^2$ be the curve of degenerate quadrics. The curve C has degree 6 and defines another K3 surface Y which is the minimal resolution of singularities of the double covering of $\mathbb{P}^2$ ramified in C. It has the natural linear system $|h|$ with $h^2 = 2$ which is preimage of lines on $\mathbb{P}^2$. This is a classical and a very beautiful example of a correspondence between K3 surfaces. It is defined by a 2-dimensional algebraic cycle $Z \subset X \times Y$.

This example is related with the moduli of vector bundles on K3 surfaces studied by Mukai [Mu1], [Mu2]. It is well-known that the K3 surface Y is the moduli of vector bundles $\mathcal{E}$ on X with the rank $r = 2$, the first Chern class $c_1(\mathcal{E}) = H$ and the Euler characteristic $\chi = \chi(\mathcal{E}) = 4$.

Let

$$H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \tag{2.1.1}$$

be the full cohomology lattice of X with the Mukai product

$$-(u_0 \cdot v_2 + u_2 \cdot v_0) + u_1 \cdot v_1 \tag{2.1.2}$$

for $u_0, v_0 \in H^0(X,\mathbb{Z})$, $u_1, v_1 \in H^2(X,\mathbb{Z})$, $u_2, v_2 \in H^4(X,\mathbb{Z})$. We naturally identify $H^0(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$ with $\mathbb{Z}$. Then the Mukai product is

$$-(u_0v_2 + u_2v_0) + u_1 \cdot v_1. \tag{2.1.3}$$

The element

$$v = (2, H, 2) = (r, H, \chi - r) \in H^*(X,\mathbb{Z}) \tag{2.1.4}$$

is called *the Mukai vector* of the net of quadrics or (more generally) of the moduli Y of vector bundles on X with these data. It is isotropic, $v^2 = 0$. Mukai [Mu1], [Mu2] showed that one has the natural identification

$$H^2(Y,\mathbb{Z}) = (v^\perp/\mathbb{Z}v) \tag{2.1.5}$$

which also gives the isomorphism of the Hodge structures of X and Y. The element $h = (-1, 0, 1) \in v^\perp$ has $h^2 = 2$, $h \mod \mathbb{Z}v$ belongs to the Picard lattice N_Y of Y, and the linear system $|h|$ defines the structure of double plane for Y for the net of quadrics.

We apply this construction to study the following questions.

Question 2.1.1. *When Y is isomorphic to X?*

We want to answer this question in terms of the Picard lattices of X and Y. Thus, our question is as follows:

Question 2.1.2. *Assume that N is a hyperbolic lattice, $\widetilde{H} \in N$ a primitive element with square 8. What are conditions on N and $\widetilde{H}$ such that for any K3 surface X with the Picard lattice $N(X)$ and a primitive polarization $H \in N(X)$ of degree 8 the corresponding K3 surface Y is isomorphic to X if the the pairs of lattices $(N(X), H)$ and $(N, \widetilde{H})$ are isomorphic as abstract lattices with fixed elements?*

Other speaking, what are conditions on $(N(X), H)$ as an abstract lattice with a primitive vector H with $H^2 = 8$ which guarantee that Y is isomorphic to X, and these conditions are necessary if X is a general K3 surfaces with this Picard lattice?

Below we formulate these results.

2.2. Formulation of general results. Below X is a K3 surface with a primitive polarization H of degree $H^2 = 8$, and Y the corresponding to X K3 surface Y with the nef element h of degree $h^2 = 2$ which was described in Sect. 2.1.

The following statement follows from the Mukai identification (2.1.5) and results from [N2]. One can consider this statement as a standard and an easy one.

Proposition 2.2.1. *If Y is isomorphic to X, then the invariant $\gamma(H)$ for H in $N(X)$ (see (1.1.1)) is equal to 1.*

Assume that $\gamma(H) = 1$ in $N(X)$. Then the Mukai identification (2.1.5) canonically identifies the transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$. It follows that the Picard lattices $N(Y)$ and $N(X)$ have the same genus. In particular, $N(Y)$ is isomorphic to $N(X)$ if the genus of $N(X)$ contains only one class. If it is true, then Y is isomorphic to X, if additionally the canonical homomorphism $O(N(X)) \to O(q_{N(X)})$ is epimorphic. Both these conditions are valid if $\rho(X) \geq 12$.

Thus, $Y \cong X$ if $\gamma(H) = 1$ and $\rho(X) \geq 12$.

Now we can assume that $\gamma(H) = 1$ in $N(X)$ (only for this case we may have $Y \cong X$).

Calculations below are valid for an arbitrary K3 surface X and a primitive vector $H \in N(X)$ with $H^2 = 8$ and $\gamma(H) = 1$. Let $K(H)$ be the orthogonal complement to H in $N(X)$. Let $H^* = H/8$. Then any element $x \in N(X)$ can be written as

$$x = aH^* + k^* \tag{2.2.1}$$

where $a \in \mathbb{Z}$ and $k^* \in K(H)^*$. Since $\gamma(H) = 1$, the map $aH^* + [H] \mapsto k^* + K(H)$ gives an isomorphism of the groups $(\mathbb{Z}/8) = [H^*]/[H] \cong [u^* + K(H)]/K(H)$ where $u^* + K(H)$ has the order 8 in $A_{K(H)} = K(H)^*/K(H)$. It follows,

$$N(X) = [\mathbb{Z}H, K(H), H^* + u^*] \tag{2.2.2}$$

where $u^* + K(H)$ is an element of order 8 in $K(H)^*/K(H)$. The element u^* is defined canonically mod $K(H)$. The element $H^* + u^*$ belongs to the even lattice $N(X)$. It follows that

$$(H^* + u^*)^2 = \frac{1}{8} + u^{*2} \equiv 0 \mod 2. \tag{2.2.3}$$

Let $\overline{H^*} = H^* \mod [H] \in [H^*]/[H] \cong (\mathbb{Z}/8)$ and $\overline{k^*} = k^* \mod K(H) \in A_{K(H)} = K(H)^*/K(H)$. We then have

$$N(X)/[H, K(H)] = (\mathbb{Z}/8)(\overline{H^*} + \overline{u^*}) \subset (\mathbb{Z}/8)\overline{H^*} + K(H)^*/K(H). \tag{2.2.4}$$

We have $N(X)^* \subset \mathbb{Z}H^* + K(H)^*$, and for $a \in \mathbb{Z}$, $k^* \in K(H)^*$ we have $x = aH^* + k^* \in N(X)$ if and only if $(aH^* + k^*)\cdot(H^* + u^*) = \frac{a}{8} + k^* \cdot u^* \in \mathbb{Z}$. It follows,

$$N(X)^* = \{aH^* + k^* \mid a \in \mathbb{Z},\ k^* \in K(H)^*,\ a \equiv -8k^* \cdot u^* \mod 8\} \subset \mathbb{Z}H^* + K(H)^*. \tag{2.2.5}$$

and

$$N(X)^*/[H, K(H)] = \{-8\overline{k^*} \cdot \overline{u^*}\, \overline{H^*} + \overline{k^*}\} \mid \overline{k^*} \in K(H)^*/K(H)\} \subset$$
$$\subset (\mathbb{Z}/8)\overline{H^*} + K(H)^*/K(H). \tag{2.2.6}$$

We introduce *the characteristic map of the polarization H*

$$\kappa(H) : K(H)^* \to (K(H)^*/K(H))/(\mathbb{Z}/8)(u^* + K(H)) \to N(X)^*/N(X) = A_{N(X)} \tag{2.2.7}$$

where for $k^* \in K(H)^*$ we have

$$\kappa(H)(k^*) = -8k^* \cdot u^* H^* + k^* + N(X) \in N(X)^*/N(X) = A_{N(X)}. \tag{2.2.8}$$

It is epimorphic, its kernel is $(\mathbb{Z}/8)(u^* + K(H))$, and it gives the canonical isomorphism

$$\overline{\kappa(H)} : (K(H)^*/K(H))/(\mathbb{Z}/8)(u^* + K(H)) \cong N(X)^*/N(X) = A_{N(X)}. \tag{2.2.9}$$

For the corresponding discriminant forms we have

$$\kappa(k^*)^2 \mod 2 = (k^*)^2 + 8(k^* \cdot u^*)^2 \mod 2. \tag{2.2.10}$$

Similar results we have for the polarization h of Y. We denote by $K(h)$ the orthogonal complement to h in $N(Y)$.

Proposition 2.2.2. *If Y is isomorphic to X, then the invariant $\gamma(H)$ of H in $N(X)$ (see (1.1.1)) is $\gamma(H)=1$. If $\gamma(H)=1$ for $H\in N(X)$, then $\gamma(h)=1$ for $h\in N(Y)$, and* $\det K(h)=\det K(H)/4$.

