Periodicity of cell pressures in periodic foams

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Abstract

A hitherto open problem of the mathematics of foams asks : are the cell pressures in a (structurally) periodic foam in equilibrium necessarily periodic ? We supply a proof that this is so. We also describe a model in which external forces can introduce a pressure gradient.

Introduction Among a long list of open problems of the mathematics of foams which has been compiled by F. Morgan, J. Sullivan and collaborators [1], one concerns the cell pressures in periodic foams. It seems obvious that if the *structure* of a foam in equilibrium is periodic, so that in particular *differences* of pressures between periodic cells are periodic, then the pressures themselves must be periodic. But it appears that a proof of this reasonable assertion has not yet been provided. We do so here. For reasons which will become clear, we should add to the conjecture "in the absence of external forces such as gravity".

Periodic foam We shall use the language of three dimensions, but the analysis does not really depend upon the dimension of the foam. We consider a foam whose unit cluster, which contains bubbles $i = 1...N$, is repeated periodically with lattice vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, thus tiling space. The bubble pressures are not taken to be periodic, so that they may be written

$$p_i(n_1, n_2, n_3). \tag{1}$$

Here n_1, n_2, n_3 define the unit cluster in which bubble i resides, according to the translation by a combination of lattice vectors $\vec{a}_i$:

$$n_1 \vec{a}_1 + n_2 \vec{a}_2 + n_3 \vec{a}_3 \tag{2}$$

of the reference unit cluster $(0, 0, 0)$. We seek to prove that p_i is independent of (n_1, n_2, n_3), that is, it is periodic. We do this by deriving an expression for the *total* force on a unit cluster, which must be zero in equilibrium.

Locally, all pressure differences must accord with the Laplace Law, that is the pressure *difference* between two bubbles is given by

$$\Delta p = \frac{4\sigma}{r} \tag{3}$$

where r is the radius of mean curvature of the boundary which separates two neighbouring bubbles. Since r is a structural constant, it is periodic and so all such pressure differences are periodic.

Proof of conjecture Consider the set of pressures in the reference cell $(0,0,0)$ and some other cell (n_1, n_2, n_3). By focusing attention on two neighbouring bubbles, indexed by i and j, we can see that their pressures must differ from one unit cluster to the other by the *same* constant

$$p_i(0,0,0) = p_i(n_1, n_2, n_3) + c(n_1, n_2, n_3) \tag{4}$$

$$p_j(0,0,0) = p_j(n_1, n_2, n_3) + c(n_1, n_2, n_3) \tag{5}$$

by the Laplace condition, provided that bubbles i and j are neighbours. But by "stepping" through the unit cluster it is clear that this extends to all of its constituent bubbles (which are assumed to be connected). We write $c_1 = c(1,0,0), c_2 = c(0,1,0)$ etc.

It is convenient at this point to consider a particular construction for the unit cluster. We take the primitive (Bravais) cell of the lattice, and assign those bubbles that overlap the boundary to a primitive cell. To make contact with the above, the unit cluster of bubbles is defined in such a way that all bubbles in cell $(0,0,0)$ which intersect the boundaries are assigned to cell $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ consistently (cf. Figure 1).

We now consider the total force on the contents of the primitive cell by examining the forces which act on its surface. By periodicity, surface terms make *zero* contribution, the individual contributions cancelling on opposite faces. By the above result for pressures, pressures on opposing faces of the cell differ by a constant c_1, c_2 or c_3. Hence the total force due to these is

$$\vec{F} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 \tag{6}$$

where $\vec{b}_i$ are the reciprocal lattice vectors which correspond to $\vec{a}_i$, according to

$$\vec{b}_i = \vec{a}_j \times \vec{a}_k. \tag{7}$$

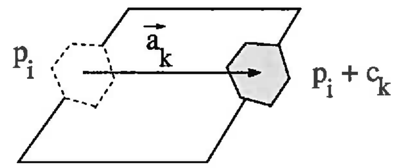

Figure 1: Equivalent bubbles on opposite sides of the cell are related through translation by a lattice vector $\vec{a}_i$. Their pressures are related by an additive constant c_i which depends, a priori, on the lattice vector.

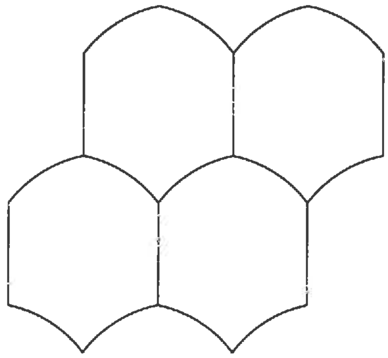

Figure 2: Two-dimensional example of a foam "loaded" by the weight of its Plateau borders.

Now the contents of this primitive cell are entirely in local equilibrium, that is they are in equilibrium under the total external force. So their combination must be so. There is no other *external* force to balance $\vec{F}$, therefore the force is zero:

$$\vec{F} = \vec{0}. \tag{8}$$

Since the $\vec{b}_i$ are linearly independent, we must have that c_1, c_2 and c_3 are zero; therefore pressures of equivalent bubbles are equal in neighbouring cells, and hence in all cells. This is what we were required to prove.

Remarks Despite this result, it is interesting to consider the hypothetical case in which c_1, c_2, c_3 are *not* zero, and there is a constant pressure *gradient* defining the variation from one cell or unit cluster to the next, so that

$$c(n_1, n_2, n_3) = c_1\, n_1 + c_2\, n_2 + c_3\, n_3. \tag{9}$$

We have shown that a foam which is in equilibrium in the absence of external forces cannot have this property, but an external force may be introduced to make it possible. In particular, our own work on drainage has led us to define the following ("loaded foam") model.

In the loaded foam, a uniform gravitational force (per unit length) acts downward on all of the edges which represent Plateau borders. In this case, the total weight force per unit volume may be balanced by a pressure gradient of the kind discussed above. The effect upon the structure is to change the angles at which the films intersect, with respect to their equilibrium values. A simple 2d example is shown in Figure 2. We believe this model will be useful in the context of drainage, and particularly in relation to drainage instabilities. We will present such an analysis in a forthcoming paper [2].

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References

[1] Burlington Mathfest (1995), organised by F. Morgan; J. Sullivan, private communication.

[2] N. Kern, D. Weaire, G. Verbist and S.J. Cox (2002) *In preparation.*