Explicit Brauer Induction for symplectic and orthogonal representations

Olaf Neiße and Victor Snaith*

1 Topological motivation

1.1 Let G_n denote one of the classical compact Lie groups, $\mathrm{U}(n), \mathrm{Sp}(n)$ or $\mathrm{O}(2n)$, of unitary, symplectic or orthogonal matrices. Since G_{n-1} embeds canonically into G_n (by adding $1 \in G_1$ at the bottom right-hand corner) we may form the mapping cone, BG_n/BG_{n-1}, of the induced map between classifying spaces. When $n = 0$ we set $BG_0/BG_{-1} = S^0$, the zero-dimensional sphere. Let X_+ denote the disjoint union of the space X and a base-point. In the stable homotopy category [Ad] there is a homotopy equivalence of the form

$$(BG_\infty)_+ \simeq \vee_{k\geq 0} BG_k/BG_{k-1}$$

which was first proved in [Sn2]. In fact, from this equivalence one can easily deduce equivalences of the form

$$(BG_n)_+ \simeq \vee_{0\leq k\leq n} BG_k/BG_{k-1}.$$

Stable decompositions of classifying spaces are important [Pr] because the factors are much simpler to work with than the whole. For example, $BU(n)/BU(n-1)$ is just the Thom space, $MU(n)$, of the universal n-plane bundle on $BU(n)$.

In this section we shall show how Explicit Brauer Induction may be used to derive these stable decompositions.

1.2 Let $R_+^U(G), R_+^{Sp}(G)$ and $R_+^O(G)$ denote, respectively, the free abelian group on the G-conjugacy classes of *representations* $\phi : H \longrightarrow G_1$ where H is a subgroup of G. Hence $R_+^U(G)$ (denoted by $R_+(G)$ in [SnEBI]) is the free abelian group on the G-conjugacy classes of homomorphisms $\phi : H \longrightarrow U(1) = S^1$. However, in the symplectic and orthogonal cases, because a representation into G_1 is a G_1-conjugacy class of a homomorphism, a free generator $\phi : H \longrightarrow G_1$ is equivalent to $X\phi(g - g^{-1})X^{-1} : H \longrightarrow G_1$ for any $g \in G$, $X \in G_1$. The equivalence class of (H, ϕ) will be denoted by $(H, \phi)^G$.

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If $J \subseteq G$ we have a restriction homomorphism

$$\mathrm{Res}_J^G : R_+^Z(G) \longrightarrow R_+^Z(J)$$

for $Z = U, Sp, O$ defined by

$$\mathrm{Res}_J^G((H,\phi)^G) = \sum_{z \in J\backslash G/H} (J \bigcap zHz^{-1}, (z^{-1})^*\phi)^J$$

where $(z^{-1})^*\phi(zhz^{-1}) = \phi(h)$. If $\pi : G \longrightarrow K$ is a surjection there is an inflation homomorphism

$$\mathrm{Inf}_K^G : R_+^Z(K) \longrightarrow R_+^Z(G)$$

given by $\mathrm{Inf}_K^G(H,\phi)^K = (\pi^{-1}(H), \phi \cdot \pi)^G$.

By means of these maps $R_+^Z(-)$ gives a functor from finite groups to abelian groups when $Z = U, Sp, O$. When $Z = U$, we even obtain a Mackey functor from finite groups to the category of rings ([Bo2], [Bo1], [BSS], [Sn4], [Sn8], [SnEBI]).

1.3 $R_+^U(G)$ *and stable homomotopy decompositions*

Let $R(G)$ denote the complex representation ring of G, so $R(G) = K_0(\mathbf{C}G)$, and let $IR(G) = Ker(\epsilon : R(G) \longrightarrow \mathbf{Z})$ denote the augmentation ideal given by the kernel of the homomorphism which sends a virtual representation to its dimension. Henceforth, following [SnEBI], we shall abbreviate $R_+^U(G)$ to $R_+(G)$.

The central result of Explicit Brauer Induction is the existence of natural transformations from representations of G to $R_+(G)$ which are right inverse to the map

$$b_G : R_+(G) \longrightarrow R(G)$$

given by $b_G(H,\phi)^G = \mathrm{Ind}_H^G(\phi) \in R(G)$. The formula of ([Bo2], [SnEBI], [Sy]) gives a natural homomorphism

$$a_G : R(G) \longrightarrow R_+(G)$$

such that $a_G(\phi : G \longrightarrow U(1)) = (G,\phi)^G$. There is only one such homomorphism and it satisfies $b_G a_G = 1$.

Now let p be a prime and consider the case when $G = GL_n\mathbf{F}_q$ with q a power of p. In [Qu] the canonical modular representations of G are used, by means of the Brauer lifting technique of [Gr], to construct a canonical element

$$\sigma_p \in \lim_{\overleftarrow{n,q}} IR(GL_n\mathbf{F}_q) \subset \lim_{\overleftarrow{n,q}} R(GL_n\mathbf{F}_q).$$

By naturality of the homomorphisms, $\{a_{GL_n\mathbf{F}_q}\}$, we obtain

$$a_{GL\overline{\mathbf{F}}_p}(\sigma_p) \in \lim_{\overleftarrow{n,q}} IR_+(GL_n\mathbf{F}_q) \subset \lim_{\overleftarrow{n,q}} R_+(GL_n\mathbf{F}_q).$$

Here $\overline{\mathbf{F}}_p$ is an algebraic closure of $\mathbf{F}_p$ and $IR_+(G)$ is the kernel of the homomorphism to the Burnside ring given by $\epsilon(H,\phi)^G = [G/H]$ ([Sn4], [SnEBI]).