Assume that $\gamma(H)=1$ for $H\in N(X)$. Then the Mukai identification canonically identifies the transcendental periods $(T(X),\ H^{2,0}(X))$ and $(T(Y),\ H^{2,0}(Y))$. It follows that the Picard lattices $N(Y)$ and $N(X)$ have the same genus. In particular, $N(X)$ is isomorphic to $N(Y)$, if the genus of $N(Y)$ contains only one class. If it is true, then Y is isomorphic to X, if additionally the canonical homomorphism $O(N(Y))\to O(q_{N(Y)})$ is epimorphic. Both above conditions are valid, if $\rho(Y)\geq 12$.

Thus, $Y\cong X$, if $\gamma(H)=1$ for $H\in N(X)$, and $\rho(Y)\geq 12$.

Calculations below are valid for an arbitrary K3 surface Y and a primitive vector $h\in N(Y)$ with $h^2=2$ and $\gamma(h)=1$. Let $K(h)$ be the orthogonal complement to h in $N(Y)$. Let $h^*=h/2$. Then any element $x\in N(Y)$ can be written as

$$x=ah^*+k^* \tag{2.2.11}$$

where $a\in\mathbb{Z}$ and $k^*\in K(h)^*$. Since $\gamma(h)=1$, the map $ah^*+[h]\mapsto k^*+K(h)$ gives the isomorphism of the groups $(\mathbb{Z}/2)=[h^*]/[h]\cong[w^*+K(h)]/K(h)$ where $w^*+K(h)$ has the order 2 in $K(h)^*/K(h)$. It follows,

$$N(Y)=[\mathbb{Z}h,K(h),h^*+w^*] \tag{2.2.12}$$

where $w^*+K(h)$ is an element of order 2 in $K(h)^*/K(h)$. The element w^* is defined canonically mod $K(h)$. The element h^*+w^* belongs to the even lattice $N(Y)$. It follows

$$(h^*+w^*)^2=\frac{1}{2}+w^{*2}\equiv 0 \mod 2. \tag{2.2.13}$$

Let $\overline{h^*}=h^*\mod [h]\in[h^*]/[h]\cong(\mathbb{Z}/2)$ and $\overline{k^*}=k^*\mod K(h)\in A_{K(h)}=K(h)^*/K(h)$. We then have

$$N(Y)/[h,K(h)]=(\mathbb{Z}/2)(\overline{h^*}+\overline{w^*})\subset(\mathbb{Z}/2)\overline{h^*}+K(h)^*/K(h). \tag{2.2.14}$$

We have $N(Y)^*\subset\mathbb{Z}h^*+K(h)^*$, and for $a\in\mathbb{Z}$, $k^*\in K(h)^*$ we have $x=ah^*+k^*\in N(Y)$ if and only if $(ah^*+k^*)\cdot(h^*+w^*)=\frac{a}{2}+k^*\cdot w^*\in\mathbb{Z}$. It follows,

$$N(Y)^*=\{ah^*+k^*\mid a\in\mathbb{Z},\ k^*\in K(h)^*,\ a\equiv -2k^*\cdot w^* \mod 2\}\subset\mathbb{Z}h^*+K(h)^*. \tag{2.2.15}$$

and

$$N(Y)^*/[h,K(h)]=\{-2\overline{k^*}\cdot\overline{w^*}\,\overline{h^*}+\overline{k^*}\}\mid\overline{k^*}\in K(h)^*/K(h)\}\subset$$
$$\subset(\mathbb{Z}/2)\overline{h^*}+K(h)^*/K(h). \tag{2.2.16}$$

We introduce *the characteristic map of the polarization* h

$$\kappa(h):K(h)^*\to(K(h)^*/K(h))/(\mathbb{Z}/2)(w^*+K(h))\to N(Y)^*/N(Y)=A_{N(Y)} \tag{2.2.17}$$

where for $k^* \in K(H)^*$ we have

$$\kappa(h)(k^*) = -2k^* \cdot w^*h^* + k^* + N(Y) \in N(Y)^*/N(Y) = A_{N(Y)}. \qquad (2.2.18)$$

It is epimorphic, its kernel is $(\mathbb{Z}/2)(w^* + K(h))$, and it gives the canonical isomorphism

$$\overline{\kappa(h)} : (K(h)^*/K(h))/(\mathbb{Z}/2)(w^* + K(h)) \cong N(Y)^*/N(Y) = A_{N(Y)}. \qquad (2.2.19)$$

For the corresponding discriminant forms we have

$$\kappa(k^*)^2 \mod 2 = (k^*)^2 + 2(k^* \cdot w^*)^2 \mod 2. \qquad (2.2.20)$$

We can now formulate our main result:

Theorem 2.2.3. *The surface Y is isomorphic to X if $\gamma(H) = 1$ and there exists $\tilde{h} \in N(X)$ with $\tilde{h}^2 = 2$, $\gamma(\tilde{h}) = 1$, and there exists an embedding $f : K(H) \to K(\tilde{h})$ of negative definite lattices such that $K(\tilde{h}) = [f(K(H)), 4f(u^*)]$, $w^* + K(\tilde{h}) = 2f(u^*) + K(\tilde{h})$ and $f^* : K(\tilde{h})^* \to K(H)^*$ commutes (up to multiplication by ± 1) with the characteristic maps $\kappa(H)$ and $\kappa(\tilde{h})$: We have*

$$\kappa(\tilde{h})(k^*) = \pm\kappa(H)(f^*(k^*)) \qquad (2.2.21)$$

for any $k^* \in K(\tilde{h})^*$.

These conditions are necessary if rk $N(X) \le 19$ and X is a general K3 surface with the Picard lattice $N(X)$ in the following sense: the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is ± 1. (Remind that $Y \cong X$ if rk $N(X) = 20$, by Proposition 2.2.1.)

We can also formulate similar result using the surface Y.

Theorem 2.2.4. *The surface Y is isomorphic to X if $\gamma(H) = 1$ for $H \in N(X)$, and there exists a primitive $\widetilde{H} \in N(Y)$ with $\widetilde{H}^2 = 8$ and $\gamma(\widetilde{H}) = 1$, and there exists an embedding $f : K(\widetilde{H}) \to K(h)$ of negative definite lattices such that $K(h) = [f(K(\widetilde{H})), 4f(u^*)]$, $w^* + K(h) = 2f(u^*) + K(h)$ and $f^* : K(h)^* \to K(\widetilde{H})^*$ commutes (up to multiplication by ± 1) with the characteristic maps $\kappa(\widetilde{H})$ and $\kappa(h)$: We have*

$$\kappa(h)(k^*) = \pm\kappa(\widetilde{H})(f^*(k^*)) \qquad (2.2.22)$$

for any $k^* \in K(h)^*$.

These conditions are necessary if rk $N(Y) \le 19$ and Y is a general K3 surface with the Picard lattice $N(Y)$ in the following sense: the automorphism group of the transcendental periods $(T(Y), H^{2,0}(Y))$ is ± 1. (Remind that $Y \cong X$ if rk $N(Y) = 20$, by Proposition 2.2.2.)

2.3. Proofs. Let us denote by e_1 the canonical generator of $H^0(X,\mathbb{Z})$ and by e_2 the canonical generator of $H^4(X,\mathbb{Z})$. They generate the sublattice U in $H^*(X,\mathbb{Z})$ with the Gram matrix U. The Mukai vector $v = (2e_1 + 2e_2 + H)$. We have

$$N(Y) = (v^{\perp})_{U\oplus N(X)}/\mathbb{Z}v. \tag{2.3.1}$$

Let us calculate $N(Y)$. Let $K(H) = (H)^{\perp}_{N(X)}$. Then we have embedding of lattices of finite index

$$\mathbb{Z}H \oplus K(H) \subset N(X) \subset \mathbb{Z}H^* \oplus K(H)^* \tag{2.3.2}$$

where $H^* = H/2$. We have the orthogonal decomposition up to finite index

$$U \oplus \mathbb{Z}H \oplus K(H) \subset U \oplus N(X) \subset U \oplus \mathbb{Z}H^* \oplus K(H)^*. \tag{2.3.3}$$

Let $s = x_1e_1 + x_2e_2 + yH^* + z^* \in v^{\perp}_{U\oplus N(X)}$, $z \in K(H)^*$. Then $-2x_1 - 2x_2 + y = 0$. Thus, $y = 2x_1 + 2x_2$ and

$$s = x_1e_1 + x_2e_2 + 2(x_1 + x_2)H^* + z^*. \tag{2.3.4}$$

Here $s \in U \oplus N(X)$ if and only if $x_1, x_2 \in \mathbb{Z}$ and $2(x_1 + x_2)H^* + z^* \in N(X)$. This orthogonal complement contains

$$[\mathbb{Z}v, K(H), \mathbb{Z}h] \tag{2.3.5}$$

where $h = -e_1 + e_2$, and this is a sublattice of finite index in $(v^{\perp})_{U\oplus N(X)}$. The generators v, generators of $K(H)$ and h are free, and we can rewrite s above using these generators with rational coefficients as follows:

$$s = \frac{-x_1 + x_2}{2}h + \frac{x_1 + x_2}{4}v + z^*, \tag{2.3.6}$$

where $2(x_1 + x_2)H^* + z^* \in N(X)$. Equivalently,

$$s = ah^* + b\frac{v}{4} + z^*, \tag{2.3.7}$$

where $a, b \in \mathbb{Z}$, $z^* \in K(H)^*$ and $a \equiv b \mod 2$, and $2bH^* + z^* \in N(X)$.

