MIXED TORIC RESIDUES AND CALABI-YAU COMPLETE INTERSECTIONS

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ABSTRACT. Using Cayley trick, we define the notions of mixed toric residues and mixed Hessians associated with r Laurent polynomials $f_1, \ldots, f_r$. We conjecture that the values of mixed toric residues on the mixed Hessians are determined by mixed volumes of the Newton polytopes of $f_1, \ldots, f_r$. Using mixed toric residues, we generalize our Toric Residue Mirror Conjecture to the case of Calabi-Yau complete intersections in Gorenstein toric Fano varieties obtained from nef-partitions of reflexive polytopes.

CONTENTS

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1. Introduction

This paper is the continuation of our previous work [BM1] where we proposed a toric mirror symmetry test using toric residues. The idea of this test has appeared in the paper of Morrison-Plesser [MP] who observed that the coefficients of some power series expansions of unnormalized Yukawa couplings for mirrors of Calabi-Yau hypersurfaces in toric varieties $\mathbb{P}$ can be interpreted as generating functions for intersection numbers of divisors on some sequences of toric varieties $\mathbb{P}_\beta$ parametrized by lattice points β in the Mori cone $K_{\text{eff}}(\mathbb{P})$ of $\mathbb{P}$. Due to results of Mavlyutov [Mav], it is known that the unnormalized Yukawa couplings can be computed using toric residues introduced by Cox [Cox]. In our paper [BM1], we formulated a general mathematical conjecture, so called *Toric Residue Mirror Conjecture*, which describes some power series expensions of the toric residues in terms of intersection numbers of divisors on a sequence of simplicial toric varieties $\mathbb{P}_\beta$ (we call them Morrison-Plesser moduli spaces). This conjecture includes all examples of mirror symmetry for Calabi-Yau hypersurfaces in Gorenstein toric varieties associated with reflexive polytopes. Since the toric mirror symmetry construction exists also for Calabi-Yau complete intersection in Gorenstein toric Fano varieties [Bo, BB1], it is natural to try to extend our conjecture to this more general situation.

The case of Calabi-Yau complete intersections of r hypersurfaces

$$f_1(t) = \cdots = f_r(t) = 0, \quad r > 1,$$

defined by Laurent polynomials $f_1(t), \ldots, f_r(t) \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ in d-dimensional toric varieties $\mathbb{P}$ was not considered by Morrison and Plesser in [MP]. We remark that in this case one *does not* get a connection to the "quantum cohomology ring" [Bat] as in the hypersurface case. This difference is explained by the consideration of a nonreflexive $(d+r-1)$-dimensional polytope $\widetilde{\Delta}$, so called Cayley polytope, and its secondary polytope $\text{Sec}(\widetilde{\Delta})$. The Cayley polytope $\widetilde{\Delta}$ appears from the Cayley trick which introduces r additional r variables $t_{d+1}, \ldots, t_{d+r}$ and a new polynomial $F(t) := \sum_{j=1}^r t_{d+j} f_j(t)$. We consider the usual toric residue Res_F associated with F and define the *k-mixed toric residue* Res_F^k corresponding to a positive integral solution $k = (k_1, \ldots, k_r)$ of the equation $k_1 + \cdots + k_r = d + r$ as a k-th homogeneous component of Res_F. We expect that the k-mixed toric residues are similar to the usual toric residues. In particular, we introduce the notion of *k-mixed Hessian H_F^k* of Laurent polynomials $f_1, \ldots, f_r$ and conjecture that the value of Res_F^k on H_F^k is exactly the mixed volume

$$V(\underbrace{\Delta_1, \ldots, \Delta_1}_{k_1 - 1}, \ldots, \underbrace{\Delta_r, \ldots, \Delta_r}_{k_r - 1}),$$

where $\Delta_1, \ldots, \Delta_r$ are Newton polytopes of $f_1, \ldots, f_r$.

Our generalization of the Toric Residue Mirror Conjecture for Calabi-Yau complete intersections uses the notions of the nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ of d-dimensional reflexive polytope Δ [Bo]. In this situation, one obtains a dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ and two more reflexive polytopes:

$$\nabla^* = conv\{\Delta_1, \ldots, \Delta_r\}, \quad \Delta^* = conv\{\nabla_1, \ldots, \nabla_r\}.$$

It is important that special coherent triangulations of ∇^* define coherent triangulations of the Cayley polytope $\widetilde{\Delta} := \Delta_1 * \cdots * \Delta_r$. Therefore the choice of such a triangulation $\mathcal{T}$ of ∇^* determines a vertex $v_{\mathcal{T}}$ of the secondary polytope $\mathrm{Sec}(\widetilde{\Delta})$ and a partial projective simplicial crepant desingularization $\mathbb{P} := \mathbb{P}_{\Sigma(\mathcal{T})}$ of the Gorenstein toric variety $\mathbb{P}_{\nabla^*}$. So one obtains a sequence of simplicial toric varieties $\mathbb{P}_\beta$ associated with the lattice points β in the Mori cone $K_{\mathrm{eff}}(\mathbb{P})$ of $\mathbb{P}$. We conjecture that the generating function of intersection numbers

$$I_P(a) = \sum_{\beta \in K_{\mathrm{eff}}(\mathbb{P})} I(P, \beta)\, a^\beta$$

coincides with the power series expansion of the k-mixed toric residue

$$R_P(a) = \mathrm{Res}^k_F(P(a,t))$$

at the vertex $v_{\mathcal{T}} \in \mathrm{Sec}(\widetilde{\Delta})$. The precise formulation of this conjecture is given in Section 4.

In Sections 5, 6 we check our conjecture for nef-partitions corresponding to Calabi-Yau complete intersections in weighted projective spaces $\mathbb{P}(w_1, \ldots, w_n)$ and in product of projective spaces $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_p}$. The final section is devoted to applications of the Toric Residue Mirror Conjecture to the computation of Yukawa couplings for Calabi-Yau complete intersections.

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2. Toric residues

In this section we remind necessary well-known facts about toric residues (see [Cox, CDS, BM1]).

Let $\widetilde{M}$ and $\widetilde{N} = \mathrm{Hom}(\widetilde{M}, \mathbb{Z})$ be two free abelian groups of rank $\widetilde{d}$ dual to each other. We denote by

$$\langle *, * \rangle \, : \, \widetilde{M} \times \widetilde{N} \to \mathbb{Z}$$

the natural bilinear pairing, and by $\widetilde{M}_{\mathbb{R}}$ (resp. $\widetilde{N}_{\mathbb{R}}$) the real scalar extension of $\widetilde{M}$ (resp. $\widetilde{N}$).

Definition 2.1 ([BB2]). A $\widetilde{d}$-dimensional rational polyhedral cone C ($\widetilde{d} > 0$) in $\widetilde{M}_{\mathbb{R}}$ is called *Gorenstein* if it is strongly convex (i.e., $C + (-C) = \{0\}$), there exists an element $n_C \in \widetilde{N}$ such that $\langle x, n_C \rangle > 0$ for any nonzero $x \in C$, and all vertices of the $(\widetilde{d} - 1)$-dimensional convex polytope

$$\Delta(C) = \{x \in C \, : \, \langle x, n_C \rangle = 1\}$$

belong to $\widetilde{M}$. The polytope $\Delta(C)$ is called the *supporting polytope of* C. For any $m \in C \cap \widetilde{M}$, we define the *degree of* m as

$$\deg m = \langle m, n_C \rangle.$$

Definition 2.2. Let $\widetilde{\Delta} = \Delta(C)$ be the supporting polytope for a Gorenstein cone $C \subset \widetilde{M}_{\mathbb{R}}$. We denote by $S_{\widetilde{\Delta}}$ the semigroup $\mathbb{C}$-algebra of the monoid of lattice points $C \cap \widetilde{M}$. In order to transform the additive semigroup operation in $C \cap \widetilde{M}$ into a multiplicative form in $S_{\widetilde{\Delta}}$, we write t^m for the element in $S_{\widetilde{\Delta}}$ corresponding to $m \in C$. One can consider $S_{\widetilde{\Delta}}$ as a graded $\mathbb{C}$-algebra:

$$S_{\widetilde{\Delta}} = \bigoplus_{l=0}^{\infty} S_{\widetilde{\Delta}}^{(l)},$$

where the l-th homogeneous component $S_{\widetilde{\Delta}}^{(l)}$ has a $\mathbb{C}$-basis consisting of all t^m such that $m \in C \cap \widetilde{M}$ and $\deg m = l$. We define also the homogeneous ideal

$$I_{\widetilde{\Delta}} = \bigoplus_{l=0}^{\infty} I_{\widetilde{\Delta}}^{(l)}$$

in $S_{\widetilde{\Delta}}$ whose $\mathbb{C}$-basis consists of all t^m such that m is a lattice point in the interior of C.

Definition 2.3. An element

$$g := \sum_{m \in \widetilde{\Delta} \cap \widetilde{M}} a_m t^m \in S_{\widetilde{\Delta}}^{(1)}, \quad a_m \in \mathbb{C}$$

is called $\widetilde{\Delta}$-regular if for some $\mathbb{Z}$-basis $n_1, \ldots, n_{\widetilde{d}}$ of $\widetilde{N}$ the elements

$$g_i := \sum_{m \in \widetilde{\Delta} \cap \widetilde{M}} a_m \langle m, n_i \rangle t^m, \quad i = 1, \ldots, \widetilde{d}$$

form a regular sequence in $S_{\widetilde{\Delta}}$. We define the matrix $G := (g_{ij})_{1 \leq i,j \leq \widetilde{d}}$, where

$$g_{ij} := \sum_{m \in \widetilde{\Delta} \cap \widetilde{M}} a_m \langle m, n_i \rangle \langle m, n_j \rangle t^m, \quad i, j = 1, \ldots, \widetilde{d}.$$

The element

$$H_g := \det G$$

is called *Hessian of* g.

Remark 2.4. a) The definition of $\widetilde{\Delta}$-regularity does not depend on the choice of $\mathbb{Z}$-basis $n_1, \ldots, n_{\tilde{d}}$ of $\widetilde{N}$. In many applications the lattice vector n_C will be included in $\{n_1, \ldots, n_{\tilde{d}}\}$.
b) If $\widetilde{\Delta} \cap \widetilde{M} = \{m_1, \ldots, m_\mu\}$, then by [CDS, Proposition 1.2], one has

$$H_g = \sum_{1 \le i_1 < \ldots < i_{\tilde{d}} \le \mu} \left(\det(m_{i_1}, \ldots, m_{i_{\tilde{d}}})\right)^2 t^{m_{i_1} + \cdots + m_{i_{\tilde{d}}}}.$$

In particular, H_g is independent on the choice of the $\mathbb{Z}$-basis $n_1, \ldots, n_{\tilde{d}}$ and $H_g \in I_{\widetilde{\Delta}}^{(\tilde{d})}$.
c) The graded $\mathbb{C}$-algebra $S_{\widetilde{\Delta}}$ is Cohen-Macaulay and $I_{\widetilde{\Delta}}$ is its dualizing module. If g is $\widetilde{\Delta}$-regular in $S_{\widetilde{\Delta}}$, then

$$S_g := S_{\widetilde{\Delta}} / \langle g_1, \ldots, g_{\tilde{d}} \rangle S_{\widetilde{\Delta}},$$

is a graded finite-dimensional ring and

$$I_g := I_{\widetilde{\Delta}} / \langle g_1, \ldots, g_{\tilde{d}} \rangle I_{\widetilde{\Delta}}$$

is a graded S_g-module together with a non-degenerate pairing

$$S_g^{(l)} \times I_g^{(\tilde{d}-l)} \to I_g^{(\tilde{d})} \simeq \mathbb{C}, \quad l = 0, \ldots, \tilde{d} - 1.$$

induced by the S_g-module structure.