If X and Y are base-pointed spaces let $\{X, Y\}$ denote the stable homotopy classes of maps from X to Y; that is, the morphisms from X to Y in the stable homotopy category [Ad]. If G is a finite group there exists a natural transformation

$$T : IR_+(G) \longrightarrow \{BG_+, BU(1)_+\}$$

given by sending $(H,\phi)^G - (H,1)^G$ to the composition

$$BG_+ \xrightarrow{\tau} BH_+ \xrightarrow{B\phi_+} BU(1)_+$$

where τ is the stable homotopy transfer ([Sn7] pp.163-4; see also [BeGo], [BeSch], [KP], [MSZ]). In fact, if $IA(G)$ is the augmentation ideal of the Burnside ring, $A(G)$, then the $IA(G)$-adic completion of T is an isomorphism. We shall not need this result, which was first proved (with $U(1)$ replaced by any torus) in ([Sn7] Ch.V Theorem 1.17) and was extended to all Lie groups in [MSZ].

Hence we have a canonical element

$$T(a_{GL\overline{\mathbf{F}}_p}(\sigma_p)) \in \lim_{\overleftarrow{n,q}} \{(BGL_n\mathbf{F}_q)_+, BU(1)_+\} \cong \{(BGL\overline{\mathbf{F}}_p)_+, BU(1)_+\}.$$

Let $Q(X_+)$ denote the iterated loopspace $Q(X_+) = \lim_n \Omega^n\Sigma^n(X_+)$. Then, if Y is a base-pointed space there is an adjunction isomorphism of the form

$$adj : \{Y, BU(1)_+\} \xrightarrow{\cong} [Y, Q(BU(1)_+)]$$

whose range is the set of based homotopy classes of maps from Y to $Q(BU(1)_+)$. Therefore we obtain a (homotopy class of a) map of the form

$$\widetilde{T}_p = adj(T(a_{GL\overline{\mathbf{F}}_p}(\sigma_p))) : (BGL\overline{\mathbf{F}}_p)_+ \longrightarrow Q(BU(1)_+).$$

Now we shall examine how the properties of a_G translate into useful properties of $\widetilde{T}_p$. Direct sum of matrices makes $BGL\overline{\mathbf{F}}_p$ into an H-space with multiplication

$$m : BGL\overline{\mathbf{F}}_p \times BGL\overline{\mathbf{F}}_p \longrightarrow BGL\overline{\mathbf{F}}_p.$$

The iterated loopspace $Q(BU(1)_+)$ is also an H-space and additivity of the Brauer lifting together with the fact that a_G is a *homomorphism* implies that $\widetilde{T}_p$ is an H-map so that

$$m(\widetilde{T}_p \times \widetilde{T}_p) \simeq \widetilde{T}_p \cdot m : BGL\overline{\mathbf{F}}_p \times BGL\overline{\mathbf{F}}_p \longrightarrow Q(BU(1)_+).$$

On the other hand, in the stable homotopy category, there is a Snaith splitting ([Sn1], [Sn3]; see also [CMT1], [CMT2]) of the form

$$\bigvee_{k\geq 0} j_k : \ Q(BU(1)_+) \xrightarrow{\simeq} \bigvee_{k\geq 0} (B\Sigma_k \int U(1))/(B\Sigma_{k-1} \int U(1))$$

where $\Sigma_k \wr U(1)$ is the wreath product given by the normaliser of the diagonal maximal torus in $U(k)$. As usual, when $k = 0$ we adopt the convention that mapping cone of $B\Sigma_{-1} \wr U(1) \longrightarrow B\Sigma_0 \wr U(1)$ is the zero-sphere, S^0. Composing with the map

$$(B\Sigma_k \wr U(1))/(B\Sigma_{k-1} \wr U(1)) \longrightarrow BU(k)/BU(k-1) = MU(k)$$

induced by the inclusion of $\Sigma_k \wr U(1)$ into $U(k)$ we obtain a stable map of the form

$$\bigvee_{k \geq 0} \hat{j}_k : \ Q(BU(1)_+) \longrightarrow \bigvee_{k \geq 0} MU(k).$$

Furthermore, as explained in ([Sn3]; see also [CMT1], [CMT2]), the maps $\bigvee_{k \geq 0} j_k$ and $\bigvee_{k \geq 0} \hat{j}_k$ are exponential with respect to the pairings

$$\begin{array}{c} (B\Sigma_k \wr U(1))/(B\Sigma_{k-1} \wr U(1)) \wedge (B\Sigma_l \wr U(1))/(B\Sigma_{l-1} \wr U(1)) \\ \downarrow \\ (B\Sigma_{k+l} \wr U(1))/(B\Sigma_{k+l-1} \wr U(1)) \end{array}$$

and

$$MU(k) \wedge MU(l) \longrightarrow MU(k+l)$$

induced by direct sum of matrices. That is, we have a homotopy-commutative diagram of stable maps of the following form.

$$\begin{array}{ccc} QBU(1)_+ \times QBU(1)_+ & \xrightarrow{\sum_k \hat{j}_k \wedge \hat{j}_{t-k}} & \bigvee_k \ MU(k) \wedge MU(t-k) \\ \Big\downarrow m & & \Big\downarrow \\ QBU(1)_+ & \xrightarrow{\hat{j}_t} & MU(t) \end{array}$$

Now we come to our motivating topological result.

Theorem 1.4 *([Sn3] Theorem 2.2; [Sn2] Theorem 4.3)*
There exists a stable homotopy equivalence of the form

$$\overline{\sigma}_U = \bigvee_{k \geq 0} \overline{\sigma}_{U,k} : \ BU_+ \xrightarrow{\simeq} \bigvee_{k \geq 0} MU(k).$$

In addition, $\overline{\sigma}_U$ is an exponential map in the sense that, for each $t \geq 0$, the diagram

$$\begin{array}{ccc}
BU_+ \wedge BU_+ = (BU \times BU)_+ & \xrightarrow{\sum_k \overline{\sigma}_{U,k} \wedge \overline{\sigma}_{U,t-k}} & \bigvee_k \ MU(k) \wedge MU(t-k) \\
\Big\downarrow m_+ & & \Big\downarrow \\
BU_+ & \xrightarrow{\hat{j}_t} & MU(t)
\end{array}$$

commutes in the stable homotopy category.