Thus, we get the following cases:

Assume that $\gamma(H) = 8$. Then $2b \equiv 0 \mod 8$, or $b \equiv 0 \mod 4$. Then $z^* \in K(H)$ $a \equiv b \equiv 0 \mod 2$, $s \in [h, K(H)] \mod \mathbb{Z}v$. It follows,

$$N(X) = [H, K(H)] \tag{2.3.8}$$

and

$$N(Y) = [h, K(h) = K(H)]. \tag{2.3.9}$$

We have $\det N(Y) = \det N(X)/4$.

Assume that $\gamma(H)=4$. Then either $2b\equiv 0 \mod 8$ or $2b\equiv 4 \mod 8$. Equivalently, $b\equiv 0,\ 2 \mod 4$. If $b\equiv 0 \mod 4$, we get for $N(Y)$ the same elements as above. If $b\equiv 2 \mod 4$, we get an additional element $\mod \mathbb{Z}v$ equals to z^* where $z^*\in K(H)^*$ is defined by the condition that $4H^*+z^*\in N(X)$. Here $z^*+K(H)$ has order two in $K(H)^*/K(H)$. Thus, for this case

$$N(X)=[H,K(H),\frac{H}{2}+z^*], \tag{2.3.10}$$

$$N(Y)=[h,K(h)=[K(H),z^*]] \tag{2.3.11}$$

where $z^*+K(H)$ has order two in $K(H)^*/K(H)$. We have $\det N(X)=2\det K(H)$ and $\det N(Y)=2\det K(H)/4=\det N(X)/4$.

Assume that $\gamma(H)=2$. Then $2b\equiv 0,\ 4,\ \pm 2 \mod 8$. Or $b\equiv 0,\ 2,\ \pm 1 \mod 4$. If $b\equiv 0,2 \mod 4$, we get the same elements as for $\gamma(H)=4$. If $b\equiv\pm 1 \mod 4$, we get additional elements $\pm(\frac{h}{2}+z_1^*)$ where $z_1^*+K(H)$ has order 4 in $K(H)^*/K(H)$. Finally we get (changing notations) that

$$N(X)=[H,K(H),\frac{H}{4}+z^*], \tag{2.3.12}$$

$$N(Y)=[h,K(h)=[K(H),2z^*],\frac{h}{2}+z^*], \tag{2.3.13}$$

where $z^*+K(H)$ has order 4 in $K(H)^*/K(H)$. We have $\det N(X)=\det K(H)/2$, $\det N(Y)=\det K(H)/8$, and $\det N(Y)=\det N(X)/4$.

Assume that $\gamma(H)=1$. Then $2b\equiv 0,\ 4,\ \pm 2 \mod 8$, and we get the same lattice $N(Y)$ as above. Thus,

$$N(X)=[H,K(H),\frac{H}{8}+u^*], \tag{2.3.14}$$

$$N(Y)=[h,K(h)=[K(H),4u^*],\frac{h}{2}+2u^*]=[h,K(h),\frac{h}{2}+w^*], \tag{2.3.15}$$

where $u^*+K(H)$ has order 8 in $K(H)^*/K(H)$, $w^*=2u^*$, $K(h)=[K(H),2w^*=4u^*]$. Here we agreed notations with Sect. 2.2. We have $\det N(X)=\det K(H)/8$ and $\det N(Y)=\det K(H)/8$. Thus, $\det N(X)=\det N(Y)$ for this case. We can formally put here $h=\frac{H}{2}$ since $h^2=\left(\frac{H}{2}\right)^2=2$. Then

$$N(X)\cap N(Y)=[H,K(H),\frac{H}{4}+2u^*]. \tag{2.3.16}$$

We have

$$[N(X):N(X)\cap N(Y)]=[N(Y):N(X)\cap N(Y)]=2. \tag{2.3.17}$$

From these calculations, we have:

Lemma 2.3.1. *For Mukai identification (2.1.5), the sublattice $T(X) \subset T(Y)$ has index 2, if $\gamma(H) = 2, 4, 8$, and $T(X) = T(Y)$, if the $\gamma(H) = 1$ for $H \in N(X)$.*

Also $T(X) \subset T(Y)$ has index 2 if either $\gamma(h) = 2$ for $h \in N(Y)$ or $\det K(h) \neq \det K(H)/4$. If the $\gamma(h) = 1$ and $\det K(h) = \det K(H)/4$, then $\gamma(H) = 2$ or 1, and $T(X) \subset T(Y)$ has index 2, if $\gamma(H) = 2$, and $T(X) = T(Y)$, if $\gamma(H) = 1$ (we cannot get $T(X) = T(Y)$ using only the Picard lattice $N(Y)$ of Y).

Proof. Really, since $v \in N(X)$, $T(X) \perp N(X)$ and $T(X) \cap \mathbb{Z}v = \{0\}$, the Mukai identification (2.1.5) gives an embedding $T(X) \subset T(Y)$. We then have $\det T(Y) = \det T(X)/[T(Y) : T(X)]^2$. Moreover, $|\det T(X)| = |\det N(X)|$ and $|\det T(Y) = |\det N(Y)|$ because the transcendental and the Picard lattice are orthogonal to each other in a unimodular lattice $H^2(*, \mathbb{Z})$. By calculations above, we get the statement.

We remark that the first statement of Lemma 2.3.1 is a particular case of the general statement by Mukai [Mu2] that $[T(Y) : T(X)] = q$ where

$$q = \min |v \cdot x| \qquad (2.3.18)$$

for all $x \in H^0(X, \mathbb{Z}) \oplus N(X) \oplus H^4(X, \mathbb{Z})$ such that $v \cdot x \neq 0$. For our Mukai vector $v = (2, H, 2)$, it is easy to see that $q = 2$, if $\gamma(H) = 2, 4, 8$, and $q = 1$, if $\gamma(H) = 1$.

Now we can prove propositions 2.2.1 and 2.2.2. If $X \cong Y$, then $T(X) \cong T(Y)$. Then $\det T(X) = \det T(Y)$, and $[T(Y) : T(X)] = 1$ for the Mukai identification. Then, $T(X) = T(Y)$ for the Mukai identification (2.1.5). By Lemma 2.3.1, we then get first statements of Propositions 2.2.1 and 2.2.2.

Assume that $\gamma(H) = 1$ for Propositions 2.2.1 and 2.2.2. Then $T(X) = T(Y)$ for the Mukai identification (2.1.5). By the discriminant forms technique (see [N2]), then the discriminant quadratic forms $q_{N(X)} = -q_{T(X)}$ and $q_{N(Y)} = -q_{T(Y)}$ are isomorphic. Thus, lattices $N(X)$ and $N(Y)$ have the same signatures and discriminant quadratic forms. It follows (see [N2]) that they have the same genus: $N(X) \otimes \mathbb{Z}_p \cong N(Y) \otimes \mathbb{Z}_p$ for any prime p and the ring of p-adic integers $\mathbb{Z}_p$. Additionally, assume that the genus of $N(X)$ or genus of $N(Y)$ contains only one class. Since $N(X)$ and $N(Y)$ have the same genus, they are then isomorphic.

If additionally the canonical homomorphism $O(N(X)) \to O(q_{N(X)})$ (equivalently, $O(N(Y)) \to O(q_{N(Y)})$) is epimorphic, then the Mukai identification $N(X) = N(Y)$ can be extended to give an isomorphism $\phi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ of cohomology lattices. The Mukai identification is identical on $H^{2,0}(X) = H^{2,0}(Y)$. Changing ϕ by ± 1 and by elements of the reflection group $W^{(-2)}(N(X))$, if necessary, we can assume that $\phi(H^{2,0}(X)) = H^{2,0}(Y)$ and ϕ maps the Kähler cone of X to the Kähler cone of Y. By global Torelli Theorem for K3 surfaces proved by Piatetskii-Shapiro and Shafarevich [PS-S], ϕ is then defined by an isomorphism of K3 surfaces X and Y.

