Two applications of instanton numbers

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Abstract

The two applications are: 1. sometimes instanton numbers stratify moduli of bundles better than Chern numbers. 2. sometimes instanton numbers distinguish singularities better than the classical numerical invariants.

1 Introduction

In a little more detail, the two applications are: $1'$. instanton numbers give the coarsest stratification of moduli of bundles on blow-ups for which the strata are separated. $2'$. some analytically inequivalent plane curve singularities have same δ_P, Milnor number and Tjurina number, but distinct instanton numbers. The instanton numbers we use are local analytic invariants for instantons on a blow-up.

Let $\widetilde{\mathbb{C}^2}$ denote the blow-up of $\mathbb{C}^2$ at the origin. Rank 2 instantons on $\widetilde{\mathbb{C}^2}$ are built from simple algebraic data, namely, a triple (j, p, t_∞), made by an integer j, a polynomial p, and a framing at infinity, that is, a holomorphic map $t_\infty\colon \mathbb{C}^2 - \{0\} \to GL(2, \mathbb{C})$. These instantons have two holomorphic invariants: the height and the width, whose sum gives the topological charge. Here I give two applications of these instanton numbers. First I use these numbers to stratify moduli of instantons on the blown-up plane and second I use this pair as analytic invariants for plane curve singularities. I show that the pair (height, width) gives instanton invariants that are strictly finer than the topological charge of the instanton. In fact, the stratification of moduli of instantons by this pair of invariants is strictly finer that the stratification by topological charge. As applications to singularities, I show that these numbers distinguish nodes/tacnodes from cusps/higher order cusps. I also give an example of analytically inequivalent curve singularities that are not

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distinguished by the classical invariants (Milnor number, Tjurina number and the δ_P invariant which calculates the change in arithmetic genus) but have distinct instanton numbers.

The charge of a rank 2 instanton on $\Delta = (j, p, t_\infty)$ on $\widetilde{\mathbb{C}^2}$ ranges between j and j^2 depending on p. However, unlike instantons on S^4, whose charge is given locally by a unique invariant, called the multiplicity, these instantons have two local holomorphic invariants: the height and the width. These invariants do not depend on the framing, and neither does the topological charge. We can therefore calculate height, width and charge directly from the algebraic data (j, p). To use instanton numbers as invariants of curve singularities the trick is as follows. Given a plane curve $p(x, y) = 0$ with singularity at the origin, chose an integer j, and construct an instanton with data (j, p). Then use its numerical invariants as analytic invariants of the curve.

Instantons on $\widetilde{\mathbb{C}^2}$ their moduli and their topological and holomorphic invariants are described in section 2 and used to stratify moduli of instantons. In section 3, these invariants are used to distinguish plane curve singularities.

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2 Instantons on $\widetilde{\mathbb{C}^2}$

We show that every rank 2 instanton on $\widetilde{\mathbb{C}^2}$ is determined by a triple (j, p, t_∞), where j is an integer called the *splitting type* of the instanton, p a polynomial and t_∞ a trivialization at infinity. Generically, two triples (j, p, t_∞) and (j', p', t'_∞) determine the same instanton if an only if $j' = j$, $p' = \lambda p$ and $t'_\infty = A\, t_\infty$ where $\lambda \neq 0$, and $A \in \Gamma\left(\mathbb{C}^2 - \{0\}, GL(2, \mathbb{C})\right)$. An instanton $\Delta = (j, p, t_\infty)$ is generic if and only if its topological charge equals its splitting type j. Moreover, for every $j > 1$ there are nongeneric instantons (j, p, t_∞), with topological charge varying from $j + 1$ up to j^2. For each integer j, we topologize the set $\mathcal{M}_j$ of equivalence classes of instantons (j, p, t_∞) and show that the generic set is a $GL(2, \mathbb{C})$–bundle over a quasi-projective smooth variety of complex dimension $2j - 3$

The fact that an instanton on $\widetilde{\mathbb{C}^2}$ is determined by a triple (j, p, t_∞) follows essentially from putting together two results: first, the proof due to King [9] of the Hitchin–Kobayashi correspondence over the noncompact surface $\widetilde{\mathbb{C}^2}$ and second, the characterization of rank two holomorphic bundles on $\widetilde{\mathbb{C}^2}$ given in [5]. We review these two results.