Proof

It suffices to construct an H-map, τ, from BU to $QBU(1)_+$ which restricts to the canonical map on $BU(1)$. Then we set σ_U equal to the composition with $\sum_{k=0}^{\infty} \hat{j}_k$. The map $\tilde{T}_p$ has the right properties except that its domain is $(BGL_n\overline{\mathbf{F}}_p)_+$ rather than BU. However, by the technique of localisation and completion in homotopy theory ([Bou1], [Bou2], [MST], [Su]) it suffices to construct τ on the rationalisation of BU and on its completion at each prime, l. Since BU and $QBU(1)_+$ are rationally equivalent we may take the rationalisation of τ to be the identity map. The completion of BU at the prime l equal to the l-adic completion of $(BGL_n\overline{\mathbf{F}}_p)$ where p is chosen to generate $(\mathbf{Z}/l^2)^*$ if l is odd and $p = 3$ when $l = 2$. Therefore we may choose the l-adic completion of a suitable $\tilde{T}_p$ as the completion of τ at l, which completes the proof. ■

Remark 1.5 We have used the Explicit Brauer Induction map to prove Theorem 1.4. The naturality of the map is required to in order to turn Quillen's element, σ_p of §1.3, into a map from $BGL\overline{\mathbf{F}}_p$ to $QBU(1)_+$. It is the fact that a_G is a *homomorphism* which yileds an H-map and hence the exponential property of the splitting.

The Explicit Brauer Induction formulae of [Sn4] are natural in the symplectic and orthognal case, too. Using this one could give a proof of the stable decompositions of [Sn2]

$$BSp \simeq \bigvee_{k=0}^{\infty} MSp(n), \quad BO \simeq \bigvee_{k=0}^{\infty} BO(2k)/BO(2k-2).$$

similar to that of Theorem 1.4.

In ([Sn3] Theorem 2.2; see also [Sn2] Theorem 3.2) it is claimed that BSp admits an *exponential* stable decomposition of the above form. However there is a gap in the proposed proof since the cavalier reference to the existence of "an analogous symplectic vector field" in ([Sn2] Example 2.13) is not true. For a time this gap did not seem serious in view of the fact that [MP] offered an alternative construction of an exponential stable decomposition of BSp.

On Friday, 18 July 1997 one of us (VPS) learned of the argument of [Ri] which showed that none of the stable decompositions of BSp which were then in the literature were exponential. The way around this gap then seemed clear. We believed that one could use the topological approach to symplectic Explicit Brauer Induction described in ([Sy] §6) to construct a natural homomorphism of the form

$$a_G^{Sp} : RSp(G) \longrightarrow R_+^{Sp}(G)$$

and then imitate the proof of Theorem 1.4. After all, in ([Sy] p.180) one finds the remark that "the symplectic case presents no new difficulties; one simply the complex projective space by the quaternionic version." Unfortunately, as we studied the symplectic and orthogonal more closely we discovered that this remark is unfortunately false. As a result, at the moment, we do not know whether or not there is an exponential stable decomposition for BSp. In fact, our analysis, together with the topological results of [Ri] strongly suggest that no such exponential stable decomposition for BSp exists.

2 Induction formula for unitary representations

2.1 *Brauer Induction formula* In this section we recall briefly the natural explicit Brauer induction formula in the complex case.

Let G be a finite group. Let $R(G)$ denote the Grothendieck group of the category of finite-dimensional left $\mathbf{C}G$-modules. Every such module yields a matrix representation $G \to \mathrm{Gl}(n, \mathbf{C})$ which is conjugated to a unitary representation $G \to \mathrm{U}(n)$. Since such a representation is determined by its character, we may identify $R(G)$ with the character ring of G, the free abelian group on the set of irreducible characters on G. The subgroup of $R(G)$ generated by the set of linear characters $G \to \mathrm{U}(1) = S^1$ will be denoted by $L(G)$. Brauer proved [Br] that every unitary representation $\rho : G \to \mathrm{U}(n)$ is a sum of representations which are induced from linear characters on subgroups of G, so

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G(\phi_i) \quad \text{with } H_i \leq G,\ \phi_i : H_i \to \mathrm{U}(1)$$

Canonical explicit Brauer induction formulae with various properties were given by Boltje [Bo1], Snaith [Sn4] etc. We are going to use the formula which is trivial on one-dimensional representations and natural with respect to restriction and inflation, namely

$$\rho = \sum_{(H_0,\phi_0) < .. < (H_r,\phi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}_{H_r}^G(\rho), \phi_r) \mathrm{Ind}_{H_0}^G(\phi_0)$$

where $H_i \leq G$, $\phi_i : H_i \to \mathrm{U}(1)$ and $m(\theta, \phi) = \langle \theta; \phi \rangle_H$ denotes the multiplicity of $\phi : H\mathrm{U}(1)$ in $\theta \in R(H)$.

In order to work with this formula we denote $R_+(G)$ the free abelian group on the G-conjucacy classes of linear characters on subgroups of G. More precisely, let G act on the set $\mathcal{M}(G)$ of pairs (H, ϕ) with $H \leq G$ and $\phi : H \to \mathrm{U}(1)$, and let $(H, \phi)^G$ denotes the G-orbit of (H, ϕ) in $\mathcal{M}(G)$, then those orbits, collected in a set denotes $\mathcal{M}(G)/_G$, form a $\mathbf{Z}$-basis of $R_+(G)$. This is an example of a general method called the $_+$-construction (see [Bo3]). There are, for $J \leq G$, homomorphisms

$$\mathrm{Res}_J^G : R_+(G) \to R_+(J)$$

and

$$\mathrm{Ind}_J^G : R_+(J) \to R_+(G)$$

and a natural conjugation map, giving the functor R_+ the structure of a G-Mackey functor.