If $\rho(X) \geq 12$, by [N2, Theorem 1.14.4], the primitive embedding of $T(X) = T(Y)$ to the cohomology lattice $H^2(X, \mathbb{Z})$ of K3 surfaces is unique up to isomorphisms of the lattice $H^2(X, \mathbb{Z})$. Like above, it then follows that X is isomorphic to Y.

Let us prove Theorems 2.2.3 and 2.2.4.

Assume that $\gamma(H)=1$. The Mukai identification then gives the canonical identification

$$T(X)=T(Y). \tag{2.3.19}$$

Thus, it gives the canonical identifications

$$\begin{aligned} N(X)^*/N(X) =&(U\oplus N(X))^*/(U\oplus N(X))=T(X)^*/T(X)=\\ T(Y)^*/T(Y) =&N(Y)^*/N(Y). \end{aligned} \tag{2.3.20}$$

Here $N(X)^*/N(X)=(U\oplus N(X))^*/(U\oplus N(X))$ because U is unimodular, $(U\oplus N(X))^*/(U\oplus N(X))=T(X)^*/T(X)$ because $U\oplus N(X)$ and $T(X)$ are orthogonal complements to each other in the unimodular lattice $H^*(X,\mathbb{Z})$. Here $T(Y)^*/T(Y)=N(Y)^*/N(Y)$ because $T(Y)$ and $N(Y)$ are orthogonal complements to each other in the unimodular lattice $H^2(Y,\mathbb{Z})$. E. g. the identification $(U\oplus N(X))^*/(U\oplus N(X))=T(X)^*/T(X)$ is given by the canonical correspondence

$$x^*+(U\oplus N(X))\to t^*+T(X) \tag{2.3.21}$$

if $x^*\in(U\oplus N(X))^*$, $t^*\in T(X)^*$ and $x^*+t^*\in H^*(X,\mathbb{Z})$.

By (2.3.15), we also have the canonical embedding of lattices

$$K(H)\subset K(h)=[K(H),4u^*]. \tag{2.3.22}$$

We have the key statement:

Lemma 2.3.2. *The canonical embedding* (2.3.22) *(it is given by* (2.3.15)*)* $K(H)\subset K(h)$ *of lattices, and the canonical identification* $N(X)^*/N(X)=N(Y)^*/N(Y)$ *(given by* (2.3.20)*) agree with the characteristic homomorphisms* $\kappa(H):K(H)^*\to N(X)^*/N(X)$ *and* $\kappa(h):K(h)^*\to N(Y)^*/N(Y)$*, i.e.* $\kappa(h)(k^*)=\kappa(H)(k^*)$ *for any* $\kappa^*\in K(h)^*\subset K(H)^*$ *(this embedding is dual to* (2.3.22)*).*

Proof. From definitions of the identifications (2.3.20), the identification $(U\oplus N(X))^*/(U\oplus N(X))=N(Y)^*/N(Y)$ is given by the canonical embeddings

$$(U\oplus N(X))^*\supset(v^\perp)^*_0=(v^\perp/\mathbb{Z}v)^* \tag{2.3.23}$$

where $(v^\perp)^*_0=\{s^*\in(U\oplus N(X))^*\mid s^*\cdot v=0\}$.

By (2.2.5), $s^*=x_1e_1+x_2e_2+yH^*+k^*\in(U\oplus N(X))^*$ if and only if $x_1,x_2,y\in\mathbb{Z}$, $k^*\in K(H)^*$, and $y\equiv -8k^*\cdot u^* \mod 8$. Here $\kappa(k^*)=s^*+(U\oplus N(X))$. We have $s^*\cdot v=-2x_1-2x_2+y$, and $s^*\in(v^\perp)^*_0$, if additionally $y=2x_1+2x_2$. Thus, we have $2(x_1+x_2)\equiv -8k^*\cdot u^* \mod 8$. It follows, $k^*\cdot(4u^*)\in\mathbb{Z}$, and $k^*\in K(h)^*=[K(h),4u^*]$. Moreover, $x_1+x_2\equiv -2k^*\cdot w^* \mod 4$ and $-x_1+x_2\equiv x_1+x_2\equiv -2k^*\cdot w^* \mod 2$ where $w^*=2u^*$. Like in (2.3.7), we then have that

$s^* \in (v^\perp)_0^*$ if and only if $s^* = (-x_1+x_2)h^* + \frac{x_1+x_2}{4}v + k^*$ where x_1, x_2, k^* satisfy the conditions above. Finally, we have $s^* \in (v^\perp)_0^*$, if and only if

$$s^* = ah^* + b\frac{v}{4} + k^* \tag{2.3.24}$$

where $a, b \in \mathbb{Z}$, $k^* \in K(h)^*$, $a \equiv -2k^* \cdot w^* \mod 2$ and $b \equiv -2k^* \cdot w^* \mod 4$. Here s^* gives $ah^* + k^* \in N(Y)^*$, and $\kappa(h)(k^*) = ah^* + k^* + N(Y) = s^* + U \oplus N(X) = \kappa(H)(k^*)$ under the identification (2.3.20). It proves the statement.

Let us prove Theorems 2.2.3. We have the Mukai identification (it is defined by (2.1.5)) of the transcendental periods

$$(T(X),\ H^{2,0}(X)) = (T(Y),\ H^{2,0}(Y)). \tag{2.3.25}$$

For general X with the Picard lattice $N(X)$, it is the unique isomorphism of the transcendental periods up to multiplication by ± 1. If $X \cong Y$, this (up to ± 1) isomorphism can be extended to $\phi : H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$. Restriction of ϕ on $N(X)$ gives then isomorphism $\phi_1 : N(X) \cong N(Y)$ which is ± 1 on $N(X)^*/N(X) = N(Y)^*/N(Y)$ under the identification (2.3.20). The element $\tilde{h} = (\phi_1)^{-1}(h)$ and $f = \phi^{-1}$ satisfy Theorem 2.2.3 by Lemma 2.3.2.

The other way round, under conditions of Theorem 2.2.3, by Lemma 2.3.2, one can construct an isomorphism $\phi_1 : N(X) \cong N(Y)$ which is ± 1 on $N(X)^*/N(X) = N(Y)^*/N(Y)$. It can be extended to be ± 1 on on transcendental periods under the Mukai identification (2.3.25). Then it is defined by the isomorphism $\phi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$. Changing ϕ by ± 1 and by reflections from $W^{(-2)}(N(X))$, if necessary (the group $W^{(-2)}(N(X))$ acts identically on the discriminant group $N(X)^*/N(X)$), we can assume that ϕ maps the Kähler cone of X to the Kähler cone of Y. By global Torelli Theorem [PS-S], it is then defined by an isomorphism of X and Y.

To prove Theorem 2.2.4, one should argue similarly.

3. The case of Picard number 2

3.1. General results. Here we want to apply results of Sect. 2 to X and Y with the Picard number 2. We start with some preliminary considerations of K3 surfaces with the Picard number 2 and primitive polarizations of degree 8 and 2.

Here we assume that rk $N(X) = 2$. Let $H \in N(X)$ be a primitive polarization with $H^2 = 8$. Assume that $\gamma(H) = 1$ for $H \in N(X)$ (we have this condition, if $Y \cong X$). Let $H^\perp_{N(X)} = \mathbb{Z}\delta$ and $\delta^2 = -t$ where $t > 0$ is even. It then follows that $N(X) = [\mathbb{Z}H,\ \mathbb{Z}\delta,\ H^* + \mu\frac{\delta}{8}]$ where $H^* = H/8$ and $\mu = \pm 1, \pm 5$. We have $(H^* + \mu\frac{\delta}{8})^2 = 1/8 - \mu^2 t/64 \equiv 0 \mod 2$. It follows, $t = -8d$ where $d \in \mathbb{N}$ and $1 - \mu^2 d \equiv 0 \mod 16$. Then d is odd and $\mu^2 d \equiv 1 \mod 16$. It follows that $d \equiv 1 \mod 8$ and $\mu^2 \equiv d \mod 16$. Changing δ by $-\delta$, we can always assume that $\mu = 1$ or $\mu = 3$ where $\mu = 1$ if $d \equiv 1 \mod 16$, and $\mu = 3$ if $d \equiv 9 \mod 16$. These simple calculations show that the lattice $N(X)$ and H (one can even replace H by any primitive element of $N(X)$ with square 8) are defined uniquely by d where $-d = \det N(X)$. Thus, we have

Proposition 3.1.1. *Let X be a K3 surface with the Picard number $\rho = rk\, N(X) = 2$. Assume that X has a primitive polarization H of degree $H^2 = 8$ and $\gamma(H) = 1$ for $H \in N(X)$. Then the lattice $N(X)$ is defined by its determinant* $\det N(X) = -d$ *where $d \in \mathbb{N}$ and $d \equiv 1 \mod 8$. There exists a unique choice of a primitive orthogonal vector $\delta \in (H^{\perp})_{N(X)}$ such that*

$$N(X) = [H, \delta, (H + \mu\delta)/8] \qquad (3.1.1)$$

where $\delta^2 = -8d$,

$$\begin{cases} \mu = 1, & \text{if } d \equiv 1 \mod 16, \\ \mu = 3, & \text{if } d \equiv 9 \mod 16. \end{cases} \qquad (3.1.2)$$

We have

$$N(X) = \{z = (xH + y\delta)/8 \mid x, y \in \mathbb{Z} \text{ and } \mu x \equiv y \mod 8\}. \qquad (3.1.3)$$

For any primitive element $H' \in N(X)$ with $(H')^2 = H^2 = 8$, there exists a unique automorphism $\phi \in O(N(X))$ such that $\phi(H) = H'$.