Instantons on the blown-up plane are naturally identified with instantons on $\overline{\mathbb{CP}}^2$ framed at infinity; this is a simple consequence of the fact that $\overline{\mathbb{CP}}^2$ is the conformal compactification of $\widetilde{\mathbb{C}^2}$. On his Ph.D. thesis, A. King [9] identifies the moduli space $\mathrm{MI}(\widetilde{\mathbb{C}^2}; r, k)$ of instantons on the blown-up plane of rank r and charge k, with the moduli space $\mathrm{MI}(\overline{\mathbb{CP}}^2, \infty : r, k)$ of instantons on $\overline{\mathbb{CP}}^2$, framed at ∞, whose underlying vector bundle has rank r, and Chern classes $c_1 = 0$ and $c_2 = k$.

On the other hand, we may consider the canonical complex compactification of $\widetilde{\mathbb{C}^2}$, which is the the Hirzebruch surface Σ_1, obtained from $\widetilde{\mathbb{C}^2}$ by adding a line ℓ_∞ at infinity. Essentially *by definition* King identifies the moduli space $\mathrm{MH}(\widetilde{\mathbb{C}^2}; r, k)$ of "stable" holomorphic bundles on $\widetilde{\mathbb{C}^2}$ with rank r and $c_2 = k$ with the moduli space $\mathrm{MH}(\Sigma_1, \ell_\infty; r, k)$ of holomorphic bundles on Σ_1 with a trivialization along ℓ_∞ and whose underlying vector bundle has rank r, $c_1 = 0$ and $c_2 = k$. King then proves the Hitchin–Kobayashi correspondence in this case, namely that the map

$$\mathrm{MI}(\widetilde{\mathbb{C}^2}; r, k) \to \mathrm{MH}(\widetilde{\mathbb{C}^2}; r, k)$$

given by taking the holomorphic part of an instanton connection is a bijection. Therefore, a rank 2 instanton on $\widetilde{\mathbb{C}^2}$ is completely determined by a rank two holomorphic bundle on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class, together with a trivialization at infinity. The instanton has charge k if and only if the corresponding holomorphic bundle extends to a bundle on Σ_1 trivial on ℓ_∞ having $c_2 = k$.

We are thus led to study holomorphic rank two bundles on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class. It turns out that holomorphic bundles on $\widetilde{\mathbb{C}^2}$ are algebraic, they are extensions of line bundles and moreover they are trivial on the complement of the exceptional divisor (see [3] and [6]). Triviality outside the exceptional divisor in this case is very useful and is intrisically related to the fact that we have algebraic bundles. It is of course not true in general that holomorphic bundles on $\widetilde{\mathbb{C}^2} - \ell$ are trivial.

A holomorphic rank 2 bundle E on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class splits over the exceptional divisor as $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ for some positive integer j, called the *splitting type* of the bundle, and, in this case, E is an algebraic extension

$$0 \to \mathcal{O}(-j) \to E \to \mathcal{O}(j) \to 0 \tag{1}$$

(here by abuse of notation we write $\mathcal{O}(k)$ both for the line bundle $\mathcal{O}(k)$ over the exceptional divisor ℓ as well as for its pull-back to $\widetilde{\mathbb{C}^2}$). A bundle E fitting in an exact sequence (1) is determined by its extension class in $p \in Ext^1(\mathcal{O}(-j), \mathcal{O}(j))$, where p a polynomial, since as showed in [**?**] the bundle E is actually algebraic. To this bundle on we assign a *canonical form* of transition matrix. We fix, once and for all, the following charts: $\widetilde{\mathbb{C}^2} = U \cup V$ where $U = \{(z,u)\} \simeq \mathbb{C}^2 \simeq \{(\xi, v)\} = V$ with $(\xi, v) = (z^{-1}, zu)$ in $U \cap V$. Once these charts are fixed, E has the canonical transition matrix of the form (see [6] Thm. 2.1)

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \tag{2}$$

from U to V, where

$$p := \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i \tag{3}$$

is a polynomial in z, z^{-1} and u.

