Stark's Conjecture and new Stickelberger phenomena

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Abstract

We introduce a new conjecture concerning the construction of elements in the annihilator ideal associated to a Galois action on the higher-dimensional algebraic K-groups of rings of integers in number fields. Our conjecture is motivic in the sense that it involves the (transcental) Borel regulator as well as being related to l-adic étale cohomology. In addition, the conjecture generalises the well-known Coates-Sinnott conjecture. For example, for a totally real extension when $r = -2, -4, -6, \ldots$ the Coates-Sinnott conjecture merely predicts that zero annihilates K_{-2r} of the ring of S-integers while our conjecture predicts a non-trivial annihilator. By way of supporting evidence, we prove the corresponding (equivalent) conjecture for the Galois action on the étale cohomology of the cyclotomic extensions of the rationals.

1 Introduction

In 1890 Stickelberger [42] proved what might be called the first "equivariant motivic" result in number theory. Needless to say this aspect of Stickelberger's Theorem was heavily disguised! Recall ([48] p.94) that one may construct, from the values of the Dirichlet L-function, a Stickelberger element in the rational group-ring of the Galois group of a cyclotomic field. Then the product of the annihilator ideal of the roots of unity with the principal fractional ideal generated by the Stickelberger element is integral and annihilates the class-group. Since Galois groups are involved it is clear how the adjective "equivariant" might be associated with Stickelberger's Theorem. The purpose of this paper is to explain the association with "motivic" and to introduce, with supporting evidence, new conjectural Stickelberger-like phenomena.

In what follows, by a Galois representation of a field E we shall mean a continuous, finite-dimensional complex representation of the absolute Galois group of E, which amounts to saying that the representation factors through a finite Galois group $G(F/E)$ of a Galois extension F/E.

We begin with the Stark conjecture, which asserts that the function assigning to a Galois representation of number fields the value of a regulator map divided by the leading term of the Artin L-function at $s = 0$ is always algebraic and is Galois equivariant. Stark's regulator is defined on K_1 of the ring of algebraic integers.

We assume that the higher-dimensional analogue of Stark's conjecture is true – that is, replace K_1 by K_{1-2r} for $r = -1, -2, -3, \ldots$ and the Dirichlet regulator by the Borel regulator. Having posed this higher-dimensional Stark conjecture in an earlier version of this paper I learned from David Burns that it long ago been mentioned by B. Gross [17].

For a Galois extension F/E of number fields with abelian Galois group $G(F/E)$ we construct a "fractional ideal", a finitely generated $\mathbf{Z}[1/2][G(F/E)]$-submodule $\mathcal{J}_F^r$ of the rational group-ring $\mathbf{Q}[G(F/E)]$, for each $r = -1, -2, -3, \ldots$. The construction of $\mathcal{J}_F^r$ is "motivic" in the sense that transcendental techniques (Borel's regulator, Deligne cohomology etc) and l-adic techniques are involved in its construction and the derivation of its properties (for example, Theorem 7.7).

When F/E is totally real and $r = -1, -3, -5, \ldots$ the L-values at $s = r$ are non-zero and in this case $\mathcal{J}_F^r$ is equal to the higher Stickelberger ideal which appears in the Brumer-Coates-Sinnott conjectures. We conjecture that $\mathcal{J}_F^r$ participates in a new Stickelberger phenomenon. Namely, for each odd prime l and a suitable Galois invariant set of primes S',

Conjecture 1.1

$$\left(\mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l) \cdot \mathcal{J}_F^r \right) \bigcap \mathbf{Z}_l[G(F/E)]$$

$$\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(K_{-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l).$$

Here $\mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(M)$ denotes the annihilator ideal of M.

The Quillen-Lichtenbaum conjecture relates K-groups to étale cohomology, predicting that the l-adic Chern classes yield natural isomorphisms of the form

$$K_{e-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l \cong H_{\text{ét}}^{2-e}(\mathrm{Spec}(\mathcal{O}_{F,S'}); \mathbf{Z}_l(1-r))$$

when $e = 0, 1$, $r = -1, -2, -3, \ldots$ and l is an odd prime.

I believe that the validity of this conjecture follows from recent work of Rost, Suslin and Voevodsky in which case Conjecture 1.1 is equivalent to the following similar one involving étale cohomology.

Conjecture 1.2

$$\left(\mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(Tors H_{\text{ét}}^1(\mathrm{Spec}(\mathcal{O}_{F,S'}); \mathbf{Z}_l(1-r))) \cdot \mathcal{J}_F^r \right) \bigcap \mathbf{Z}_l[G(F/E)]$$

$$\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(H_{\text{ét}}^2(\mathrm{Spec}(\mathcal{O}_{F,S'}); \mathbf{Z}_l(1-r))).$$

In this paper we shall verify the second conjecture in the case of abelian extensions of the rationals. In fact, for this it suffices to treat the case of cyclotomic fields (Theorem 6.2). Even this simple case reveals a new phenomenon. Suppose that $F/\mathbf{Q}$ is a totally real, abelian Galois extension and that $r = -1, -3, -5, \ldots$. In this case the Coates-Sinnott conjecture would predict that the higher Stickelberger ideal times the annihilator of $Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l$ lies in the annihilator of $K_{-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l$. However, when $r = -2, -4, -6, \ldots$ the Stickelberger ideal is zero and the Coates-Sinnott conjecture becomes trivial but the conjecture which I have just introduced does not.

The paper is arranged in the following manner. In §2 we recall the Stark conjecture concerning the leading term at $s = 0$ of the Artin L-function and the Brumer conjecture, a generalisation of Stickelberger's theorem, concerning the relation between the value at $s = 0$ of the Artin L-function and the annihilator ideal of the S-class-group in an abelian extension of number fields. In §3 we describe the analogues of the Stark and Brumer conjectures - conjectures of Gross and Coates-Sinnott respectively - in which the algebraic K-groups K_1 and K_0 are replaced by K_{1-2r} and K_{-2r} for $r = -1, -2, -3, \ldots$. In §4, assuming the validity of the higher-dimensional Stark conjecture, we construct a finitely generated, Galois invariant subgroup - the fractional ideal $\mathcal{J}_F^r$ - of $\mathbf{Q}[G(F/E)]$ where $G(F/E)$ is an abelian Galois group of number fields with F a CM field and E totally real. We verify that $\mathcal{J}_F^r$ is well-defined and coincides with the higher Stickelberger ideal when the latter is defined and non-trivial. In §5 we introduce a new conjectural relationship between $\mathcal{J}_F^r$ and the annihilator ideals of higher-dimensional algebraic K-groups (or étale cohomology groups) of algebraic integers. In §6 we prove Conjecture 1.2, the étale cohomology version of Conjecture 1.1, for cyclotomic fields. This is sufficient to verify the conjecture for any abelian CM or totally real extension of the rationals. In §7 we use the technique of ([41]; see also [40] Chapter 6 and 7) together with results from [2] and [5] to establish the technical results which are needed in §6. §8 contains some concluding remarks about possible generalisations and the naturality of the fractional ideal.

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2   Some well-known conjectures

2.1 Let $\zeta_F(s)$ denote the Dedekind zeta function of a number field F. The analytic class number formula ([45] p.21) gives the residue at $s = 1$ in terms of the order of the class-group of $\mathcal{O}_F$, the algebraic integers of F, and the

3

Dirichlet regulator $R_0(F)$. Let d_F denote the discriminant of F. In terms of algebraic K-groups of $\mathcal{O}_F$ the class-group is equal to the torsion subgroup $TorsK_0(\mathcal{O}_F)$ of $K_0(\mathcal{O}_F)$ and the formula has the form

$$res_{s=1} \; \zeta_F(s) \; = \; \frac{2^{r_1+r_2}\pi^{r_2}R_0(F)|TorsK_0(\mathcal{O}_F)|}{|TorsK_1(\mathcal{O}_F)|\sqrt{d_F}}.$$

The Dirichlet regulator $R_0(F)$, which is a real number, is the covolume of the lattice given by the image of the Dirichlet regulator homomorphism ([45] p.25)

$$R_F^0 : \mathcal{O}_F^* = K_1(\mathcal{O}_F) \longrightarrow \mathbf{R}^{r_1+r_2-1}.$$

Here r_1 and $2r_2$ denote the number of real or complex embeddings of F respectively. Equivalently, by Hecke's functional equation ([27], [45] p.18), $\zeta_F(s)$ has a zero of order $r_1 + r_2 - 1$ at $s = 0$. Let $\zeta_F^*(s_0)$ denote the first non-zero coefficient in the Taylor series for ζ_F at $s = s_0$. Therefore at $s = 0$ the functional equation yields

$$\zeta_F^*(0) = \lim_{s \to 0} \frac{\zeta_F(s)}{s^{r_1+r_2-1}} = -\frac{R_0(F)|TorsK_0(\mathcal{O}_F)|}{|TorsK_1(\mathcal{O}_F)|}.$$

This form of the analytic class number formula prompted Lichtenbaum [26] to ask: Which number fields F satisfy the analogous equation for higher-dimensional algebraic K-groups

$$\zeta_F^*(r) = \pm 2^{\epsilon}\frac{R_r(F)|TorsK_{-2r}(\mathcal{O}_F)|}{|TorsK_{1-2r}(\mathcal{O}_F)|}$$

for $r = -1, -2, -3, \ldots$ and some integer ϵ? Here $R_r(F)$ is the covolume of the Borel regulator homomorphism defined on $K_{1-2r}(\mathcal{O}_F)$ and to which we shall return shortly. This identity has become known as the Lichtenbaum conjecture and is known to be true in many cases [22].

Next we shall recall how Stark [45] refined the analytic class number formula into a conjecture dealing with $L_F^*(0, V)$, the leading coefficient of the Taylor series at $s = 0$ of the Artin L-function associated to a Galois representation V of F [27].

Let $\Sigma(F)$ denote the set of embeddings of F into the complex numbers. For $r = -1, -2, -3, \ldots$ set

$$Y_r(F) = \prod_{\Sigma(F)} (2\pi i)^{-r}\mathbf{Z} = \text{Map}(\Sigma(F), (2\pi i)^{-r}\mathbf{Z})$$

endowed with the $G(\mathbf{C}/\mathbf{R})$-action diagonally on $\Sigma(F)$ and on $(2\pi i)^{-r}$. If c_0 denotes complex conjugation $c_0((\ldots, (2\pi i)^{-r}n_\sigma, \ldots)_{\sigma \in \Sigma(L)})$ has $(-1)^r(2\pi i)^{-r}n_\sigma$ in the $c_0 \cdot \sigma$-coordinate. Therefore the fixed points of $Y_r(F)$ under c_0, denoted

4

by $Y_r(F)^+$, correspond to elements $\{(2\pi i)^{-r}n_\sigma\}_{\sigma\in\Sigma(F)}$ such that $(-1)^r n_\sigma = n_{c_0\cdot\sigma}$. Hence if $\sigma(F) \subset \mathbf{R}$ and r is odd then $n_\sigma = 0$. When $r = 0$ we define $Y_0(F)^+$ as the c_0-fixed points of

$$Y_0(F) = Ker(\alpha : (\prod_{\Sigma(F)} \mathbf{Z}) \longrightarrow \mathbf{Z})$$

where α is the homomorphism defined by $\alpha((\ldots, n_\sigma, \ldots)_{\sigma\in\Sigma(F)}) = \sum_{\sigma\in\Sigma(F)} n_\sigma$.

This discussion shows that the rank of $Y_r(F)^+$ is given by

$$rank_{\mathbf{Z}}(Y_r(F)^+) = \begin{cases} r_2 & \text{if } r \text{ is odd,} \\ r_1 + r_2 & \text{if } r > 0 \text{ is even} \\ r_1 + r_2 - 1 & \text{if } r = 0. \end{cases}$$

where $|\Sigma(F)| = r_1 + 2r_2$ and r_1 is the number of real embeddings of F.