Theorem 2.2 *[SnEBI, 2.2.45]*
The map

$$a_G : R(G) \ni \rho \mapsto \sum_{(H_0,\phi_0) < .. < (H_r,\phi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}_{H_r}^G(\rho), \phi_r)(H_0, \phi_0)^G$$

takes values in $R_+(G)$ and this is a homomorphism.

This homomorphism $a_G : R(G) \to R_+(G)$ satisfies the following natural properties:

Proposition 2.3 *[SnEBI, 2.3.2]*
i) For $J \le G$ the following diagram commutes

$$\begin{array}{ccc} R(G) & \xrightarrow{a_G} & R_+(G) \\ {\scriptstyle \mathrm{Res}^G_J}\downarrow & & {\scriptstyle \mathrm{Res}^G_J}\downarrow \\ R(J) & \xrightarrow{a_J} & R_+(J) \end{array}$$

ii) For $\phi \in L(G)$, $a_G(\phi) = (G, \phi)^G$.
iii) For $N \lhd G$ the following diagram commutes

$$\begin{array}{ccc} R(G/N) & \xrightarrow{a_{G/N}} & R_+(G/N) \\ {\scriptstyle \mathrm{Infl}^G_{G/N}}\downarrow & & {\scriptstyle \mathrm{Infl}^G_{G/N}}\downarrow \\ R(G) & \xrightarrow{a_G} & R_+(G) \end{array}$$

Theorem 2.4 *[SnEBI, 2.3.2]*
If $b_G : R_+(G) \to R(G)$ is the homomorphism defined by

$$(H, \phi)^G \mapsto \mathrm{Ind}^G_H(\phi)$$

that a_G is a splitting, that is $b_G \circ a_G = \mathrm{id} : R(G) \to R(G)$.

2.5 *Example*
Let Q_{4n} denote the generalised quaternion group

$$Q_{4n} = \left\langle x, y \mid x^n = y^2,\ y^4 = 1, yxy^{-1} = x^{-1} \right\rangle$$

and let Ψ denote the symplectic representation

$$\Psi : Q_{4n} \longrightarrow Sp(1) \subset \mathbf{H}^*$$

given by $\Psi(x) = \xi_{2n}$ and $\Psi(y) = j$. Here $\xi_n = e^{2\pi\sqrt{-1}/n}$ and j is the usual quaternion.
We wish to evaluate

$$a_{Q_{4n}}(c(\Psi))$$

$$= \sum_{(H,\phi)<(H_1,\phi_1)<\ldots<(H_r,\phi_r)} (-1)^r \frac{|H|}{4n} < c(\Psi), \phi_r >_{H_r} \cdot (H, \phi)^{Q_{4n}} \in R_+(Q_{4n}).$$

If the multiplicity $< c(\Psi), \phi_r >_{H_r}$ is non-zero then $\mathrm{Res}^{Q_{4n}}_{H_r}(c(\Psi)) = \phi_r \oplus \overline{\phi}_r$ so that ϕ_r must be injective on H_r, because Ψ is. Hence H_r must be cyclic. The cyclic subgroups contained in $< x >\cong \mathbf{Z}/2n$ are given by $< x^m >\cong \mathbf{Z}/(2n/m)$ for each m dividing $2n$. Otherwise each x^iy satisfies $(x^iy)^2 = y^2$ and generators a cyclic subgroup of order four. When n is odd, all the subgroups, $< x^iy >$, are conjugate in Q_{4n} but when n is even, there are two conjugacy classes, $< y >$ and $< xy >$. Also $\Psi(y^2) = -1$ so that, if χ is the non-trivial character on $< y^2 >$, $< y^2, \chi) >\leq (H, \phi)$ for any $(H, \phi)^{Q_{4n}}$ which has a non-zero coefficient in $a_{Q_{4n}}(c(\Psi))$ ([SnEBI] Corollary 2.2.40).

Now consider the coefficient of $(H, \phi)^{Q_{4n}}$ when $< y^2 >\subseteq H \subseteq< x >$ with $|H| = t > 2$. In this case $(H, \phi) \neq (H, \overline{\phi})$ but $(H, \phi)^{Q_{4n}} = (H, \overline{\phi})^{Q_{4n}}$. If $\lambda \in \{\phi, \overline{\phi}\}$ then we have $1 =< c(\Psi), \phi_r >_{H_r}$ for any chain of the form $(H, \lambda) < (H_1, \phi_1) < \ldots < (H_r, \phi_r)$. The chains starting with ϕ are distinct from those starting with $\overline{\phi}$ so that the coefficient of $(H, \phi)^{Q_{4n}}$ is equal to

$$\frac{2 \cdot |H|}{4n} \sum_{\{1\}<A_1<A_2<\ldots<A_r\leq<x>/H} (-1)^r.$$

Here the sum of taken over all proper chains of subgroups, $A_i = H_i/H \subseteq< x > /H$ or, when $< x >= H$, just the trivial chain. By ([SnEBI] Exercise 2.5.1)

$$\frac{2 \cdot |H|}{4n} \sum_{\{1\}<A_1<A_2<\ldots<A_r\leq<x>/H} (-1)^r = \frac{2 \cdot |H|}{4n} \sum_{d| \ |<x>/H|} \mu(d)$$

where $\mu(n)$ denotes the classical Möbius function. This expression is zero unless $H =< x >$ in which case it equals one.

Now consider the possibly non-trivial coefficients of the basis elements o$(< y >, \phi)^{Q_{4n}}$ and $(< xy >, \phi)^{Q_{4n}}$. If g has order four let ρ_g denote the character on $< g >$ given by $\rho_g(g) = \sqrt{-1}$. When n is odd we must evaluate the coefficients of $(< y >, \rho_y)^{Q_{4n}}$ and $(< y >, \overline{\rho}_y)^{Q_{4n}}$. These coefficients are both equal to one since $< y >$ is a maximal cyclic subgroup of Q_{4n} and, for example, there are $2n$ (H, ϕ)'s which are conjugate to $(< y >, \rho_y)$. When n is even the distinct $(H, \phi)^{Q_{4n}}$'s with $H =< x^iy >$ are $(< y >, \rho_y)^{Q_{4n}}$ and $(< xy >, \rho_{xy})^{Q_{4n}}$, each of which has coefficient equal to one.

