Explicit Brauer Induction and the Glauberman Correspondence

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1 Introduction

Let S and G be finite groups of coprime order such that S acts on G. If S is solvable, Glauberman [6] proves the existence of a bijection between the $S-$fixed irreducible representations of G and the irreducible representations of G^S. A second proof using Brauer's results in block theory was given by Alperin [1]. Boltje [5] assumes Glauberman's theorem and gives an explicit version of the correspondence using Explicit Brauer Induction (see [15]) in the case in which S is a $p-$group.

We shall use Explicit Brauer Induction to give another proof of Glauberman's result and also show how Boltje's result follows immediately from our method.

Section Two gives some preliminary results: the 'canonical' extensions, Explicit Brauer Induction and results in Tate cohomology and non-abelian cohomology. Section Three contains a proof of Glauberman's theorem, and Section Four moves on to demonstrate how this extends to give Boltje's explicit correspondence.

It would be very interesting to explain the Shintani correspondence [9] (a correspondence in which $|S|$ and $|G|$ are not coprime) in terms of Explicit Brauer Induction.

2 Preliminary Results

Throughout this section let S and G be finite groups of coprime order such that S is a solvable group acting on G. We denote the set of S-fixed points of a set X on which S acts by X^S.

2.1 Extensions of Representations

For an irreducible complex representation ρ of G, which is fixed under the action of S, the existence of a unique (canonical) extension $\tilde{\rho}$ to the semi-direct product $S \propto G$ is well known, and characterised by the following theorem (for example, see [6]):

Theorem 2.1

Let $\rho \in Irr(G)^S$. Then there exists a unique $\tilde{\rho} \in Irr(S \propto G)$ such that $Res_G^{S\propto G}(\tilde{\rho}) = \rho$ and $det(\tilde{\rho}(s)) = 1$ for all $s \in S$. Moreover, if $\tilde{\rho}$ is such an extension then $\mathbf{Q}(\tilde{\rho}) = \mathbf{Q}(\rho)$ and $\tilde{\rho}(s) \in \mathbf{Q}$ for all $s \in S$ (where $\mathbf{Q}(\rho)$ is the field obtained by adjoining the character values of ρ to $\mathbf{Q}$).

The second part of the theorem leads to the following observation:

Lemma 2.2

Let S be a cyclic group and $\rho \in Irr(G)^S$. If $s \in S$ and a an integer prime to $|S|$, then the character values of $\tilde{\rho}$ satisfy $\tilde{\rho}(s,t) = \tilde{\rho}(s^a, t)$ for all $t \in G^S$.

2.2 Explicit Brauer Induction

We briefly recall Explicit Brauer Induction from [15].

2.3 Let G be any finite group and let $R(G)$ denote the complex representation ring of G. Denote by $\mathcal{M}_G$ the poset of characters (or one-dimensional complex representations) on subgroups, (H, ϕ), where $H \leq G$ and $\phi : H \longrightarrow \mathbf{C}^*$. Then $\mathcal{M}_G$ is a poset if we define the partial ordering by

$$(H, \phi) \leq (H', \phi') \Longleftrightarrow H \leq H' \text{ and } Res_H^{H'}(\phi') = \phi.$$

In addition, $g \in G$ acts on $\mathcal{M}_G$ by the formula

$$g(H, \phi) = (gHg^{-1}, (g^{-1})^*(\phi))$$

where $(g^{-1})^*(\phi)(ghg^{-1}) = \phi(h)$ for all $h \in H$.

Let $R_+(G)$ denote the free abelian group on the set $(\mathcal{M}_G)/G$. Hence the free generators are the G-conjugacy classes of characters $\phi : H \longrightarrow \mathbf{C}^*$ where $H \leq G$. We shall denote this character by (H, ϕ) and its G-conjugacy class by $(H, \phi)^G \in R_+(G)$.