Now let $G(F/E)$ denote the Galois group of an extension of number fields F/E. Then, for $g \in G(F/E)$, let $g((\ldots, (2\pi i)^{-r}n_\sigma, \ldots)_{\sigma\in\Sigma(F)}) \in Y_r(F)$ have $(2\pi i)^{-r}n_\sigma$ in the $\sigma\cdot g^{-1}$-coordinate. This defines a left $G(F/E)$-action on $Y_r(F)$ which commutes with that of c_0 so that $Y_r(F)^+$ is a $\mathbf{Z}[G(F/E)]$-lattice. The Dirichlet regulator homomorphism induces an $\mathbf{R}[G(F/E)]$-module isomorphism of the form

$$R_F^0 : K_1(\mathcal{O}_F) \otimes \mathbf{R} = \mathcal{O}_F^* \otimes \mathbf{R} \xrightarrow{\cong} Y_0(F)^+ \otimes \mathbf{R} \cong \mathbf{R}^{r_1+r_2-1}.$$

The existence of this isomorphism implies that ([37] §12.1, [45] p.26) there exists at least one $\mathbf{Q}[G(F/E)]$-module isomorphism of the form

$$f_{0,F} : K_1(\mathcal{O}_F) \otimes \mathbf{Q} \xrightarrow{\cong} Y_0(F)^+ \otimes \mathbf{Q}.$$

For any choice of $f_{0,F}$ Stark forms the composition

$$R_F^0 \cdot (f_{0,F})^{-1} : Y_0(F)^+ \otimes \mathbf{C} \xrightarrow{\cong} Y_0(F)^+ \otimes \mathbf{C}$$

which is an isomorphism of complex representations of $G(F/E)$. Let V be a finite-dimensional complex representation of $G(F/E)$ whose contragredient is denoted by $V^\vee$. The Stark regulator is defined to be the exponential homomorphism, $(V \mapsto R(V, f_{0,F}))$, from representations to non-zero complex numbers given by

$$R(V, f_{0,F}) = det((R_F^0 \cdot f_{0,F}^{-1})_* \in Aut_{\mathbf{C}}(Hom_{G(F/E)}(V^\vee, Y_0(F)^+ \otimes \mathbf{C})))$$

where $(R_F^0 \cdot f_{0,F}^{-1})_*$ is composition with $R_F^0 \cdot f_{0,F}^{-1}$.

Let $R(G(F/E))$ denote the complex representation ring of the finite group $G(F/E)$; that is, $R(G(F/E)) = K_0(\mathbf{C}[G(F/E)])$. Since V determines a Galois representation of E, we have a non-zero complex number $L_E^*(0, V)$ given the leading coefficient of the Taylor series at $s = 0$ of the Artin L-function associated to V [27]. We may modify $R(V, f_{0,F})$ to give another exponential homomorphism

$$\mathcal{R}_{f_0,F} \in \operatorname{Hom}(R(G(F/E)), \mathbf{C}^*)$$

defined by

$$\mathcal{R}_{f_0,F}(V) = \frac{R(V, f_{0,F})}{L_E^*(0, V)}.$$

Let $\overline{\mathbf{Q}}$ denote the algebraic closure of the rationals in the complex numbers and let $\Omega_{\mathbf{Q}}$ denote the absolute Galois group of the rationals, which acts continuously on $R(G(F/E))$ and $\overline{\mathbf{Q}}^*$. The Stark conjecture asserts that

$$\mathcal{R}_{f_0,F} \in \operatorname{Hom}_{\Omega_{\mathbf{Q}}}(R(G(F/E)), \overline{\mathbf{Q}}^*) \subseteq \operatorname{Hom}(R(G(F/E)), \mathbf{C}^*).$$

In other words, $\mathcal{R}_{f_0,F}(V)$ is an algebraic number for each V and for all $z \in \Omega_{\mathbf{Q}}$ we have $z(\mathcal{R}_{f_0,F}(V)) = \mathcal{R}_{f_0,F}(z(V))$. Since any two choices of $f_{0,F}$ differ by multiplication by a $\mathbf{Q}[G(F/E)]$-automorphism, the truth of the conjecture is independent of the choice of $f_{0,F}$.

We shall be particularly interested in the case when $G(F/E)$ is abelian in which case the following observation is important. Let $\hat{G} = \operatorname{Hom}(G, \overline{\mathbf{Q}}^*)$ denote the set of characters on G and let $\mathbf{Q}(\chi)$ denote the field generated by the character values of a representation χ.

Proposition 2.2

Let G be a finite abelian group. Then there exists an isomorphism

$$\lambda_G : \operatorname{Hom}_{\Omega_{\mathbf{Q}}}(R(G), \overline{\mathbf{Q}}^*) \xrightarrow{\cong} \mathbf{Q}[G]^*$$

given by

$$\lambda_G(h) = \sum_{\chi \in \hat{G}} h(\chi) e_\chi$$

where

$$e_\chi = |G|^{-1} \sum_{g \in G} \chi(g) g^{-1} \in \mathbf{Q}(\chi)[G].$$

Proof

This follows by combining the isomorphisms $\mathbf{Q}[G]^* \cong K_1(G)$ and $K_1(G) \cong \operatorname{Hom}_{\Omega_{\mathbf{Q}}}(R(G), \overline{\mathbf{Q}}^*)$, which are part of Fröhlich's Hom-description machinery described in [10]. In fact the second isomorphism, proved originally in [32], is true for arbitrary finite groups G.

When G is abelian the proof is very simple. There is a well-known isomorphism of rings ([24] p.648)

$$\psi : \overline{\mathbf{Q}}[G] \longrightarrow \prod_{\chi \in \hat{G}} \overline{\mathbf{Q}} = \mathrm{Map}(\hat{G}, \overline{\mathbf{Q}})$$

given by $\psi(\sum_{g \in G} \lambda_g g)(\chi) = \sum_{g \in G} \lambda_g \chi(g)$. If $\Omega_{\mathbf{Q}}$ acts on $\overline{\mathbf{Q}}$ and $\hat{G}$ in the canonical manner then ψ is Galois equivariant and induces an isomorphism of $\Omega_{\mathbf{Q}}$-fixed units of the form

$$\mathbf{Q}[G]^* = (\overline{\mathbf{Q}}[G]^*)^{\Omega_{\mathbf{Q}}} \cong \mathrm{Map}_{\Omega_{\mathbf{Q}}}(\hat{G}, \overline{\mathbf{Q}}^*) \cong \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R(G), \overline{\mathbf{Q}}^*).$$

It is straightforward to verify that this isomorphism is the inverse of λ_G. $\square$

2.3 *Stickelberger elements and annihilators*

Now we are going to turn our attention to some conjectures concerning annihilator ideals which appear in [3], [8] and [50]. Suppose that F/E is a Galois extension of number fields with $G(F/E)$ abelian. Suppose also that E is totally real and that F is totally real or is a CM field (i.e. F is a totally imaginary quadratic extension of a totally real field (see [48] p.38)). Let S be a finite set of primes of $\mathcal{O}_E$ including those which ramify in F/E. The reciprocity map of class field theory sends the class of a proper ideal prime to S, $\mathcal{A} \lhd \mathcal{O}_E$, to its Artin symbol $(\mathcal{A}, F/E) \in G(F/E)$. The associated partial zeta function is defined for complex numbers s having $\mathcal{R}e(s) > 1$ by

$$\zeta_{E,S}(g, s) = \sum_{(\mathcal{A}, F/E) = g, \ \mathcal{A} \ prime \ to \ S} N\mathcal{A}^{-s}.$$

Here $g \in G(F/E)$ and the sum is over all ideals coprime to all primes in S. These functions have a meromorphic continuation to the whole complex plane and the corresponding Stickelberger elements are defined to be

$$\Theta_{F/E,S}(1 - r) = \sum_{g \in G(F/E)} \zeta_{E,S}(g, r) \cdot g^{-1} \in \mathbf{C}[G(F/E)]$$

for $r = 0, -1, -2, -3, \ldots$. These elements are characterised by the relation

$$\chi(\Theta_{F/E,S}(1 - r)) = L_{E,S}(r, \chi^{-1})$$

for all one-dimensional complex representations χ of $G(F/E)$, where $L_{E,S}(r, \chi^{-1})$ is the Artin L-function with all the Euler factors associated to elements of S removed. By a result of Klingen and Siegel [38] $\Theta_{F/E,S}(1 - r)$ lies in $\mathbf{Q}[G(F/E)]$ for $r = 0, -1, -2, -3, \ldots$.

Let $\mu(F)$ denote the roots of unity in F so that $\mu(F) = Tors K_1(\mathcal{O}_F)$ in the notation of §2.1. The Stickelberger elements $\Theta_{F/E,S}(1)$ satisfy the integrality relation

$$\mathrm{ann}_{\mathbf{Z}[G(F/E)]}(\mu(F)) \cdot \Theta_{F/E,S}(1) \subseteq \mathbf{Z}[G(F/E)]$$

where $\mathrm{ann}_{\mathbf{Z}[G(F/E)]}(\mu(F)) \lhd \mathbf{Z}[G(F/E)]$ denotes the annihilator ideal of $\mu(F)$. When $E = \mathbf{Q}$ this was proved in [23], for K real quadratic in [9] and in general in [6], [13].

The Brumer conjecture goes further than the mere integrality statement, asserting that

$$\mathrm{ann}_{\mathbf{Z}[G(F/E)]}(Tors K_1(\mathcal{O}_{F,S'})) \cdot \Theta_{F/E,S}(1) \subseteq \mathrm{ann}_{\mathbf{Z}[G(F/E)]}(Tors K_0(\mathcal{O}_{F,S'}))$$

where S' is the set of primes of F above those of S and $\mathcal{O}_{F,S'}$ denotes the S'-integers of F. When $E = \mathbf{Q}$ this is Stickelberger's Theorem ([7] p.298; [48] p.94). In general there are only partial results (for example, [16]).

3   Analogous conjectures for higher K-groups

3.1 *Higher dimensional Stark conjectures*

Lots of interesting progress has been made by simply taking some phenomenon involving class-groups or Picard groups, such as the analytic class number formula, and asking the question: What happens when K_0 is replaced by K_n? The Lichtenbaum conjecture of §2.1 is a prime example. It was B.H. Gross [17] who first asked this question about the Stark conjecture of §2.1.

For any negative integer $r < 0$ we have the Borel regulator ([4], [19])

$$R_F^r : K_{1-2r}(\mathcal{O}_F) \otimes \mathbf{R} \xrightarrow{\cong} Y_r(F)^+ \otimes \mathbf{R}$$

which is an $\mathbf{R}[G(F/E)]$-isomorphism. Now we mechanically imitate Stark's procedure with the Dirichlet regulator replaced by Borel's. We choose a $\mathbf{Q}[G(F/E)]$-isomorphism of the form

$$f_{r,F} : K_{1-2r}(\mathcal{O}_F) \otimes \mathbf{Q} \xrightarrow{\cong} Y_r(F)^+ \otimes \mathbf{Q}$$

so that

$$R_F^r \cdot (f_{r,F})^{-1} : Y_r(F)^+ \otimes \mathbf{R} \xrightarrow{\cong} Y_r(F)^+ \otimes \mathbf{R}$$

is an $\mathbf{R}[G(F/E)]$-isomorphism. Then, as in §2.1, we form the Stark regulator defined, for each representation V of $G(F/E)$, by

$$R(V, f_{r,F}) = det((R_F^r \cdot f_{r,F}^{-1})_* \in Aut_{\mathbf{C}}(Hom_{G(F/E)}(V^{\vee}, Y_r(F)^+ \otimes \mathbf{C}))).$$

Let S be a finite set of primes of E which includes all the primes which ramify in F/E. Let $L_{E,S}^*(r, V)$ denote the leading term of the Taylor expansion of the Artin L-function associated to S and V at $s = r$. We define a function $\mathcal{R}_{f_{r,F}}$ given on a finite-dimensional complex representation V by

$$\mathcal{R}_{f_{r,F}}(V) = \frac{R(V, f_{r,F})}{L_{E,S}^*(r, V)}.$$

8

Then the higher-dimensional analogue of the Stark conjecture of §2.1 asserts that

$$\mathcal{R}_{f,F} \in Hom_{\Omega_{\mathbf{Q}}}(R(G(F/E)), \overline{\mathbf{Q}}^*) \subseteq Hom(R(G(F/E)), \mathbf{C}^*)$$

and the truth of this conjecture is independent of the choice of $f_{r,F}$.

The calculations of Beilinson ([2]; see also [5] §4.2, [19] and [30]) show that the higher-dimensional analogue of the Stark conjecture is true when F/E is a subextension of any abelian extension of the rationals (see Theorem 7.7(proof)).

3.2 *Higher dimensional annihilator conjectures*

In this section we study the case F/E when the subfield E is the rational numbers. In this case it will be convenient to use Dirichlet L-functions [48]. For our purposes this is equivalent to the use of Artin L-functions $L_{\mathbf{Q},S}(r,\chi)$ when S is the set of primes dividing the conductor of F.

Now let us examine the higher-dimensional analogues of the Brumer conjecture of §2.3. These analogues were first posed Coates and Sinnott in the case of abelian extensions of the rationals and were expressed in terms of Stickelberger elements constructed from the Dirichlet L-function. Since we are going to return to this case as a source of crucial examples in §6.1 and Theorem 6.2 we shall recall the situation of [8].