We shall denote the described above (unique up to isomorphisms) hyperbolic lattice $N(X)$ of the determinant $-d$ as N_d^8 where $d \in \mathbb{N}$ and $d \equiv 1 \mod 8$.

Proof. Let H' be a primitive element of $N(X) \cong N_d^8$ with square 8. It is easy to see that, if $\gamma(H') \neq 1$, then $\det N(X)$ is even. These is impossible. The rest calculations are very elementary.

Now we assume that rk $N(Y) = 2$. Let $h \in N(Y)$ be a primitive polarization of degree $h^2 = 2$. Assume that $\gamma(h) = 1$ (we have this condition if $Y \cong X$. Let $h^{\perp}_{N(Y)} = \mathbb{Z}\alpha$ and $\alpha^2 = -t$ where $t > 0$ is even. It then follows that $N(Y) = [\mathbb{Z}h, \mathbb{Z}\alpha, h^* + \frac{\alpha}{2}]$ Where $h^* = h/2$. We have $(h^* + \frac{\alpha}{2})^2 = 1/2 - t/4 \equiv 0 \mod 2$. It follows that $t = -2d$ where $d \in \mathbb{N}$ and $d \equiv 1 \mod 4$. Like for X above, we then get

Proposition 3.1.2. *Let Y be a K3 surface with the Picard number $\rho = rk\, N(Y) = 2$. Assume that Y has a primitive polarization h of degree $h^2 = 2$ and $\gamma(h) = 1$ for $h \in N(Y)$. Then the lattice $N(Y)$ is defined by its determinant* $\det N(Y) = -d$ *where $d \in \mathbb{N}$ and $d \equiv 1 \mod 4$. For a primitive orthogonal vector $\alpha \in (h^{\perp})_{N(Y)}$ (it is unique up to changing by $-\alpha$), the lattice*

$$N(Y) = [h,\ \alpha,\ (h + \alpha)/2] \qquad (3.1.4)$$

where $\alpha^2 = -2d$. We have

$$N(Y) = \{z = (xh + y\alpha)/2 \mid x, y \in \mathbb{Z} \text{ and } x \equiv y \mod 2\}. \qquad (3.1.5)$$

For any primitive vector $h' \in N(Y)$ with square $h' = h^2 = 2$, there exists an automorphism $\phi \in O(N(X))$ such that $\phi(h) = h'$. It is unique up to changing to ϕs_α where $s_\alpha(h) = h$, $s_\alpha(-\alpha) = -\alpha$.

We shall denote the described above unique (up to isomorphisms) hyperbolic lattice $N(Y)$ of determinant $-d$ as N_d^2 where $d \in \mathbb{N}$ and $d \equiv 1 \mod 4$.

Proof. It is trivial.

From Propositions 3.1.1 and 3.1.2, we then get

Proposition 3.1.3. *We follow conditions and notations of Propositions 3.1.1 and 3.1.2.*

All elements $h' = (xH + y\delta)/8 \in N(X)$ with square $(h')^2 = 2$ are in one to one correspondence with solutions (x, y) of the equation

$$x^2 - dy^2 = 16 \tag{3.1.6}$$

with odd x, y and $\mu x \equiv y \mod 8$. Changing an odd solution (x, y) of the equation to $(x, -y)$, if necessary, one can always satisfy the last congruence.

The Picard lattices of X and Y are isomorphic, $N(X) \cong N(Y)$, if and only if $\det N(X) = \det N(Y) = -d$ (it follows that $d \equiv 1 \mod 8$), and an element $h' \in N(X)$ with $(h')^2 = 2$ from above does exist. Equivalently, the equation $x^2 - dy^2 = 16$ has a solution with odd x and y.

Proof. Assume, $h' = (xH + y\delta)/8 \in N(X)$ and $(h')^2 = 2$. This equivalent to $x, y \in \mathbb{Z}$, $\mu x \equiv y \mod 8$ and $2 = (x^2 - dy^2)/8$, equivalently, $x^2 - dy^2 = 16$. Thus, the elements h' are in one to one correspondence with integral solutions (x, y) of the equation $x^2 - dy^2 = 16$ which satisfy the condition $x\mu \equiv y \mod 8$.

Let (x, y) be an integral solution of the equation. Clearly, then $x \equiv y \mod 2$. Let (x, y) are even. Then $(x, y) = 2(x_1, y_1)$ where (x_1, y_1) is an integral solution of the equation $x_1^2 - dy_1^2 = 4$. Again $x_1 \equiv y_1 \mod 2$. If $x_1 \equiv y_1 \equiv 1 \mod 2$, we get a contradiction with $d \equiv 1 \mod 8$. If $x_1 \equiv y_1 \equiv 0 \mod 2$, we get that $(x, y) = 4(x_2, y_2)$ where (x_2, y_2) are integral solutions of the equation $x_2^2 - dy_2^2 = 1$. It then follows that $x_2 \mod 2$ and $y_2 \mod 2$ are different. Then the congruence $\mu x \equiv y \mod 8$ is not satisfied, and the solution (x, y) does not give an element of $N(X)$.

Assume that (x, y) is a solution of $x^2 - dy^2 = 16$ with odd x, y. We have $d \equiv \mu^2 \mod 16$ and $(x - y\mu)(x + y\mu) \equiv 0 \mod 16$. If $x - y\mu \equiv 0 \mod 4$ and $x + y\mu \equiv 0 \mod 4$, then $2x \equiv 0 \mod 4$ which is impossible for odd x. Thus, we have that one of congruences: $x - y\mu \equiv 0 \mod 8$ or $x + y\mu \equiv 0 \mod 8$ is valid. It follows that exactly one of solutions (x, y) or $(x, -y)$ gives an element of $N(X)$. It proves first statement.

If $N(Y) \cong N(X)$, the lattices should have the same determinant. By Proposition 3.1.2, $N(X) \cong N(Y)$, if and only if $N(X)$ has an element h' with square 2. This finishes the proof.

We have a similar statements in terms of Y.

Proposition 3.1.4. *We follow conditions and notations of Propositions 3.1.1 and 3.1.2.*

All primitive elements $H' = (xh + y\alpha)/2 \in N(Y)$ with square $(H')^2 = 8$ are in one to one correspondence with solutions (x, y) of the equation

$$x^2 - dy^2 = 16$$

with odd x, y.

The Picard lattices of X and Y are isomorphic, $N(X) \cong N(Y)$, if and only if $\det N(X) = \det N(Y) = -d$ (it follows that $d \equiv 1 \mod 8$), and an element $H' \in N(Y)$ with $(H')^2 = 8$ from above does exist. Equivalently, the equation $x^2 - dy^2 = 16$ has a solution with odd x and y. It follows that $d \equiv 1 \mod 8$.

Proof. Let $H' = (xh + y\alpha)/2$ where $x, y \in \mathbb{Z}$ and $x \equiv y \mod 2$. Then $2 = (x^2 - dy^2)/8$ or $x^2 - dy^2 = 16$. The element H' is not primitive if and only if $x/2, y/2 \in \mathbb{Z}$ and $x/2 \equiv y/2 \mod 2$. This is equivalent that $x \equiv y \equiv 0 \mod 2$ and $x \equiv y \mod 4$. Thus, the elements H' are in one to one correspondence with integral solutions (x, y) of the equation $x^2 - dy^2 = 16$ which satisfy the condition that either they are both odd or both even, but $x \mod 4$ and $y \mod 4$ are different.

Assume that (x, y) is an integral solution of the equation $x^2 - dy^2 = 16$. Clearly, then $x \equiv y \mod 2$. If $x \equiv y \equiv 1 \mod 2$, then (x, y) gives a primitive elements H' by previous considerations. Let (x, y) are even. Then $(x, y) = 2(x_1, y_1)$ where (x_1, y_1) is an integral solution of the equation $x_1^2 - dy_1^2 = 4$. Again $x_1 \equiv y_1 \mod 2$. It then follows that $x \equiv y \mod 4$, and the solution (x, y) does not give a primitive element H'.