If $J \leq G$ we define a restriction homomorphism

$$Res_J^G : R_+(G) \longrightarrow R_+(J)$$

by the double coset formula

$$Res_J^G((H, \phi)^G) = \sum_{z \in J\backslash G/H} (J \cap zHz^{-1}, (z^{-1})^*(\phi))^J.$$

If $\pi : J \longrightarrow G$ is a surjection we may define an inflation homomorphism

$$Inf_G^J : R_+(G) \longrightarrow R_+(J)$$

by the formula

$$Inf_G^J((H, \phi)^G) = (\pi^{-1}(H), \phi\pi)^J.$$

By means of restriction and inflation we may define

$$f^{*} : R_{+}(G) \longrightarrow R_{+}(J)$$

for any homomorphism $f : J \longrightarrow G$ by factorising f as $f : J \longrightarrow im(f) \subseteq G$ and setting

$$f^{*} = Inf^{J}_{im(f)} \cdot Res^{G}_{im(f)} : R_{+}(G) \longrightarrow R_{+}(im(f)) \longrightarrow R_{+}(J).$$

In addition, we may define a product on $R_{+}(G)$ by the formula

$$(K,\phi)^{G} \cdot (H,\psi)^{G} = \sum_{z \in K\backslash G/H} ((z^{-1}Kz) \cap H, z^{*}(\phi)\psi)^{G}.$$

These definitions make $R_{+}(G)$ into a functor from the category of finite groups to the category of commutative rings.

If $J \leq G$ one may also define an induction homomorphism

$$Ind^{G}_{J} : R_{+}(J) \longrightarrow R_{+}(G)$$

by the formula $Ind^{G}_{J}((H,\phi)^{J}) = (H,\phi)^{G}$. As in the case of the representation ring induction and restriction homomorphisms satisfy Frobenius reciprocity making $R_{+}(G)$ into a ring-valued Mackey functor. There is a natural transformation of ring-valued Mackey functors

$$b_{G} : R_{+}(G) \longrightarrow R(G)$$

defined on generators by the formula

$$b_{G}((H,\phi)^{G}) = Ind^{G}_{H}(\phi) \in R(G).$$

The following result was first discovered in [11] (see also [10]), improved upon in ([2],[3]) and developed and applied in ([4],[12],[13],[14],[15],[17]).

Theorem 2.4 *(Explicit Brauer Induction; [15] Theorem 2.3.9)*
Let G be any finite group then, there exists an additive homomorphism

$$a_{G} : R(G) \longrightarrow R_{+}(G)$$

satisfying the following properties:
(i) For $H \leq G$ the following diagram commutes:

$$\begin{array}{ccc} R(G) & \xrightarrow{a_{G}} & R_{+}(G) \\ {\scriptstyle Res^{G}_{H}} \downarrow & & \downarrow {\scriptstyle Res^{G}_{H}} \\ R(H) & \xrightarrow{a_{H}} & R_{+}(H) \end{array}$$

(ii) Let $\rho : G \longrightarrow GL_n(\mathbf{C})$ be a representation and suppose that

$$a_G(\rho) = \sum \alpha_{(H,\phi)^G}(H,\phi)^G \in R_+(G)$$

then $\alpha_{(G,\phi)^G} = <\rho, \phi>$ for each $(H,\phi)^G$ such that $H = G$. Here $<\rho,\phi>$ denotes the Schur inner product (i.e. the multliplicity of the irreducible representation ϕ in ρ). In particular, if ρ is one-dimensional then $a_G(\rho) = (G,\rho)^G$.

(iii) The homomorphism, a_G, is uniquely characterised by (i)-(ii) and satisfies $b_G a_G = 1$.

(iv) In terms of the Schur inner product and the Möbius function of the poset, $\mathcal{M}_G$, $a_G(\rho)$ is given by the following formula:

$$a_G(\rho) = |G|^{-1} \sum_{\substack{(H,\phi)\le(H',\phi') \\ \text{in } \mathcal{M}_G}} |H|\mu^{\mathcal{M}_G}_{(H,\phi),(H',\phi')} < \phi', Res^G_{H'}(\rho) > (H,\phi)^G.$$

2.3 Cohomological Results

Lemma 2.5

Let p be a prime not dividing $|G|$ and let S be a p-group acting on G. Then $H^1(S;G) = \{*\}$, the set with one element.

Proof

If $f : S \to G$ is a $1-$cocycle, define a homomorphism $\Phi : S \to S \propto G$ by $\Phi(s) = (s, f(s))$ [13]. By Sylow's Theorem $Im(\Phi)$ is conjugate to S in $S \propto G$ and it follows that there exists $g \in G$ such that $f(s) = gs(g^{-1})$ for all $s \in S$. □

Lemma 2.6

Let S be a solvable group acting on G, with $(|S|,|G|) = 1$. Then $H^1(S;G) = \{*\}$.