Definition 2.5. By *toric residue* corresponding to a $\widetilde{\Delta}$-regular element $g \in S_{\widetilde{\Delta}}^{(1)}$ we mean the $\mathbb{C}$-linear mapping

$$\mathrm{Res}_g \; : \; I_{\widetilde{\Delta}}^{(\tilde{d})} \to \mathbb{C}$$

which is uniquely determined by two conditions:
(i) $\mathrm{Res}_g(h) = 0$ for any $h \in \langle g_1, \ldots, g_{\tilde{d}} \rangle I_{\widetilde{\Delta}}$;
(ii) $\mathrm{Res}_g(H_g) = \mathrm{Vol}(\widetilde{\Delta})$, where $\mathrm{Vol}(\widetilde{\Delta})$ denotes the volume of the $(\tilde{d}-1)$-dimensional polytope $\widetilde{\Delta}$ multiplied by $(\tilde{d}-1)!$.

Let $\mathbb{P}_{\widetilde{\Delta}} := \mathrm{Proj}\, S_{\widetilde{\Delta}}$ be $(\tilde{d}-1)$-dimensional toric variety associated with the polytope $\widetilde{\Delta}$ and $\mathcal{O}_{\mathbb{P}_{\widetilde{\Delta}}}(1)$ the corresponding ample sheaf on $\mathbb{P}_{\widetilde{\Delta}}$. Then one has the canonical isomorphisms of graded rings

$$S_{\widetilde{\Delta}} \cong \bigoplus_{l \ge 0} H^0(\mathbb{P}_{\widetilde{\Delta}}, \mathcal{O}_{\mathbb{P}_{\widetilde{\Delta}}}(l))$$

and graded modules

$$I_{\widetilde{\Delta}} \cong \bigoplus_{l \ge 0} H^0(\mathbb{P}_{\widetilde{\Delta}}, \omega_{\mathbb{P}_{\widetilde{\Delta}}}(l)),$$

where $\omega_{\mathbb{P}_{\widetilde{\Delta}}}$ is the dualizing sheaf on $\mathbb{P}_{\widetilde{\Delta}}$. In particular, we obtain a canonical isomorphism

$$I_{\widetilde{\Delta}}^{(\widetilde{d})} \cong H^0(\mathbb{P}_{\widetilde{\Delta}}, \omega_{\mathbb{P}_{\widetilde{\Delta}}}(\widetilde{d})).$$

The following statement is a simple reformulation of Theorem 2.9(i) in [BM1]:

Proposition 2.6. *Let $n_1, \ldots, n_{\widetilde{d}}$ be a $\mathbb{Z}$-basis of $\widetilde{N}$ such that $n_1 = n_C$. Denote by $m_1, \ldots, m_{\widetilde{d}}$ the dual $\mathbb{Z}$-basis of $\widetilde{M}$. For any elements $h \in I_{\widetilde{\Delta}}^{(\widetilde{d})}$ and $g \in S_{\widetilde{\Delta}}^{(1)}$, we define a rational differential $(\widetilde{d}-1)$-form on $\mathbb{P}_{\widetilde{\Delta}}$:*

$$\Omega(h,g) := \frac{h}{g_1 \cdots g_{\widetilde{d}}} \frac{dt^{m_2}}{t^{m_2}} \wedge \cdots \wedge \frac{dt^{m_{\widetilde{d}}}}{t^{m_{\widetilde{d}}}}.$$

If g is $\widetilde{\Delta}$-regular, then

$$\mathrm{Res}_g(h) = \sum_{\xi \in V_g} \mathrm{res}_\xi \left(\Omega(h,g)\right),$$

where $V_g = \{\xi \in \mathbb{P}_{\widetilde{\Delta}} : g_2(\xi) = \cdots = g_{\widetilde{d}}(\xi) = 0\}$ is the set of common zeros of $g_2, \ldots, g_{\widetilde{d}}$ and $\mathrm{res}_\xi(\Omega(h,g))$ is the local Grothendieck residue of the form $\Omega(h,g)$ at the point $\xi \in V_g$.

In particular, if all the common roots of $g_2, \ldots, g_{\widetilde{d}}$ are simple and contained in the open dense $(\widetilde{d}-1)$-dimensional torus $\mathbb{T} \subset \mathbb{P}_{\widetilde{\Delta}}$, then

$$\mathrm{Res}_g(h) = \sum_{\xi \in V_g} \frac{p(\xi)}{g_1(\xi) H_g^1(\xi)},$$

where H_g^1 is the determinant of the matrix $G^1 := (g_{ij}))_{2 \le i,j \le \widetilde{d}}$.

Definition 2.7 ([BB1]). A Gorenstein cone C is called *reflexive* if the dual cone

$$\check{C} = \{y \in \widetilde{N}_{\mathbb{R}} : \langle x, y \rangle \ge 0 \quad \forall x \in C\}$$

is also Gorenstein, i.e., there exists $m_{\check{C}} \in \widetilde{M}$ such that $\langle m_{\check{C}}, y \rangle > 0$ for all $y \in \check{C} \setminus \{0\}$, and all vertices of the supporting polytope

$$\Delta(\check{C}) = \{y \in \check{C} : \langle m_{\check{C}}, y \rangle = 1\}$$

belong to $\widetilde{N}$. We will call the integer $r = \langle m_{\check{C}}, n_C \rangle$ the *index of C* (or $\check{C}$). A $(\widetilde{d}-1)$-dimensional lattice polytope $\widetilde{\Delta}$ is called *reflexive* if it is a supporting polytope of some $\widetilde{d}$-dimensional reflexive Gorenstein cone C of index 1. Moreover, the supporting polytope $\widetilde{\Delta}^*$ of the dual cone $\check{C}$ is also reflexive polytope which is called *dual* (or *polar*) to $\widetilde{\Delta}$.

If C is a reflexive Gorenstein cone of index r, then $I_{\widetilde{\Delta}}$ is a principal ideal generated by the element $t^{m_{\check{c}}}$ of degree r. So one obtains the canonical isomorphism $I^{(l)}_{\widetilde{\Delta}} \cong S^{(l-r)}_{\widetilde{\Delta}}$. In particular, there exists the toric residue mapping

$$\mathrm{Res}_g \,:\, S^{(\widetilde{d}-r)}_{\widetilde{\Delta}} \to \mathbb{C}$$

which is uniquely determined by the conditions:
(i) $\mathrm{Res}_g(h) = 0$ for any $h \in \langle g_1, \ldots, g_{\widetilde{d}} \rangle S_{\widetilde{\Delta}}$;
(ii) $\mathrm{Res}_g(H'_g) = \mathrm{Vol}(\widetilde{\Delta})$, where $H_g = t^{m_{\check{c}}} H'_g$.

3. Cayley trick and mixed toric residues

Let M be a free abelian group of rank d, $M_{\mathbb{R}} := M \otimes \mathbb{R}$, and $\Delta \subset M_{\mathbb{R}}$ a convex d-dimensional polytope with vertices in M. We assume that there exist r convex polytopes $\Delta_1, \ldots, \Delta_r$ with vertices in M such that Δ can be written as the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ (here we do not require that all polytopes $\Delta_1, \ldots, \Delta_r$ have maximal dimension d).

Definition 3.1. We set $\widetilde{M} := M \oplus \mathbb{Z}^r$, $\widetilde{d} := d + r$ and define the $\widetilde{d}$-dimensional Gorenstein cone $C = C(\Delta_1, \ldots, \Delta_r)$ in $\widetilde{M_{\mathbb{R}}} := M_{\mathbb{R}} \oplus \mathbb{R}^r$ as follows

$$C := \{(\lambda_1 x_1 + \cdots + \lambda_r x_r, \lambda_1, \ldots, \lambda_r) \in \widetilde{M_{\mathbb{R}}} \,:\, \lambda_i \geq 0,\, x_i \in \Delta_i,\, i = 1, \ldots, r\}.$$

The $(d + r - 1)$-dimensional polytope $\Delta_1 * \cdots * \Delta_r$ defined as the intersection of the cone C with the affine hyperplane $\sum_{i=1}^r \lambda_i = 1$

$$\Delta_1 * \cdots * \Delta_r := \{(\lambda_1 x_1 + \cdots + \lambda_r x_r, \lambda_1, \ldots, \lambda_r) \,:\, \lambda_i \geq 0,\, \sum_{i=1}^r \lambda_i = 1,\, x_i \in \Delta_i\},$$

will be called *Cayley polytope associated with the Minkowski sum decomposition* $\Delta = \Delta_1 + \cdots + \Delta_r$. It is clear that all vertices of $\Delta_1 * \cdots * \Delta_r$ are contained in $\widetilde{M}$ and

$$\Delta_1 * \cdots * \Delta_r = conv((\Delta_1 \times \{b_1\}) \cup \cdots \cup (\Delta_r \times \{b_r\})),$$

where $\{b_1, \ldots, b_r\}$ is the standard basis of $\mathbb{Z}^r$. For fixed polytopes $\Delta_1, \ldots, \Delta_r$ we denote $\Delta_1 * \cdots * \Delta_r$ simply by $\widetilde{\Delta}$.

Definition 3.2. Define $S_{\widetilde{\Delta}} := \mathbb{C}[C \cap \widetilde{M}]$ to be the semigroup algebra of the monoid $C \cap \widetilde{M}$ over complex numbers. The algebra $S_{\widetilde{\Delta}}$ has a natural $\mathbb{Z}^r_{\geq 0}$-grading defined by the last r coordinates of lattice points in $\widetilde{M}$. By choosing an isomorphism $M \cong \mathbb{Z}^d$, we can identify $S_{\widetilde{\Delta}}$ with a $\mathbb{Z}^r_{\geq 0}$-graded monomial subalgebra in

$$\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, t_{d+1}, \ldots, t_{d+r}],$$

where the $\mathbb{Z}^r_{\geq 0}$-grading is considered with respect to the last r variables $t_{d+1}, \dots, t_{d+r}$. We denote by $I_{\widetilde{\Delta}}$ the $\mathbb{Z}^r_{\geq 0}$-graded monomial ideal in $S_{\widetilde{\Delta}}$ generated by all lattice points in the interior of C. For any $k = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geq 0}$, we denote by $S^k_{\widetilde{\Delta}}$ (resp. by $I^k_{\widetilde{\Delta}}$) the k-homogeneous component of $S_{\widetilde{\Delta}}$ (resp. of $I_{\widetilde{\Delta}}$). We will use also the total $\mathbb{Z}_{\geq 0}$-grading on $S_{\widetilde{\Delta}}$ and $I_{\widetilde{\Delta}}$. For any nonnegative integer l, we denote the corresponding l-homogeneous components of $S_{\widetilde{\Delta}}$ and $I_{\widetilde{\Delta}}$ by $S^{(l)}_{\widetilde{\Delta}}$ and $I^{(l)}_{\widetilde{\Delta}}$ respectively. So one has:

$$S^{(l)}_{\widetilde{\Delta}} = \bigoplus_{|k|=l} S^k_{\widetilde{\Delta}}, \quad I^{(l)}_{\widetilde{\Delta}} = \bigoplus_{|k|=l} I^k_{\widetilde{\Delta}},$$

where $|k| := k_1 + \cdots + k_r$.