It follows that a rank 2 holomorphic bundle E on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class is completely determined by a pair (j, p) where j is a nonnegative integer and a p is a polynomial of the form (3). According to King's results, to have an instanton we need also a trivialization at infinity. However, it follows from [3] Cor. 4.2, that the bundle E is trivial outside the exceptional divisor. Therefore, to any bundle over $\widetilde{\mathbb{C}^2}$ represented by a pair (t, p) we may assign a trivialization at infinity $t_\infty \in \mathrm{G}L(2, \mathbb{C}^2 - \{0\})$ thus obtaining an instanton. As a consequence every rank–two instanton Δ on $\widetilde{\mathbb{C}^2}$ is determined by a triple

$$\Delta := (j, p, t_\infty). \tag{4}$$

To define the topological charge of the instanton we need to extend Δ to a bundle on a compact surface. The charge is independent of the chosen compactification (and in fact it only depends on an infinitesimal neighborhood of the exceptional divisor), but for simplicity we may take the compactification of $\widetilde{\mathbb{C}^2}$ be the Hirzebruch surface Σ_1. This extension is obtained as follows. Let M be a complex manifold and N a complex submanifold of M. We denote by $\mathcal{E}_r(M,N)$ the set of equivalence classes of pairs (E,η) where E is a rank r holomorphic bundle over M such that $E|_N$ is trivial, and η is a trivialization of $E|_N$. Here (E,η) is equivalent to (E',η') if there is a bundle equivalence $\alpha\colon E \to E'$ such that $\alpha\,\eta = \eta'$. Recall that ℓ denotes the exceptional divisor in $\widetilde{\mathbb{C}^2}$ and that the Hirzebruch surface $\Sigma_1 = \mathbb{P}(\mathcal{O}(-1)\oplus\mathcal{O})$ is the complex compactification of $\widetilde{\mathbb{C}^2}$ obtained by adding a line at infinity ℓ_∞. The following lemma is easy to prove.

Lemma 2.1 *There is a bijection between the sets $\mathcal{E}_2(\widetilde{\mathbb{C}^2},\widetilde{\mathbb{C}^2}-\ell)$ and $\mathcal{E}_2(\Sigma_1,\ell_\infty)$.*

The proof is in the appendix.

2.1 Moduli spaces

We wish to study moduli of instantons. We say that two triples $\Delta = (j,p,t_\infty)$ and $\Delta' = (j',p',t'_\infty)$ are equivalent if they represent the same instanton. In terms of holomorphic bundles, this means that two triples are equivalent if their corresponding holomorphic bundles E and E' over $\widetilde{\mathbb{C}^2}$ framed at infinity are isomorphic, via an isomorphism taking t_∞ into t'_∞. In particular these bundles give isomorphic restrictions over the exceptional divisor, hence E and E' must have the same splitting type, that is, $j = j'$.

Let us consider triples (j,p,t_∞) and (j,p',t'_∞), with the same integer j, representing holomorphic bundles E and E' over $\widetilde{\mathbb{C}^2}$ trivialized at infinity. We know from proposition 2.1 that these bundles may be looked upon as bundles (E,t_∞) and (E',t'_∞) over Σ_1 trivialized over ℓ_∞. An isomorphism for framed bundles is a bundle isomorphism $\Phi\colon E \to E'$ such that $\Phi(t_\infty) = t'_\infty$. Two framings t_∞ and t'_∞ for the same same underling bundle E over Σ_1 differ by a holomorphic map $\Phi\colon \ell_\infty \to GL(2,\mathbb{C})$ and, since ℓ_∞ is compact, Φ must be constant. Hence, projecting (E,t_∞) on the first coordinate we obtain a fibration of the space of framed bundles over Σ_1 over the space of bundles over Σ_1 which are trivial on the line at infinity, with fibre $GL(2,\mathbb{C})$.

$$
\begin{array}{c}
\mathrm{G}L(2,\mathbb{C}) \\
\downarrow \\
\{framed\, rank-2\, bundles\, over\, \Sigma_1\} \\
\downarrow \\
\{rank-2\, bundles\, over\, \Sigma_1\, trivial\, on\, \ell_\infty\}\,.
\end{array}
\tag{5}
$$

We are thus led to study the base space of this fibration, or equivalently, the space of isomorphism classes of bundles on $\widetilde{\mathbb{C}^2}$ which are trivial on the line at infinity. We define $\mathcal{M}_j$ to be space of rank two holomorphic bundles on the $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class and with splitting type j, modulo isomorphism, that is,

$$
\mathcal{M}_j = \left\{ \begin{array}{l} E \;\; hol.\ bundle\ over\ \widetilde{\mathbb{C}^2}: \\ E|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j) \end{array} \right\} \Big/ \sim\,. \tag{6}
$$

Fix the splitting type j and set $J = (j-1)(2j-1)$, then the polynomial p has J coefficients. Identifying the polynomial p with the $J-$tuple formed by its coefficients written in lexicographical order, we may define in $\mathbb{C}^J$ the equivalence relation $p \sim p'$ if (j,p) and (j,p') represent isomorphic bundles. Set-theoretically there is an identification

$$
\mathcal{M}_j = \mathbb{C}^J / \sim\,. \tag{7}
$$

We give $\mathbb{C}^J/\sim$ the quotient topology and $\mathcal{M}_j$ the topology induced by (7). $\mathcal{M}_j$ is generically a complex projective space of dimension $2j-3$ ([6] Thm. 3.5). However the topology of $\mathcal{M}_j$ is quite complex, and, in particular, is non-Hausdorff for any $j \geq 2$.