Proof

Let T be an abelian normal subgroup of S. The result follows from lemma 2.5 and the following well-known exact sequence of pointed sets (see for example [8]):

$$H^1(S/T; G^T) \longrightarrow H^1(S;G) \longrightarrow H^1(T;G)^{S/T}$$

□

2.7 Consider the $\mathbf{Z}[S]-$module given by $R_+(G)$ where the action of $s \in S$ given by setting $s \circ (H,\phi)^G$ equal to the $G-$conjugacy class of $s(\phi) : s(H) \longrightarrow \mathbf{C}^*$ given by $s(\phi)(s(h)) = \phi(h)$, for $h \in H$.

Recall that ([16] Definition 1.1.2 p.3), if M is a $\mathbf{Z}[S]-$module and $N_S = \sum_{s\in S} s \in \mathbf{Z}[S]$ is the norm element, the $0-$th Tate cohomology group is defined by:

$$\hat{H}^0(S;M) = M^S/(N_S M).$$

For any $\mathbf{Z}[S]$-permutation module of the form $M = \bigoplus_i Ind_{S_i}^S(\mathbf{Z})$ we set

$$M_0 = \bigoplus_{S_i = S} Ind_{S_i}^S(\mathbf{Z})$$

and we have $\mathbf{Z}[S]$-module homomorphisms:

$$M_0 \xrightarrow{j} M \xrightarrow{\pi} M_0$$

with j the inclusion map and $\pi j = 1$, which induce

$$\bigoplus_i \mathbf{Z}/|S_i| \cong \hat{H}^0(S; M) \underset{j_*}{\overset{\pi_*}{\rightleftarrows}} \hat{H}^0(S, M_0) \cong \bigoplus_{S_i = S} \mathbf{Z}/|S|.$$

Specifically, we shall apply this to:

$$R_+(G) \cong \bigoplus_{(H,\phi)^G, J = Stab_S(H,\phi)^G} Ind_J^S(\mathbf{Z}).$$

Hence, summing over the same elements as above,

$$\begin{aligned} \hat{H}^0(S; R_+(G)) &\cong \hat{H}^0(S; \bigoplus_J Ind_J^S(\mathbf{Z})) \\ &\cong \bigoplus_J \hat{H}^0(S; Ind_J^S(\mathbf{Z})) \\ &\cong \bigoplus_J \mathbf{Z}/|J| \end{aligned}$$

From this we see:

$$\hat{H}^0(S; R_+(G)_0) \cong \bigoplus_{(J,\phi)^G \in R_+(G), S = Stab_S(J,\phi)^G} \mathbf{Z}/|S|$$

and

$$\hat{H}^0(S; R_+(G^S)) \cong \bigoplus_{(J,\phi)^{G^S} \in R_+(G^S)} \mathbf{Z}/|S|.$$

Theorem 2.8

Let S be a p-group acting on G of coprime order as in §2.7. Then the restriction and induction homomorphisms induce inverse isomorphisms

$$\hat{H}^0(S; R_+(G)_0) \underset{\pi_*}{\overset{j_*}{\rightleftarrows}} \hat{H}^0(S; R_+(G)) \underset{Ind_{G^S}^G}{\overset{Res_{G^S}^G}{\rightleftarrows}} \hat{H}^0(S; R_+(G^S)).$$

Corollary 2.9

With S and G as Theorem 2.8, the restriction homomorphism induces an isomorphism

$$Res_{G^S}^G : \hat{H}^0(S; R(G)_0) \longrightarrow \hat{H}^0(S; R(G^S)).$$

Proof

Consider the following diagram (the summation is as in §2.7).

$$\begin{array}{ccc}
\bigoplus_{(J',\phi')^G} \mathbf{Z}/|Stab_S((J',\phi')^G)| & & \bigoplus_{(J,\phi)^{G^S}} \mathbf{Z}/|S| \\
\downarrow \cong & & \downarrow \cong \\
\hat{H}^0(S; R_+(G)) & \longrightarrow & \hat{H}^0(S; R_+(G^S)) \\
a_G \uparrow \; \downarrow b_G & & b_{G^S} \downarrow \; \uparrow a_{G^S} \\
\hat{H}^0(S; R(G)) & \longrightarrow & \hat{H}^0(S; R(G^S))
\end{array}$$