Suppose that $F/\mathbf{Q}$ is a finite Galois extension of number fields with abelian Galois group, $G(F/\mathbf{Q})$, and F totally real. Then, for each negative integer $r = -1, -2, -3, \ldots$, there is a unique unit of the rational group-ring

$$\Theta_{F/\mathbf{Q}}^{Dir}(1-r) \in \mathbf{Q}[G(F/\mathbf{Q})]^*$$

such that

$$\chi(\Theta_{F/\mathbf{Q}}^{Dir}(1-r)) = L_{\mathbf{Q}}(r, \chi^{-1})$$

for each one-dimensional complex representation χ where $L_{\mathbf{Q}}(s, \chi^{-1})$ denotes the Dirichlet L-function of the character χ^{-1} (more precisely, the Dirichlet L-function of the primitive character associated to χ^{-1}; see [48] Ch.4). The rationality of $\Theta_{F/\mathbf{Q}}^{Dir}(n)$ is seen by writing the L-function in terms of partial zeta functions

$$L_{\mathbf{Q}}(r, \chi^{-1}) = \sum_{g \in G(L^{Ker(\chi)}/\mathbf{Q})} \chi(g)^{-1} \zeta_{\mathbf{Q}}(g, r)$$

and recalling that $\zeta_{\mathbf{Q}}(g, r)$ is a rational number, by a result of Klingen and Siegel [38].

Define $\mu_{1-r}(F)$ to be the $\mathbf{Z}[G(F/\mathbf{Q})]$-module given by

$$\mu_{1-r}(F) = \varinjlim_{M/\mathbf{Q}} (\mu(M)^{\otimes^{1-r}})^{G(M/F)} \qquad \vdots$$

9

where the limit is taken over Galois extensions $M/\mathbf{Q}$ containing F. Hence $\mu_1(F) = \mu(F) = TorsK_1(\mathcal{O}_F)$ and the Quillen-Lichtenbaum conjectures in algebraic K-theory predict that $\mu_{1-r}(F) = TorsK_{1-2r}(\mathcal{O}_F)$ ([22], [34]).

Inspired by Stickelberger's Theorem ([48] p.94), the Coates-Sinnott conjecture ([8]; see also [9]) asserts that for any prime l

$$\Theta^{Dir}_{F/\mathbf{Q}}(1-r) \cdot \operatorname{ann}_{\mathbf{Z}_l[G(F/\mathbf{Q})]}(\mu_{1-r}(F) \otimes \mathbf{Z}_l)$$

$$\subseteq \operatorname{ann}_{\mathbf{Z}_l[G(F/\mathbf{Q})]}(K_{-2r}(\mathcal{O}_F) \otimes \mathbf{Z}_l).$$

Actually the conjecture in [8] incorporated an extra factor denoted by $w_{n+1}(\mathbf{Q})$ which we have omitted because it was unnecessary (at least when l is odd; see [41] §1). Also the annihilator of $\mu_{1-r}(F)$ is known ([40] Proposition 7.2.5 and [7]).

The higher-dimensional analogue of the Brumer conjecture, posed and discussed in [40] Chapters 6 and 7), asserts for $r = -1, -2, -3, \ldots$ and $F/E, S, \Theta_{F/E,S}(1-r)$ as in §2.3 that

$$\operatorname{ann}_{\mathbf{Z}[G(F/E)]}(TorsK_{1-2r}(\mathcal{O}_{F,S'})) \cdot \Theta_{F/E,S}(1-r) \subseteq \operatorname{ann}_{\mathbf{Z}[G(F/E)]}(K_{-2r}(\mathcal{O}_{F,S'}))$$

where S' is the set of primes of F above those in S and $\mathcal{O}_{F,S'}$ denotes the S'-integers of F. When $E = \mathbf{Q}$ this is equivalent to the conjecture of [8] mentioned above (see also [1]). Note that $K_{-2r}(\mathcal{O}_{F,S'}) = TorsK_{-2r}(\mathcal{O}_{F,S'})$, being a finite group.

4 The canonical fractional ideal

4.1 As in §2.3, let F/E be a Galois extension of number fields with abelian Galoup group $G(F/E)$. Suppose also that E is totally real, F being arbitrary. Let S be a finite set of primes of $\mathcal{O}_E$ including those which ramify in F/E. Throughout this section we shall assume that the higher-dimensional Stark conjecture of §3.1 is true. Therefore, by Proposition 2.2, we have an element

$$\mathcal{R}_{f_r,F} \in Hom_{\Omega_\mathbf{Q}}(R(G(F/E)), \overline{\mathbf{Q}}^*) \cong \mathbf{Q}[G(F/E)]^*$$

which depends upon the choice of a $\mathbf{Q}[G(F/E)]$-isomorphism $f_{r,F}$ in §3.1 where $r = -1, -2, -3, \ldots$.

The following result is an observation concerning the naturality of the Stark conjecture.

Proposition 4.2

Suppose that F/E is an abelian extension for which the higher-dimensional Stark conjecture §3.1 holds and suppose that F/E_1 is a subextension. If we

choose for E_1 the set of primes S_1 over those of S then the conjecture holds for F/E_1 also.

Let M/E be an intermediate Galois extension of F/E. Then the conjecture is also true for M/E and S if it is true for F/E and S in §3.1.

Proof

Let χ denote a character of $G(F/E)$. We may choose the same $f_{r,F}$ for E and E_1 then

$$L_{E_1,S_1}(r, Res^{G(F/E)}_{G(F/E_1)}(\chi)) = L_{E,S}(r, \chi \otimes Ind^{G(F/E)}_{G(F/E_1)}(1))$$

and

$$Hom_{G(F/E_1)}(Res^{G(F/E)}_{G(F/E_1)}(\chi)^{\vee}, Y_r(F)) \cong Hom_{G(F/E)}((\chi \otimes Ind^{G(F/E)}_{G(F/E_1)}(1))^{\vee}, Y_r(F)).$$

Therefore, since $(R^r_F \cdot f^{-1}_{r,F})_*$ is the same for E and E_1 we find that

$$\mathcal{R}_{f_{r,F}}(Res^{G(F/E)}_{G(F/E_1)}(\chi)) = \mathcal{R}_{f_{r,F}}(\chi \otimes Ind^{G(F/E)}_{G(F/E_1)}(1)).$$

Therefore the conjecture §3.1 holds for F/E_1 if it holds for F/E because $Res^{G(F/E)}_{G(F/E_1)} : R(G(F/E)) \longrightarrow R(G(F/E_1))$ is surjective.

The proof for intermediate extensions M/E is similar and will be left to the reader. $\square$

4.3 $Det_P(\alpha)$

Here is a simple, probably familiar, algebraic construction. Let l be a prime and G a finite abelian group. Suppose that P is a finitely generated projective $\mathbf{Z}_l[G]$-module and that

$$\alpha \in End_{\mathbf{Q}_l[G]}(P \otimes \mathbf{Q}_l).$$

Choose a finitely generated projective $\mathbf{Z}_l[G]$-module R together with a $\mathbf{Z}_l[G]$-module isomorphism of the form

$$\phi : P \oplus R \xrightarrow{\cong} \mathbf{Z}_l[G]^n.$$

Tensoring with the l-adics we may form

$$\phi \cdot (\alpha \oplus 1)\phi^{-1} : \mathbf{Q}_l[G]^n \longrightarrow (P \otimes \mathbf{Q}_l) \oplus (R \otimes \mathbf{Q}_l) \longrightarrow \mathbf{Q}_l[G]^n$$

and taking the determinant with respect to any $\mathbf{Z}_l[G]$-basis for $\mathbf{Z}_l[G]^n$ yields a well-defined element

$$det(\phi \cdot (\alpha \oplus 1)\phi^{-1}) \in \mathbf{Q}_l[G]/\simeq$$

11

where $a \simeq b$ in $\mathbf{Q}_l[G]$ if and only if $a = ub$ for some $u \in \mathbf{Z}_l[G]^*$. We shall denote this determinant by

$$Det_P(\alpha) \in \mathbf{Q}_l[G]/ \simeq .$$

Sometimes it will be convenient to replace $\mathbf{Z}_l[G]$, $\mathbf{Q}_l$ and $\mathbf{Q}_l[G]$ by $\mathbf{Z}[1/2][G]$, $\mathbf{Q}$ and $\mathbf{Q}[G]$, respectively, in the construction of $Det_P(\alpha)$.

Theorem 4.4

(i) In §7.1 $Det_P(\alpha)$ depends only on α.

(ii) If α is an automorphism that $det_P(\alpha)$ defines an element of

$$\mathbf{Q}_l[G]^*/\mathbf{Z}_l[G]^* \cong K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$$

corresponding to $[P, \alpha, P]$. Here the isomorphism is the one described in ([41] §2).

(iii) If l is an odd prime and $c_0 \in G$ has order two with $P = ((1 \pm c_0)/2)\mathbf{Z}_l[G]^n \cong (\mathbf{Z}_l[G]/((1 \mp c_0)/2))^n$ then under the canonical map

$$\mathbf{Q}_l[G]/ \simeq \longrightarrow ((\mathbf{Q}_l[G]/((1 \mp c_0)/2))/ \simeq) \times (\mathbf{Q}_l[G]/((1 \pm c_0)/2))/ \simeq)$$

$Det_P(\alpha)$ maps to $(det(\alpha), 1)$, where $det(\alpha)$ is the determinant of α computed with respect to a $(\mathbf{Z}_l[G]/((1 \mp c_0)/2))$-basis for P.

(iv) Parts (i) and (iii) remain true if $\mathbf{Z}_l[G]$, $\mathbf{Q}_l$ and $\mathbf{Q}_l[G]$ are replaced by $\mathbf{Z}[1/2][G]$, $\mathbf{Q}$ and $\mathbf{Q}[G]$, respectively.

Proof

For part (i) we first observe that changing the $\mathbf{Z}_l[G]$-basis for $\mathbf{Z}_l[G]^n$ changes the determinant by a unit factor in $\mathbf{Z}_l[G]^*$. Next suppose that

$$\phi' : P \oplus R' \xrightarrow{\cong} \mathbf{Z}_l[G]^m$$

is a second $\mathbf{Z}_l[G]$-module isomorphism. Form

$$\phi \oplus \phi' : P \oplus R \oplus P \oplus R' \xrightarrow{\cong} \mathbf{Z}_l[G]^n \oplus \mathbf{Z}_l[G]^m.$$

Clearly we may compute a determinant by tensoring with the l-adic field and using the endomorphism α on the first or the third summand. The difference of these deteminants will be a factor ± 1 since the resulting automorphisms differ by the involution which switches the two copies of P. However, using the first factor is the same as using α and ϕ to commute $Det_P(\alpha)$ while using the second factor is the same as using α and ϕ'. Hence $Det_P(\alpha)$ is also independent of ϕ, as required.

Part (ii) is clear from the definition of the isomorphism described in ([41] §2). Part (iii) follows from the naturality of the construction with respect to homomorphisms of rings. Part (iv) is obvious. $\square$

Definition 4.5 *The fractional ideal $\mathcal{J}_F^r$*

Let F/E be a Galois extension of number fields with E totally real and abelian Galois group $G(F/E)$. When $r = -1, -2, -3, \ldots$ the lattice $Y_r(F)$ is a free $\mathbf{Z}[G(F/E)]$-module as is seen by identifying $Y_r(F)$ with the $G(L/F)$-fixed elements of $Y_r(L)$ for some Galois extension of the rationals L containing F. Therefore $Y_r(F)^+ \otimes \mathbf{Z}[1/2]$ is a finitely generated, projective $\mathbf{Z}[1/2]G(F/E)$-module. By the construction of §4.3, each $\mathbf{Q}[G(F/E)]$-endomorphism of $Y_r(F)^+ \otimes \mathbf{Q}$ gives rise to an element $Det_{Y_r(F)^+ \otimes \mathbf{Z}[1/2]}(\alpha) \in \mathbf{Q}[G(F/E)]$ which is well-defined up to multiplication by a unit of $\mathbf{Z}[1/2]G(F/E)$.

Define $\mathcal{I}_{f_r,F}$ to be the (finitely generated) $\mathbf{Z}[1/2][G(F/E)]$-submodule of $\mathbf{Q}[G(F/E)]$ generated by all the elements $Det_{Y_r(F)^+ \otimes \mathbf{Z}[1/2]}(\alpha)$ where $\alpha \in End_{\mathbf{Q}[G(F/E)]}(Y_r(F)^+ \otimes \mathbf{Q})$ satisfies the integrality condition

$$\alpha \cdot f_{r,F}(K_{1-2r}(\mathcal{O}_F)) \subseteq Y_r(F)^+.$$

Define $\mathcal{J}_F^r$ to be the finitely generated $\mathbf{Z}[1/2][G(F/E)]$-submodule of $\mathbf{Q}[G(F/E)]$ given by

$$\mathcal{J}_F^r = \mathcal{I}_{f_r,F} \cdot \tau(\mathcal{R}_{f_r,F}^{-1})$$

where τ is the automorphism of the group-ring induced by sending each $g \in G(F/E)$ to its inverse. Recall that throughout this section (see §4.1) we are assuming the validity of the higher-dimensional Stark conjecture of §3.1 in order for $\mathcal{R}_{f_r,F} \in \mathbf{Q}[G(F/E)]^*$ to be defined.