There is a topological embedding taking $\mathcal{M}_j$ into the least generic strata of $\mathcal{M}_{j+1}$. We write it out explicitly in coordinates, representing an element $E \in \mathcal{M}_j$, by its canonical form of transition matrix according to (2).

Proposition 2.2 *: The following map defines a topological embedding*

$$
\begin{array}{rcl}
\Phi_j : \mathcal{M}_j & \to & \mathcal{M}_{j+1} \\
(j,p) & \mapsto & (j+1, zu^2p)
\end{array}\,.
$$

The proof is in the appendix.

The map Φ_j takes $\mathcal{M}_j$ into the least generic strata of $\mathcal{M}_{j+1}$. In fact, imΦ is the subset of $\mathcal{M}_{j+1}$ consisting of bundles that split in the second formal neighborhood of the exceptional divisor. The complexity of the topology of $\mathcal{M}_j$ increases with j according to these embeddings. $\mathcal{M}_2$ is non-Hausdorff and therefore this property persists in $\mathcal{M}_j$ for $j \geq 2$. Explicitly $\mathcal{M}_2 \simeq \mathbb{P}^1 \cup \{A, B\}$ where the generic set $\mathbb{C}P^1$ consists of bundles that do not split on the first formal neighborhood, A and B are special points corresponding to two special bundles with splitting type 2; the one that splits on the first formal neighborhood but not on higher neighborhoods, and the split bundle (see [6])

2.2 Instanton numbers

We now define the instanton numbers that stratify the spaces $\mathcal{M}_j$ into Hausdorff components.

We consider a compact complex (smooth) surface X together with the blow–up $\pi\colon \widetilde{X} \to X$ of a point $x \in X$ and once again denote by ℓ the exceptional divisor. Let $\widetilde{E}$ be a rank 2 holomorphic bundle over $\widetilde{X}$ satisfying $\det \widetilde{E} \simeq \mathcal{O}_{\widetilde{X}}$. The splitting type of $\widetilde{E}$ is by definition the integer $j \geq 0$ such that $\widetilde{E}|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$. Set $E = \pi_* \widetilde{E}^{\vee\vee}$. Assuming X compact, Friedman and Morgan [2], p. 393 gave the following estimate relating the second Chern classes to the splitting type

$$j \leq c_2(\widetilde{E}) - c_2(E) \leq j^2.$$

Sharpness of these bounds was proven in [4]. Let F be a bundle on $\widetilde{\mathbb{C}^2}$ with vanishing first Chern class. If X is a compact complex surface, then there exist holomorphic bundles $\widetilde{E} \to \widetilde{X}$ which are isomorphic to F on a neighborhood of the exceptional divisor. In fact, following [5], given a bundle $E \to X$, we can construct bundles $\widetilde{E} \to \widetilde{X}$ satisfying:

ι) $\widetilde{E}|_{\widetilde{X}-\ell} = \pi^*(E|_{X-p})$, where π is the blow–up map, and

$\iota\iota$) $\widetilde{E}|_V \simeq F|_U$ for small neighborhoods V and U of the exceptional divisor in $\widetilde{X}$ and $\widetilde{\mathbb{C}^2}$ respectively.

Moreover, every bundle $\widetilde{E}$ on $\widetilde{X}$ is obtained this way [8, Cor. 3.4]. The isomorphism class of $\widetilde{E}$ depends on the attaching map $\phi\colon (\widetilde{X}-\ell) \cap V \to GL(2, \mathbb{C})$, however, the topological type of $\widetilde{E}$ is independent of ϕ. Therefore the charge does not depend upon the choice of ϕ. Since the $n-$th infinitesimal neighborhood of the exceptional divisor on a compact complex surface is isomorphic (as a scheme) to the $n-$th infinitesimal neighborhood of the exceptional divisor on $\widetilde{\mathbb{C}^2}$ we are able to use an explicit description for bundles on $\widetilde{\mathbb{C}^2}$, even though $x \in \widetilde{X}$ might not have an open neighborhood analytically equivalent to $\widetilde{\mathbb{C}^2}$. We quote:

Proposition 2.3 *([5], Cor. 4.1) Let X be a compact surface and $\widetilde{X}$ denote the blow up of X at x. Every holomorphic rank 2 vector $\widetilde{E}$ bundle over $\widetilde{X}$ with vanishing first Chern class is topologically determined by a triple (E, j, p) where E is a rank 2 holomorphic bundle on X with vanishing first Chern class, j is a nonnegative integer, and p is a polynomial.*

If $\widetilde{E}$ is as in the above proposition, we denote

$$\widetilde{E} := (E, j, p). \tag{8}$$

The pair (j, p) gives an explicit description of $\widetilde{E}$ on a neighborhood of the exceptional divisor, and determines the charge of $\widetilde{E}$. To calculate the charge, we actually compute two finer numerical invariants of $\widetilde{E}$, which we now describe. Following Friedman and Morgan ([2], p. 302), we define a sheaf Q by the exact sequence,

$$0 \to \pi_* \widetilde{E} \to \pi_* (\widetilde{E})^{\vee\vee} \to Q \to 0.$$

Note that Q is supported only at the point x. From the exact sequence (1.6) it follows immediately that $c_2(\pi_* \widetilde{E}) - c_2(E) = l(Q)$, where l stands for length. An application of Grothendieck–Riemann–Roch (see [8], p. 392) gives that

$$c_2(\widetilde{E}) - c_2(E) = l(Q) + l(R^1 \pi_* \widetilde{E}).$$

We call $w := l(Q)$ the *width* and $h := l(R^1 \pi_* \widetilde{E})$ the *height* of the instanton $\widetilde{E}$.

2.3 Holomorphic instanton patching

Let X be a surface with polarization L. Choose $N >> 0$, then $\widetilde{L} = NL - \ell$ is ample and it is natural to choose $\widetilde{L}$ as a polarization of $\widetilde{X}$. We fix these choices of polarizations and by stable bundle we mean stable with respect to the fixed polarization. By the Hitchin–Kobayashi correspondence for compact surfaces, instantons correspond to stable bundles. A stable bundle $\widetilde{E}$ on $\widetilde{X}$ such that $\widetilde{E}|_{\widetilde{X}-\ell} \simeq \pi^*(E|_{X-\{p\}})$ with E stable on X and $\widetilde{E}|_{V(\ell)} \simeq \Delta = (j, p, t_\infty)$ on some neighborhood $V(\ell)$ of the exceptional divisor is said to be obtained by obtained by *holomorphic patching* of the instantons of Δ to E. The reason for this terminology is that given E and Δ any choice of gluing $\Phi : \mathbb{C}^2 - \{0\} \to GL(2, \mathbb{C})$ gives a holomorphic way to construct a new instanton. Equivalently, it is enough to choose a framing at the point p.

Lemma 2.4 *Every instanton on $\widetilde{X}$ is obtained by* "holomorphic" *patching of an instanton on $\widetilde{\mathbb{C}^2}$ to an instanton on X.*

Proof: By [5] Corollary 3.4 every holomorphic rank two vector bundle $\widetilde{E}$ over $\widetilde{X}$ with vanishing first Chern class is completely determined (up to isomorphism) by a 4-tuple $\widetilde{E} := (E, j, p, \Phi)$ where E is a rank two holomorphic bundle on X with vanishing first Chern class, j is a nonnegative integer, p is a polynomial, and $\Phi : \mathbb{C}^2 - \{0\} \to GL(2, \mathbb{C})$ is a holomorphic map. The bundle $\widetilde{E}$ has splitting type j over the exceptional divisor, and satisfy the property $\widetilde{E}|_{\widetilde{X}-\ell} \simeq \pi^*(E|_{X-\{p\}})$. If $\widetilde{E}$ is stable, then so is E (see [2]). □

Remark 2.5 *The charge addition given by the patching of Δ can be calculated by a Macaulay2 program written by Irena Swanson and the author [7]. The program has as input j and p and as output the height and the width of an instanton (j, p).*

2.4 Stratification of Moduli of Instantons

The following theorem shows that instanton numbers provide good stratifications for moduli of instantons on $\widetilde{\mathbb{C}}^2$. In fact, these numbers stratify the spaces $\mathcal{M}_j$ into Hausdorff components, and this is the coarsest stratification of $\mathcal{M}_j$ for which the strata are Hausdorff. In [1] it is shown that stratification by Chern numbers is not fine enough to have this property. We cite.