By naturality, each of the homomorphisms a_G, b_G, a_{G^S} and b_{G^S} is a $\mathbf{Z}[S]$-module homomorphism. By Theorem 2.4 and functoriality of b_G all these homomorphisms commute with the restriction homomorphisms. Theorem 2.8 implies that $Res^G_{G^S}$ induces an isomorphism of the form

$$\hat{H}^0(S; R_+(G)_0) \cong \bigoplus_{(J,\phi)^G \in R_+(G), S = Stab_S(J,\phi)^G} \mathbf{Z}/|S| \xrightarrow{\cong} \bigoplus_{(J,\phi)^{G^S}} \mathbf{Z}/|S|.$$

Since $b_G a_G = 1$ and $b_{G^S} a_{G^S} = 1$, the restriction homomorphism on $\hat{H}^0(S; R(G)_0)$ is a natural summand of the restriction homomorphism on $\hat{H}^0(S; R_+(G)_0)$ and is therefore an isomorphism modulo p, as required. □

Theorem 2.8 will be proved in §2.12 after a series of preliminary results. Unless specified, S is assumed to be a solvable group acting on G with $(|G|, |S|) = 1$.

Proposition 2.10

Suppose that $J \subseteq G$ is a subgroup such that $s(J) = J$ for all $s \in S$. Then

$$(G^S \backslash G/J)^S = G^S \cdot 1 \cdot J,$$

the identity double coset.

Proof

Assume S is cyclic of order m, generated by an element s. If S fixes a double coset $G^S \cdot z \cdot J$ then there exists $\alpha \in G^S$, $\beta \in J$ such that $s(z) = \alpha z \beta$. By repeated action of s, we see that

$$z = s^m(z) = \alpha^m z \beta s(\beta) s^2(\beta) \ldots s^{m-1}(\beta)$$

and so $z^{-1}\alpha^{-m}z \in J$. Since $|S|$ is prime to the order of α we find that $z^{-1}\alpha z = z^{-1}s(z)\beta^{-1} \in J$ and so $z^{-1}s(z) \in J$. This holds for all elements of S which means we may define a 1-cocycle, $f : S \longrightarrow J$, by $f(s) = z^{-1}s(z)$ for all $s \in S$. By Lemma 2.6 there exists $j \in J$ such that $j^{-1}s(j) = f(s) = z^{-1}s(z)$ for all $s \in S$ and so $zj^{-1} \in G^S$. Hence $\alpha z \beta = \alpha(zj^{-1})(j\beta)$ and this implies that $G^S \cdot z \cdot J = G^S \cdot 1 \cdot J$ as required.

Assume now that S is non-cyclic and take $S' \lhd S$ such that S/S' is cyclic and let

$$X_{S'} = \{G^S \cdot z \cdot J | z \in G^{S'}\}.$$

We see S/S' acts on $X_{S'}$ and the proof follows by induction on $|S|$. □

Lemma 2.11

If $J', J \subseteq G^S$ and $(J,\phi)^G = (J',\phi')^G \in R_+(G)$ then

$$(J,\phi)^{G^S} = (J',\phi')^{G^S} \in R_+(G^S).$$

Proof

By definition, there exists $g \in G$ such that $gJg^{-1} = J'$ and $\phi(j) = \phi'(gjg^{-1})$ for all $j \in J$. Now consider the function, $f : S \longrightarrow G$, given by $f(s) = g^{-1}s(g)$. Define the normaliser of (J,ϕ) in G, $N_G(J,\phi)$, to be the subgroup given by

$$N_G(J,\phi) = \{z \in N_G J \mid \phi(zjz^{-1}) = \phi(j) \ for \ all \ j \in J\}.$$

Then f is a 1-cocycle with values in $N_G(J,\phi)$ and the result follows by application of Lemma 2.6. □

2.12 *Proof of Theorem 2.8*

Since the groups we are considering are both direct sums of copies of the cyclic p-group $\mathbf{Z}/|S|$, we have only to show that $Ind^G_{G^S}$ induces a modulo p inverse to $Res^G_{G^S}$ on Tate cohomology in dimension zero.