Example 4.6 (i) In the situation of §4.1 and Definition 4.5 suppose that F is totally real and $r = -1, -2, -3, \ldots$. In this case $Y_r(F)^+$ is a free $\mathbf{Z}[G(F/E)]$-module of rank equal to $[E : \mathbf{Q}]$ when r is even and is trivial when r is odd.

Assume for the moment that $r = -1, -3, -5, \ldots$. Bearing in mind that the determinant of the zero automorphism of the zero module is 1, for each character χ we have

$$1 = R(\chi, f_{r,F}) = det((R_F^r \cdot f_{r,F}^{-1})_* = det(0) \in Aut_{\mathbf{C}}(Hom_{G(F/E)}(\chi^{-1}, 0)).$$

Next we form

$$\mathcal{R}_{f_r,F}(\chi) = \frac{R(\chi, f_{r,F})}{L_{E,S}^*(r, \chi)} = L_{E,S}(r, \chi)^{-1},$$

since $L_{E,S}(r, \chi)$ is non-zero.

Therefore $\mathcal{J}_F^r$ is the $\mathbf{Z}[1/2][G(F/E)]$-submodule of $\mathbf{Q}[G(F/E)]$ given by

$$\mathcal{J}_F^r = \mathbf{Z}[1/2][G(F/E)] \cdot \tau(\mathcal{R}_{f_r,F}^{-1}) = \mathbf{Z}[1/2][G(F/E)] \cdot (\chi \mapsto L_{E,S}(r, \chi^{-1})).$$

Hence $\mathcal{J}_F^r$ is the principal $\mathbf{Z}[1/2][G(F/E)]$-submodule generated by the Stickelberger element $\Theta_{F/E,S}(1 - r)$ of §2.3

$$\mathcal{J}_F^r = \mathbf{Z}[1/2][G(F/E)]\langle\Theta_{F/E,S}(1 - r)\rangle \in \mathbf{Q}[G(F/E)].$$

This fact can also be deduced directly from Theorem 4.4 (iii) and (iv).

(ii) In the situation of §4.1 and Definition 4.3 suppose that F is a CM field and $c_0 \in G(F/E)$ denotes complex conjugation. Then c_0 acts on $Y_r(F)^+$ like multiplication by $(-1)^r$. Hence $((1 - (-1)^r c_0)/2)Y_r(F)^+ = 0$. Remembering once again that the determinant of the zero automorphism of the zero module is 1 or by means of Theorem 4.4 (iii) and (iv) we find that

$$\mathcal{I}_{f_r,F}((1 - (-1)^r c_0)/2) = \mathbf{Z}[1/2][G(F/E)]((1 - (-1)^r c_0)/2).$$

Therefore, since $1 = R(\chi, f_{r,F})$ for all characters χ such that $\chi(c_0) = (-1)^{r+1}$, we find that

$$\begin{aligned}
\mathcal{J}_F^\tau((1 - (-1)^r c_0)/2) &= \mathcal{I}_{f_r,F}((1 - (-1)^r c_0)/2) \cdot \tau(\mathcal{R}_{f_r,F}^{-1}) \\
&= \mathbf{Z}[1/2][G(F/E)]((1 - (-1)^r c_0)/2) \cdot \tau(\mathcal{R}_{f_r,F}^{-1}) \\
&= \mathbf{Z}[1/2][G(F/E)]((1 - (-1)^r c_0)/2) \cdot \Theta_{F/E,S}(1 - r)).
\end{aligned}$$

Notice that $\Theta_{F/E,S}(1 - r))((1 - (-1)^r c_0)/2) \in \mathbf{Q}[G(F/E)]$ is characterised by the relation that $\chi(\Theta_{F/E,S}(1 - r))((1 - (-1)^r c_0)/2) = L_{E,S}(r, \chi^{-1})$ for all characters of $G(F/E)$ satisfying $\chi(c_0) = (-1)^{r+1}$ and is zero otherwise.

4.7 Suppose that F is totally real, that $r < 0$ is even and that $Y_r(F)^+$ has a $\mathbf{Z}[G(F/E)]$-basis $v_1, v_2, \ldots, v_t$.

Now suppose that we change $f_{r,F}$ by composition with

$$U \in Aut_{\mathbf{Q}[G(F/E)]}(Y_r(F)^+ \otimes \mathbf{Q})$$

so that there is an invertible matrix $U = (U_{u,v}) \in GL_t\mathbf{Q}[G(F/E)]$ given by

$$U(v_i) = \sum_{s=1}^{t} U_{s,i} v_s.$$

Now let $\chi : G(F/E) \longrightarrow \mathbf{C}^*$ be a one-dimensional representation. The subspace of $Y_r(F)^+ \otimes \mathbf{C}$ on which $G(F/E)$ acts via χ^{-1} is

$$e_{\chi^{-1}} Y_r(F)^+ \otimes \mathbf{C} = \oplus_{i=1}^{t} \mathbf{C}[G(F/E)]e_{\chi^{-1}} v_i.$$

and

$$U(e_{\chi^{-1}} v_i) = \sum_{s=1}^{t} \chi^{-1}(U_{s,i}) e_{\chi^{-1}} v_s$$

so that

$$\mathcal{R}_{f_r,F \cdot U}(\chi) = \mathcal{R}_{f_r,F}(\chi) \cdot det(\chi^{-1}(U))^{-1}.$$

Observe that $\chi^{-1}(U) = \chi(\tau(U))$ where τ is the involution of $\mathbf{Q}[G(F/E)]$ sending each group element to its inverse.

Suppose now that F is a CM field and that $w_1, \ldots, w_u$ is a $\mathbf{Z}[1/2][G(F/E)]/(c_0 + (-1)^{r+1})$-basis for $Y_r(F)^+ \otimes \mathbf{Z}[1/2]$.

Since c_0 acts like $(-1)^r$, if χ is a one-dimensional representation such that $\chi(c_0) = (-1)^{r-1}$ then $Hom_{G(F/E)}(\chi^{-1}, Y_r(F)^+ \otimes \mathbf{C}) = 0$ and $R(\chi, f_{r,F}) = 1$. Since $L_{E,S}(r, \chi)$ is zero if and only if $\chi(c_0) = (-1)^r$ we find that

$$\mathcal{R}_{f_{r,F}}(\chi) = \frac{1}{L_{E,S}(r, \chi)},$$

which is independent of $f_{r,F}$.

On the other hand, suppose that we change $f_{r,F}$ by composition with

$$V \in Aut_{\mathbf{Q}[G(F/E)]}(Y_r(F)^+ \otimes \mathbf{Q}) = Aut_{\mathbf{Q}[G(F/E)]/(c_0 + (-1)^{r+1})}(Y_r(F)^+ \otimes \mathbf{Q}).$$

There is a matrix $V = (V_{a,b}) \in GL_u\mathbf{Q}[G(F/E)/(c_0 + (-1)^{r+1})]$ given by $V(w_i) = \sum_{a=1}^u V_{a,i} w_a$. If $\chi(c_0) = (-1)^r$ then the subspace of $Y_r(F)^+ \otimes \mathbf{C}$ on which $G(F/E)$ acts via χ^{-1} is

$$e_{\chi^{-1}} Y_r(F)^+ \otimes \mathbf{C} = \oplus_{i=1}^u \mathbf{C}[G(F/E)/(c_0 + (-1)^{r+1})]e_{\chi^{-1}} w_i.$$

and

$$V(e_{\chi^{-1}} v_i) = \sum_{a=1}^u \chi^{-1}(V_{a,i}) e_{\chi^{-1}} w_a$$

so that

$$\mathcal{R}_{f_{r,F} \cdot V}(\chi) = \mathcal{R}_{f_{r,F}}(\chi) \cdot det(\chi^{-1}(V))^{-1}.$$

As in the totally real case, we have $\chi^{-1}(V) = \chi(\tau(V))$ where τ is the involution of $\mathbf{Q}[G(F/E)]$ sending each group elements to its inverse.

Via the homomorphism $\lambda_{G(F/E)}$ of Proposition 2.2 we may consider $\mathcal{R}_{f_{r,F} \cdot V}$ and $\mathcal{R}_{f_{r,F}}$ as elements of $\mathbf{Q}[G(F/E)]$ given by

$$\mathcal{R}_{f_{r,F} \cdot V} = \Sigma_{\chi \in \hat{G}(F/E), \, \chi(c_0)=(-1)^{r-1}} \, \mathcal{R}_{f_{r,F}}(\chi) e_\chi$$

$$+ \Sigma_{\chi \in \hat{G}(F/E), \, \chi(c_0)=(-1)^r} \, \mathcal{R}_{f_{r,F}}(\chi) \chi(\tau(det(V)))^{-1} e_\chi.$$

Next we observe that

$$(1 + (-1)^{r-1} c_0)/2 + ((1 + (-1)^r c_0)/2)\tau(det(V))^{-1}$$

is a well-defined element of $\mathbf{Q}[G(F/E)]$ since the difference between any two liftings of $\tau(det(V))^{-1} \in \mathbf{Q}[G(F/E)]/(c_0 + (-1)^{r+1})$ lies in $(c_0 + (-1)^{r+1})\mathbf{Q}[G(F/E)]$ and $((1 + (-1)^r c_0)/2)(c_0 + (-1)^{r+1}) = 0$.

Furthermore, calculating in $\mathbf{C}[G(F/E)]$ using the relation $ze_\chi = \chi(z)e_\chi$ we find that

$$\mathcal{R}_{f_{r,F}} \cdot ((1 + (-1)^{r-1}c_0)/2 + ((1 + (-1)^r c_0)/2)\tau(det(V))^{-1})$$

$$= (\textstyle\sum_{\chi \in \hat{G}(F/E)} \mathcal{R}_{f_{r,F}}(\chi)e_\chi) \cdot (1 + (-1)^{r-1}c_0)/2$$

$$+ (\textstyle\sum_{\chi \in \hat{G}(F/E)} \mathcal{R}_{f_{r,F}}(\chi)e_\chi)((1 + (-1)^r c_0)/2)\tau(det(V))^{-1})$$

$$= \textstyle\sum_{\chi \in \hat{G}(F/E),\ \chi(c_0)=(-1)^{r-1}} \mathcal{R}_{f_{r,F}}(\chi)e_\chi$$

$$+ \textstyle\sum_{\chi \in \hat{G}(F/E),\ \chi(c_0)=(-1)^r} \mathcal{R}_{f_{r,F}}(\chi)\chi(\tau(det(V)))^{-1}e_\chi$$

$$= \mathcal{R}_{f_{r,F} \cdot V}.$$

Therefore

$$\mathcal{R}_{f_{r,F} \cdot V} = \mathcal{R}_{f_{r,F}} \cdot ((1 + (-1)^{r-1}c_0)/2 + ((1 + (-1)^r c_0)/2)\tau(det(V))^{-1}).$$

We have proved the following result.

Lemma 4.8

Let F/E be a Galois extension of number fields with abelian Galois group $G(F/E)$. Assume that the higher-dimensional Stark conjecture §3.1 holds for F/E.

(i) If F is totally real and $r < 0$ is even then

$$\mathcal{R}_{f_{r,F} \cdot U} = \mathcal{R}_{f_{r,F}} \cdot \tau(det(U))^{-1} \in \mathbf{Q}[G(F/E)]^*$$

in §4.5.

(ii) If E is totally real and F is a CM field then, in §4.5,

$$\mathcal{R}_{f_{r,F} \cdot V} = \mathcal{R}_{f_{r,F}} \cdot ((1 + (-1)^{r-1}c_0)/2 + ((1 + (-1)^r c_0)/2)\tau(det(V))^{-1}).$$

Proposition 4.9

Let F/E be a Galois extension of number fields with abelian Galois group $G(F/E)$. Suppose also that E is totally real and that F is totally real or is a CM field. Then, assuming that the higher-dimensional Stark conjecture of §3.1 holds for F/E, the finitely generated $\mathbf{Z}[1/2][G(F/E)]$-submodule of $\mathbf{Q}[G(F/E)]$, $\mathcal{J}_F^r$ defined in §4.3, is independent of the choice of $f_{r,F}$.

Proof

Let $r < 0$ be even and assume that F is totally real. If we change $f_{r,F}$ to $f_{r,F} \cdot U$ as in Lemma 4.8 then $\mathcal{I}_{f_{r,F}}$ changes to $\mathcal{I}_{f_{r,F}}det(U)^{-1}$ and $\tau\mathcal{R}_{f_{r,F}}$ changes to

$$\tau(\mathcal{R}_{f_{r,F}} \cdot \tau(det(U))^{-1}) = \tau(\mathcal{R}_{f_{r,F}})det(U)^{-1}$$

16

and $\mathcal{J}_F^r$ is remains unchanged, as required.

The case when F is totally real and $r < 0$ is odd is trivial by Example 4.6(i).