If $N(Y) \cong N(X)$, the lattices should have the same determinant. By Proposition 3.1.1, $N(X) \cong N(Y)$, if and only if the lattice $N(Y)$ has a primitive element H' with square 8.

It finishes the proof.

Now we can apply Theorems 2.2.3 and 2.2.4 to find out when $X \cong Y$. We have

Theorem 3.1.5. *Let X be a K3 surface and $\rho(X) = rk\, N(X) = 2$. Assume that X is an intersection of three quadrics (more generally, X has a primitive polarization H of degree 8). Let Y be a K3 surface which is the double covering of the net $\mathbb{P}^2$ of the quadrics ramified in the curve of degenerate quadrics and h is preimage of line of $\mathbb{P}^2$ (more generally, Y is the moduli space of vector bundles on X with the Mukai vector $v = (2, H, 2)$ and the canonical polarization $h = (-1, 0, 1) \mod \mathbb{Z}v$).*

If $Y \cong X$, then

$$\gamma(H) = 1,\ \det N(X) = -d \text{ where } d \equiv 1 \mod 8. \qquad (3.1.7)$$

Further we assume (3.1.7) *and follow notations of Propositions 3.1.1 and 3.1.3 for X and H.*

We have that all elements $\tilde{h} = (xH + y\delta)/8 \in N(X)$ with square $\tilde{h}^2 = 2$ satisfying Theorem 2.2.3 are in one to one correspondence with integral solutions (x, y) of the equation

$$x^2 - dy^2 = 16 \qquad (3.1.8)$$

with odd x, y, and $x \equiv \pm 4 \mod d$, and $\mu x \equiv y \mod 8$ (one can always satisfy the congruence $\mu x \equiv y \mod 8$ changing y to $-y$ if necessary).

In particular (by Theorem 2.2.3), for a general K3 surface X with $\rho(X) = 2$ and $\det N(X) = -d$ where $d \equiv 1 \mod 8$, we have $Y \cong X$, if and only if the equation $x^2 - dy^2 = 16$ has an integral solution (x, y) with odd (x, y) and $x \equiv \pm 4 \mod d$; a *polarization $h=(xH+y\delta)/8$ of X with $h^2=2$ defines a structure of double plane on X which is isomorphic to the double plane Y, if and only if $x\equiv \pm 4 \mod d$.*

Proof. If $Y\cong X$, then $\gamma(H)=1$ by Proposition 2.2.1. By Proposition 3.1.1, $N(X)\cong N_d^8$ where $d\equiv 1 \mod 8$ and $\det N(X)=-d$ where $d\equiv 1 \mod 8$.

Assume that $Y\cong X$ for any general X with the Picard lattice N_d^8. Let $\tilde{h}\in N(X)$ satisfies conditions of Theorem 2.2.3.

By Proposition 3.1.3, all primitive

$$\tilde{h}=(xH+y\delta)/8 \tag{3.1.9}$$

are in one to one correspondence with odd (x,y) which satisfy the equation $x^2-dy^2=16$ and $y\equiv \mu x \mod 8$, and any integral solution of the equation $x^2-dy^2=16$ with odd x, y gives such $\tilde{h}$ after replacing y to $-y$ if necessary, which does not matter for the statement of Theorem.

Let $k=aH+b\delta\in \tilde{h}^\perp=\mathbb{Z}\alpha$. Then $ax-byd=0$ and $(a,b)=\lambda(yd,x)$. We have $(\lambda(ydH+x\delta))^2=\lambda^2(8y^2d^2-8dx^2)=8\lambda^2d(y^2d-x^2)=-2^7\lambda^2d$. Since $\alpha^2=-2d$, we get $\lambda=2^{-3}$ and $\alpha=(ydH+x\delta)/8$. There exists a unique (up to ± 1) embedding $f:K(H)=\mathbb{Z}\delta\to K(h)=\mathbb{Z}\alpha$ of one-dimensional lattices. It is given by $f(\delta)=2\alpha$ up to ± 1. Thus, its dual is defined by $f^*(\alpha^*)=2\delta^*$ where $\alpha^*=\alpha/2d$ and $\delta^*=\delta/8d$. To satisfy the conditions of Theorem 2.2.3, we should have

$$\kappa(\tilde{h})(\alpha^*)=\pm 2\kappa(H)(\delta^*). \tag{3.1.10}$$

We have $u^*=\mu d\delta^*$, $w^*=2f(u^*)=\mu d\alpha^*=\mu\alpha/2$, and

$$\kappa(\tilde{h})(\alpha^*)=(-2\alpha^*\cdot w^*)\tilde{h}^*+\alpha^*+N(X) \tag{3.1.11}$$

by (2.2.18). Here $\tilde{h}^*=\tilde{h}/2$. We then have $\alpha^*\cdot w^*=\mu(\alpha/2d)\cdot(\alpha/2)=-\mu/2$, and $\kappa(\tilde{h})(\alpha^*)=\mu\tilde{h}^*+\alpha^*+N(X)=\mu(x(H/16)+y(\delta/16))+y(H/16)+x\delta/16d$. It follows,

$$\kappa(\tilde{h})(\alpha^*)=\frac{\mu x+y}{16}H+\frac{x+\mu yd}{16d}\delta+N(X)=\frac{\mu x+y}{2}H^*+\frac{x+\mu yd}{2}\delta^*+N(X). \tag{3.1.12}$$

where $H^*=H/8$. We have $u^*=\mu d\delta^*=\mu\delta/8$. By (2.2.8), $\kappa(H)(\delta^*)=(-8\delta^*\cdot u^*)H^*+\delta^*+N(X)$. We have $\delta^*\cdot u^*=-\mu/8$. It follows,

$$\kappa(H)(\delta^*)=\mu H^*+\delta^*. \tag{3.1.13}$$

By (3.1.12) and (3.1.13), we then get that $\kappa(H)(2\delta^*)=\pm\kappa(\tilde{h})(\alpha^*)$ is equivalent to $(x+\mu yd)/2\equiv \pm 2 \mod d$ or $x+\mu yd\equiv \pm 4 \mod d$ since the group $N(X)^*/N(X)$ is cyclic of order d and it is generated by $\mu H^*+\delta^*+N(X)$. Thus, finally we get $x\equiv \pm 4 \mod d$.

The condition to have an odd solution (x, y) of (3.1.18) with $x \equiv \pm 4 \mod d$ does not depend on the choice of X and the polarization H. If this condition is satisfied, it is valid for any X with the Picard lattice $N(X)$ and any its primitive polarization H of degree 8. By Theorem 2.2.3, $Y \cong X$ for all of them. This finishes the proof.

We have a similar statement in terms of Y.

Theorem 3.1.6. *Let X be a K3 surface and $\rho(X) = rk\, N(X) = 2$. Assume that X is an intersection of three quadrics (more generally, X has a primitive polarization H of degree 8). Let Y be a K3 surface which is the double covering of the net $\mathbb{P}^2$ of the quadrics ramified in the curve of degenerate quadrics and $h \in N(Y)$ is preimage of line of $\mathbb{P}^2$ (more generally, Y is the moduli space of vector bundles on X with the Mukai vector $v = (2, H, 2)$ and the canonical polarization $h = (-1, 0, 1) \mod \mathbb{Z}v$).*

If $Y \cong X$, then

$$\gamma(H) = 1, \ \gamma(h) = 1, \ \det N(Y) = -d \text{ where } d \equiv 1 \mod 8. \tag{3.1.14}$$

Further we assume (3.1.14) and follow notations of Propositions 3.1.2 and 3.1.4 for Y and h.

We have that all elements $\widetilde{H} = (xh + y\alpha)/2 \in N(Y)$ with square $\widetilde{H}^2 = 8$ satisfying Theorem 2.2.4 are in one to one correspondence with integral solutions (x, y) of the equation

$$x^2 - dy^2 = 16 \tag{3.1.15}$$

with odd x, y, and $x \equiv \pm 4 \mod d$.

In particular (by Theorem 2.2.4), for a general K3 surface Y with $\rho(Y) = 2$ and $\det N(Y) = -d$ where $d \equiv 1 \mod 8$, we have $Y \cong X$, if and only if the equation $x^2 - dy^2 = 16$ has an integral solution (x, y) with odd (x, y) and $x \equiv \pm 4 \mod d$; a primitive polarization $H = (xh + y\alpha)/2$ of X with $H^2 = 8$ defines a structure of intersection of three quadrics on Y which is isomorphic to the structure of intersection of three quadrics of X, if and only if $x \equiv \pm 4 \mod d$.

Proof. If $Y \cong X$, then $\gamma(H) = 1$ and $\gamma(h) = 1$, by Proposition 2.2.2. By Propositions 3.1.2 and 3.1.4, $N(Y) \cong N_d^2$ where $d \equiv 1 \mod 8$ and $\det N(Y) = -d$.