Given $(H,\psi)^G \in R_+(G)^S$, Glauberman's fixed point lemma (see for example Lemma 13.8 of [7]) implies that we can find an element (J,ϕ) such that $(H,\psi)^G =$

$(J,\phi)^G$ and $Stab_S(J,\phi) = S$. Proposition 2.10 implies that $(G^S\backslash G/J)^S$ is the identity double coset. The following composition

$$\hat{H}^0(S;R_+(G)_0) \xrightarrow{j_*} \hat{H}^0(S;R_+(G)) \xrightarrow{Res^G_{G^S}} \hat{H}^0(S;R_+(G^S)) \xrightarrow{Ind^G_{G^S}} \hat{H}^0(S;R_+(G))$$

sends $(H,\psi)^G$ to

$$\begin{aligned} Ind^G_{G^S}(Res^G_{G^S}(H,\psi)^G) &= Ind^G_{G^S}(Res^G_{G^S}(J,\phi)^G) \\ &= Ind^G_{G^S}\left(\textstyle\sum_{z\in G^S\backslash G/J}(G^S\cap zJz^{-1},(z^{-1})^*(\phi))^{G^S}\right) \\ &= \textstyle\sum_{z\in G^S\backslash G/J}(G^S\cap zJz^{-1},(z^{-1})^*(\phi))^G \end{aligned}$$

The action of S permutes the terms in this sum so that we can separate the fixed and non-fixed double-cosets and apply Proposition 2.10 to write

$$Ind^G_{G^S}(Res^G_{G^S}(J,\phi)^G) = (J,\phi)^G + \sum_{s\in S} s(\sum_z (G^S\cap zJz^{-1},(z^{-1})^*(\phi))^G)$$

where the final sum is taken over S-orbit representatives of non-fixed double cosets. We see that all the terms in this sum are fixed by the action of S. Since S is a p-group, all the S-orbits in the sum have orbit size a multiple of p, which implies that

$$Ind^G_{G^S}(Res^G_{G^S}(J,\phi)^G) \equiv (J,\phi)^G \text{ (modulo p)}.$$

Therefore $Ind^G_{G^S}$ induces a split surjection modulo p on $\hat{H}^0$. However Lemma 2.11 implies that $Ind^G_{G^S}$ is one-to-one (both integrally and modulo p) which completes the proof. □

Corollary 2.13
Let S be a cyclic group acting on G with $(|S|,|G|)=1$. Then

$$|Irr(G)^S| = |Irr(G^S)|.$$

Proof

Let S be a p-group, let $\lambda_1,\ldots,\lambda_t$ denote the irreducible representations of the fixed group G^S and let $\rho_1,\ldots,\rho_{t'}$ denote the S-fixed irreducible representations of G. Then t equals the rank of $\hat{H}^0(S;R(G)_0)$ and t' is the rank of $\hat{H}^0(S;R(G^S))$. By Corollary 2.9, these numbers are equal. If S is cyclic, let $S = S_1\times S_2$ where $(|S_1|,|S_2|)=1$. Then, by induction,

$$|Irr(G)^S| = |(Irr(G)^{S_1})^{S_2}| = |Irr(G^{S_1})^{S_2}| = |Irr(G^{S_1\times S_2})|.$$

□

3 Proof of Glauberman's Theorem

Let $\lambda_1, \ldots, \lambda_t$ denote the irreducible representations of the fixed group G^S. By Corollary 2.13, the set of S-fixed irreducible representations of G may be written $\rho_1, \ldots, \rho_t$. Let $\tilde{\rho}_i$ be the canonical extension of ρ_i to the semi-direct product $S \propto G$, as in §2.1. Let $C_G(g)$ denote the centraliser of g in G so that the order of the G-conjugacy class of g is equal to $|G|/|C_G(g)|$. From Glauberman ([6] Lemma 2) we have the following result.

Lemma 3.1

Let $T =< s >$ be a cyclic subgroup of S of order n, coprime to $|G|$.

(i) For any $g \in G$ the $T \propto G$-conjugacy class of (s, g) contains an element of the form (s, g') with $g' \in G^T$.

(ii) Let $z, z' \in G^T$. Then (s, z) and (s, z') are conjugate in $T \propto G$ if and only if z and z' are conjugate in G^T.