Finally we must evaluate the coefficient of $(< y^2 >, \chi)^{Q_{4n}}$, which is given by

$$\sum_{(<y^2>,\chi)<(H_1,\phi_1)<\ldots<(H_r,\phi_r)} (-1)^r \frac{|H|}{4n} < c(\Psi), \phi_r >_{H_r} .$$

The $2n$ chains of length one of the form $(< y^2 >, \chi) < (< x^iy >, \phi_1)$ contribute -1 to this coefficient. The remaining terms contribute zero, as is seen by the argument used on $(H, \phi)^{Q_{4n}}$'s with $H \subseteq< x >$ together with the observation that, in this case, for the trivial chain, the multiplicity $< c(\Psi), \chi >_{<y^2>}= 2$.

The preceding discussion establishes the following result:

Proposition 2.6

In the notation of 2.5

$$a_{Q_{4n}}(c(\Psi)) = \begin{cases} (< x >, \phi_x)^{Q_{4n}} + (< y >, \rho_y)^{Q_{4n}} + (< y >, \overline{\rho}_y)^{Q_{4n}} - (< y^2 >, \chi)^{Q_{4n}} \\ \qquad \text{if } n \text{ is odd,} \\ (< x >, \phi_x)^{Q_{4n}} + (< y >, \rho_y)^{Q_{4n}} + (< xy >, \rho_{xy})^{Q_{4n}} - (< y^2 >, \chi)^{Q_{4n}} \\ \qquad \text{if } n \text{ is even} \end{cases}$$

where $\rho_x(x) = \xi_{2n}$.

Remark 2.7 The formula for $a_{Q_{4n}}(c(\Psi))$ is determined by the projective representation associated to $c(\Psi)$ [Sy] which is the same projection representation as the one associated to the dihedral representation

$$\nu : D_{2n} \longrightarrow GL_2\mathbf{C}$$

given by

$$\nu(x) = \begin{pmatrix} \xi_n & 0 \\ 0 & \overline{\xi}_n \end{pmatrix}, \quad \nu(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where

$$D_{2n} = \{x, y \mid x^n = 1 = y^2, yxy = x^{-1}\}.$$

This implies that $a_{D_{2n}}(\nu)$ is given by the formulae of Proposition 2.6.

3 Induction formula for symplectic representations

3.1 *Symplectic representations*

Let G be a finite group. Let $R^{sp}(G)$ denote the Grothendieck group of finite-dimensional $\mathbf{H}G$-modules, or in language of matrix representations, the Grothendieck group of equivalent classes of symplectic representations

$$\rho : G \longrightarrow \mathrm{Sp}(n) := \mathrm{Sp}(n, \mathbf{H}).$$

By an induction theorem of Martinet, every such symplectic representation ρ is a $\mathbf{Z}$-linear combination of representations induced from one-dimensionals; that is $\rho : G \to \mathrm{Sp}(n)$ can be written as

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G(\Psi_i)$$

with $H_i \leq G$, $\Psi_i : H_i \to \mathrm{Sp}(1)$ and $n_i \in \mathbf{Z}$. In order to make this formula explicit, we apply the machinery of Mackey functors as described in [Bo3]. Endowed with the usual conjugation, restriction and induction maps, $H \mapsto R^{sp}(H)$ is a $\mathbf{Z}$-Mackey functor on G. In $R^{sp}(H)$ (with $H \leq G$) we study the free abelian group $L^{sp}(H)$ generated by the classes of one-dimensional $\mathbf{H}H$-modules, that is by Sp(1)-conjugacy classes of homomorphisms

$$\Psi : H \longrightarrow \mathrm{Sp}(1) = S^3.$$

Let $H \mapsto R_+^{sp}(H)$ denote the Mackey functor obtained by the $_+$-construction on the subfunctor $H \mapsto L^{sp}(H)$. So $R^{sp}(G)$ is the free abelian group of $G-\mathrm{Sp}(1)$-conjugacy classes of elements in $L^{sp}(H)$ for $H \leq G$; these classes correspond to G-conjugacy classes of a one-dimensional symplectic representation (up to isomorphisms) of the subgroup, H, and will be denoted by $(H, \Psi)^G \in R_+^{sp}(G)$. For $J \leq G$ define homomorphisms

$$\mathrm{Res}_J^G : R_+^{sp}(G) \longrightarrow R_+^{sp}(J)$$

and

$$\mathrm{Ind}_J^G : R_+^{sp}(J) \longrightarrow R_+^{sp}(G)$$

in a manner which is analogous to the complex case (or given by the $_+$-construction). For $N \lhd G$ we have the inflation map

$$\mathrm{Infl}_{G/N}^G : R_+^{sp}(G/N) \longrightarrow R_+^{sp}(G)$$

defined by mapping $(HN/N, \overline{\Psi})^{G/N}$ to $(HN, \Psi)^G$. Let

$$b_G^{sp} : R_+^{sp}(G) \longrightarrow R^{sp}(G)$$

be the homomorphism defined by $b_G^{sp}((H, \Psi)^G) = \mathrm{Ind}_H^G(\Psi)$. Our aim is to define a map $\hat{a}_G^{sp} : R^{sp}(G) \to R_+^{sp}(G)$ which is a splitting of b_G^{sp}, that is $b_G^{sp} \circ \hat{a}_G^{sp} = \mathrm{id} : R^{sp}(G) \to R^{sp}(G)$, and behaves naturally with respect to restriction. But this will not be possible in this integral form, as we will see.