Let $f_1(t), \dots, f_r(t)$ be Laurent polynomials in $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ such that Δ_i is the Newton polytope of f_i $(1 \leq i \leq r)$. We set

$$F(t) := t_{d+1} f_1(t) + \cdots + t_{d+r} f_r(t).$$

It is easy to see that $\widetilde{\Delta} = \Delta_1 * \cdots * \Delta_r$ is the Newton polytope of F. Moreover, using the decomposition

$$S^{(1)}_{\widetilde{\Delta}} = \bigoplus_{|k|=1} S^k_{\widetilde{\Delta}} = \bigoplus_{i=1}^{r} S^{b_i}_{\widetilde{\Delta}},$$

we see that every Laurent polynomial G in $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}, t_{d+1}, \dots, t_{d+r}]$ with the Newton polytope $\widetilde{\Delta}$ can be obtained from the sequence of arbitrary Laurent polynomials $g_1, \dots, g_r \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ by the formula $G = t_{d+1} g_1 + \cdots + t_{d+r} g_r$, where Δ_i is the Newton polytope of g_i $(1 \leq i \leq r)$. The above correspondence $\{f_1, \dots, f_r\} \mapsto F$ is usually called *Cayley trick.* We call $F = t_{d+1} f_1 + \cdots + t_{d+r} f_r$ the *Cayley polynomial associated with* $f_1, \dots, f_r$.

Definition 3.3. Let $\Delta_1, \dots, \Delta_r \subset M_{\mathbb{R}}$ be convex polytopes with vertices in M such $\Delta = \Delta_1 + \cdots + \Delta_r$ has dimension d. We say that r Laurent polynomials

$$f_i(t) = \sum_{m \in \Delta_i \cap M} a^{(i)}_m t^m, \quad i = 1, \dots, r$$

form a $\widetilde{\Delta}$*-regular sequence* if the corresponding Cayley polynomial F is $\widetilde{\Delta}$-regular, i.e., the polynomials

$$F_i := t_i \partial / \partial t_i F, \quad i = 1, \dots, d + r$$

form a regular sequence in $S_{\widetilde{\Delta}}$.

Definition 3.4. Let $f_1(t), \dots, f_r(t)$ be Laurent polynomials with Newton polytopes $\Delta_1, \dots, \Delta_r$ as above, $F = t_{d+1} f_1 + \cdots + t_{d+r} f_r$ the corresponding Cayley polynomial, and

$$H_F := \det \left(t_i \frac{\partial F_j}{\partial t_i} \right)_{1 \leq i,j \leq d+r} = \det \left(\left(t_i \frac{\partial}{\partial t_i} \right) \left(t_j \frac{\partial}{\partial t_j} \right) F \right)_{1 \leq i,j \leq d+r} \in I^{(d+r)}_{\widetilde{\Delta}} \subset S^{(d+r)}_{\widetilde{\Delta}}$$

the Hessian of F. For any $k=(k_1,\dots,k_r)$ with $|k|=d+r$ we define $H_F^k \in I_{\tilde\Delta}^k$ to be the k-homogeneous component of H_F. The polynomial H_F^k will be called *k-mixed Hessian of $f_1,\dots,f_r$*.

Remark 3.5. Since the last r rows of the matrix

$$\left(\left(t_i\frac{\partial}{\partial t_i}\right)\left(t_j\frac{\partial}{\partial t_j}\right)F\right)_{1\le i,j\le d+r}$$

are divisible respectively by $t_{d+1},\dots,t_{d+r}$, the Hessian H_F is divisible by the monomial $t_{d+1}\cdots t_{d+r}$. Therefore $H_F^k=0$ if one of the coordinates k_i of $k=(k_1,\dots,k_r)$ is zero. In particular, one has

$$H_F=\sum_{\substack{k\in\mathbb{Z}^r_{>0}\\ |k|=d+r}} H_F^k.$$

Let $k=(k_1,\dots,k_r)\in\mathbb{Z}^r_{>0}$ be a solution of the linear Diophantine equation

$$|k|=k_1+\cdots+k_r=d+r.$$

For any r subsets $S_i\subset\Delta_i\cap M$ such that $|S_i|=k_i$ $(1\le i\le r)$ we define the nonnegative integer $\nu(S_1,\dots,S_r)$ as follows: choose an element s_i in each S_i $(1\le i\le r)$, define S to be the $d\times d$-matrix whose rows are all possible nonzero vectors $s-s_i$, where $s\in S_i$, $1\le i\le r$, and set $\nu(S_1,\dots,S_r):=(\det S)^2$. It is easy to see that up to sign $\det S$ does not depend on the choice of elements $s_i\in S_i$ and therefore $\nu(S_1,\dots,S_r)$ is well defined.

Proposition 3.6. *Let $k=(k_1,\dots,k_r)\in\mathbb{Z}^r_{>0}$ be a positive integral solution of the linear Diophantine equation*

$$|k|=k_1+\cdots+k_r=d+r.$$

Then the mixed Hessian can be computed by the following formula

$$H_F^k=t_{d+1}^{k_1}\cdots t_{d+r}^{k_r}\sum_{(S_1,\dots,S_r)}\nu(S_1,\dots,S_r)\prod_{i=1}^r\prod_{s_i\in S_i}a_{s_i}^{(i)}t^{s_i},$$

where the sum runs over all r-tuples $(S_1,\dots,S_r)$ of subsets $S_i\subset\Delta_i\cap M$ such that $|S_i|=k_i$ $(1\le i\le r)$.

Proof. The formula for H_F^k follows immediately from the formula in 2.4(b) applied to the Cayley polytope $\tilde\Delta$. □

Definition 3.7. Let $k=(k_1,\dots,k_r)\in\mathbb{Z}^r_{>0}$ be a positive integral solution of the equation

$$|k|=k_1+\cdots+k_r=d+r.$$

Consider the toric residue

$$\mathrm{Res}_F\ :\ I_{\tilde\Delta}^{(d+r)}\to\mathbb{C}$$

defined as a $\mathbb{C}$-linear map which vanishes on $\langle F_1, \ldots, F_{d+r}\rangle I_{\widetilde{\Delta}}$ and sends H_F to $\mathrm{Vol}(\widetilde{\Delta}) = \mathrm{Vol}(\Delta_1 * \cdots * \Delta_r)$. The restriction Res_F^k of Res_F to the k-th homogeneous component $I_{\widetilde{\Delta}}^k$:

$$\mathrm{Res}_F^k \;:\; I_{\widetilde{\Delta}}^k \to \mathbb{C}$$

will be called the *k-mixed toric residue associated with $f_1, \ldots, f_r$.*

Since H_F^k is an element of $I_{\widetilde{\Delta}}^k$, it is natural to ask about the value of $\mathrm{Res}_F^k(H_F^k)$.

Conjecture 3.8. *Let $k = (k_1, \ldots, k_r) \in \mathbb{Z}_{>0}^r$ be a positive integral solution of $k_1 + \cdots + k_r = d + r$. We set $\overline{k} = (\overline{k_1}, \ldots, \overline{k_r}) := (k_1 - 1, \ldots, k_r - 1)$. Then*

$$\mathrm{Res}_F^k(H_F^k) = V(\underbrace{\Delta_1, \ldots, \Delta_1}_{\overline{k_1}}, \ldots, \underbrace{\Delta_r, \ldots, \Delta_r}_{\overline{k_r}}),$$

where $V(\Theta_1, \ldots, \Theta_d)$ denotes the mixed volume of convex polytopes $\Theta_1, \ldots, \Theta_d$ multiplied by $(d + r - 1)!$.

Our conjecture agrees with a result of Danilov and Khovanskii:

Proposition 3.9. [DK, §6] *The normalized volume of the Cayley polytope $\widetilde{\Delta} = \Delta_1 * \cdots * \Delta_r$ can be computed by the following formula:*

$$\mathrm{Vol}(\Delta_1 * \cdots * \Delta_r) = \sum_{|\overline{k}|=d} V(\underbrace{\Delta_1, \ldots, \Delta_1}_{\overline{k_1}}, \ldots, \underbrace{\Delta_r, \ldots, \Delta_r}_{\overline{k_r}}).$$

Remark 3.10. Let $r = d$ and $k = (d + 1, 1, \ldots, 1)$. It follows from 3.6 and 2.4(b) that

$$H_F^k = H_{t_{d+1}f_1}(t_{d+2}f_2) \cdots (t_{2d}f_d),$$

where

$$H_{t_{d+1}f_1} = \det\left(\left(t_i\frac{\partial}{\partial t_i}\right)\left(t_j\frac{\partial}{\partial t_j}\right)t_{d+1}f_1\right)_{1\le i,j\le d+1}$$

On the other hand, we have

$$V(\underbrace{\Delta_1, \ldots, \Delta_1}_{\overline{k_1}}, \ldots, \underbrace{\Delta_r, \ldots, \Delta_r}_{\overline{k_r}}) = V(\underbrace{\Delta_1, \ldots, \Delta_1}_{d}) = \mathrm{Vol}(\Delta_1).$$

Therefore, Conjecture 3.8 can be considered as a "generalization" of 2.5(ii).

It is easy to show that the cone C from 3.1 is a reflexive Gorenstein cone of index r if and only if $\Delta = \Delta_1 + \cdots + \Delta_r$ is a reflexive polytope. In this situation, we have

$$I_{\widetilde{\Delta}} = t_{d+1} \cdots t_{d+r} S_{\widetilde{\Delta}}.$$

Therefore one has canonical isomorphisms:

$$I_{\widetilde{\Delta}}^k \cong S_{\widetilde{\Delta}}^{\overline{k}}, \;\; \forall\, k \in \mathbb{Z}_{>0}^r,$$

where the monomial basis in $S^{\overline{k}}_{\overline{\Delta}}$ can be identified with the set of all lattice points in $\overline{k_1}\Delta_1 + \cdots + \overline{k_r}\Delta_r$. The $\overline{k}$-homogeneous component of corresponding toric residue map

$$\mathrm{Res}^{\overline{k}}_F \;:\; S^{\overline{k}}_{\overline{\Delta}} \to \mathbb{C}.$$

will be also called $\overline{k}$-*mixed toric residue.*

4. Toric Residue Mirror Conjecture

Let M and $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual to each other abelian groups of rank d, $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ their $\mathbb{R}$-scalar extensions and $\Delta \subset M_{\mathbb{R}}$ a reflexive polytope with the unique interior lattice point $0 \in M$. Denote by $\mathbb{P}_\Delta$ a Gorenstein toric Fano variety associated with Δ. Let $D_1, \ldots, D_s$ be the toric divisors on $\mathbb{P}_\Delta$ corresponding to the codimension-1 faces $\Theta_1, \ldots, \Theta_s$ and $e_1, \ldots, e_s$ the vertices of the dual reflexive polytope $\Delta^* \subset N_{\mathbb{R}}$ such that

$$\Delta = \{x \in M_{\mathbb{R}} \;:\; \langle x, e_j\rangle \geq -1, \;\; j = 1, \ldots, s\},$$

$$\Theta_j = \Delta \cap \{x \in M_{\mathbb{R}} \;:\; \langle x, e_j\rangle = -1\}, \;\; j \in \{1, \ldots, s\}.$$

Definition 4.1. A Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ is called a *nef-partition* of the reflexive polytope Δ if all vertices of $\Delta_1, \ldots, \Delta_r$ belong to M, and

$$\min_{x \in \Delta_i} \langle x, e_j\rangle \in \{0, -1\}, \;\; \forall 1 \leq i \leq r, \; \forall 1 \leq j \leq s.$$

Since $\min_{x\in\Delta}\langle x, e_j\rangle = -1$ for all $j \in \{1, \ldots, s\}$, the equality $\min_{x\in\Delta_i}\langle x, e_j\rangle = -1$ holds exactly for one index $i \in \{1, \ldots, r\}$ if we fix a vertex $e_j \in \Delta^*$. Therefore, we can split the set of vertices $\{e_1, \ldots, e_s\} \subset \Delta^*$ into a disjoint union of subsets $B_1, \ldots, B_r$ where

$$B_i := \{e_j \;:\; j \in \{1, \ldots, s\}, \;\; \min_{x \in \Delta_i} \langle x, e_j\rangle = -1\}.$$

Now we can define r nef Cartier divisors

$$E_i := \sum_{j\,:\,e_j \in B_i} D_j, \;\; i = 1, \ldots, r.$$

Therefore, a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ of polytopes induces a partition of the anti-canonical divisor $-K_{\mathbb{P}_\Delta} = D_1 + \cdots + D_n$ of $\mathbb{P}_\Delta$ into a sum of r nef Cartier divisors:

$$-K_{\mathbb{P}_\Delta} = E_1 + \cdots + E_r.$$

Now it is easy to see that the above definition of the nef-partition is equivalent to the definition given in [Bo].