Assume that $Y \cong X$ for a general Y with the Picard lattice N_d^2. Let $\widetilde{H} \in N(Y)$ satisfies conditions of Theorem 2.2.4.

By Proposition 3.1.4, all primitive

$$\widetilde{H} = (xh + y\alpha)/2 \tag{3.1.16}$$

with $\widetilde{H}^2 = 8$ are in one to one correspondence with odd (x, y) which satisfy the equation $x^2 - dy^2 = 16$.

Let $k = ah + b\alpha \in \widetilde{H}^{\perp} = \mathbb{Z}\delta$. Then $ax - byd = 0$ and $(a, b) = \lambda(yd, x)$. We have $(\lambda(ydh + x\alpha))^2 = \lambda^2(2y^2d^2 - 2dx^2) = 2\lambda^2 d(y^2d - x^2) = -2^5\lambda^2 d$. Since $\delta^2 = -8d$, we get $\lambda = 2^{-1}$, and

$$\delta = (ydh + x\alpha)/2. \tag{3.1.17}$$

There exists a unique (up to ± 1) embedding $f : K(\widetilde{H}) = \mathbb{Z}\delta \to K(h) = \mathbb{Z}\alpha$ of one-dimensional lattices. It is given by $f(\delta) = 2\alpha$ up to ± 1. Thus, its dual is defined by $f^*(\alpha^*) = 2\delta^*$ where $\alpha^* = \alpha/2d$ and $\delta^* = \delta/8d$. To satisfy the conditions of Theorem 2.2.4, we should have

$$\kappa(h)(\alpha^*) = \pm 2\kappa(\widetilde{H})(\delta^*). \tag{3.1.18}$$

Like for the proof above, we have

$$\kappa(h)(\alpha^*) = \mu h^* + \alpha^* + N(Y) \tag{3.1.19}$$

where $h^* = h/2$, and

$$\kappa(\widetilde{H})(\delta^*) = \mu \widetilde{H}^* + \delta^* \tag{3.1.20}$$

where $\widetilde{H}^* = \widetilde{H}/8$. By (3.1.16) and (3.1.17), we then have

$$\kappa(\widetilde{H})(\delta^*) = \mu(xh + y\alpha)/16 + (ydh + x\alpha)/16d + N(Y). \tag{3.1.21}$$

It follows,

$$\kappa(\widetilde{H})(\delta^*) = \frac{\mu x + y}{8} h^* + \frac{x + \mu dy}{8}\alpha^* + N(Y). \tag{3.1.22}$$

Thus, $\kappa(\widetilde{H})(2\delta^*) = \pm\kappa(h)(\alpha^*)$ is equivalent to $(x + \mu yd)/4 \equiv \pm 1 \mod d$ or $x + \mu yd \equiv \pm 4 \mod d$ since the group $N(Y)^*/N(Y)$ is cyclic of the order d, and it is generated by $\mu h^* + \alpha^* + N(Y)$. Thus, finally we get $x \equiv \pm 4 \mod d$.

The condition to have an odd solution (x, y) of (3.1.15) with $x \equiv \pm 4 \mod d$ does not depend on the choice of Y and the polarization h. If this condition is satisfied, it is valid for any Y with the Picard lattice $N(Y)$ and any its polarization h of degree 2. By Theorem 2.2.4, $Y \cong X$ for all of them. This finishes the proof.

By Theorems 3.1.5 and 3.1.6 general X and Y with $\rho = 2$ and $X \cong Y$ are parametrized by $d \in \mathbb{N}$ such that the conditions

$$x^2 - dy^2 = 16,\ x \equiv y \mod 2,\ x \equiv \pm 4 \mod d \tag{3.1.23}$$

are satisfied for some integers x, y. Since (x, y) are odd, it follows that $d \equiv 1 \mod 8$, which we had known. Since $x \equiv \pm 4 \mod d$ and x is odd, we can write down all these x as

$$x = \pm 4 + kd,\ k \in \mathbb{Z},\ k \equiv 1 \mod 2. \tag{3.1.24}$$

The equation (2.1.23) gives then $16 \pm 8kd + k^2d^2 - dy^2 = 16$ or $\pm 8k + k^2d - y^2 = 0$. It follows that

$$d = \frac{y^2 + 8k}{k^2} \tag{3.1.25}$$

where k is any odd integer. Let p is odd prime, $p^{2t+1}|k$ and p^{2t+2} does not divide k for some $t \geq 0$. Since $k|(y^2+8k)$, it follows that $k|y^2$ and $p^{t+1}|y$. It follows $p^{2t+2}|y^2$. Since $p^{4t+2}|k^2$, it then follows that $p^{2t+2}|y^2+8k$ and $p^{2t+2}|y^2$. Thus, $p^{2t+2}|k$. We get a contradiction. It shows that an odd prime p my divide k only in even power. Thus $k = \mp b^2$ for some $b \in \mathbb{N}$. Then $b|y$ (since $k|y^2$), and $y = ab$ for some integer a. Thus, finally we get that

$$d = \frac{a^2 \mp 8}{b^2} \tag{3.1.26}$$

for some odd integers a and b. Thus, a, b are odd solutions of one of equations

$$a^2 - b^2 d = 8, \tag{3.1.27}$$

$$a^2 - b^2 d = -8. \tag{3.1.28}$$

If the equations have a solution with odd (a, b), then $d \equiv 1 \mod 8$. Further we suppose that. Simple considerations show that then any solution (a, b) of the equations (3.1.27) and (3.1.28) has odd a and b. Our considerations show that solutions (x, y) of (3.1.23) are in one to one correspondence with solutions (a, b) of the equations (3.1.27) and (3.1.28). More exactly, we get solutions

$$(x, y) = \pm\left(4 + b^2 d,\ ab\right), \text{ if } a^2 - b^2 d = 8 \tag{3.1.29}$$

and

$$(x, y) = \pm\left(b^2 d - 4,\ ab\right), \text{ if } a^2 - b^2 d = -8 \tag{3.1.30}$$

of (3.1.23), and any solution of (3.1.23) can be written in this form. We call solutions (3.1.29) and (3.1.30) of (3.1.23) as *associated solutions with solutions of the equations* (3.1.27) *and* (3.1.28) *respectively.*

Thus, we get the final result

Theorem 3.1.7. *Let X be a K3 surface and $\rho(X) = rk\, N(X) = 2$. Assume that X is an intersection of three quadrics (more generally, X has a primitive polarization H of degree 8). Let Y be a K3 surface which is the double covering of the net $\mathbb{P}^2$ of the quadrics ramified in the curve of degenerate quadrics and h is preimage of line of $\mathbb{P}^2$ (more generally, Y is the moduli space of vector bundles on X with the Mukai vector $v = (2, H, 2)$ and the canonical polarization $h = (-1, 0, 1) \mod \mathbb{Z}v$).*

Then $Y \cong X$ for a general X with $\rho(X) = 2$, if and only if $\det N(X) = -d$ where $d \equiv 1 \mod 8$, and one of equations

$$a^2 - b^2 d = 8, \tag{3.1.31}$$

or

$$a^2 - b^2 d = -8 \tag{3.1.32}$$

has an integral solutions. All solutions of these equations have odd a and b, and the set of possible d is union of two infinite sets

$$d \in \mathcal{D}_+ = \{\frac{a^2-8}{b^2} \in \mathbb{N} \mid \textit{where } a,\ b \in \mathbb{N} \textit{ are odd}\} \tag{3.1.33}$$

for the equation (3.1.31), *and*

$$d \in \mathcal{D}_- = \{\frac{a^2+8}{b^2} \in \mathbb{N} \mid \textit{where } a,\ b \in \mathbb{N} \textit{ are odd}\} \tag{3.1.34}$$

for the equation (3.1.32)

Solutions of (3.1.31) *and* (3.1.32) *give all solutions of* (3.1.23) *as associated solutions* (3.1.29) *and* (3.1.30), *and all primitive elements* $\widetilde{h} \in N(X)$ *with* $\widetilde{h}^2 = 2$ *of Theorem 3.1.5, and* $\widetilde{H} \in N(Y)$ *with* $\widetilde{H}^2 = 8$ *of Theorem 3.1.6.*

We can also interpret solutions of (3.1.31) and (3.1.32) as appropriate elements of $N(X)$ and $N(Y)$.