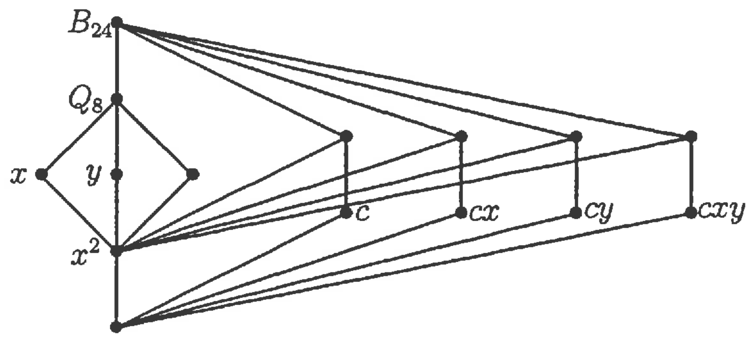
3.2 *One-dimensional symplectic representations*

Besides the cyclic and the quaternion type groups there are three more types of finite subgroups in the unit group Sp(1) of length 1 in $\mathbf{H}^\times$, namely the binary tetrahedral, octahedral and icosahedral groups. They arise from finite groups in $SO(3)$ given by rigid solids centered in the origin. These groups can be pulled back via $\pi : \mathrm{Sp}(1) \to SO(3)$, the map defined by letting Sp(1) act on the pure quaternion space (a 3-dimensional real space with standard inner product) via conjugation. The kernel of this map is $\{\pm 1\}$.

The binary tetrahedral group B_{24} is the preimage of the group of motions of a regular tetrahedron, which is isomorphic to the alternating group A_4. So B_{24} is an extension of a cyclic group of order 2 with A_4. In fact, B_{24} turns out to be $SL_2(3)$, and can be expressed as the semidirect product of the quaternion group Q_8 of order 8 and a cyclic group of order 3 acting faithfully on Q_8, so

$$B_{24} =< x, y, c | x^4 = 1, y^2 = x^2, c^3 = 1, x^y = x^{-1}, x^c = y, y^c = xy > \ .$$

($x \mapsto \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $c \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ defines an isomorphism $B_{24} \xrightarrow{\cong} SL_2(3)$.) The lattice of subgroups can be pictured as follows:



With this notation let Ψ denote the representation

$$\Psi : B_{24} \to \mathrm{Sp}(1) \subset \mathbf{H}^\times$$

defined by

$$x \mapsto i \ , \ y \mapsto j \ , \ c \mapsto -\frac{1}{2}(1 - i - j - k) \ .$$

This is, up to Sp(1)-conjugation, the unique faithful symplectic representation of B_{24}.

The binary octahedral group B_{48} is the preimage of the group of motions of a regular cube, which is isomorphic to the symmetric group S_4. So B_{48} is an extension of a cyclic group of order 2 with S_4. This group of order 48 appears as the nonsplit extension of the quaternion group of order 8 with the symmetric group S_3 acting as the full automorphism group. We can describe B_{48} as an extension of the binary tetrahedral group with a cyclic group $\langle d \rangle$ of order 2, acting as described in this group presentation:

$$B_{48} =< x, y, c, d | x^4 = c^3 = 1, d^2 = y^2 = x^2, x^y = x^3, x^c = y, y^c = xy, x^d = x^3 y, y^d = y^3, c^d = c^2 > \ .$$

The maximal subgroups of B_{48} are the normal subgroup B_{24} of index 2 (the binary tetrahedral group), three conjugate groups of quaternion type Q_{16} of order 16 and four conjugate groups of quaternion type Q_{12} of order 12.
With this notation let Ψ denote the (up to Sp(1)-conjugation) unique faithful symplectic representation

$$\Psi : B_{48} \to \mathrm{Sp}(1) \subset \mathbf{H}^{\times}$$

given by

$$x \mapsto i \ , \ y \mapsto j \ , \ c \mapsto -\frac{1}{2}(1-i-j-k) \ , \ d \mapsto \frac{\sqrt{2}}{2}(i-k) \ .$$

The binary icosahedral group B_{120} is the preimage of the group of motions of a regular icosahedron, which is isomorphic to the alternation group A_5. In fact, B_{120} is isomorphic to $SL_2(5)$, and can be described as [CCNW]

$$B_{120} = < r, s, t | r^2 = s^3 = t^5 = rst > \quad .$$

B_{120} has order 120. The maximal subgroups of B_{120} are six conjugate groups of quaternion type Q_{20} of order 20, five conjugate groups of type binary tetrahedral group of order 24 and ten conjugate groups of quaternion type Q_{12} of order 12.

There are actually two types of faithful symplectic representations

$$\Psi : B_{120} \to \mathrm{Sp}(1) \subset \mathbf{H}^{\times}$$

given by

$$\begin{aligned} r &\mapsto i \\ s &\mapsto \frac{1}{2}(1-(\zeta_5+\zeta_5^4)i-(\zeta_5^2+\zeta_5^3)j) \\ t &\mapsto \frac{1}{2}(-(\zeta_5+\zeta_5^4)+i-(\zeta_5^2+\zeta_5^3)k) \ . \end{aligned}$$

depending on the choice of ζ_5 (e.g. $\zeta_5 = \exp(\frac{2\pi i}{5})$ or $\zeta_5 = \exp(2\frac{2\pi i}{5})$).

3.3 *Complexification*

There is an obvious homomorphism

$$c = c_G : R^{sp}(G) \longrightarrow R(G)$$

given by complexifying; so $\Psi : G \to \mathrm{Sp}(1)$ in $R^{sp}(G)$ maps to

$$(c(\Psi) : G \longrightarrow \mathrm{Sp}(1) \longrightarrow \mathrm{U}(2)) \in R(G) \, .$$

Of course this homomorphism c is injective. Define the homomorphism

$$c_+ = c_{+,G} : R_+^{sp}(G) \longrightarrow R_+(G)$$

by the formula

$$c_{+,G}((H,\Psi)^G) = \mathrm{Ind}_H^G(a_H(c(\Psi))) \in R_+(G).$$

The homomorphism was first defined in ([SnEBI] §5.4.40 p.213). There is an a_H missing in the formula of ([SnEBI] §5.4.41) but not in the proof of the following result, which is part of the proof of Theorem 5.4.42 of [SnEBI]. Obviously b_G and b_G^{sp} are naturally connected via complexification, so that $b_G \circ c_+ = c \circ b_G^{sp} : R_+^{sp}(G) \to R(G)$.