Now consider the case when F is a CM field. In the situation and notation of Definition 4.5, and Example 4.6(ii) we have

$$
\begin{aligned}
\mathcal{I}_{f_r,F} \cdot v &= \mathcal{I}_{f_r,F} \cdot v((1-(-1)^r c_0)/2) + \mathcal{I}_{f_r,F} \cdot v((1+(-1)^r c_0)/2) \\
&= \mathcal{I}_{f_r,F}((1-(-1)^r c_0)/2) + \mathcal{I}_{f_r,F} \cdot v((1+(-1)^r c_0)/2) \\
&= \mathcal{I}_{f_r,F}((1-(-1)^r c_0)/2) + \mathcal{I}_{f_r,F}((1+(-1)^r c_0)/2) det(V)^{-1} \\
&= \mathcal{I}_{f_r,F}((1+(-1)^{r-1} c_0)/2 + ((1+(-1)^r c_0)/2) det(V)^{-1}).
\end{aligned}
$$

while, by Lemma 4.8,

$$
\tau(\mathcal{R}_{f_r,F} \cdot v) = \tau(\mathcal{R}_{f_r,F} \cdot v)((1+(-1)^{r-1} c_0)/2 + ((1+(-1)^r c_0)/2) det(V)^{-1}),
$$

as required. $\square$

5 Annihilators, a new conjecture

5.1 As in §2.3, let F/E be a Galois extension of number fields with abelian Galois group $G(F/E)$. Suppose also that E is totally real and that F is totally real or is a CM field. Let S be a finite set of primes of $\mathcal{O}_E$ including those which ramify in F/E and let S' denote the primes of F above those of S. Throughout this section we shall assume that the higher-dimensional Stark conjecture of §3.1 is true. Therefore from §4.3 we have a "fractional ideal" (that is, a well-defined, finitely generated $\mathbf{Z}[1/2][G(F/E)]$-module)

$$
\mathcal{J}_F^r \subseteq \mathbf{Q}[G(F/E)]
$$

for each negative integer $r = -1, -2, -3, \ldots$.

Now we come to the most important part of the paper, a conjecture for which some supporting evidence will be presented in §6.1 and Theorem 6.2.

Conjecture 5.2 Let l be an odd prime. Then, in the situation and notation of §5.1,

$$
(\mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l) \cdot \mathcal{J}_F^r) \bigcap \mathbf{Z}_l[G(F/E)]
$$

$$
\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(K_{-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l).
$$

$$
\vdots
$$

Remark 5.3 In Conjecture 5.2 when F is totally real and $r = -1, -3, -5, \ldots$ then, by Example 4.6(i),

$$\mathcal{J}_F^r = \mathbf{Z}[G(F/E)]\langle \Theta_{F/E,S}(1-r) \rangle.$$

In addition, the Quillen-Lichtenbaum conjecture (see §6.1; known to be true when $l = 3$ for example[1]) predicts that

$$\mu_{1-r}(F) \otimes \mathbf{Z}_l \cong Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l$$

so that

$$\mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l) \cdot \mathcal{J}_F^r$$

is expected to contain

$$\Theta_{F/\mathbf{Q}}^{Dir}(1-r) \cdot \mathrm{ann}_{\mathbf{Z}_l[G(F/\mathbf{Q})]}(\mu_{1-r}(F) \otimes \mathbf{Z}_l)$$

of §3.2. By ([13]; see also [7]) this finitely generated $\mathbf{Z}_l[G(F/E)]$-submodule of $\mathbf{Q}_l[G(F/E)]$ actually lies in $\mathbf{Z}_l[G(F/E)]$.

In any case, this discussion shows that Conjecture 5.2 coincides with one of the well-known conjecture of §2.2 when F is totally real and $r = -1, -3, -5, \ldots$.

On the other hand, if F is totally real in §5.1 and $r = -2, -4, -6, \ldots$ then $\mu_{1-r}(F)$ is trivial because the action of complex conjecture on the $\mathbf{Z}_l$-module $\mu(M)^{\otimes 1-r} \otimes \mathbf{Z}_l$ of §3.2 is multiplication by $(-1)^{1-r}$. In this case the Quillen-Lichtenbaum conjecture predicts that $Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l$ is trivial and Conjecture 5.2 reduces to

$$\mathcal{J}_F^r \bigcap \mathbf{Z}_l[G(F/E)] \subseteq \mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(K_{-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l).$$

Question 5.4 *Integrality*

Perhaps, by analogy with the totally real case when r is odd,

$$\mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(Tors K_{1-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l) \cdot \mathcal{J}_F^r \lhd \mathbf{Z}_l[G(F/E)]$$

in general?

Perhaps

$$\mathcal{J}_F^r \subseteq \mathrm{ann}_{\mathbf{Z}_l[G(F/E)]}(K_{-2r}(\mathcal{O}_{F,S'}) \otimes \mathbf{Z}_l)$$

when F is totally real in §5.1 and $r = -2, -4, -6, \ldots$?

---

[1] I understand that the Quillen-Lichtenbaum conjecture follows for all odd primes from the results of Voevodsky, Suslin and Rost.

6 Supporting evidence

6.1 Throughout this section let l be an odd prime, m a positive integer prime to l and $r = -1, -2, -3, -4, -5, \ldots$. We are going to study Conjecture 5.2 for the cyclotomic extension $\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q}$ where $s \geq 0$ and $\xi_t = e^{2\pi\sqrt{-1}/t}$. We shall study the case where $\mathcal{O}_{F,S}$ is equal to $\mathbf{Z}(\xi_{ml^{s+1}})[1/ml]$, to get from this case to a larger set S is straightforward using the localisation exact sequence.

For $\epsilon = 1, 2$ there are étale cohomology Chern classes ([11], [12]) of the form

$$K_{2-2r+\epsilon}(\mathbf{Z}(\xi_{ml^{s+1}})[1/ml]) \otimes \mathbf{Z}_l \xrightarrow{c_{1-r,\epsilon}} H^{\epsilon}_{\text{ét}}(\text{Spec}(\mathbf{Z}(\xi_{ml^{s+1}})[1/ml]); \mathbf{Z}_l(1-r))$$

which the Quillen-Lichtenbaum conjecture predicts to be isomorphisms. This was proved for K_2 in [44] and for K_3 in ([25], [29]). As a corollary of the fundamental results of Voevodsky [46] [47], the corresponding Chern classes when $l = 2$ are nearly isomorphisms in all dimensions [34]. Voevodsky's method requires the existence of suitable "norm varieties" which is not yet established for all odd primes. However many partial results are known at odd primes (see footnote in Remark 5.3). For example, recent work by Rost combined with that of Suslin-Voevodsky shows that $c_{1-r,\epsilon}$ is always an isomorphism for $l = 3$.

Observe that, if $c_{1-r,1}$ is an isomorphism, we have

$$Tors(K_{1-2r}(\mathbf{Z}[\xi_{ml^{s+1}}]) \otimes \mathbf{Z}_l)$$

$$\cong Tors H^1_{\text{ét}}(\text{Spec}(\mathbf{Z}(\xi_{ml^{s+1}})[1/ml]); \mathbf{Z}_l(1-r))$$

$$\cong H^0_{\text{ét}}(\text{Spec}(\mathbf{Z}(\xi_{ml^{s+1}})[1/ml]); (\mathbf{Q}_l/\mathbf{Z}_l)(1-r))$$

$$\cong \mu_{1-r}(\mathbf{Q}(\xi_{ml^{s+1}}))$$

where $\mu_{1-r}(F)$ is as in §3.2 and Question 5.3.

We are now almost ready to state our main result. Since we are studying cyclotomic fields I shall follow the example of [8] and state the result in terms of Dirichlet L-functions. We shall need to recall the corresponding higher Stickelberger elements and leading terms.

Suppose that $L/\mathbf{Q}$ is a finite Galois extension of number fields with abelian Galois group, $G(L/\mathbf{Q})$. Then, for each negative integer $r = -1, -2, -3, \ldots$, there is a unique element of the rational group-ring (cf. §2.3)

$$\Theta^{Dir}_{L/\mathbf{Q}}(1-r) \in \mathbf{Q}[G(L/\mathbf{Q})]$$

such that

$$\chi(\Theta^{Dir}_{L/\mathbf{Q}}(1-r)) = L_{\mathbf{Q}}(r, \chi^{-1})$$

for each one-dimensional complex representation χ where $L_{\mathbf{Q}}(s, \chi^{-1})$ denotes the Dirichlet L-function of the character χ^{-1} ([48] Ch.4). The rationality of $\Theta_{L/\mathbf{Q}}(n)$ is seen by writing the L-function in terms of partial zeta functions

$$L_{\mathbf{Q}}(r, \chi^{-1}) = \sum_{g \in G(L^{Ker(\chi)}/\mathbf{Q})} \chi(g)^{-1} \zeta_{\mathbf{Q}}(g, r)$$

and recalling that $\zeta_{\mathbf{Q}}(g, r)$ is a rational number, by a result of Klingen and Siegel (cf. [38]).

We have $0 = L_{\mathbf{Q}}(r, \chi^{-1})$ precisely when $\chi(c_0) = (-1)^r$, c_0 being complex conjugation as in Question 5.3. In this case there is a zero of order one and the leading term is defined by the formula

$$L_{\mathbf{Q}}^*(r, \chi) = \frac{d}{dz} L_{\mathbf{Q}}(z, \chi)|_{z=r} \cdot \prod_{\substack{p \text{ prime} \\ p | ml, \ (p, f(\chi))=1}} (1 - \chi(\sigma_p)^{-1} p^{-r}).$$

Now let $f_{r, \mathbf{Q}(\xi_{ml^s+1})}$, $\mathcal{I}_{f_r, \mathbf{Q}(\xi_{ml^s+1})}$ and $R(\chi, f_{r, \mathbf{Q}(\xi_{ml^s+1})})$ be as in §2, §4.1 and Definition 4.5. Imitating Definition 4.5, set

$$\mathcal{R}_{f_r, \mathbf{Q}(\xi_{ml^s+1})}^{Dir}(\chi) = \frac{R(\chi, f_{r, \mathbf{Q}(\xi_{ml^s+1})})}{L_{\mathbf{Q}}^*(r, \chi)}$$

and

$$\mathcal{J}_{Dir, \mathbf{Q}(\xi_{ml^s+1})}^r = \mathcal{I}_{f_r, \mathbf{Q}(\xi_{ml^s+1})}^{Dir} \cdot \tau((\mathcal{R}_{f_r, \mathbf{Q}(\xi_{ml^s+1})})^{-1}).$$

It is important to realise that, although in general the fractional ideal of Definition 4.5 was constructed under the assumption of the higher-dimensional Stark conjecture of §3.1, in the case of abelian extensions of the rationals the calculations of [2] (see also [5] and [19]) show 985 that the higher-dimensional Stark conjecture is true (see the proofs of Theorem 7.7 and [41] Theorem 4.9).

Theorem 6.2

Let l be an odd prime. Then, in the situation and notation of §6.1,

$$(\text{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r}) \cdot \mathcal{J}_{Dir, \mathbf{Q}(\xi_{ml^s+1})}^r) \bigcap \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]$$

$$\subseteq \text{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H_{\text{ét}}^2(\text{Spec}(\mathbf{Z}[\xi_{ml^s+1}][1/ml]); \mathbf{Z}_l(1-r)))$$

where $\mu_{1-r} = \mu_{1-r}(\mathbf{Q}(\xi_{ml^s+1})) \otimes \mathbf{Z}_l$ as in §6.1.

Remark 6.3 It is possible to deduce from Theorem 6.2 the corresponding result in which $\mathcal{J}_{Dir, \mathbf{Q}(\xi_{ml^s+1})}^r$ is replaced by $\mathcal{J}_{\mathbf{Q}(\xi_{ml^s+1})}^r$, in the spirit of Conjecture 5.2. The passage from Dirichlet L-functions to Artin L-functions seems fairly straightforward [22], if a little delicate.