Definition 4.2. If $\Delta = \Delta_1 + \cdots + \Delta_r$ is a nef-partition, then for any $i = 1, \ldots, r$ we denote

$$\nabla_i := \{y \in N_{\mathbb{R}} \,:\, \langle x, y\rangle \geq -\delta_{ij},\ \ x \in \Delta_j,\ j = 1, \ldots, r\}.$$

The lattice polytopes $\nabla_1, \ldots, \nabla_r$ define another nef-partition $\nabla := \nabla_1 + \cdots + \nabla_r$ of the reflexive polytope $\nabla \subset N_{\mathbb{R}}$ which is called *dual nef-partition.*

The lattice polytopes $\nabla_1, \ldots, \nabla_r$ can be also defined as

$$\nabla_j := conv(\{0\} \cup B_j) \subset M_{\mathbb{R}}, \quad j = 1, \ldots, r.$$

Moreover, one has two dual reflexive polytopes

$$\Delta^* = conv(\nabla_1 \cup \cdots \cup \nabla_r) \subset N_{\mathbb{R}}$$

$$\nabla^* = conv(\Delta_1 \cup \cdots \cup \Delta_r) \subset M_{\mathbb{R}}.$$

Nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ defines a family of $(d-r)$-dimensional Calabi-Yau complete intersections defined by vanishing of r Laurent polynomials $f_1, \ldots, f_r$ with Newton polytopes $\Delta_1, \ldots, \Delta_r$. According to [Bo], the dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ defines the mirror dual family of Calabi-Yau complete intersections.

Define $\mathcal{A}_j$ to be a subset in $\Delta_j \cap M$ containing all vertices of Δ_j and set $A_j := \mathcal{A}_j \setminus \{0\}$, $j = 1, \ldots, r$. It is easy to see that $A_i \cap A_j = \emptyset$ for all $i \neq j$. We set $A_1 \cup \cdots \cup A_r := \{v_1, \ldots, v_n\}$ and define $a_1, \ldots, a_n \in \mathbb{C}$ to be the coefficients of the Laurent polynomials

$$f_j(t) := 1 - \sum_{i\,:\,v_i \in A_j} a_i t^{v_i}, \quad j = 1, \ldots, r.$$

Let $A := \{0\} \cup A_1 \cup \cdots \cup A_r$ and $\widetilde{\Delta} = \Delta_1 * \cdots * \Delta_r$ be the Cayley polytope. Denote by π the injective mapping

$$A_1 \cup \cdots \cup A_r \to \widetilde{\Delta} \cap \widetilde{M}$$

which sends a nonzero lattice point $m \in A_j$ to (m, b_j) $(1 \leq j \leq r)$ and define

$$\widetilde{A} := \pi(A_1 \cup \cdots \cup A_r) \cup \{(0, b_1), \ldots, (0, b_r)\}.$$

We hold notations from [BM1, §4].

Definition 4.3. Choose a coherent triangulation $\mathcal{T} = \{\tau_1, \ldots, \tau_p\}$ of the reflexive polytope $\nabla^* = conv(\Delta_1 \cup \cdots \cup \Delta_r)$ associated with A such that 0 is a vertex of all its d-dimensional simplices $\tau_1, \ldots, \tau_p$. Define a coherent triangulation $\widetilde{\mathcal{T}} = \{\widetilde{\tau_1}, \ldots, \widetilde{\tau_p}\}$ of $\widetilde{\Delta} = \Delta_1 * \cdots * \Delta_r$ associated with $\widetilde{A}$ as follows: a $(d + r - 1)$-dimensional simplex $\widetilde{\tau_i} \in \widetilde{\mathcal{T}}$ is the convex hull of π-images of all nonzero vertices of τ and $\{(0, b_1), \ldots, (0, b_r)\}$. We call $\widetilde{\mathcal{T}}$ the **induced triangulation** of $\widetilde{\Delta}$.

Let $\mathbb{P} := \mathbb{P}_{\Sigma(\mathcal{T})}$ be the d-dimensional simplicial toric variety defined by the fan $\Sigma(\mathcal{T}) \subset M_{\mathbb{R}}$ ($\mathbb{P}$ is a partial crepant desingularization of the Gorenstein toric Fano variety $\mathbb{P}_{\nabla}$) and denote by $\mathbb{P}_\beta$ the *Morrison-Plesser moduli space* [BM1, Definition 3.3] corresponding to a lattice point

$$\beta = (\beta_1, \dots, \beta_n) \in R(\Sigma) = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \, : \, x_1 v_1 + \cdots + x_n v_n = 0\}$$

in the Mori cone $K_{\mathrm{eff}}(\mathbb{P})$. One has a canonical surjective homomorphism

$$\psi_\beta \, : \, H^2(\mathbb{P}, \mathbb{Q}) \to H^2(\mathbb{P}_\beta, \mathbb{Q}).$$

Definition 4.4. By abuse of notations, let us denote by $[D_j] \in H^2(\mathbb{P}_\beta, \mathbb{Q})$ ($1 \leq j \leq n$) the image of $[D_j] \in H^2(\mathbb{P}, \mathbb{Q})$ under ψ_β. Using the multiplication in the cohomology ring $H^*(\mathbb{P}_\beta, \mathbb{Q})$, we define the intersection product

$$\Phi_\beta := [E_1]^{(E_1, \beta)} \cdots [E_r]^{(E_r, \beta)} \prod_{i:(D_i, \beta) < 0} [D_i]^{-(D_i, \beta) - 1}$$

considered as a cohomology class in $H^{2(\dim \mathbb{P}_\beta - d)}(\mathbb{P}_\beta, \mathbb{Q})$ and call Φ_β the **Morrison-Plesser class** corresponding to the nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$.

Definition 4.5. Let $k = (k_1, \dots, k_r) \in \mathbb{Z}^r_{>0}$ be a positive integral solution of

$$|k| = k_1 + \cdots + k_r = d + r.$$

A polynomial $P(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ is called $\overline{k}$**-homogeneous** if it is homogeneous of degree $\overline{k_i} = k_i - 1$ with respect to every group of $|A_i|$ variables x_j ($v_j \in A_i$) ($1 \leq i \leq r$).

Now we are able to formulate a generalized Toric Residue Mirror Conjecture:

Conjecture 4.6. *Let $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\nabla = \nabla_1 + \cdots + \nabla_r$ be two arbitrary dual nef-partitions. Choose any coherent triangulation $\mathcal{T} = \{\tau_1, \dots, \tau_p\}$ of ∇^* associated with A such that 0 is a vertex of all the simplices $\tau_1, \dots, \tau_p$ as above. Then for any $\overline{k}$-homogeneous polynomial $P(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ of degree d the Laurent expansion of the $\overline{k}$-mixed toric residue*

$$R_P(a) := (-1)^d \operatorname{Res}^{\overline{k}}_F(t_{d+1}^{\overline{k_1}} \cdots t_{d+r}^{\overline{k_r}} \, P(a_1 t^{v_1}, \dots, a_n t^{v_n}))$$

at the vertex $v_{\widetilde{\mathcal{T}}} \in \operatorname{Sec}(\widetilde{\Delta})$ corresponding to the induced triangulation $\widetilde{\mathcal{T}} = \{\widetilde{\tau}_1, \dots, \widetilde{\tau}_p\}$ coincides with the generating function of intersection numbers

$$I_P(a) := \sum_{\beta \in K_{\mathrm{eff}}(\mathbb{P})} I(P, \beta) \, a^\beta,$$

where the sum runs over all integral points $\beta = (\beta_1, \dots, \beta_n)$ of the Mori cone $K_{\mathrm{eff}}(\mathbb{P})$, $a^\beta := a_1^{\beta_1} \cdots a_n^{\beta_n}$,

$$I(P, \beta) = \int_{\mathbb{P}_\beta} P([D_1], \dots, [D_n]) \Phi_\beta = \langle P([D_1], \dots, [D_n]) \Phi_\beta \rangle_\beta,$$

and $\Phi_\beta \in H^{2(\dim \mathbb{P}_\beta - d)}(\mathbb{P}_\beta, \mathbb{Q})$ *is the Morrison-Plesser class of* $\mathbb{P}_\beta$. *We assume* $I(P,\beta)$ *to be zero if* $\mathbb{P}_\beta$ *is empty.*

5. Complete intersections in weighted projective spaces

Let $\mathbb{P} = \mathbb{P}(w_1, \dots, w_n)$ be a d-dimensional weighted projective space, $n = d+1$. The fan Σ of $\mathbb{P}(w_1, \dots, w_n)$ is determined by n vectors $v_1, \dots, v_n \in M \simeq \mathbb{Z}^d$ which generate M and satisfy the relation

$$w_1 v_1 + \cdots + w_n v_n = 0.$$

If we assume that $\gcd(w_1, \dots, w_n) = 1$ and

$$w_i | (w_1 + \cdots + w_n), \quad i = 1, \dots, n,$$

then $\mathbb{P}$ is a Gorenstein toric Fano variety with the anticanonical divisor $-K_{\mathbb{P}} = D_1 + \cdots + D_n$, where D_i is the toric divisor corresponding to the vector v_i. These divisors are related modulo rational equivalence as

$$\frac{[D_1]}{w_1} = \cdots = \frac{[D_n]}{w_n} =: [D_0].$$

Consider a decomposition $\{v_1, \dots, v_n\}$ into a disjoint union of r nonempty subsets $A_1, \dots, A_r$ and define the divisors $E_i := \sum_{j\,:\,v_j \in A_i} D_j$ on $\mathbb{P}$ such that $[E_i] = d_i[D_0]$, where $d_j = \sum_{i \in A_j} w_i$, $j = 1, \dots, r$. Note that the integers d_i satisfy $d_1 + \cdots + d_r = w_1 + \cdots + w_n$. Let $\Delta_i := conv(\{0\} \cup A_i)$ $(1 \le i \le r)$. The polytopes $\Delta_1, \dots, \Delta_r$ define a nef-partition $\Delta := \Delta_1 + \cdots + \Delta_r$ if and only if

$$w_i | d_j, \quad i = 1, \dots, n, \; j = 1, \dots, r.$$

The following result generalizes [BM1, Theorem 7.3]:

Theorem 5.1. *Let $P \in \mathbb{Q}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d. Then the generating function of intersection numbers on the Morrison-Plesser moduli spaces has the form*

$$I_P(y) = \nu \cdot P(w_1, \dots, w_n) \sum_{b \ge 0} \mu^b y^b = \frac{\nu \cdot P(w_1, \dots, w_n)}{1 - \mu y},$$

where

$$\nu := \frac{1}{w_1 \cdots w_n}, \quad \mu := \frac{d_1^{d_1} \cdots d_r^{d_r}}{w_1^{w_1} \cdots w_n^{w_n}}, \quad y := a_1^{w_1} \cdots a_n^{w_n}.$$

Proof. The lattice points β in the Mori cone of $\mathbb{P}$ correspond to the linear relations $bw_1v_1 + \cdots + bw_nv_n = 0$, $b \in \mathbb{Z}_{\ge 0}$. Therefore we set $y := a_1^{w_1} \cdots a_n^{w_n}$.