Proposition 3.4

The homomorphism $c_{+,G}$ is natural with respect to restriction so that, if $J \leq G$,

$$\mathrm{Res}_J^G \circ c_{+,G} = c_{+,J} \circ \mathrm{Res}_J^G : R_+^{sp}(G) \longrightarrow R_+(J).$$

Proof

We recall that there is a double coset formula for the composition

$$\mathrm{Res}_J^G \mathrm{Ind}_H^G : R_+(H) \longrightarrow R_+(J)$$

if $H, J \leq G$. Explicitly we have

$$\mathrm{Res}_J^G(\mathrm{Ind}_H^G((K,\Psi)^H)) = \sum_{w \in J\backslash G/H} \mathrm{Ind}_{J\cap wHw^{-1}}^J(w^*(\mathrm{Res}_{H\cap w^{-1}Jw}^H((K,\Psi)^H)))$$

whose proof is similar to the proof of the product formula of ([SnEBI] Exercise 2.5.7).

Hence, if $(H,\Psi)^G \in R_+^{sp}(G)$ with $\Psi : H \longrightarrow \mathrm{Sp}(1)$, then

$$\begin{aligned}
&\mathrm{Res}_J^G(c_{+,G}((H,\Psi)^G)) \\
&= \mathrm{Res}_J^G(\mathrm{Ind}_H^G(a_H(c(\Psi) : H \longrightarrow U(2)))) \\
&= \textstyle\sum_{w\in J\backslash G/H} \mathrm{Ind}_{J\cap wHw^{-1}}^J(w^*(\mathrm{Res}_{H\cap w^{-1}Jw}^H(a_H(c(\Psi))))) \\
&= \textstyle\sum_{w\in J\backslash G/H} \mathrm{Ind}_{J\cap wHw^{-1}}^J(w^*(a_{J\cap wHw^{-1}}(c(\mathrm{Res}_{H\cap w^{-1}Jw}^H(\Psi))))) \\
&= \textstyle\sum_{w\in J\backslash G/H} \mathrm{Ind}_{J\cap wHw^{-1}}^J(a_{J\cap wHw^{-1}}(c(\mathrm{Res}_{J\cap wHw^{-1}}^{wHw^{-1}}((w^{-1})^*(\Psi))))) \\
&= c_{+,J}(\mathrm{Res}_J^G((H,\Psi)^G)),
\end{aligned}$$

which completes the proof. ■

Proposition 3.5

Suppose that

$$x = \sum_i n_i(H_i,\psi_i)^G \in Ker(c_{+,G})$$

and that each image, $\psi_i(H_i)$, is abelian. Then $x = 0$.
In particular, for G abelian $c_{+,G}$ is injective.

Proof

Amongst the H_i's which appear in x, choose an H which is maximal in the poset of conjugacy classes of subgroups of G. Then we may write

$$x = \sum_i n_i(H, \psi_i)^G + \sum_j n_j(H_j, \psi_j)^G$$

where the $(H,\psi_i)^G$'s are all distinct and where none of the H_j in the second sum satisfies $H_j \geq gHg^{-1}$ for any $g \in G$. Then, if $c(\psi_i) = \phi_i \oplus \overline{\phi}_i$,

$$0 = c_{+,G}(x) = \sum_i n_i((H,\phi_i)^G + (H,\overline{\phi}_i)^G) + \sum_j n_j \mathrm{Ind}_{H_j}^G(a_{H_j}(c(\psi_j))) \in R_+(G)$$

and it is clear that no term from the second sum can cancel any term from the first sum, so that

$$0 = \sum_i n_i((H,\phi_i)^G + (H,\overline{\phi}_i)^G) \in R_+(G).$$

This can only happen if, for distinct i_0, i_1, $(H,\phi_{i_0})^G = (H,\phi_{i_1})^G$ or $(H,\phi_{i_0})^G = (H,\overline{\phi}_{i_1})^G$. In turn, this can only happen if there exists $g \in N_G H$ such that $\phi_{i_0} = g^*(\phi_{i_1})$ or $\phi_{i_0} = g^*(\overline{\phi}_{i_1})$. Both these relations imply that $\psi_{i_0} = g^*(\psi_{i_1})$ and so $(H,\psi_{i_0})^G = (H,\psi_{i_1})^G$, which is a contradiction. ■

3.6 *Example*

We calculate explicitly the complexification of the faithful irreducible representations Ψ of the binary tetrahedral, octahedral and icosahedral groups as described in (3.2).

With the notation of (3.2), for the binary tetrahedral group B_{24},

$$\begin{aligned} a_{B_{24}}(c(\Psi)) &= \frac{1}{6}(\langle x\rangle,\zeta_4)^{B_{24}} + \frac{1}{6}(\langle x\rangle,\overline{\zeta_4})^{B_{24}} + \frac{1}{6}(\langle y\rangle,\zeta_4)^{B_{24}} + \frac{1}{6}(\langle y\rangle,\overline{\zeta_4})^{B_{24}} + \\ &\quad \frac{1}{6}(\langle xy\rangle,\zeta_4)^{B_{24}} + \frac{1}{6}(\langle xy\rangle,\overline{\zeta_4})^{B_{24}} + \frac{1}{4}(\langle -c\rangle,\zeta_6)^{B_{24}} + \frac{1}{4}(\langle -c\rangle,\overline{\zeta_6})^{B_{24}} + \\ &\quad \frac{1}{4}(\langle -cx\rangle,\zeta_6)^{B_{24}} + \frac{1}{4}(\langle -cx\rangle,\overline{\zeta_6})^{B_{24}} + \frac{1}{4}(\langle -cy\rangle,\zeta_6)^{B_{24}} + \frac{1}{4}(\langle -cy\rangle,\overline{\zeta_6})^{B_{24}} + \\ &\quad \frac{1}{4}(\langle -cxy\rangle,\zeta_6)^{B_{24}} + \frac{1}{4}(\langle -cxy\rangle,\overline{\zeta_6})^{B_{24}} + \frac{1}{12}(2-(3\cdot 2+4\cdot 2))(\langle x^2\rangle,\varepsilon)^{B_{24}} \\ &= (C_4,\zeta_4)^{B_{24}} + (C_6,\zeta_6)^{B_{24}} + (C_6,\overline{\zeta_6})^{B_{24}} - (C_2,\varepsilon)^{B_{24}} \end{aligned}$$