6.4 *Proof of Theorem 6.2*

Arguing as in Example 4.6(ii) we have

$$\mathcal{J}^r_{Dir,\mathbf{Q}(\xi_{ml^s+1})} = \mathbf{Z}[1/2][G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]((1-(-1)^r c_0)/2)\Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1-r)$$

$$+\mathcal{I}_{f_r,\mathbf{Q}(\xi_{ml^s+1})}((1+(-1)^r c_0)/2) \cdot \tau(\mathcal{R}^{Dir}_{f_r,\mathbf{Q}(\xi_{ml^s+1})})^{-1}.$$

Furthermore it is shown in Theorem 7.7(proof) that $f_{r,\mathbf{Q}(\xi_{ml^s+1})}$ may be chosen so that

$$\mathcal{R}^{Dir}_{f_r,\mathbf{Q}(\xi_{ml^s+1})}(1+(-1)^r c_0)/2 = (1+(-1)^r c_0)/2.$$

Making this choice ensures that

$$\mathcal{J}^r_{Dir,\mathbf{Q}(\xi_{ml^s+1})} = \mathbf{Z}[1/2][G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]((1-(-1)^r c_0)/2)\Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1-r)$$

$$+\mathcal{I}_{f_r,\mathbf{Q}(\xi_{ml^s+1})}((1+(-1)^r c_0)/2).$$

Since c_0 acts on μ_{1-r} like multiplication by $(-1)^{r+1}$ we have

$$\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r})$$

$$= \mathbf{Z}_l[G(F/E)]((1+(-1)^r c_0)/2)$$

$$+ \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r})((1-(-1)^r c_0)/2)$$

and therefore

$$(\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r}) \cdot \mathcal{J}^r_{Dir,\mathbf{Q}(\xi_{ml^s+1})}) \bigcap \mathbf{Z}_l[G(F/E)]$$

is generated by

$$(((1-(-1)^r c_0)/2)\Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1-r)\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r})) \bigcap \mathbf{Z}_l[G(F/E)]$$

and

$$(\mathcal{I}_{f_r,\mathbf{Q}(\xi_{ml^s+1})}((1+(-1)^r c_0)/2)) \bigcap \mathbf{Z}_l[G(F/E)]$$

which both lie in $\mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^2_{\text{ét}}(\mathrm{Spec}(\mathbf{Z}[\xi_{ml^s+1}][1/ml]); \mathbf{Z}_l(1-r)))$, by §7.6 and Theorem 7.7. $\square$

7 Annihilators and $K_0(\mathbf{Z}[G], \mathbf{Q}_l)$

The results of this section extend the results for the totally real subfield of a cyclotomic field, proved in [41], to the full cyclotomic field in order to establish the results which were required in the proof of Theorem 6.2.

7.1 Let l be a prime, G a finite group and let $f : \mathbf{Z}_l[G] \longrightarrow \mathbf{Q}_l[G]$ denote the homomorphism of group-rings induced by the inclusion of the l-adic integers into the fraction field, the l-adic rationals. Write $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ for the relative K-group of f, denoted by $K_0(\mathbf{Z}_l[G], f)$ in ([43] p.214; see also [40] Definition 2.1.5). By ([43] Lemma 15.6) elements of $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ are represented by triples $[A, g, B]$ where A, B are finitely generated, projective $\mathbf{Z}_l[G]$-modules and g is a $\mathbf{Q}_l[G]$-module isomorphism of the form $g : A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \stackrel{\cong}{\to} B \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$. Defining an exact sequence of triples in the obvious manner, the relations between these elements are generated by the following two types:

(i) $[A, g, B] = [A', g', B'] + [A'', g'', B'']$ if there exists an exact sequence

$$0 \longrightarrow (A', g', B') \longrightarrow (A, g, B) \longrightarrow (A'', g'', B'') \longrightarrow 0$$

and

(ii) $[A, gh, C] = [A, h, B] + [B, g, C]$.

This group fits into a localisation sequence of the form ([33] §5 Theorem 5; see also [15] p.233)

$$K_1(\mathbf{Z}_l[G]) \xrightarrow{f_*} K_1(\mathbf{Q}_l[G]) \xrightarrow{\partial} K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \xrightarrow{\pi} K_0(\mathbf{Z}_l[G]) \xrightarrow{f_*} K_0(\mathbf{Q}_l[G]).$$

Assume now that G is abelian. In this case $K_1(\mathbf{Q}_l[G]) \cong \mathbf{Q}_l[G]^*$ because $\mathbf{Q}_l[G]$ is a product of fields and $K_1(\mathbf{Z}_l[G]) \cong \mathbf{Z}_l[G]^*$ ([10]I p.179 Theorem (46.24)). Under these isomorphisms f_* is identified with the canonical inclusion.

The homomorphism, $K_0(\mathbf{Z}_l[G]) \xrightarrow{f_*} K_0(\mathbf{Q}_l[G])$, is injective for all finite groups G ([37] Theorem 34 p.131; [10]II p.47 Theorem 39.10). Thus the localisation sequence yields an isomorphism of the form

$$K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \cong \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

when G is abelian. From the explicit description of ∂ ([43] p.216) this isomorphism sends the coset of $\alpha \in \mathbf{Q}_l[G]^*$ to $[\mathbf{Z}_l[G], (\alpha \cdot -), \mathbf{Z}_l[G]]$. The inverse isomorphism sends $[A, g, B]$, where A and B may be assumed to be **free** $\mathbf{Z}_l[G]$-modules, to the coset of $det(g) \in \mathbf{Q}_l[G]^*$ with respect to any choice of $\mathbf{Z}_l[G]$-bases for A and B.

We shall be particularly interested in the following source of elements of $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$. Let l be a prime and let G be a finite abelian group. Suppose that

$$0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a bounded complex of finitely generated, projective $\mathbf{Z}_l[G]$-modules (i.e. a *perfect* complex of $\mathbf{Z}_l[G]$-modules), having all its homology groups finite. As usual, let $Z_t = \text{Ker}(d_t : F_t \longrightarrow F_{t-1})$ and $B_t = d_{t+1}(F_{t+1}) \subseteq F_t$ denote the

22

$\mathbf{Z}_l[G]$-modules of t-dimensional cycles and boundaries, respectively. We have short exact sequences of the form

$$0 \longrightarrow B_i \xrightarrow{\phi_i} Z_i \longrightarrow H_i(F_*) \longrightarrow 0$$

and

$$0 \longrightarrow Z_{i+1} \xrightarrow{\psi_{i+1}} F_{i+1} \xrightarrow{d_{i+1}} B_i \longrightarrow 0.$$

Applying $(- \otimes \mathbf{Q}_l)$ we obtain isomorphisms

$$\phi_i : B_i \otimes \mathbf{Q}_l \xrightarrow{\cong} Z_i \otimes \mathbf{Q}_l$$

and we may choose $\mathbf{Q}_l[G]$-module splittings of the form

$$\eta_i : B_i \otimes \mathbf{Q}_l \longrightarrow F_{i+1} \otimes \mathbf{Q}_l$$

such that $(d_{i+1} \otimes 1)\eta_i = 1 : B_i \otimes \mathbf{Q}_l \longrightarrow B_i \otimes \mathbf{Q}_l$. Then, using these splittings, we form a $\mathbf{Q}_l[G]$-module isomorphism of the form

$$X : \oplus_j F_{2j} \otimes \mathbf{Q}_l \xrightarrow{\cong} \oplus_j F_{2j+1} \otimes \mathbf{Q}_l.$$

This construction defines a class, $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}]$, in $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ which is well-known to be independent of the choices of the splittings used to define X ([43] Ch. 15; see also [40] Propositions 2.5.35 and 7.1.8).

We shall denote by

$$det(X) \in \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

the element which corresponds to $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ under the isomorphism mentioned above. We may modify the F_i to be free finitely generated $\mathbf{Z}_l[G]$-modules without changing the homology modules or the class in $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$. Then $det(X)$ is explicitly represented by the determinant of the $\mathbf{Q}_l[G]$-isomorphism X with respect any $\mathbf{Z}_l[G]$-basis for F_*.

Let us recall from ([28] Appendix; see also [49]) the properties of the Fitting ideal (referred to as the initial Fitting invariant in [31]). Let R be a commutative ring with identity and let M be a finitely presented R-module, in our applications M will actually be finite. Suppose that M has a presentation of the form

$$R^a \xrightarrow{f} R^b \longrightarrow M \longrightarrow 0$$

with $a \geq b$ then the Fitting ideal of the R-module M, denoted by $F_R(M)$, is the ideal of R generated by all $b \times b$ minors of any matrix representing f.

The Fitting ideal $F_R(M)$ is independent of the presentation chosen for M and is contained in the annihilator ideal of M, $F_R(M) \subseteq \operatorname{ann}_R(M)$. If M is generated by n elements then $\operatorname{ann}_R(M)^n \subseteq F_R(M)$ and if $\pi : M \to M'$ is a surjection of finitely presented R-modules then $F_R(M) \subseteq F_R(M')$.

The following result yields relations between the annihilator ideals and Fitting ideals of the homology modules in previous example in the special case when each $H_i(F_*)$ is finite and zero except for $i = 0, 1$.

Theorem 7.2 *([41] Theorem 2.4)*

Let G be a finite abelian group and l a prime. Suppose that

$$0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a bounded, perfect complex of $\mathbf{Z}_l[G]$-modules, as in Example 2.2, having $H_i(F_*)$ finite for $i = 0, 1$ and zero otherwise. Let

$$[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \cong \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

be as in §7.1. Then:

(i) if $t_i \in \text{ann}_{\mathbf{Z}_l[G]}(H_i(F_*))$,

$$det(X)^{(-1)^i} t_i^{m_i} \in \text{ann}_{\mathbf{Z}_l[G]}(H_{1-i}(F_*)) \lhd \mathbf{Z}_l[G]$$

for $i = 0, 1$. Here m_0, m_1 is the minimal number of generators required for the $\mathbf{Z}_l[G]$-module $H_0(F_*)$, $Hom(H_1(F_*), \mathbf{Q}_l/\mathbf{Z}_l)$, respectively,

(ii) if the Sylow l-subgroup of G is cyclic then in (i) $\text{ann}_{\mathbf{Z}_l[G]}(H_{1-i}(F_*))$ may be replaced by $F_{\mathbf{Z}_l[G]}(H_{1-i}(F_*))$.

7.3 Now we shall apply Theorem 7.2 to a very important perfect complex constructed in [5].

If F_* is a bounded, perfect complex of R-modules, with R commutative and reduced, let $D_R(F_*)$ denote the determinant of F_*, as developed in [21]. Hence $D_R(F_*)$ is a graded invertible R-module (i.e. projective of rank one). For every distinguished triangle of perfect complexes $C_1 \to C_2 \to C_3 \to C_1[1]$ there is a canonical isomorphism of the form

$$D_R(C_1) \otimes D_R(C_3) \xrightarrow{\cong} D_R(C_2)$$

which is functorial in the triangle ([21] Proposition 7). Also, if C is an acyclic perfect complex, there exists a canonical isomorphism of the form

$$D_R(C) \xrightarrow{\cong} R.$$

If G is a finite abelian group and F_* is a bounded perfect complex of $\mathbf{Z}_l[G]$-modules with finite homology groups, as in §7.1 and Theorem 7.2, we have the relation ([41] §4.1)

$$D_{\mathbf{Z}_l[G]}(F_*) = \mathbf{Z}_l[G] < det(X)^{-1} > \subset \mathbf{Q}_l[G] \cong D_{\mathbf{Q}_l[G]}(F_* \otimes \mathbf{Q}_l).$$

We recall now some crucial results from [5]. Let l be an odd prime, m a positive integer not divisible by l and r a strictly negative integer. If $\xi_t = e^{2\pi i/t}$ we have a canonical projection of the form $G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q}) \longrightarrow$

24

$G(\mathbf{Q}(\xi_{ml^s})/\mathbf{Q})$ and, taking the inverse limit over the induced homomorphisms of l-adic group-rings, we define

$$\Lambda_m^\infty = \varprojlim_n \, \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{n+1}})/\mathbf{Q})].$$

Write $Q(\Lambda_m^\infty)$ for the total quotient ring of Λ_m^∞ ([14] p.60). In ([5] §7 (48)) a bounded perfect complex of Λ_m^∞-modules, denoted by $C_m(r)$, is constructed for which ([5] Lemma 7.2) $C_m(r) \otimes_{\Lambda_m^\infty} Q(\Lambda_m^\infty)$ is acyclic. Therefore we have a canonical isomorphism of the form

$$D_{\Lambda_m^\infty}(C_m(r)) \otimes_{\Lambda_m^\infty} Q(\Lambda_m^\infty) \xrightarrow{\cong} Q(\Lambda_m^\infty).$$

We shall recall the definition of $C_m(r)$ presently. Furthermore ([5] Theorem 7.1) calculates a Λ_m^∞-basis for the free, rank one Λ_m^∞-module $D_{\Lambda_m^\infty}(C_m(r))$ by showing that the image of $D_{\Lambda_m^\infty}(C_m(r))$ under the canonical homomorphism into $Q(\Lambda_m^\infty)$ is $\Lambda_m\langle(e_+ + e_- \cdot g_m)\rangle$ in the notation of [5]. Fix an integer $s \geq 0$ and a topological generator $\gamma_s \in G(\mathbf{Q}(\xi_{ml^\infty})/\mathbf{Q}(\xi_{ml^{s+1}}))$ and form the bounded perfect complex of $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]$-modules

$$C_{0,m}(r) = C(r) \otimes_{\Lambda_m^\infty}^{\mathbf{L}} \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})].$$

so that

$$C_m(r) \xrightarrow{1-\gamma_s} C_m(r) \longrightarrow C_{0,m}(r) \longrightarrow C_m(r)[1]$$

is a distinguished triangle in the derived category of the homotopy category of Λ_m^∞-modules.