Theorem 2.6 *([1] Thm. 4.1) The numerical invariants w and h provide a decomposition $\mathcal{M}_j = \cup S_i$ where each S_i is homeomorphic to an open subset of a complex projective space of dimension at most $2j - 3$. The lower bounds* *for these invariants are* $(1, j-1)$ *and this pair of invariants takes place on the generic part of* $\mathcal{M}_j$ *which is homeomorphic to* $\mathbb{CP}^{2j-3}$ *minus a closed subvariety of codimension at least 2. The upper bounds for these invariants are* $(j(j-1)/2, j(j+1)/2)$ *and this pair occurs at one single point of* $\mathcal{M}_j$ *which represents the split bundle.*

3 Curve singularities

Here is how to use instanton numbers to distinguish curve singularities. Start with a curve $p(x,y) = 0$ on $\mathbb{C}^2$. Choose your favorite integer j and construct an instanton on $\widetilde{\mathbb{C}^2}$i having data (j,p). Calculate the height and the width of the instanton, use them as analytic invariants of the curve, and use the charge as a topological invariant. In other words, we are using the polynomial defining the plane curve as an extension class in $\mathrm{Ext}^1\mathcal{O}(j), \mathcal{O}(-|))$. This defines a bundle $E(j,p)$ as in 8. We then calculate the instanton numbers of this bundle, as defined in section 2, and regard them as being associated to the curve.

Note that to perform the computations we must choose a representative for the curve and coordinates for the bundle. I use the canonical choice of coordinates for $\widetilde{\mathbb{C}^2}$ as in section 2. Taking into account that the blow-up map in these coordinates is given by $x \mapsto u$ and $y \mapsto zu$ the bundle $E(j,p)$ is then given canonically in these coordinates by

$$E(j,p) := \begin{pmatrix} z^j & p(u,zu) \\ 0 & z^{-j} \end{pmatrix}.$$

If a second representative $\bar{p}$ for the same singularity is given, there is a holomorphic change of coordinates ϕ taking p to $\bar{p}$. To compute the invariants using this second representative, the coordinate change has to be applied to the bundle as well. In this paper I give only a couple of results to illustrate the behavior of the instanton numbers applied to singularities. Explicit hand-made computations of these invariants for small values of j appear in [1] and [4]. The invariants can be computed by a Macaulay2 algorithm written by Irena Swanson and the author, see Remark **??**.

The next theorems show that instanton numbers distinguish the most basic singularities and also give some examples where instanton numbers are finer than classical invariants.

Theorem 3.1 *Instanton numbers distinguish nodes/tacnodes from cusps/higher order cusps.*

Proof: These singularities have quasi–homogeneous representatives of the form $y^n - x^m$, $n < m$, n even for nodes and tacnodes, and n odd for cusps and higher order cusps. We want to show that instanton numbers detect the parity of the smallest these exponents. In fact, more is true, instanton numbers detect the multiplicity itself.

Suppose $n_1 < n_2$. We claim that if $j > n_2$ then $w(j, p_1) \neq w(j, p_2)$. In fact, for $n < m$ and large enough j the width takes the value

$$w(j, y^n - x^m) = n(n+1)/2.$$

Alternatively, by vector bundle reasons we have that $w(j, p_1) < w(j, p_2)$. The second assertion is easier to show. The holomorphic bundle $E(j, p_1)$ restricts as a non-trivial extension on the n_1th formal neighborhood l_{n_1} whereas $E(j, p_2)$ splits on l_{n_1}. These bundle therefore belong to different strata of $\mathcal{M}_j$ and by theorem 2.6 must have distinct instanton numbers. □

Theorem 3.2 *In some cases instanton numbers are finer than classical invariants.*

Proof: See tables I and II below. □

The classical invariants we consider are:

- $\delta_P = dim(\widetilde{\mathcal{O}}_P/\mathcal{O}_P)$
- Milnor number $\mu = \mathcal{O}/ < J(P) >$
- Tjurina number $\tau = \mathcal{O}/ < P, J(P) >$

Note: The first table is motivated by exercise 3.8 of Hartshorne [8] page 395. However, in the statement of the problem, the first polynomial contains an incorrect exponent. It is written as "$x^4y - y^4$" but it should be "$x^5y - y^4$."