(iii) If H is a group and $z \in H$, let $Cl_H(z)$ denote the H-conjugacy class of z. Suppose that $z \in G^T$. Then

$$|Cl_{T \propto G}(s, z)| = \frac{|G|}{|G^T|} |Cl_{G^T}(z)|$$

3.2 Let $S =< s >$ be cyclic of order n where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ for distinct primes p_i not dividing $|G|$. The irreducible representations of S are given by powers of the one-dimensional representation y given by $y(s) = e^{2\pi i/n} = \xi_n$.

We first consider the Galois orbits of elements of the set $\{y^j \mid 0 \leq j \leq n\}$ under the action of elements of the Galois group $Gal(\mathbf{Q}(\xi_n)/\mathbf{Q})$. There are $(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_r + 1)$ such orbits. For β dividing n the "Galois orbit" of y^β, denoted by G_β, is defined by

$$G_\beta = \sum_{(k, \frac{n}{\beta})=1,\ 1 \leq k \leq \frac{n}{\beta}-1} y^{\beta k}.$$

We may use the Artin induction formula (see [15] Theorem 2.1.3) to rewrite each G_β as a sum of induced representations

$$G_\beta = Ind_{C_\beta}^{C_n}(1) - \sum_{C_\gamma > C_\beta} \mu_\gamma Ind_{C_\gamma}^{C_n}(1)$$

where, for each γ, μ_γ is an integer (calculated in terms of Möbius coefficients but this is not necessary here) with $\mu_\beta = 1$. Also the C_γ are cyclic groups. Using Lemma 2.2, summing G_β over all β dividing n and gathering together terms, we find

$$Res_{S \times G^S}^{S \propto G}(\tilde{\rho}_i) = \sum_{\beta | n} a_{i\beta}(Ind_{C_\beta}^{C_n}(1) \otimes A_{i\beta})$$

for integers $a_{i\beta}$ and some (yet to be determined) $A_{i\beta} \in R(G^S)$.

Choose a such that $(a,n)=1$. We see all the terms in the sum above vanish on s^a except the term $\beta = n$ so that

$$Res^{S\propto G}_{S\times G^S}(\tilde{\rho}_i)(s^a,g) = \alpha_{in}A_{in}(g).$$

Hence

$$\tfrac{1}{|G^S|}\textstyle\sum_{g\in G^S} Res^{S\propto G}_{S\times G^S}(\tilde{\rho}_i)(s^a,g)\overline{Res^{S\propto G}_{S\times G^S}(\tilde{\rho}_j)(s^a,g)}$$

$$= \tfrac{1}{|G^S|}\textstyle\sum_{g\in G^S} \alpha_{in}A_{in}(g)\alpha_{jn}\overline{A_{jn}(g)}$$

$$= \alpha_{in}\alpha_{jn}\langle A_{in}(g), A_{jn}(g)\rangle_{G^S}$$

Proposition 3.3

If $S = < s >$ is a cyclic group order n and let a be an integer satisfying $1 \le a \le n-1$ and $(a,n)=1$. Define T_{ij} by the formula

$$T_{ij} = \frac{1}{|G^S|}\sum_{g\in G^S} Res^{S\propto G}_{S\times G^S}(\tilde{\rho}_i)(s^a,g)\overline{Res^{S\propto G}_{S\times G^S}(\tilde{\rho}_j)(s^a,g)}.$$

Then $T_{ij} = \delta_{ij}$, the Kronecker delta.

Proof

For β dividing n, let k_β be an integer satisfying $1 \le k_\beta \le \frac{n}{\beta}-1$ and $(k_\beta, \frac{n}{\beta}) = 1$. Let $S_\beta = < s^{\beta k_\beta} >$, which is independent of the choice of k_β. By Lemma 3.1, if $\phi(m)$ denotes Euler's totient function, we have

$$\delta_{ij}$$

$$= \langle \tilde{\rho}_i, \tilde{\rho}_j\rangle_{S\propto G}$$

$$= \tfrac{1}{n|G|}\textstyle\sum_{\beta|n}\phi\left(\frac{n}{\beta}\right)\frac{|G|}{|G^{S_\beta}|}\sum_{g\in G^{S_\beta}} Res^{S\propto G}_{S_\beta\times G^{S_\beta}}(\tilde{\rho}_i)(s^{\beta k_\beta},g)\overline{Res^{S\propto G}_{S_\beta\times G^{S_\beta}}(\tilde{\rho}_j)(s^{\beta k_\beta},g)}$$