In addition, the cohomology groups of $C_{0,m}(r)$ are finite so that we have a canonical isomorphism

$$D_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]}(C_{0,m}(r)) \otimes_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]} \mathbf{Q}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]$$

$$\xrightarrow{\cong} \mathbf{Q}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})].$$

As in Example 4.6(ii), let c_0 denote complex conjugation and let

$$\chi : G(\mathbf{Q}(\xi_{ml^\infty})/\mathbf{Q}) \longrightarrow \overline{\mathbf{Q}_l^*}$$

denote an l-adic character. Suppose that $z_{m,r}$ is a basis element for the free rank one $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]$-module

$$D_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]}(C_{0,m}(r)) \subset \mathbf{Q}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})].$$

By the descent lemma ([5] Lemma 8.1) with C replaced by $C_m(r)$) $z_{m,r}$ may be chosen to satisfy

$$\chi(z_{m,r}) = \chi((1 + (-1)^r c_0)/2) + \chi((1 + (-1)^{r-1} c_0)/2)\chi(g_m(r))$$

$$= (1 + (-1)^r \chi(c_0))/2 - (1 + (-1)^{r-1}\chi(c_0)/2)L_\mathbf{Q}(r, \tilde\chi)$$

since $-\chi(g_m(r)) = L_{\mathbf{Q}}(r, \tilde{\chi})$, by ([5] Proposition 5.5), the Dirichlet L-function at $s = r$ of the primitive Dirichlet character $\tilde{\chi}$ associated to χ^{-1}.

Let $X_l = \mathrm{Spec}(\mathbf{Z}[\xi_{ml^s+1}][1/ml])$ denote the spectrum of the S_{ml}-integers of $\mathbf{Q}(\xi_{ml^s+1})$, where S_{ml} equals the set of primes dividing ml.

Now we must examine the construction of $C_m(r)$ prior to applying Theorem 7.2. Following [5] let $\Sigma_{L_m^\infty}$ denote the set of embeddings of $\mathbf{Q}(\xi_{ml^\infty})$ into the complex numbers. Then complex conjugation acts diagonally on $\prod_{\Sigma_{L_m^\infty}} \mathbf{Z}_l(-r)$ so that the fixed points $(\prod_{\Sigma_{L_m^\infty}} \mathbf{Z}_l(-r))^+$ naturally form a projective Λ_m^∞-module generated by $e_+(\tilde{\sigma}_m^\infty \nu_r)$ in the notation of ([5] §7). In ([5] §7 (48)) a Λ_m^∞-module homomorphism $c_m(r)$ from $(\prod_{\Sigma_{L_m^\infty}} \mathbf{Z}_l(-r))^+$ to $H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))$ sending $e_+(\tilde{\sigma}_m^\infty \nu_r)$ to a "cyclotomic element" denoted by $\eta_m(r)$. This uniquely determines a homomorphism in the derived category

$$c_m(r) \in Hom_{\mathcal{D}(\Lambda_m^\infty)}((\prod_{\Sigma_{L_m^\infty}} \mathbf{Z}_l(-r))^+[-1], R\Gamma(X_l; \mathbf{Z}_l(1-r)))$$

and $C_m(r)$ is the bounded perfect complex of Λ_m^∞-modules given by the mapping cone of $c_m(r)$. Hence there is a distinguished triangle of perfect complexes in $\mathcal{D}(\Lambda_m^\infty)$ of the form

$$(\prod_{\Sigma_{L_m^\infty}} \mathbf{Z}_l(-r))^+[-1] \xrightarrow{c_m(r)} R\Gamma(X_l; \mathbf{Z}_l(1-r)) \longrightarrow C_m(r).$$

The complex $C_{0,m}(r)$ is obtained by applying $(- \otimes_\Lambda^{\mathbf{L}} \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})])$. Complex conjugation acts on $\mathbf{Z}_l(-r)$ as multiplication by $(-1)^r$ so that $(\prod_{\Sigma_{L_m^\infty}} \mathbf{Z}_l(-r))^+[-1]$ gives rise to a projective module abstractly isomorphic to $Y_r(\mathbf{Q}(\xi_{ml^s+1}))^+ \otimes \mathbf{Z}_l \cong \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]/(1 + (-1)^{r+1}c_0)$ in dimension one. Therefore there is a short exact sequence of the form

$$0 \longrightarrow \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]/(1 + (-1)^{r+1}c_0)\langle e_+(\tilde{\sigma}_m^\infty \nu_r)\rangle \xrightarrow{c_m(r)} H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))$$

$$\longrightarrow H^1(C_{0,m}(r)) \longrightarrow 0$$

and a $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]$-module isomorphism

$$H^2(C_{0,m}(r)) \cong H^2_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))$$

while $H^i(C_{0,m}(r)) = 0$ otherwise.

For $\epsilon = \pm 1$,

$$\Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1-r)((1-\epsilon c_0)/2) \in \mathbf{Q}[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]$$

is the unique element of the $(-\epsilon)$-eigenspace which satisfies

$$\chi(\Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1-r)((1-\epsilon c_0)/2))) = \begin{cases} L_{\mathbf{Q}}(r, \tilde{\chi}) & \text{if } \chi(c_0) = -\epsilon, \\ 0 & \text{if } \chi(c_0) = \epsilon \end{cases}$$

26

for all characters χ, where $L_{\mathbf{Q}}(r, \tilde{\chi})$ is the value of the Dirichlet L-function at $s = r$ of the primitive character $\tilde{\chi}$ associated to χ^{-1}. Therefore the class in

$$K_0(\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})], \mathbf{Q}_l) \cong \mathbf{Q}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]^*/\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]^*$$

is represented by the unit $z_{m,r} \in \mathbf{Q}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]^*$ given by

$$z_{m,r} = (1 + (-1)^r c_0)/2 - \Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1 - r)((1 - (-1)^r c_0)/2).$$

In the notation of Theorem 7.2 $z_{m,r} = det(X)^{-1}$ where $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})], \mathbf{Q}_l)$ is the class associated to the perfect complex $F_* = (C_{0,m}(r))^{2-*}$ by the construction given in §7.1. Hence Theorem 7.2 yields parts (i) and (ii) of the following result.

Theorem 7.4

Let l be an odd prime, m a positive integer prime to l and $r = -1, -2, -3, -4, -5, \ldots$ as in §6.1. Then, in the notation of §7.1 and §7.3:

(i) If $t_i \in \text{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^{2-i}(C_{0,m}(r)))$, the element

$$((1 + (-1)^r c_0)/2 - \Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1 - r)((1 - (-1)^r c_0)/2))^{(-1)^{i+1}} t_i^{m_i}$$

lies in $\text{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^{1+i}(C_{0,m}(r)))$ for $i = 0, 1$. Here $m_1 = \max(m_{1,+}, m_{1,-})$ and $m_0, m_{1,\pm}$ is the minimal number of generators required for the $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]$-module $H^2(C_{0,m}(r))$, $Hom(H^1(C_{0,m}(r))^{\pm}, \mathbf{Q}_l/\mathbf{Z}_l)$, respectively, and $A^{\pm}$ is the ± 1-eigenspace of complex conjugation.

(ii) If l does not divide $m-1$ then in (i) $\text{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^{1+i}(C_{0,m}(r)))$ may be replaced by the Fitting ideal $F_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^{1+i}(C_{0,m}(r)))$.

(iii) Furthermore $m_1 = 1$.

Proof

Parts (i) and (ii) are immediate from Theorem 7.2, applied to each of the ± 1-eigenspaces of complex conjugation, so it remains to prove (iii).

Since l is odd, it suffices to show that each of the eigenspaces of complex conjugation c_0 are generated by one element. From the long exact étale cohomology sequence associated to $\mathbf{Z}_l(1-r) \longrightarrow \mathbf{Q}_l(1-r) \longrightarrow (\mathbf{Q}_l/\mathbf{Z}_l)(1-r)$ we see that

$$Tors H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1 - r)) \cong H^0_{\text{ét}}(X_l; (\mathbf{Q}_l/\mathbf{Z}_l)(1 - r)) \cong \mu_{1-r}(\mathbf{Q}(\xi_{ml^s+1})) \otimes \mathbf{Z}_l$$

on which c_0 acts as multiplication by $(-1)^{1-r}$. Dividing out the short exact sequence of §7.3 by $Tors H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1 - r))$ we obtain a short exact sequence of the form

$$0 \longrightarrow \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]/(1 + (-1)^{r+1} c_0)\langle e_+(\tilde{\sigma}^{\infty}_m \nu_r) \rangle \xrightarrow{c_m(r)} \frac{H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))}{Tors H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))}$$

$$\longrightarrow \frac{H^1(C_{0,m}(r))}{Tors H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))} \longrightarrow 0.$$

Since c_0 acts like multiplication by $(-1)^r$ on the left-hand module it must also act by $(-1)^r$ on the central module, since $c_m(r)$ induces a rational isomorphism. Hence if we set

$$\mathcal{H}^1_{cyclo}(X_l) = \frac{H^1(C_{0,m}(r))}{TorsH^1_{\acute{e}t}(X_l; \mathbf{Z}_l(1-r))}$$

then $\mathcal{H}^1_{cyclo}(X_l)$ is the $(-1)^r$ eigenspace of the finite group $H^1(C_{0,m}(r))$ and the finite cyclic group $\mu_{1-r}(\mathbf{Q}(\xi_{ml^{s+1}})) \otimes \mathbf{Z}_l$ is the $(-1)^{1-r}$-eigenspace. Of course, the Pontrjagin dual of a finite cyclic group is again a cyclic group so it remains to show that

$$Ext^1_{\mathbf{Z}_l}(\mathcal{H}^1_{cyclo}(X_l), \mathbf{Z}_l) \cong Hom(\mathcal{H}^1_{cyclo}(X_l), \mathbf{Q}_l/\mathbf{Z}_l).$$

is generated by one element.

We have a short exact sequence of $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]/(1+(-1)^{r+1}c_0)$-modules

$$0 \longrightarrow Hom_{\mathbf{Z}_l}\left(\frac{H^1_{\acute{e}t}(X_l; \mathbf{Z}_l(1-r))}{TorsH^1_{\acute{e}t}(X_l; \mathbf{Z}_l(1-r))}, \mathbf{Z}_l\right) \longrightarrow$$

$$Hom_{\mathbf{Z}_l}(\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]/(1+(-1)^{r+1}c_0)\langle e_+(\tilde{\sigma}^\infty_m \nu_r)\rangle, \mathbf{Z}_l) \longrightarrow$$

$$Ext^1_{\mathbf{Z}_l}(\mathcal{H}^1_{cyclo}(X_l), \mathbf{Z}_l) \longrightarrow 0$$

in which the central module is cyclic, because $\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]/(1+(-1)^{r+1}c_0)$ is self-dual. Hence the right-hand module is cyclic, as required. $\square$

Corollary 7.5

In the situation and notation of Theorem 7.4

$$\{t_0^{m_0} \mid t_0 \in \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]}(H^2(C_{0,m}(r)))$$

$$\subseteq z_{m,r} \cdot \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]}(H^1(C_{0,m}(r)))$$

$$\subseteq \mathrm{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]}(H^2(C_{0,m}(r)))$$

where, in $\mathbf{Q}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]^*$,

$$z_{m,r} = (1+(-1)^r c_0)/2 - \Theta^{Dir}_{\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q}}(1-r)((1-(-1)^r c_0)/2).$$

7.6 From the proof of Theorem 7.4

$$H^1(C_{0,m}(r)) = \mathcal{H}^1_{cyclo}(X_l) \oplus \mu_{1-r}(\mathbf{Q}(\xi_{ml^{s+1}})) \otimes \mathbf{Z}_l$$

where c_0 acts like multiplication by $(-1)^r$ on the first summand and by $(-1)^{r+1}$ on the second.

Now suppose that $t \in \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r}(\mathbf{Q}(\xi_{ml^s+1})) \otimes \mathbf{Z}_l)$ then

$$t \cdot (1 - (-1)^r c_0) \in \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^1(C_{0,m}(r)))$$

and, by Corollary 7.5,

$$t \cdot ((1 - (-1)^r c_0)/2)((1 + (-1)^r c_0)/2 - \Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1 - r)((1 - (-1)^r c_0)/2))$$

$$= -t \cdot ((1 - (-1)^r c_0)/2)\Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1 - r)$$

lies in $\operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^2(C_{0,m}(r)))$. Therefore

$$\operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mu_{1-r}(\mathbf{Q}(\xi_{ml^s+1})) \otimes \mathbf{Z}_l) \cdot \Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1 - r)((1 - (-1)^r c_0)/2)$$

$$\subseteq \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^2_{\text{ét}}(X_l; \mathbf{Z}_l(1 - r))).$$

Similarly, if $t' \in \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mathcal{H}^1_{cyclo}(X_l))$ then

$$t' \cdot (1 - (-1)^{1-r} c_0) \in \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^1(C_{0,m}(r)))$$

and

$$t' \cdot ((1 - (-1)^{1-r} c_0)/2)((1 + (-1)^r c_0)/2 - \Theta^{Dir}_{\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q}}(1 - r)((1 - (-1)^r c_0)/2))$$

$$= t' \cdot ((1 - (-1)^{1-r} c_0)/2)$$

lies in $\operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^2(C_{0,m}(r)))$. Therefore

$$\operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mathcal{H}^1_{cyclo}(X_l)) \cdot ((1 + (-1)^r c_0)/2)$$

$$\subseteq \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(H^2_{\text{ét}}(X_l; \mathbf{Z}_l(1 - r))).$$

The following result explains the relation between the fractional ideal of Definition 4.5 and the annihilator ideal of $\mathcal{H}^1_{cyclo}(X_l)$.