TABLE I				$j = 4$	
polynomial	δ_P	μ	τ	w	h
$x^5y - y^4$	9	17	17	10	6
$x^8 - x^5y^2 - x^3y^2 + y^4$	9	17	15	8	6

The second table shows an example where instanton numbers are finer than δ_P, μ, and τ.

TABLE II				$j = 4$	
polynomial	δ_P	μ	τ	w	h
$x^2 - y^7$	3	6	6	3	5
$x^3 - y^4$	3	6	6	6	6

Remark 3.3 *The idea of using the polynomial defining a singularity as the extension class of a holomorphic bundle can be further generalized in several ways. For curves themselves, one can use other base spaces. For instance, constructing bundles on the total space of $\mathcal{O}_{\mathbb{P}^1}(-k)$ requires very little modifications, but give quite different results. One can also generalize to hypersurfaces in higher dimensions.*

4 Appendix

This appendix contains proofs of two technical but straightforward results used in the text.

Proof of Lemma 2.1: Given that $\Sigma_1 = \widetilde{\mathbb{C}^2} \cup \ell_\infty$, there exists an open neighborhood W of ℓ_∞ satisfying:

ι) $\Sigma_1 - W = \ell$

$\iota\iota$) $W - \ell_\infty \simeq \widetilde{\mathbb{C}^2} - \ell$

$\iota\iota\iota$) we have a commutative diagram

$$\begin{array}{ccc} \ell_\infty & \to & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ W & \to & \mathcal{O}(1) \end{array}$$

where the vertical arrows are inclusions and the horizontal arrows are isomorphisms. Now, given $(E, \eta) \in \mathcal{E}_2(\widetilde{\mathbb{C}^2}, \widetilde{\mathbb{C}^2} - \ell)$, i.e., $\eta = (a, b)$; where $a, b \colon \widetilde{\mathbb{C}^2} - \ell \to E|_{\widetilde{\mathbb{C}^2} - \ell}$ are linearly independent sections, define

$$\Phi(E, \eta) := (F, \mu) \in \mathcal{E}_2(\Sigma_1, \ell_\infty)$$

by gluing E with $W \times \mathbb{C}^2$ along η, that is, define

$$F = E \sqcup (W \times \mathbb{C}^2)/\sim$$

where $\alpha\, a(x) + \beta\, b(x) \sim (x, (\alpha, \beta))$ for $x \in \widetilde{\mathbb{C}^2} - \ell$ and $(\alpha, \beta) \in \mathbb{C}^2$.

Let $\mu = (e_1, e_2)$ be the canonical section of the trivial bundle over ℓ_∞, that is, $\mu(y) = (y, (1, 0), (0, 1))$ where $y \in \ell_\infty \subset W$. Now, given $(F, \mu) \in \mathcal{E}_2(\Sigma_1, \ell_\infty)$ define

$$\Psi(F, \mu) := (E, \eta)$$

where $E = F|_{\widetilde{\mathbb{C}^2}}$ and η is defined as follows. Since $F|_W$ is trivial (because $F\,|_{W|_{\ell_\infty}} = F|_{\ell_\infty}$ is trivial) there is a unique trivialization $\widetilde{\eta}$ over W such that $\widetilde{\eta}|_{\ell_\infty} = \mu$ (this is because $H^0(W, \mathcal{O}) = H^0(\mathcal{O}(1), \mathcal{O}) = \mathbb{C}$). Define

$$\eta := \widetilde{\eta}|_{W_{\ell_\infty}} = \widetilde{\eta}|_{\widetilde{\mathbb{C}} - \{0\}}.$$

It is straightforward to prove that $\Psi\Phi$ and $\Phi\Psi$ are the identities. □

Proof of Proposition 2.2: We first show that the map is well defined. Suppose $\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix}$ and $\begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix}$ represent isomorphic bundles. Then there are coordinate changes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ holomorphic in z, u and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ holomorphic in z^{-1}, zu for which the following equality holds (compare [?] pg. 587)

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix}.$$

Therefore these two bundles are isomorphic exactly when the system of equations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a + z^{-j}p'c & z^{2j}b + z^j(p'd - ap) - pp'c \\ z^{-2j}c & d - z^{-j}pc \end{pmatrix} \qquad (*)$$

can be solved by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ holomorphic in z, u which makes $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ holomorphic in z^{-1}, zu.