The Morrison-Plesser moduli space $\mathbb{P}_\beta$ is the $(\sum_{i=1}^n w_i)b+d$-dimensional weighted projective space:

$$\mathbb{P}(\underbrace{w_1,\ldots,w_1}_{b+1},\ldots,\underbrace{w_n,\ldots,w_n}_{b+1}).$$

It is easy to see that the Morrison-Plesser class defined by the nef-partition is

$$\Phi_\beta = (d_1[D_0])^{d_1 b}\cdots(d_r[D_0])^{d_r b}.$$

Using $\langle [D_0]^{\dim \mathbb{P}_\beta}\rangle_\beta = 1/w_1^{w_1 b+1}\cdots w_n^{w_n b+1}$, we obtain

$$\begin{aligned}
I_P(y) &= \sum_{b\geq 0}\langle P([D_1],\ldots,[D_n])(d_1[D_0])^{d_1 b}\cdots(d_r[D_0])^{d_r b}\rangle_\beta\, y^b \\
&= P(w_1,\ldots,w_n)\sum_{b\geq 0}(d_1^{d_1}\cdots d_r^{d_r})^b\langle [D_0]^{\dim \mathbb{P}_\beta}\rangle_\beta\, y^b \\
&= P(w_1,\ldots,w_n)\sum_{b\geq 0}(d_1^{d_1}\cdots d_r^{d_r})^b\frac{1}{w_1^{w_1 b+1}\cdots w_n^{w_n b+1}}\, y^b \\
&= \nu\cdot P(w_1,\ldots,w_n)\sum_{b\geq 0}\mu^b\, y^b \\
&= \frac{\nu\cdot P(w_1,\ldots,w_n)}{1-\mu y}.
\end{aligned}$$

□

The convex hull of the vectors $v_1,\ldots,v_n$ is a reflexive polytope (simplex) $\nabla^*\subset M_\mathbb{R}\cong\mathbb{R}^d$. Let $\widetilde{M} := M\oplus\mathbb{Z}^r$ be an extension of the lattice M and $\{b_1,\ldots,b_r\}$ the standard basis of $\mathbb{Z}^r$. The $(d+r-1)$-dimensional Cayley polytope

$$\widetilde{\Delta} = \Delta_1 * \cdots * \Delta_r$$

is the convex hull of $(d+r+1)$ points: $(0,b_1),\ldots,(0,b_r)$ and (v_k,b_j) $(k=1,\ldots,d+1)$, where $v_k\in A_j$. We denote this set of points by $\widetilde{A}$. The points from $\widetilde{A}$ are affinely dependent, while any proper subset of $\widetilde{A}$ is affinely independent, i.e., defines a circuit (see [GKZ, Chapter 7]). It is easy to see that the only affine relation (up to a real multiple) between the points from A is

$$d_1e_1+\cdots+d_re_r-w_1u_1-\cdots-w_nu_n=0.$$

Thus by [GKZ, Chapter 7, Proposition 1.2], polytope $\widetilde{\Delta}$ has exactly two triangulations: the triangulation $\mathcal{T}=\mathcal{T}_1$ with the simplices $conv(A\setminus\{e_i\})$, $i=1,\ldots,r$, and the triangulation $\mathcal{T}_2$ with the simplices $conv(A\setminus\{u_k\})$, $k=1,\ldots,n$. Note that

$$\text{(1)}\qquad \mathrm{Vol}(conv(A\setminus\{e_i\})) = d_i,\quad i=1,\ldots,r,$$

$$\text{(2)}\qquad \mathrm{Vol}(conv(A\setminus\{v_k\})) = w_k,\quad k=1,\ldots,n.$$

Therefore $\mathrm{Vol}(\widetilde{\Delta}) = \sum_{i=1}^{r} d_i = \sum_{k=1}^{n} w_k$.

Let

$$f_j(t) := 1 - \sum_{i\,:\,v_i \in A_j} a_i t^{v_i} \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}], \quad j = 1, \dots, r$$

be generic Laurent polynomials. Denote by

$$F(t) = t_{d+1} f_1(t) + \cdots + t_{d+r} f_r(t)$$

a Laurent polynomial whose support polytope is $\widetilde{\Delta}$.

The next statement follows directly from [GKZ, Chapter 9, Proposition 1.8] and from the equalities (1), (2).

Proposition 5.2. *The A-discriminant of F is equal (up to sign) to the binomial*

$$D_A(F) = \prod_{k=1}^{n} w_k^{w_k} - \prod_{i=1}^{r} d_i^{d_i} \prod_{k=1}^{n} a_k^{w_k} = \prod_{k=1}^{n} w_k^{w_k}(1 - \mu y),$$

where $y = \prod_{k=1}^{n} a_k^{w_k}$ and the first summand in $D_A(F)$ corresponds to the triangulation $\mathcal{T}$.

Theorem 5.3. *Let $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ be a $\overline{k}$-homogeneous polynomial with $|\overline{k}| = d$. Then*

$$R_P(a) = (-1)^d \mathrm{Res}_F^k \left(t_{d+1}^{\overline{k_1}+1} \cdots t_{d+r}^{\overline{k_r}+1} P(a_1 t^{v_1}, \dots, a_n t^{v_n}) \right) = \frac{\nu \cdot P(w_1, \dots, w_n)}{1 - \mu y},$$

where $y := a_1^{w_1} \cdots a_n^{w_n}$.

Proof. By Proposition 2.6 the toric residue $R_P(a)$ is the following sum over the critical points ξ of the polynomial $F_1(t,y) := f_1(t) + y_2 f_2(t) + \cdots + y_r f_r(t)$, where $(t,y) \in (\mathbb{C}^*)^d \times (\mathbb{C}^*)^{r-1}$:

$$R_P(a) = (-1)^d \sum_{\xi \in V_{F_1}} \frac{P(a_1 \xi^{v_1}, \dots, a_n \xi^{v_n})}{F_1(\xi) H_{F_1}^1(\xi)}.$$

We rewrite polynomial F_1 as

$$F_1 = y_2 + \cdots + y_r + 1 - \sum_{i=1}^{n} c_i t^{v_i},$$

where $c_i = y_j \cdot a_i$ if $v_i \in A_j$. Then at the critical point ξ, we have

$$c_1 \frac{\xi^{v_1}}{w_1} = \cdots = c_n \frac{\xi^{v_n}}{w_n} = z$$

and

$$z^{w_1 + \cdots + w_n} = \left(\frac{c_1}{w_1} \right)^{w_1} \cdots \left(\frac{c_n}{w_n} \right)^{w_n}.$$

Moreover, at the critical points one has:

$$f_2(\xi) = \cdots = f_r(\xi) = 0,$$

which is equivalent to

$$f_j(\xi) = 1 - \sum_{i:v_i \in A_j} a_i \xi^{v_i} = 1 - \left(\sum_{i:v_i \in A_j} w_i \right) \frac{z}{\eta_j} = 1 - \frac{d_j z}{\eta_j} = 0, \quad j = 2, \ldots, r,$$

where η_j is the value of y_j at the critical point. Hence, it is easy to see that $\eta_j = d_j z$, $(j = 2, \ldots, r)$ and $F_1 = 1 - d_1 z$, which implies

$$z^{w_1+\cdots+w_n} = \left(\frac{a_1}{w_1}\right)^{w_1} \cdots \left(\frac{a_n}{w_n}\right)^{w_n} d_2^{d_2} \cdots d_r^{d_r} \cdot z^{d_2+\cdots+d_r}, \tag{3}$$

or, equivalently,

$$z^{d_1} = \left(\frac{a_1}{w_1}\right)^{w_1} \cdots \left(\frac{a_n}{w_n}\right)^{w_n} d_2^{d_2} \cdots d_r^{d_r}.$$

The value of the Hessian $H^1_{F_1}$ at ξ equals

$$H^1_{F_1}(\xi) = (-1)^d w_1 \cdots w_n d_1 \cdots d_r z^{d+r-1}.$$

Since there are exactly $w_1 + \cdots + w_n = d_1 + \cdots + d_r$ critical points of F, the summation over the critical points is equivalent to the summation over the roots of (3), we get

$$\begin{aligned} R_P(y) &= \sum_{z^{d_1} = \left(\frac{a_1}{w_1}\right)^{w_1} \cdots \left(\frac{a_n}{w_n}\right)^{w_n} d_2^{d_2} \cdots d_r^{d_r}} \frac{P(w_1, \ldots, w_n)}{w_1 \cdots w_n d_1 (1 - d_1 z)} \\ &= \frac{P(w_1, \ldots, w_n)}{w_1 \cdots w_n} \sum_{b \geq 0} \left(d_1^{d_1} \cdots d_r^{d_r}\right)^b \left(\frac{a_1^{w_1} \cdots a_n^{w_n}}{w_1^{w_1} \cdots w_n^{w_n}}\right)^b \\ &= \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y}. \end{aligned}$$

□

6. Complete intersections in product of projective spaces

In this section we check the Toric Residue Mirror Conjecture for nef-partitions corresponding to mirrors of complete intersections in product of projective spaces $\mathbb{P} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_p}$ of dimension $d = d_1 + \cdots + d_p$. We set $n_i := d_i + 1$ and denote by $N = (n_{ij})$ an integral $p \times r$-matrix with non-negative elements having columns $\mathbf{n}_1, \ldots, \mathbf{n}_r \in \mathbb{Z}^p_{\geq 0}$. A complete intersection V of r hypersurfaces $V_1, \ldots, V_r$ in $\mathbb{P}$ of multidegrees $\mathbf{n}_1, \ldots, \mathbf{n}_r$ is a Calabi-Yau $(d-r)$-fold if and only if $\sum_{j=1}^r n_{ij} = n_i$ $(i = 1, \ldots, p)$. We will use the standard notation

$$\left(\begin{array}{c||ccc} \mathbb{P}^{d_1} & n_{11} & \cdots & n_{1r} \\ \vdots & \vdots & & \vdots \\ \mathbb{P}^{d_p} & n_{p1} & \cdots & n_{pr} \end{array} \right)$$

to denote this complete intersection.