Theorem 7.7
In the notation of §4.3 and §§7.4-7.6

$$\mathcal{J}^r_{Dir,\mathbf{Q}(\xi_{ml^s+1})} \cap \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})] \cdot (1 + (-1)^r c_0)/2$$

$$\subseteq \operatorname{ann}_{\mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^s+1})/\mathbf{Q})]}(\mathcal{H}^1_{cyclo}(X_l)).$$

Proof
Firstly we show that $f_{r,\mathbf{Q}(\xi_{ml^s+1})}$ may be chosen so that

$$\mathcal{R}^{Dir}_{f_r,\mathbf{Q}(\xi_{ml^s+1})}(1 + (-1)^r c_0)/2 = (1 + (-1)^r c_0)/2$$

in $(\mathbf{Q}[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]/(1+(-1)^r c_0)/2)^*$.

It is known that $K_{1-2r}(X_l) \otimes \mathbf{Q}$ is the free $\mathbf{Q}[G(\mathbf{Q}(\xi_{ml^{s+1}})^+/\mathbf{Q})]/(1-(-1)^r c_0)$-module on the Beilinson element, which is denoted by $(-r)!(ml^{s+1})^{-r} b_r(ml^{s+1})$ in the notation of [5]. Therefore we may choose f_r to satisfy

$$f_r((-r)!(ml^{s+1})^{-r} b_r(ml^{s+1})) = y_r \in Y_r(\mathbf{Q}(\xi_{ml^{s+1}}))^+ \otimes \mathbf{Q}$$

where, in the notation of ([5] §8), $Y_r(\mathbf{Q}(\xi_{ml^{s+1}}))^+ \otimes \mathbf{Z}[1/2]$ is a free $\mathbf{Z}[1/2][G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]/((1-(-1)^r c_0)/2)$-module of rank one with a generator y_r.

However, as recapitulated in ([5] §4.2), Beilinson proved ([30] Part I Theorem 4.3(ii) and Part II Theorem 1.1) that

$$R^r_{\mathbf{Q}(\xi_{ml^{s+1}})^+}(f_r^{-1}(y_r)) = R^r_{\mathbf{Q}(\xi_{ml^{s+1}})^+}((-r)!(ml^{s+1})^{-r} b_r(ml^{s+1}))$$

$$= \sum_{\chi(c_0)=(-1)^r} L^*_{\mathbf{Q}}(r, \chi^{-1}) e_\chi y_r$$

which establishes the claim concerning $\mathcal{R}^{Dir}_{f_r,\mathbf{Q}(\xi_{ml^{s+1}})}(1+(-1)^r c_0)/2$.

For the remainder of the proof, assume that $f_{r,\mathbf{Q}(\xi_{ml^{s+1}})}$ has been chosen in this manner. Hence

$$\mathcal{H}^1_{cyclo}(X_l) = \frac{H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))}{Tors H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r)) + Im(c_m(r))}$$

$$\cong \frac{H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))((1+(-1)^r c_0)/2)}{Im(c_m(r))}$$

and, by Definition 4.5,

$$\mathcal{J}^r_{Dir,\mathbf{Q}(\xi_{ml^{s+1}})}(1+(-1)^r c_0)/2 = \mathcal{I}_{f_r,\mathbf{Q}(\xi_{ml^{s+1}})}(1+(-1)^r c_0)/2$$

is generated as a $\mathbf{Z}[1/2][G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]$-module by the elements $\alpha \in \mathbf{Q}[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})](1+(-1)^r c_0)/2$ such that

$$\alpha \cdot f_{r,\mathbf{Q}(\xi_{ml^{s+1}})}(K_{1-2r}(X_l)) \subseteq Y_r(\mathbf{Q}(\xi_{ml^{s+1}}))^+.$$

Let $x \in H^1_{\text{ét}}(X_l^+; \mathbf{Z}_l(1-r))$ satisfy $c_0(x) = (-1)^r x$, every element in $\mathcal{H}^1_{cyclo}(X_l)$ may be represented by such an x, and let

$$\alpha \in \mathcal{J}^r_{Dir,\mathbf{Q}(\xi_{ml^{s+1}})} \bigcap \mathbf{Z}_l[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})] \cdot (1+(-1)^r c_0)/2.$$

We must show that $\alpha x \in Im(c_m(r)) \subseteq H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))$. The image of the Chern class

$$c_{1-r,1} : K_{1-2r}(X_l) \longrightarrow H^1_{\text{ét}}(X_l; \mathbf{Z}_l(1-r))$$

30

(denoted by $c^r_{\mathbf{Q}(\xi_{ml^{s+1}})}$ in ([5] Lemma 8.16)) is dense. Choose $x_m \in K_{1-2r}(X_l)$ so that $lim_{\overrightarrow{m}}c_{1-r,1}(x_m) = x$. Then $f_{r,\mathbf{Q}(\xi_{ml^{s+1}})}(\alpha x_m)$ lies in

$$Y_r(\mathbf{Q}(\xi_{ml^{s+1}}))^+ = f_{r,\mathbf{Q}(\xi_{ml^{s+1}})}(\mathbf{Z}[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]\langle(-r)!(ml^{s+1})^{-r}b_r(ml^{s+1})\rangle)$$

so that

$$\alpha x_m \in \mathbf{Z}[G(\mathbf{Q}(\xi_{ml^{s+1}})/\mathbf{Q})]\langle(-r)!(ml^{s+1})^{-r}b_r(ml^{s+1})\rangle.$$

However ([5] Lemma 8.16; [19] Theorem 6.4 (proof))) $c_{1-r,1}((-r)!(ml^{s+1})^{-r}b_r(ml^{s+1}))$ is equal to an element $\eta_m(r)$ which generates $Im(c_m(r))$. Hence $lim_{\overrightarrow{m}}\alpha c_{1-r,1}(x_m) = \alpha x \in Im(c_m(r))$, since $c_m(r)$ is continuous in the l-adic topology. $\square$

8 Concluding observations

8.1 *Galois Descent*

Whenever one predicts a new phenomenon concerning Galois actions on some number theoretic Mackey functor such as a cohomology group or an algebraic K-group the question of Galois descent arises. For example, in the case of the Brumer conjecture Hayes, Popescu and Sands ([18], [35], [36]) have shown that the Stickelberger ideal is natural with respect to passing from F/E to a subextension F/L.

As explained in ([40] Chapters VI and VII, particularly p.287) the method of proof of Theorem 6.2, given in §7, would predict nice functorial behaviour for Conjecture 5.2 under all types of passage to Galois subextensions. This is because the perfect complexes from which the annihilator relations were derived arose first in the construction of Chinburg-type invariants (see [40] Chapter III) and are natural with respect to change of fields.

I do not know very much about the functorial behaviour of the fractional ideal $\mathcal{J}_F^r$. Here is a modest example of the difficulties.

Consider the special case when $F = \mathbf{Q}(\xi_{ml^{s+1}})^+$, $E = \mathbf{Q}$ and $E_1 \subset F$ is quadratic over $\mathbf{Q}$. Suppose also that $r = -2, -4, -6, \ldots$ since otherwise the fractional ideal is just one of the higher Stickelberger ideals. In this case we may choose $f_{r,F}$ so that $\mathcal{R}_{f_{r,F}}(\chi) = 1$ for all characters χ of $G(F/\mathbf{Q})$ (see Theorem 7.7(proof)). Hence, using the same $f_{r,F}$ for F/E_1 we see that $\mathcal{R}_{f_{r,F}}$ is trivial for F/E_1 also.

There exists $g \in G(F/\mathbf{Q})$ such that $g^2 \in G(F/E_1)$ and

$$Y_r(F)^+ = \mathbf{Z}[G(F/\mathbf{Q})]\langle\sigma_0\rangle \cong \mathbf{Z}[G(F/E_1)]\langle\sigma_0\rangle \oplus \mathbf{Z}[G(F/E_1)]\langle g\sigma_0\rangle.$$

Now suppose that $a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbf{Q}[G(F/E_1)]$ are such that for all $z \in K_{1-2r}(\mathcal{O}_F)$ with $f_{r,F}(z) = f_1(z)\sigma_0 \oplus f_2(z)g\sigma_0$ and $f_i(z) \in \mathbf{Z}[G(F/E_1)]$ both $y_1(z) = a_{1,1}f_1(z)+a_{1,2}f_2(z)$ and $y_2(z) = a_{2,1}f_1(z)+a_{2,2}f_2(z)$ lie in $\mathbf{Z}[G(F/E_1)]$.

In other words, the $\mathbf{Q}[G(F/E_1)]$-module endomorphism of $Y_r(F)^+ \otimes \mathbf{Q}$ with matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

satisfies the integrality condition of Definition 4.5 for the extension F/E_1. Therefore we have

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_1(z) = a_{2,2}y_1(z) - a_{1,2}y_2(z),$$

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_2(z) = -a_{2,1}y_1(z) + a_{1,1}y_2(z).$$

In this totally real situation Conjecture 5.2 for F/E_1 (see [41] Theorem 4.9) predicts, in the notation of Theorem 6.2, that when $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})$ lies in $\mathbf{Z}_l[G(F/E_1)]$ it annihilates $H^2_{\text{ét}}(\text{Spec}(\mathcal{O}_F[1/ml]); \mathbf{Z}_l(1-r))$. If we could show for all z that the expressions for $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_1(z)$ and $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})f_2(z)$ both lie in $\mathbf{Z}[G(F/E_1)]$ then $(a_{1,1}a_{2,2}-a_{1,2}a_{2,1})f_{r,F}(K_{1-2r}(\mathcal{O}_F)) \in Y_r(F)^+$ and then $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \in \mathbf{Q}[G(F/\mathbf{Q})]$ is one of the generators for $\mathcal{J}_F^r$ for $F/\mathbf{Q}$. Then, by Theorem 6.2, if $(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \in \mathbf{Z}_l[G(F/\mathbf{Q}]$ it would annihilate $H^2_{\text{ét}}(\text{Spec}(\mathcal{O}_F[1/ml]); \mathbf{Z}_l(1-r))$.

However, such integrality does not happen automatically. Here is a purely algebraic example.

Let $G = \mathbf{Z}/2 = \langle g \rangle$ and let $f_r \in Aut_{\mathbf{Q}[G]}(\mathbf{Q}[G])$ be $f_r = ((a+bg)^{-1} \cdot -)$ where $a+bg \in \mathbf{Z}[G] \cap \mathbf{Q}[G]^*$. Hence we clearly have

$$\mathcal{I} = \{\alpha \in \mathbf{Q}[G] \mid \alpha f_r(\langle 2, 1-g \rangle) \subseteq \mathbf{Z}[G]\} = \langle a+bg, (a+bg)(1-g)/2 \rangle.$$

Now let $\beta : \mathbf{Q}[G] \longrightarrow \mathbf{Q}[G]$ be a homomorphism of $\mathbf{Z}$-modules satisfying $\beta f_r(\mathbf{Z}[G]) \subseteq \langle 2, 1-g \rangle$ and suppose that

$$\beta = \begin{pmatrix} u & w \\ v & z \end{pmatrix}$$

meaning $\beta(1) = u + vg$, $\beta(g) = w + zg$. Taking the values $a = 1, b = 2$ with $u = v = 9/2, w = 3/2 = -z$ one finds that β satisfies the integrality condition but $det(\beta) = -27/2$ which does not lie in $\mathcal{I} = \langle 1 + 2g, (1 + 2g)(1 - g)/2 \rangle$ because $\mathbf{Z} \cap \mathcal{I} = 3\mathbf{Z}$.

8.2 The non-CM case

In §5.1 and in the formulation of Conjecture 5.2 we have concentrated on the case when F/E is an Galois extension of number fields with E totally real and F a CM field. This is something of a tradition in the Brumer-Coates-Sinnott conjectures. It has the convenience, away from 2, that one can use

32

the eigenspaces of complex conjugation to study the fractional ideal $\mathcal{J}_F^\tau$. On the other hand the higher dimensional Stark conjecture is not restricted to the CM case nor is the construction of $\mathcal{J}_F^\tau$ and the same should be true about Conjecture 5.2.

Despite the fact that the fractional ideal may not enjoy good descent properties, say, in this totally real cyclotomic example $\mathbf{Q} = E \subset E_1 \subset F = \mathbf{Q}(\xi_{ml^{s+1}})^+$ of §8.1 I believe the statement of Theorem 6.2 is still true in this case, because the proof does "descend".

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