On the other hand, the images of these two bundles are given by transition matrices $\begin{pmatrix} z^{j+1} & z\,u^2p \\ 0 & z^{-j-1} \end{pmatrix}$ and $\begin{pmatrix} z^{j+1} & z\,u^2p' \\ 0 & z^{-j-1} \end{pmatrix}$, which represent isomorphic bundles iff there are coordinate changes $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ holomorphic in z, u and $\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$ holomorphic in z^{-1}, zu satisfying the equality

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} z^{j+1} & z\,u^2p' \\ 0 & z^{-j-1} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} z^{-j-1} & -z\,u^2p \\ 0 & z^{j+1} \end{pmatrix}.$$

That is, the images represent isomorphic bundles if the system

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} \bar{a} + z^{-j}u^2p'\bar{c} & z^{2j+2}\bar{b} + z^{j+2}u^2(p'\bar{d} - \bar{a}p) - z^2u^4pp'\bar{c} \\ z^{-2j-2}\bar{c} & \bar{d} - z^{-j}u^2p\bar{c} \end{pmatrix} \quad (**)$$

has a solution.

Write $x = \sum x_iu^i$ for $x \in \{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ and choose $\bar{a}_i = a_{i+2}$, $\bar{b}_i = b_{i+2}u^2$, $\bar{c}_i = c_{i+2}u^{-2}$, $\bar{d}_i = d_{i+2}$. Then if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ solves (*), one verifies that $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ solves (**), which implies that the images represent isomorphic bundles and therefore Φ_j is well defined. To show that the map is injective just reverse the previous argument. Continuity is obvious. Now we observe also that the image $\Phi_j(\mathcal{M}_j)$ is a saturated set in $\mathcal{M}_{j+1}$ (meaning that if $y \sim x$ and $x \in \Phi_j(\mathcal{M}_j)$ then $y \in \Phi_j(\mathcal{M}_j)$). In fact, if $E \in \Phi_j(\mathcal{M}_j)$ then E splits in the 2nd formal neighborhood. Now if $E' \sim E$ than E' must also split in the 2nd formal neighborhood therefore the polynomial corresponding to E' is of the form u^2p' and hence $\Phi_j(z^{-1}p')$ gives E'. Note also that $\Phi_j(\mathcal{M}_j)$ is a closed subset of $\mathcal{M}_{j+1}$, given by the equations $p_{il} = 0$ for $i = 1, 2$ and $i - j + 1 \leq l \leq j - 1$. Now the fact that Φ_j is a homeomorphism over its image follows from the following easy lemma. □

Lemma 4.1 *Let $X \subset Y$ be a closed subset and $\sim$ an equivalence relation in Y, such that X is $\sim$ saturated. Then the map $I : X/\sim \to Y/\sim$ induced by the inclusion is a homeomorphism over the image.*

Proof: Denote by $\pi_X : X \to X/\sim$ and $\pi_Y : Y \to Y/\sim$ the projections. Let F be a closed subset of $X/\sim$. Then $\pi_X^{-1}(F)$ is closed and saturated in X and therefore $\pi_X^{-1}(F)$ is also closed and saturated in Y. It follows that $\pi_Y(\pi_X^{-1}(F))$ is closed in $Y/\sim$. □

References

[1] Ballico, E. and Gasparim,E. *Numerical Invariants for Bundles on Blow-ups, Proc. Amer. Math. Soc.,* **130** *(2002), n.1, 23-32*

[2] Friedman, R. and Morgan, J. *On the diffeomorphism types of certain algebraic surfaces II. J.Differ. Geom. 27,371-398(1988)*

[3] Gasparim, E. *Holomorphic Bundles on $\mathcal{O}(-k)$ are algebraic, Comm. Algebra,***25** *(1997), n.9, 3001-3009*

[4] Gasparim, E.*Chern Classes of Bundles on Blown-up Surfaces, Comm. Algebra,* **28** *(2000), n.10, 4919-4926*

[5] Gasparim, E.*On the Topology of Holomorphic Bundles, Bol. Soc. Parana. Mat.* **18** *(1998) n. 1-2, 113-119*

[6] Gasparim, E. *Rank Two Bundles on the Blow up of* $\mathbb{C}^2$, *J. Algebra,* **199** *(1998), 581-590*

[7] Gasparim, E. and Swanson, I. *Computing instanton numbers of curve singularities, preprint*

[8] Hartshorne, R. *Algebraic Geometry. Graduate Texts in Mathematics 56, Springer Verlag (1977)*

[9] King, A. *Ph.D. Thesis, Oxford (1989)*

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