The cone of effective curves $K_{\rm eff}(\mathbb{P})$ is isomorphic to $\mathbb{R}^p_{\geq 0}$ and its integral part $K_{\rm eff}(\mathbb{P})_{\mathbb{Z}}$ consists of the points $\beta = (b_1, \ldots, b_p) \in \mathbb{Z}^p_{\geq 0}$. Thus, the Morrison-Plesser moduli spaces are the products of projective spaces: $\mathbb{P}_\beta = \mathbb{P}^{n_1 b_1 + d_1} \times \cdots \times \mathbb{P}^{n_p b_p + d_p}$ and the generating function for intersection numbers may be written

$$I_P(y) = \sum_{b_1, \ldots, b_p \geq 0} I(P, \beta)\, y_1^{b_1} \cdots y_p^{b_p}.$$

Theorem 6.1. *The generating function for intersection numbers associated with monomial* $x^k = x_1^{k_1} \cdots x_p^{k_p}$ *can be written as the integral*

$$I_{x^k}(y) = \left(\frac{1}{2\pi i} \right)^p \int_\Gamma \frac{z_1^{k_1} \cdots z_p^{k_p} dz_1 \wedge \cdots \wedge dz_p}{G_1(z) \cdots G_p(z)}, \tag{4}$$

where the polynomials G_i *have the form*

$$G_i = z_i^{n_i} - \prod_{j=1}^r (n_{1j} z_1 + \cdots + n_{pj} z_p)^{n_{ij}} y_i, \quad i = 1, \ldots, p,$$

and Γ *is the compact cycle in* $\mathbb{C}^p$ *defined by* $\Gamma = \{|G_1| = \cdots = |G_p| = \varepsilon\}$ *for small positive* ε.

Proof. Let $[H_i]$ denotes the class of hyperplane section in $\mathbb{P}^{d_i}$. The class of the divisor E_j defining hypersurface V_j equals

$$[E_j] = n_{1j}[H_1] + \cdots + n_{pj}[H_p], \quad j = 1, \ldots, r.$$

Hence, the coefficients of the series $I_{x^k}(y)$ are

$$\langle [H_1]^{k_1} \cdots [H_p]^{k_p} \prod_{j=1}^r (n_{1j}[H_1] + \cdots + n_{pj}[H_p])^{n_{1j} b_1 + \cdots + n_{pj} b_p} \rangle_\beta.$$

The lattice points $\beta = (b_1, \ldots, b_p) \in \mathbb{Z}^r_{\geq 0}$ in the integral part of the Mori cone $K_{\rm eff}(\mathbb{P})$ correspond to the p linear relations

$$b_i v_{i1} + \cdots + b_i v_{in_i} = 0, \quad i = 1, \ldots, p,$$

where $v_{j1}, \ldots, v_{jn_j}$ generate lattice M_j of rank d_j $(1 \leq j \leq p)$. Therefore we set $y_i := a_{i1} \cdots a_{in_i}$. Using the property of the integral

$$\left(\frac{1}{2\pi i}\right)^p \int_{\gamma_\rho} z_1^{m_1-1} \cdots z_p^{m_p-1}\, dz = \begin{cases} 1, & m_1 = \cdots = m_p = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\gamma_\rho = \{|z_1| = \cdots = |z_p| = \rho\}$ is the cycle winding around the origin ($\rho > 0$ is small) and the fact that the intersection numbers on $\mathbb{P}$ are

$$\langle [H_1]^{l_1} \cdots [H_p]^{l_p} \rangle_\beta = \begin{cases} 1, & l_j = n_j b_j + d_j, \quad j = 1, \ldots, r, \\ 0, & \text{otherwise,} \end{cases}$$

we can represent the functions $I(x^k, \beta)$ by integrals

$$I(x^k, \beta) = \left(\frac{1}{2\pi i}\right)^p \int_{\gamma_\rho} \frac{z_1^{k_1} \cdots z_p^{k_p} \prod_{j=1}^r (n_{1j} z_1 + \cdots + n_{pj} z_p)^{n_{1j} b_1 + \cdots + n_{pj} b_p}\, dz}{z_1^{n_1 b_1 + d_1 + 1} \cdots z_p^{n_p b_p + d_p + 1}}.$$

Denote

$$F_i(z) := \prod_{j=1}^r (n_{1j} z_1 + \cdots + n_{pj} z_p)^{n_{ij}}, \quad i = 1, \ldots, p.$$

If $z \in \gamma_\rho$ for some fixed ρ, then the geometric series

$$z_1^{k_1 - n_1} \cdots z_p^{k_p - n_p} \sum_{b_1, \ldots, b_p \geq 0} \left(\frac{F_1(z) y_1}{z_1^{n_1}}\right)^{b_1} \cdots \left(\frac{F_p(z) y_p}{z_p^{n_p}}\right)^{b_p} = \frac{z_1^{k_1} \cdots z_p^{k_p}}{\prod_{i=1}^p [z_i^{n_i} - F_i(z) y_i]}$$

converges absolutely and uniformly for all y from the neighbourhood $\mathcal{U}_\varepsilon = \{y : ||y|| < \varepsilon\}$, where $0 < \varepsilon < \min_{i=1,\ldots,p}(\rho^{n_i}/M_i)$, $M_i = \max_{z \in \gamma_\rho} |F_i(z)|$. Integrating the last expression and changing the order of integration and summation, we get

$$I_{x^k}(y) = \left(\frac{1}{2\pi i}\right)^p \int_{\gamma_\rho} \frac{z_1^{k_1} \cdots z_p^{k_p}\, dz_1 \wedge \cdots \wedge dz_p}{\prod_{i=1}^p [z_i^{n_i} - F_i(z) y_i]}.$$

The cycle γ_ρ for fixed $y \in \mathcal{U}_\varepsilon$ can be replaced by its homologous by Rouché's principle for residues (see [Ts, Chapter 2, §8] or [AY, Lemma 4.9])

$$\gamma_\rho \sim \Gamma = \{z : |z_1^{n_1} - F_1(z) y_1| = \cdots = |z_p^{n_p} - F_p(z) y_p| = \delta\}.$$

Therefore, we have

$$I_{x^k}(y) = \left(\frac{1}{2\pi i}\right)^p \int_{\Gamma} \frac{z_1^{k_1} \cdots z_p^{k_p}\, dz_1 \wedge \cdots \wedge dz_p}{\prod_{i=1}^p [z_i^{n_i} - F_i(z) y_i]}$$

which finishes the proof. □

The Conjecture 4.6 follows now from a general result in [BM2] which identifies

$$\left(\frac{1}{2\pi i}\right)^p \int_{\Gamma} \frac{z_1^{k_1} \cdots z_p^{k_p} dz_1 \wedge \cdots \wedge dz_p}{G_1(z) \cdots G_p(z)}$$

with the toric residue.

7. Computation of Yukawa $(d-r)$-point functions

Let $\Delta = \Delta_1 + \cdots + \Delta_r$ be a nef-partition of a reflexive polytope Δ, $A_i \subset \partial\nabla^* \cap \Delta_i \cap M$ a subset containing all nonzero vertices of Δ_i $(1 \le i \le r)$. We set $A_1 \cup \cdots \cup A_r := \{v_1, \ldots, v_n\}$ and consider a $\Delta_1 * \cdots * \Delta_r$-regular sequence of Laurent polynomials

$$f_j(t) := 1 - \sum_{i: v_i \in A_j} a_i t^{v_i} \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}], \quad j = 1, \ldots, r,$$

which define r affine hypersurfaces

$$Z_{f_j} := \{t \in \mathbb{T} \cong (\mathbb{C}^*)^d \ : \ f_j(t) = 0\}, \quad j = 1, \ldots, r,$$

The compactification $\overline{Z}_f$ in $\mathbb{P}_\Delta$ of the affine complete intersection $Z_f := Z_{f_1} \cap \cdots \cap Z_{f_r}$ is a $(d-r)$-dimensional projective Calabi-Yau variety with at worst Gorenstein canonical singularities. Using the Poincaré residue mapping

$$\mathbf{Res} : H^d(\mathbb{T} \setminus Z_{f_1} \cup \cdots \cup Z_{f_r}) \to H^{d-r}(Z_{f_1} \cap \cdots \cap Z_{f_r})$$

one can construct a nowhere vanishing section of the canonical bundle of $\overline{Z}_f$ as

$$\Omega := \mathbf{Res} \left(\frac{1}{f_1 \cdots f_r} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \right).$$

Definition 7.1. Let $Q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d-r$. The *Q-Yukawa $(d-r)$-point function* is defined by the formula

$$Y_Q(a_1, \ldots, a_n) := \frac{(-1)^{\frac{(d-r)(d-r-1)}{2}}}{(2\pi i)^{d-r}} \int_{Z_f} \Omega \wedge Q\left(a_1 \frac{\partial}{\partial a_1}, \ldots, a_n \frac{\partial}{\partial a_n}\right) \Omega,$$

where the differential operators $a_1 \partial/\partial a_1, \ldots, a_n \partial/\partial a_n$ are determined by the Gauß-Manin connection. If $\overline{k} = (\overline{k_1}, \ldots, \overline{k_r})$ is a nonnegative integral vector with $|\overline{k}| = d-r$ and $Q(x_1, \ldots, x_n)$ is a $\overline{k}$-homogeneous polynomial ($\deg x_j = k_i \Leftrightarrow v_j \in A_i$), then

$$Q\left(a_1 \frac{\partial}{\partial a_1}, \ldots, a_n \frac{\partial}{\partial a_n}\right) \Omega = (-1)^{d-r} \mathbf{Res} \left(\frac{Q(a_1 t^{v_1}, \ldots, a_n t^{v_n})}{f_1^{\overline{k_1}+1} \cdots f_r^{\overline{k_r}+1}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \right).$$

Theorem 7.2. *Let $Q(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ be a $\overline{k}$-homogeneous polynomial with $|\overline{k}| = d-r$. We define*

$$P(x_1, \ldots, x_n) := \prod_{j=1}^{r} \left(\sum_{i: v_i \in A_j} x_i \right) Q(x_1, \ldots, x_n).$$

Then the Yukawa $(d-r)$-point function is equal to the k-mixed toric residue

$$Y_Q(a_1,\dots,a_n) = (-1)^d \mathrm{Res}^k_F\left(t_{d+1}^{\overline{k_1}+1}\cdots t_{d+r}^{\overline{k_r}+1} P(a_1 t^{v_1},\dots,a_n t^{v_n})\right).$$

Proof. We sketch only the idea of the proof. The hypersurface

$$Z_F = \{t_{d+1}f_1 + \cdots + t_{d+r}f_r = 1\}$$

in $(\mathbb{C}^*)^d \times \mathbb{C}^r$ is a $\mathbb{C}^{r-1}$-bundle over $(\mathbb{C}^*)^d \setminus (Z_{f_1}\cap\cdots\cap Z_{f_r})$. This fact allows to identify primitive parts of the cohomology groups $H^{d-r}(Z_{f_1}\cap\cdots\cap Z_{f_r})$ and $H^{d-1}(Z_F)$ together with their intersection forms. By the result of Mavlyutov [Mav], one can compute the intersection form on $H^{d-1}(Z_F)$ using toric resides. □

Example 7.3. Consider the mirror family V^* to Calabi-Yau complete intersections V of r hypersurfaces of degrees $d_1,\dots,d_r$ respectively in $\mathbb{P}^d$, $d_1+\cdots+d_r = d+1$. Its nef-partition can be constructed as follows. Let $v_1,\dots,v_d$ be a basis vectors of the lattice M and

$$v_{d+1} := -v_1 - \cdots - v_d.$$

We divide the set $\{v_1,\dots,v_{d+1}\}$ into a disjoint union of r subsets $A_1,\dots,A_r$ such that $|A_i| = d_i$. For $j=1,\dots,r$, we define Laurent polynomials

$$f_j(t) := 1 - \sum_{i:v_i\in A_j} a_i t^{v_i} \in \mathbb{C}[t_1^{\pm 1},\dots,t_d^{\pm 1}].$$

Then the affine part of V^* is the complete intersection $Z_f \subset \mathbb{T}$ of hypersurfaces $Z_{f_1},\dots,Z_{f_r} \subset \mathbb{T}$ defined by polynomials $f_1,\dots,f_r$. The Yukawa coupling for V^* has been computed in [BvS, Proposition 5.1.2]:

$$Y_Q(y) = \frac{d_1\cdots d_r Q(1,\dots,1)}{1-\mu y},$$

where $y = a_1\cdots a_n$ and $\mu = \prod_{i=1}^r d_i^{d_i}$.