$$= \tfrac{1}{n}\textstyle\sum_{\beta|n}\phi\left(\frac{n}{\beta}\right)\frac{1}{|G^{S_\beta}|}\sum_{g\in G^{S_\beta}} Res^{S_\beta\propto G}_{S_\beta\times G^{S_\beta}}(\tilde{\rho}_i)(s^{\beta k_\beta},g)\overline{Res^{S_\beta\propto G}_{S_\beta\times G^{S_\beta}}(\tilde{\rho}_j)(s^{\beta k_\beta},g)}.$$

By induction, we can assume the result true for all cases above except the case $\beta = 1$, which implies that

$$\delta_{ij} = \frac{\phi(n)}{n}T_{ij} + \frac{\delta_{ij}}{n}\sum_{\substack{\beta|n\\ \beta\neq 1}}\phi\left(\frac{n}{\beta}\right).$$

Thus the identity $\sum_{\beta|n}\phi(\beta) = \sum_{\beta|n}\phi\left(\frac{n}{\beta}\right) = n$ implies that $T_{ij} = \delta_{ij}$ as required.
□

From Proposition 3.3 we see that

$$\delta_{ij} = \alpha_{in}\alpha_{jn}\langle A_{in}(g), A_{jn}(g)\rangle_{G^S}.$$

Since α_{in} and α_{jn} are integers, we must have $\langle A_{in}, A_{jn}\rangle_{G^S} = \delta_{ij}$ and $\alpha_{in}^2 = 1$. By Corollary 2.13, the distinct irreducible representations $\{A_{1n}, \ldots A_{tn}\}$, of G^S must be precisely $\{\lambda_1, \ldots \lambda_t\}$ which implies that there exists a permutation, σ, and a sign $\epsilon_i \in \{\pm 1\}$ such that

$$A_{in} = \epsilon_i \lambda_{\sigma(i)}$$

for $1 \leq i \leq t$. If we choose a coprime to n and $t \in G^S$ we see that

$$Res^{S \propto G}_{S \times G^S}(\tilde{\rho}_i)(s^a, t) = \tilde{\rho}_i(s^a, t) = A_{in}(t) = \epsilon_i \lambda_{\sigma(i)}$$

This proves the following lemma, and is exactly Glauberman's characterisation of the correspondence.

Lemma 3.4

Let S be a cyclic group and let $\rho \in Irr(G)^S$. Then there exists a unique sign $\epsilon = \pm 1$ and a unique $\lambda \in Irr(G^S)$ such that

$$\tilde{\rho}(s, t) = \epsilon \lambda(t),$$

for all s which generate S and all $t \in G^S$. Moreover, for every $\lambda \in Irr(G^S)$, there exists a unique S–fixed $\rho \in Irr(G)$ which corresponds to λ in this manner.

From the above equations, we also obtain

$$Res^G_{G^S}(\rho_i) = \epsilon_i \lambda_{\sigma(i)} + \sum_{\substack{\beta | n \\ \beta \neq n}} \binom{n}{\beta} \alpha_{i\beta} A_{i\beta}$$

If S is a p–group we note that all terms except the first term disappear modulo p. Hence we immediately obtain the following result.

Lemma 3.5

Let S be a cyclic p-group and $\rho \in Irr(G)^S$. Let

$$Res^G_{G^S}(\rho) = n_1 \lambda_1 + \ldots + n_t \lambda_t,$$

where λ_i are distinct irreducible representations. Then there exists a unique i such that $p \not| n_i$. Moreover, $n_i \equiv \pm 1$ (modulo p).

This gives a complete characterisation of Glauberman's correspondence for cyclic groups. It is straightforward to extend Lemma 3.4 to all solvable S, using the results above with a composition series (see [6], §4 pp.1477-1479). Similarly the result of Lemma 3.5 is easily strengthened by induction on $|S|$ to yield the following:

Lemma 3.6

Lemma 3.5 holds for any finite p-group S.