We have

Theorem 3.1.8. *In conditions and notations of Proposition 3.1.1, the elements*

$$h_1 = (2aH + 2b\delta)/8 \in N(X), \tag{3.1.35}$$

where (a, b) *is any integral solution of* $a^2 - db^2 = \pm 8$ *satisfying the congruence* $\mu a \equiv b \mod 4$ *(one can always satisfy the congruence changing* b *to* $-b$ *if necessary) are all elements* $h_1 \in N(X)$ *with*

$$(h_1)^2 = \pm 4 \textit{ and } h_1 \cdot H \equiv 0 \mod 2. \tag{3.1.36}$$

In particular, if (a, b) *is a solution of* $a^2 - db^2 = 8$, *the surface* X *has a nef element* h_1 *with square* 4 *and the structure of quartic, if the* h_1 *is very ample.*

Existence of an element $h_1 \in N(X)$ *satisfying* (3.1.36) *(for one of signs + or −) is equivalent to* $Y \cong X$ *for a general* X *with* $\rho(X) = 2$. *In particular,* $Y \cong X$ *if* X *has a structure of quartic with the linear system* $|h_1|$ *of planes of even degree with respect to the hyperplane* H *of the intersection of quadrics (i. e.* $H \cdot h_1 \equiv 0 \mod 2$*). It is even sufficient to have an element* $\widetilde{h_1} \in W^{(-2)}(N(X))(h_1)$ *with* $\widetilde{h_1} \cdot H \equiv 0 \mod 2$.

Proof. Let (a, b) be a solution of $a^2 - db^2 = \pm 8$. It follows that a, b are odd. We have $(a - b\mu)(a + b\mu) \equiv 0 \mod 8$. Since a and b are odd, it follows that either $a - b\mu \equiv 0 \mod 4$ or $a + b\mu \equiv 0 \mod 4$. Changing b by $-b$ if necessary we can assume that $a \equiv b\mu \mod 4$. Then $2a \equiv 2b\mu \mod 8$ and $h_1 \in N(X)$. We have $(h_1)^2 = 4(a^2 - db^2)/8 = \pm 8/2 = \pm 4$. Vice versa, if $h_1 \in N(X)$ satisfies (3.1.36), it can be written in the form (3.1.35), where (a, b) satisfy $a^2 - b^2 d = \pm 8$.

A similar statement for Y is simpler.

Theorem 3.1.9. *In notations of Theorem 3.1.2, assume that* $d \equiv 1 \mod 8$. *Then elements*

$$h_1 = (ah + b\delta)/2 \in N(Y), \tag{3.1.37}$$

where (a, b) *is any integral solution of* $a^2 - db^2 = \pm 8$ *are all elements* $h_1 \in N(Y)$ *with*

$$(h_1)^2 = \pm 4. \tag{3.1.38}$$

In particular, if (a, b) *is a solution of* $a^2 - db^2 = 8$, *the surface* X *has a nef element* h_1 *with square* 4, *and a structure of quartic, if the* h_1 *is very ample.*

The equality $\gamma(H) = 1$ *(equivalently,* $d = -\det N(X)$ *is odd) and existence of an elements* $h_1 \in N(Y)$ *satisfying one of conditions:* $(h_1)^2 = 4$ *or* $(h_1)^2 = -4$ *are equivalent to* $Y \cong X$ *for a general* Y *with* $\rho(Y) = 2$. *In particular,* $Y \cong X$, *if* $\gamma(H) = 1$ *and* Y *has a structure of quartic.*

Proof. It is similar and simpler.

We remark that if $N(X)$ has elements h with $h^2 = 2$, then $N(X) \cong N(Y)$ by Proposition 3.1.3. By Theorem 3.1.9, then $Y \cong X$ if $N(X)$ has elements h_1 with $h_1^2 = 4$ or $h_1^2 = -4$ (one does not need the congruence $h_1 \cdot H \equiv 0 \mod 2$).

3.2. The list of first elements from $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$.

The binary equations (3.1.31) and (3.1.32) are very classical (e.g. see [BSh], [Ca], [CS]. Below we give the list of first elements from $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$. We have

Theorem 3.2.1. *The first elements of* $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$ *are*

$$\begin{gathered}1, 9, 17, 33, 41, 57, 73, 89, 97, 113, 129, 137, 153, 161, 177, 193, 201, 209, 217, 233, 241,\\ 249, 281, 297, 313, 329, 337, 353, 369, 393, 409, 417, 433, 449, 457, 489, 497, 513, 521,\\ 537, 553, 561, 569, 593, 601, 617, 633, 641, 649, 657, 673, 681, 713, 721, 737, 753, 769,\\ 801, 809, 833, 849, 857, 873, 881, 889, 913, 921, 929, 937, 953, 969, 977, 1017, 1033,\\ 1041, 1049, 1057, 1081, 1097, 1121, 1137, 1153, 1161, 1169, 1177, 1193, 1201, 1217,\\ 1233, 1241, 1249, 1273, 1289, 1321, 1329, 1337, 1353, 1361, 1377, 1401, 1409, 1433,\\ 1441, 1457, 1473, 1481, 1497, 1513, 1529, 1553, 1561, 1569, 1577\end{gathered}$$

Proof. These calculations are standard. We mention that the genus of the lattice N_d^2 represents ± 4 (it is equivalent to have a solution of (3.1.31) or (3.1.32)), if and only if for any prime $p|d$ (we assume that $d \equiv 1 \mod 8$) the number ± 2 is a square $\mod p$. Thus, these conditions are sufficient to have a solution of (3.1.31) or (3.1.32), if the genus of N_d^2 has only one class.

We hope to study geometry of K3 surfaces X and Y for $X \cong Y$ with the Picard lattices $N_d^8 \cong N_d^2$ in further publications.

3.3. An application to moduli of X and Y.

Calculations above can be interpreted from the point of view of moduli of intersections of three quadrics as follows.

It is well-known that the moduli space of the K3 surfaces X which are intersections of three quadrics is 19-dimensional. If X is general, i.e. $\rho(X) = 1$, then the surface Y cannot be isomorphic to X because $N(X) = \mathbb{Z}H$ where $H^2 = 8$, and $N(X)$ does not have elements with square 2 which is necessary if $Y \cong X$. Thus, if $Y \cong X$, then $\rho(X) \geq 2$, and X belongs to codimension 1 submoduli space of K3 surfaces which is a divisor in moduli (up to cidimension 2). The set of numbers $\mathcal{D}$ labels connected components of the divisor. Each $d \in \mathcal{D}$, gives a connected 18-dimensional moduli space of K3 surfaces with the Picard lattice $N(X) = N_d^8 \cong N_d^2$ (more generally, $N_d^8 \subset N(X)$, but the polarization $H \in N_d^8$). See [N1], [N2] and also [J] for the corresponding results.

E.g. it is well-known that $Y \cong X$ if X has a line. This is a divisorial condition on moduli of X. This component is labeled by $d = 17 \in \mathcal{D}$. Really, let $l \in N(X)$ be the class of line. Then the intersection matrix is

$$\begin{pmatrix} H^2 & H\cdot l \\ H\cdot l & l^2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 1 & -2 \end{pmatrix}$$

which has the determinant -17. The projection from the line l gives an embedding of X to $\mathbb{P}^3$ as a quartic. The element $h_1 = H - l$ has $h_1^2 = 4$. It has $H \cdot h_1 = 7$ which is odd. But its reflection in l gives $\tilde{h}_1 = h_1 + (h_1 \cdot l)l = h_1 + 3l = H + 2l$. We have $(\tilde{h}_1)^2 = 4$ and $\tilde{h}_1 \cdot H = 10 \equiv 0 \mod 2$. Then $Y \cong X$ by Theorem 3.1.8. Of course, there exists a direct classical geometrical isomorphim between X and Y for this case.

Our results show that *There exists an infinite number of similar divisorial conditions on moduli of intersections of three quadrics in $\mathbb{P}^5$ when $Y \cong X$. They are labeled by elements of the infinite set $\mathcal{D}$ which was described above. The $d = 17$ corresponds to the classical example above of the line in X.*

It is easy to see that for $d \in \mathcal{D}$, the lattice $N(X) \cong N_d^8$ has a non-zero element with square 0 (then d should be a square) only for $d = 1$ and $d = 9$. By Propositions 1.3.1 and 1.3.2, the first $d = 1$ only does not give an intersection of three quadrics.

4. A GENERAL PERSPECTIVE

Similar methods and calculation can be applied for much more general situation. We hope to consider that in further publications.

Let X and Y are K3 surfaces, which are general for their Picard lattices, and

$$\phi : (T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y)) \tag{4.1}$$

an isomorphism of their transcendental periods over $\mathbb{Q}$, and

$$(a_1, H_1, b_1)^{\pm}, \dots (a_k, H_k, b_k)^{\pm}, \tag{4.2}$$

a sequence of types of isotropic Mukai vectors of moduli of vector bundles on K3, and $\pm$ shows the direction of the correspondence.

Similar methods and calculations can be applied to study the following question: *When there exists a correspondence between X and Y which is given by the sequence* (4.2) *of Mukai vectors and which gives the isomorphism* (4.1) *between their transcendental periods?*

In [N4], some necessary and sufficient conditions for (4.1) are given when there exists at least one such a sequence (4.2) with comprime Mukai vectors (a_i, H_i, b_i).

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