Example 7.4. Consider Calabi-Yau varieties V obtained as complete intersection of hypersurfaces V_1, V_2, V_3 in $\mathbb{P}^3\times\mathbb{P}^3$ of degrees $(3,0)$, $(0,3)$ and $(1,1)$ respectively of type

$$\left(\begin{array}{c||ccc} \mathbb{P}^3 & 3 & 0 & 1 \\ \mathbb{P}^3 & 0 & 3 & 1 \end{array}\right).$$

Let $M\cong\mathbb{Z}^6$ and $\nabla^* = conv(\Delta_1\cup\Delta_2\cup\Delta_3)\subset M_{\mathbb{R}}$ be a reflexive polytope defined by the polytopes $\Delta_1 := conv\{0,v_1,v_2,v_3\}$, $\Delta_2 := conv\{0,v_5,v_6,v_7\}$ and $\Delta_3 := conv\{0,v_4,v_8\}$, where

$$v_1 = (1,0,0,0,0,0),\ v_2 = (0,1,0,0,0,0),\ v_3 = (0,0,1,0,0,0),$$
$$v_4 = (-1,-1,-1,0,0,0),\ v_5 = (0,0,0,1,0,0),\ v_6 = (0,0,0,0,1,0),$$
$$v_7 = (0,0,0,0,0,1),\ v_8 = (0,0,0,-1,-1,-1).$$

The nef-partition $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ corresponds to mirrors V^* of $V = V_1 \cap V_2 \cap V_3$. We define the disjoint sets: $A_1 := \{v_1, v_2, v_3\}$, $A_2 := \{v_5, v_6, v_7\}$, $A_3 := \{v_4, v_8\}$ and the Laurent polynomials

$$f_1(t) := 1 - \sum_{i:v_i \in A_1} a_i t^{v_i} = 1 - a_1 t_1 - a_2 t_2 - a_3 t_3,$$

$$f_2(t) := 1 - \sum_{i:v_i \in A_2} a_i t^{v_i} = 1 - a_5 t_4 - a_6 t_5 - a_7 t_6,$$

$$f_3(t) := 1 - \sum_{i:v_i \in A_3} a_i t^{v_i} = 1 - a_4 t_1^{-1} t_2^{-1} t_3^{-1} - a_8 t_4^{-1} t_5^{-1} t_6^{-1}.$$

The complete intersection $Z_f := Z_{f_1} \cap Z_{f_2} \cap Z_{f_3}$ of the affine hypersurfaces

$$Z_{f_j} = \{t \in (\mathbb{C}^*)^6 \; : \; f_j(t) = 0\}, \quad j = 1, 2, 3$$

is an affine part of V^*.

Denote by $y_1 = 3^3 a_1 a_2 a_3 a_4$, $y_2 = 3^3 a_5 a_6 a_7 a_8$ the new variables and by $\theta_1 := y_1 \partial / \partial y_1$, $\theta_2 := y_2 \partial / \partial y_2$ the corresponding logarithmic partial derivations. Given a form-residue

$$\Omega := \mathbf{Res} \left(\frac{1}{f_1 f_2 f_3} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_6}{t_6} \right) \in H^3(Z_f),$$

we compute the 2-parameter Yukawa couplings defined as integrals

$$Y^{(k_1,k_2)}(y_1, y_2) = \frac{-1}{(2\pi i)^3} \int_{Z_f} \Omega \wedge \theta_1^{k_1} \theta_2^{k_2} \Omega, \quad k_1 + k_2 = 3.$$

Proposition 7.5. *The Yukawa couplings are*

$$Y^{(3,0)}(y_1, y_2) = \frac{9 y_1}{(1 - y_1 - y_2)(1 - y_1)^2}, \; Y^{(2,1)}(y_1, y_2) = \frac{9}{(1 - y_1 - y_2)(1 - y_1)},$$
$$Y^{(1,2)}(y_1, y_2) = \frac{9}{(1 - y_1 - y_2)(1 - y_2)}, \; Y^{(0,3)}(y_1, y_2) = \frac{9 y_2}{(1 - y_1 - y_2)(1 - y_2)^2}.$$

Remark 7.6. Note that the functions in Proposition 7.5 are completely consistent with Yukawa couplings from [BvS, §8.3]. Indeed, if we put $K(y_1, y_2) = Y^{(3,0)} + 3Y^{(2,1)} + 3Y^{(1,2)} + Y^{(0,3)}$ and consider restriction to the diagonal subfamily $y = y_1 = y_2$, then we get the same expression as in [BvS]:

$$K(y, y) = \frac{18(3 - 2y)}{(1 - 2y)(1 - y)^2}.$$

Denote by $F(t) := t_7 f_1(t) + t_8 f_2(t) + t_9 f_3(t)$ the Cayley polynomial associated with Laurent polynomials f_1, f_2, f_3, and by $\widetilde{\Delta} = \Delta_1 * \Delta_2 * \Delta_3 \subset \widetilde{M_{\mathbb{R}}} = M_{\mathbb{R}} \oplus \mathbb{R}^3$ its supporting polytope which is the Cayley polytope associated with $\Delta_1, \Delta_2, \Delta_3$.

Proposition 7.7. *Let $A := \widetilde{\Delta} \cap \widetilde{M}$ and $F(t)$ be the Cayley polynomial as above. Then the principal A-determinant of F has the form*

$$E_A(F) = (a_1 \cdots a_8)^{12}(1-y_1)^3(1-y_1)^3(1-y_1-y_2).$$

Remark 7.8. It is easy to see that the products of $Y^{(k_1,k_2)}$ by $E_A(F)$ are polynomials in $a_1, \dots, a_8$.

Let us find the generating function $I_P(y)$ for the monomial $P(x) = x_1^{k_1}x_2^{k_2}$. There are two linear independent integral relations between $v_1, \dots, v_8$:

$$v_1 + \cdots + v_4 = 0, \quad v_5 + \cdots + v_8 = 0.$$

Hence the Mori cone $K_{\text{eff}}(\mathbb{P})$ is spanned by the vectors

$$l^{(1)} = (1,1,1,1,0,0,0,0), \quad l^{(2)} = (0,0,0,0,1,1,1,1)$$

and the Morrison-Plesser moduli spaces are $\mathbb{P}_\beta = \mathbb{P}^{4b_1+3} \times \mathbb{P}^{4b_2+3}$ ($b_1, b_2 \in \mathbb{Z}_{\geq 0}$). The cohomology ring of $\mathbb{P}_\beta$ is generated by two hyperplane classes: $[H_1]$ and $[H_2]$. We set $E_1 := 3[H_1]$, $E_2 := 3[H_2]$ and $E_3 := [H_1] + [H_2]$. Then the nef-partition of the anticanonical divisor $-K_{\mathbb{P}}$ corresponding to the nef-partition $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ is defined by $-K_{\mathbb{P}} = E_1 + E_2 + E_3$. Therefore, the Morrison-Plesser cohomology class associated with the nef-partition of $-K_{\mathbb{P}}$ equals

$$\Phi_\beta = (3[H_1])^{3b_1}(3[H_2])^{3b_2}([H_1]+[H_2])^{b_1+b_2}$$

and the generating function for intersection numbers can be written

$$I_P(y) = \sum_{b_1,b_2 \geq 0} \langle [H_1]^{k_1+3b_1+1}[H_2]^{k_2+3b_2+1}([H_1]+[H_2])^{b_1+b_2+1} \rangle_\beta \, y_1^{b_1} y_2^{b_2}.$$

Intersection theory on $\mathbb{P}_\beta$ implies

$$I_P(y) = 9 \sum_{b_1,b_2 \geq 0} \frac{(b_1+b_2+1)!}{(b_1-k_1+2)!(b_2-k_2+2)!} y_1^{b_1} y_2^{b_2}.$$

By Theorem 6.1 we can write $I_P(y)$ as the integral

$$I_P(y) = \frac{1}{(2\pi i)^2} \int_\Gamma \frac{9 z_1^{k_1+1} z_2^{k_2+1} (z_1+z_2)\, dz_1 \wedge dz_2}{(z_1^4 - z_1^3(z_1+z_2)y_1)(z_2^4 - z_2^3(z_1+z_2)y_2)}$$

with the cycle $\Gamma = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1^4 - z_1^3(z_1+z_2)y_1| = \varepsilon_1, |z_2^4 - z_2^3(z_1+z_2)y_2| = \varepsilon_2\}$, $\varepsilon_1, \varepsilon_2 > 0$. Computing the last integrals, we get the same rational functions as in Proposition 7.5.

Example 7.9. Consider an example of Calabi-Yau variety V obtained as complete intersections of two hypersurfaces of degrees $(4,0)$, $(1,2)$ in $\mathbb{P} = \mathbb{P}^4 \times \mathbb{P}^1$ which corresponds to the configuration

$$\left(\begin{array}{c||cc} \mathbb{P}^4 & 4 & 1 \\ \mathbb{P}^1 & 0 & 2 \end{array} \right).$$

This example was investigated in details by Hosono, Klemm, Theisen and Yau (cf. [HKTY]). The corresponding nef-partition $\Delta = \Delta_1 + \Delta_2 \subset M_{\mathbb{R}} \cong \mathbb{R}^5$ consists of polytopes $\Delta_1 := conv\{0, v_1, v_2, v_3, v_4\}$ and $\Delta_2 := conv\{0, v_5, v_6, v_7\}$, where

$$v_1 = (1,0,0,0,0), \quad v_2 = (0,1,0,0,0), \quad v_3 = (0,0,1,0,0), \quad v_4 = (0,0,0,1,0),$$
$$v_5 = (-1,-1,-1,-1,0), \quad v_6 = (0,0,0,0,1), \quad v_7 = (0,0,0,0,-1).$$

We have two disjoint sets: $A_1 := \{v_1, v_2, v_3, v_4\}$ and $A_2 := \{v_5, v_6, v_7\}$ which are the vertices of the reflexive polytope ∇^* and define the Laurent polynomials

$$f_1(t) = 1 - \sum_{i:v_i \in A_1} a_i t^{v_i} = 1 - a_1t_1 - a_2t_2 - a_3t_3 - a_4t_4,$$
$$f_2(t) = 1 - \sum_{i:v_i \in A_2} a_i t^{v_i} = 1 - a_5(t_1t_2t_3t_4)^{-1} - a_6t_5 - a_7t_5^{-1}.$$

Denote by $y_1 := a_1 \cdots a_5$, $y_2 := a_6a_7$ the new variables and $\theta_1 := y_1\partial/\partial y_1$, $\theta_2 := y_2\partial/\partial y_2$ the corresponding logarithmic partial derivations. Let Ω be a form defined by

$$\Omega := \mathbf{Res}\left(\frac{1}{f_1f_2}\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}\right) \in H^3(Z_{f_1} \cap Z_{f_2}).$$

Then the Yukawa coupling associated with f_1, f_2 is the integral

$$Y^{(k_1,k_2)}(y_1, y_2) := \frac{-1}{(2\pi i)^3}\int_{Z_f} \Omega \wedge \theta_1^{k_1}\theta_2^{k_2}\Omega, \quad k_1 + k_2 = 3,$$

where $Z_f := Z_{f_1} \cap Z_{f_2}$ is an affine Calabi-Yau complete intersection which compactification forms a mirror dual family to V.