4 Boltje's Explicit Map

Let S be a p–group. Boltje [5] defines a map $bol^{S,G} : R(G)^S \to R(G^S)$ as a composition of the following form

$$bol^{S,G} : R(G)^S \xrightarrow{a_G} R_+(G)^S \xrightarrow{bol_+^{S,G}} R_+(G^S) \xrightarrow{b_{G^S}} R(G^S).$$

We define the map $bol_+^{S,G} : R_+(G)^S \to R_+(G^S)$. Let $T = Stab_S((H,\lambda)^G)$, then the S-orbit sums

$$\sum_{s\in S/T} s\circ(H,\lambda)^G, \qquad (H,\lambda) \in S \propto G\backslash\mathcal{M}(G)$$

form a $\mathbf{Z}$-basis of $R_+(G)^S$. By Glauberman's fixed point lemma (see for example Lemma 13.8 of [7]), there is a T-fixed point (H',λ') in the G-orbit of (H,λ) and we can define the map (Boltje [5] proves this map is well-defined):

$$bol_+^{S,G}\left(\sum_{s\in S/T} s\circ(H,\lambda)^G\right) = \begin{cases} (G^S\cap H', Res^{H'}_{G^S\cap H'}(\lambda'))^{G^S} & if\ T=S, \\ 0 & if\ T<S. \end{cases}$$

We conclude this section by deducing the main theorem of [5] from the results of §2 and §3.

Theorem 4.1
When S is a p-group with p prime then the homomorphism $bol^{S,G}$ is congruent to Glauberman's correspondence modulo p.

Proof
If $\rho \in Irr(G)^S$ then, by Lemma 3.6, $Res^G_{G^S}(\rho) \equiv \epsilon\lambda$ (modulo p) for some irreducible representation $\lambda \in Irr(G^S)$. Applying a_G and the formulae of §2.3 to this we obtain

$$Res^G_{G^S}(a_G(\rho)) = a_{G^S}(Res^G_{G^S}(\rho)) \equiv a_{G^S}(\epsilon\lambda) \text{ (modulo p)}$$

On the other hand, if $a_G(\rho) = \sum_i n_i(H_i,\phi_i)^G$ then

$$\begin{aligned} Res^G_{G^S}(a_G(\rho)) &= Res^G_{G^S}(\textstyle\sum_i n_i(H_i,\phi_i)^G) \\ &= \textstyle\sum_i \sum_{z\in G^S\backslash G/H_i}(G^S\cap zH_iz^{-1},(z^{-1})^*(\phi_i))^{G^S} \end{aligned}$$

We can split $a_G(\rho)$ in the form

$$a_G(\rho) = \sum_i (H_i,\phi_i)^G = \sum_j (J_j,\phi_j)^G + \sum_{s\in S}\sum_k s(K_k,\psi_k)$$

where the $(J_j, \phi_j)^G$'s are S-fixed, and the (K_k, ψ_k)'s are not. Further from Glauberman's fixed point lemma, we may assume that $s \circ (J_j, \phi_j) = (J_j, \phi_j)$ for all $s \in S$ and for each j. Applying Proposition 2.10, we have:

$$\begin{aligned} Res^G_{G^S}(a_G(\rho)) &= \textstyle\sum_j (G^S \cap J_j, Res^{J_j}_{G^S \cap J_j}(\phi_j))^{G^S} \\ &\quad + \textstyle\sum_k \sum_{s \in S} \sum_{z \in G^S \backslash G / K_k} s(G^S \cap zK_k z^{-1}, (z^{-1})^*(\psi_k))^{G^S}. \end{aligned}$$

The non-fixed terms all restrict to subgroups and representations of G^S, so the action of s leaves the element unchanged in its G^S orbit and each term will therefore appear a multiple of p times (cf. the argument of §2.12) so that

$$Res^G_{G^S}(a_G(\rho)) \equiv \sum_j (G^S \cap J_j, Res^{J_j}_{G^S \cap J_j}(\phi_j))^{G^S} \equiv bol^{S,G}_+(a_G(\rho)) \text{ (modulo p)}.$$

Applying b_{G^S} yields the following congruences modulo p

$$\begin{aligned} \epsilon\lambda &= b_{G^S}(a_{G^S}(\epsilon\lambda)) \\ &\equiv b_{G^S}(Res^G_{G^S}(a_G(\rho))) \\ &\equiv b_{G^S}(bol^{S,G}_+(a_G(\rho))), \end{aligned}$$

as required. □

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