Proposition 7.10. [HKTY] *The Yukawa couplings $Y^{(k_1,k_2)}(y)$ are:*

$$Y^{(3,0)}(y) = \frac{8}{D_0}, \quad Y^{(2,1)}(y) = \frac{4(1 - 256y_1 + 4y_2)}{D_0D_1},$$
$$Y^{(1,2)}(y) = \frac{8y_2(3 - 512y_1 + 4y_2)}{D_0D_1^2},$$
$$Y^{(0,3)}(y) = \frac{4y_2(1 - 256y_1 + 24y_2 - 3072y_1y_2 + 16y_2^2)}{D_0D_1^3}.$$

Let $F(t) := t_6f_1(t) + t_7f_2(t)$ be the Cayley polynomial associated with $f_1(t)$ and $f_2(t)$. Its support polytope is the Cayley polytope $\widetilde{\Delta} = \Delta_1 * \Delta_2 \subset \widetilde{M_{\mathbb{R}}} = M_{\mathbb{R}} \oplus \mathbb{R}^2$ which is the convex hull of the vectors:

$$u_1 = (0,0,0,0,0;1,0),\ u_2 = (1,0,0,0,0;1,0),\ u_3 = (0,1,0,0,0;1,0),$$
$$u_4 = (0,0,1,0,0;1,0),\ u_5 = (0,0,0,1,0;1,0),\ u_6 = (0,0,0,0,0;0,1),$$
$$u_7 = (-1,-1,-1,-1,0;0,1),\ u_8 = (0,0,0,0,1;0,1),\ u_9 = (0,0,0,0,-1;0,1).$$

Proposition 7.11. *Let* $A := \{u_1, \ldots, u_9\} \subset \widetilde{M}$ *and*

$$D_0 := (1 - 256y_1)^2 - 4y_2, \quad D_1 := 1 - 4y_2.$$

Then the principal A*-determinant of* $F(t)$ *has the following form:*

$$\begin{aligned}
&E_A(F) = a_1^8 a_2^8 a_3^8 a_4^8 a_5^8 a_6^5 a_7^5 D_0 D_1^4 = \\
&-640a1^8a2^8a3^8a4^8a5^8a6^8a7^8 - 16777216a1^{10}a2^{10}a3^{10}a4^{10}a5^{10}a6^8a7^8 + \\
&160a1^8a2^8a3^8a4^8a5^8a6^7a7^7 + \underline{16777216a1^{10}a2^{10}a3^{10}a4^{10}a5^{10}a6^9a7^9} + \\
&\underline{65536a1^{10}a2^{10}a3^{10}a4^{10}a5^{10}a6^5a7^5} + 6291456a1^{10}a2^{10}a3^{10}a4^{10}a5^{10}a6^7a7^7 - \\
&1048576a1^{10}a2^{10}a3^{10}a4^{10}a5^{10}a6^6a7^6 - \underline{1024a1^8a2^8a3^8a4^8a5^8a6^{10}a7^{10}} + \\
&1280a1^8a2^8a3^8a4^8a5^8a6^9a7^9 - 512a1^9a2^9a3^9a4^9a5^9a6^5a7^5 - \\
&20a1^8a2^8a3^8a4^8a5^8a6^6a7^6 + 131072a1^9a2^9a3^9a4^9a5^9a6^8a7^8 + \\
&\underline{a1^8a2^8a3^8a4^8a5^8a6^5a7^5} + 8192a1^9a2^9a3^9a4^9a5^9a6^6a7^6 - \\
&49152a1^9a2^9a3^9a4^9a5^9a6^7a7^7 - 131072a1^9a2^9a3^9a4^9a5^9a6^9a7^9,
\end{aligned}$$

where the terms corresponding to the vertices of Newton polytope of $E_A(F)$ *are underlined.*

Proof. The principal A-determinant can be found by using the algorithm proposed by A. Dickenstein and B. Sturmfels [DS] via the computation of the corresponding Chow forms. □

Remark 7.12. We note that D_0 is the principal component of the discriminant locus $E_A(F) = 0$ and the component D_1 corresponds to the edge Γ of $\widetilde{\Delta}$ with

$$\Gamma \cap \widetilde{M} = \{u_6, u_8, u_9\} = \{(0,0,0,0,0;0,1), (0,0,0,0,1;0,1), (0,0,0,0,-1;0,1)\}.$$

The Newton polytope of $E_A(F)$ is the secondary polytope $\mathrm{Sec}(A)$ depicted in Figure 1. The vertices of $\mathrm{Sec}(A)$ are in one-to-one correspondence with coherent triangulations $\mathcal{T}_1, \ldots, \mathcal{T}_4$ of $\widetilde{\Delta}$ which are:

$$\begin{aligned}
\mathcal{T}_1 &= \{\langle u_1,u_3,u_4,u_5,u_6,u_7,u_8\rangle, \langle u_1,u_2,u_4,u_5,u_6,u_7,u_8\rangle, \langle u_1,u_2,u_3,u_5,u_6,u_7,u_8\rangle, \\
&\quad \langle u_1,u_2,u_3,u_4,u_6,u_7,u_8\rangle, \langle u_1,u_2,u_3,u_4,u_5,u_6,u_8\rangle, \langle u_1,u_3,u_4,u_5,u_6,u_7,u_9\rangle, \\
&\quad \langle u_1,u_2,u_4,u_5,u_6,u_7,u_9\rangle, \langle u_1,u_2,u_3,u_5,u_6,u_7,u_9\rangle, \langle u_1,u_2,u_3,u_4,u_6,u_7,u_9\rangle, \\
&\quad \langle u_1,u_2,u_3,u_4,u_5,u_6,u_9\rangle\}. \\
\mathcal{T}_2 &= \{\langle u_1,u_3,u_4,u_5,u_7,u_8,u_9\rangle, \langle u_1,u_2,u_4,u_5,u_7,u_8,u_9\rangle, \langle u_1,u_2,u_3,u_5,u_7,u_8,u_9\rangle, \\
&\quad \langle u_1,u_2,u_3,u_4,u_7,u_8,u_9\rangle, \langle u_1,u_2,u_3,u_4,u_5,u_8,u_9\rangle\}. \\
\mathcal{T}_3 &= \{\langle u_2,u_3,u_4,u_5,u_7,u_8,u_9\rangle, \langle u_1,u_2,u_3,u_4,u_5,u_7,u_9\rangle, \langle u_1,u_2,u_3,u_4,u_5,u_7,u_8\rangle\}. \\
\mathcal{T}_4 &= \{\langle u_1,u_2,u_3,u_4,u_5,u_7,u_8\rangle, \langle u_1,u_2,u_3,u_4,u_5,u_7,u_9\rangle, \langle u_2,u_3,u_4,u_5,u_6,u_7,u_8\rangle, \\
&\quad \langle u_2,u_3,u_4,u_5,u_6,u_7,u_9\rangle\}.
\end{aligned}$$

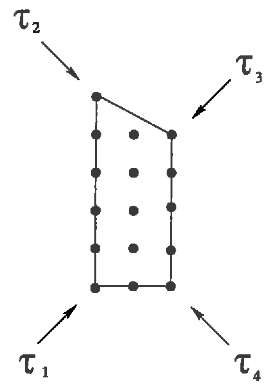

FIGURE 1. Secondary polytope and coherent triangulations

The generating function $I_P(y)$ for intersection numbers corresponding to the monomial $P(x) = x_1^{k_1}x_2^{k_2}$ can be computed from the intersection theory on the Morrison-Plesser moduli spaces. Using two independent integral relations between $v_1, \ldots, v_7$

$$v_1 + \cdots + v_5 = 0, \quad v_6 + v_7 = 0,$$

we see that the Mori cone $K_{\text{eff}}(\mathbb{P})$ is spanned by two vectors

$$l^{(1)} = (1,1,1,1,1,0,0), \quad l^{(2)} = (0,0,0,0,0,1,1).$$

The Morrison-Plesser moduli spaces are $\mathbb{P}_\beta = \mathbb{P}^{5b_1+4} \times \mathbb{P}^{2b_2+1}$ $(b_1, b_2 \in \mathbb{Z}_{\geq 0})$. The cohomology of $\mathbb{P}_\beta$ are generated by the hyperplane classes $[H_1]$ and $[H_2]$. Let $E_1 := [H_1] + 2[H_2]$ and $E_2 := 4[H_1]$. Then the nef-partition $\Delta = \Delta_1 + \Delta_2$ of polytopes induces the nef-partition of the anticanonical divisor

$$-K_{\mathbb{P}} = E_1 + E_2 = ([H_1] + 2[H_2]) + (4[H_1]).$$

It is straightforward to see that the corresponding Morrison-Plesser class is

$$\Phi_\beta = ([H_1] + 2[H_2])^{b_1+2b_2}(4[H_1])^{4b_1}.$$

So we get

$$I_P(y) = \sum_{b_1,b_2\geq 0} \langle [H_1]^{k_1}[H_2]^{k_2}([H_1] + 2[H_2])^{b_1+2b_2+1}(4[H_1])^{4b_1+1}\rangle_\beta \, y_1^{b_1}y_2^{b_2}.$$

Using the intersection theory on $\mathbb{P}_\beta$, we obtain

$$I_P(y) = \sum_{b_1,b_2\geq 0} 2^{8b_1+2b_2-k_2+3}\frac{(b_1+2b_2+1)!}{(b_1-k_1+3)!(2b_2-k_2+1)!}\, y_1^{b_1}y_2^{b_2}.$$

By Theorem 6.1 the function $I_P(y)$ admits the integral representation:

$$I_P(y) = \frac{1}{(2\pi i)^2}\int_\Gamma \frac{4z_1^{k_1+1}z_2^{k_2}(z_1+2z_2)dz_1\wedge dz_2}{(z_1^5 - (z_1+2z_2)(4z_1)^4y_1)(z_2^2-(z_1+2z_2)^2y_2)}$$

with the cycle

$$\Gamma = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1^5 - (z_1 + 2z_2)(4z_1)^4 y_1| = \varepsilon_1,\, |z_2^2 - (z_1 + 2z_2)^2 y_2| = \varepsilon_2\},$$

where $\varepsilon_1, \varepsilon_2$ are positive. These integrals can be easily computed and yield the same rational functions as in Proposition 7.10.

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