Here C_4 denotes one of the cyclic groups of order 4 (e.g. $C_4 = \langle x\rangle$) and ζ_4 one of the faithful unitary representation on C_4 sending a generator to a primitive fourth root of unity, C_6 denotes one of the cyclic groups of order 6 (e.g. $C_6 = \langle x^2c\rangle$) and ζ_6 one of the faithful unitary representation on C_6 sending a generator to a primitive sixth root of unity, and $C_2 = \langle x^2\rangle$ denotes the unique cyclic subgroup of order 2, with representation ε defined by $\varepsilon(x^2) = -1$.

For the binary octahedral group B_{48}

$$a_{B_{48}}(c(\Psi)) = (C_8,\zeta_8)^{B_{48}} + (C_6,\zeta_6)^{B_{48}} + (C_4,\zeta_4)^{B_{48}} - (C_2,\varepsilon)^{B_{48}}$$

Here C_8 denotes one of the three cyclic groups of order 8 (e.g. $C_8 = \langle xd \rangle$) and ζ_8 one of the faithful unitary representation on C_8 sending a generator to a primitive eighth root of unity, C_6 denotes one of the four cyclic groups of order 6 (e.g. $C_6 = \langle x^2 c \rangle$) and ζ_6 one of the faithful unitary representation on C_6 sending a generator to a primitive sixth root of unity, C_4 denotes one of the six cyclic groups of order 4which are not in N (e.g. $C_4 = \langle d \rangle$) and ζ_4 one of the faithful unitary representation on C_4 sending a generator to a primitive fourth root of unity, and $C_2 = \langle x^2 \rangle$ the unique cyclic subgroup of order 2 with representation ε defined by $\varepsilon(x^2) = -1$.

For the two symplectic representations on the binary icosahedral B_{120} we derive

$$a_{B_{120}}(c(\Psi)) \quad = \quad (C_{10}, \zeta_{10})^{B_{120}} + (C_6, \zeta_6)^{B_{120}} + (C_4, \zeta_4)^{B_{120}} - (C_2, \varepsilon)^{B_{120}}.$$

Here C_{10} denotes one of the six cyclic groups of order 10 (e.g. $C_{10} = \langle t \rangle$) and ζ_{10} one of the faithful unitary representation on C_{10} sending a generator to a primitive tenth root of unity such that the square is the fifth root of unity choosen to define Ψ, C_6 denotes one of the ten cyclic groups of order 6 (e.g. $C_6 = \langle s \rangle$) and ζ_6 one of the faithful unitary representation on C_6 sending a generator to a primitive sixth root of unity, C_4 denotes one of the fifteen cyclic groups of order 4 (e.g. $C_4 = \langle r \rangle$) and ζ_4 one of the faithful unitary representation on C_4 sending a generator to a primitive fourth root of unity, and $C_2 = \langle rst \rangle$ the unique cyclic subgroup of order 2 with representation ε defined by $\varepsilon(rst) = -1$.

3.7 *Some elements in $Ker(c_{+,G})$*

Type τ:

If $H \subseteq G$ and $\psi : H \longrightarrow Sp(1)$ has non-abelian image which is isomorphic to a generalised quaternion group, Q_{4n} for some n. Then, by the formula of (2.2),

$$a_H(c(\psi)) = \sum_\alpha m_\alpha (H_\alpha, \phi_\alpha)^H = \sum_\alpha m_\alpha (H_\alpha, \overline{\phi}_\alpha)^H \in R_+(H)$$

in which each image, $\phi_\alpha(H_\alpha)$, is abelian. Also there exists $\psi_\alpha : H_\alpha \longrightarrow Sp(1)$, unique up to $H - Sp(1)$-conjugation, such that $c(\psi_\alpha) = \phi_\alpha \oplus \overline{\phi}_\alpha$. Then

$$\begin{aligned} & c_{+,G}(2(H,\psi)^G - \textstyle\sum_\alpha m_\alpha (H_\alpha, \psi_\alpha)^G) \\ & = 2 \textstyle\sum_\alpha m_\alpha (H_\alpha, \phi_\alpha)^G - \textstyle\sum_\alpha m_\alpha ((H_\alpha, \phi_\alpha)^G + (H_\alpha, \overline{\phi}_\alpha)^G) \\ & = 0 \in R_+(G). \end{aligned}$$

Denote by $\tau_G(H, \psi)$ the element

$$\tau_G(H,\psi) = 2(H,\psi)^G - \sum_\alpha (H_\alpha, \psi_\alpha)^G \in Ker(c_{+,G}) \subseteq R_+^{sp}(G).$$

Notice that each of the images, $\psi_\alpha(H_\alpha)$ is abelian.

Type β:

Let B_{24}, B_{48}, B_{120} denote the binary tetrahedral, octahedral and icosahedral groups, respectively. Let $\Psi_n : B_n \longrightarrow Sp(1)$ denote the faithful representation described in 3.2.

From 3.6, we have an inclusion $Q_8 \subset B_{24}$ under which all C_4's are conjugate. Therefore

$$\mathrm{Ind}_{Q_8}^{B_{24}}(c_{+,Q_8}(Q_8, \Psi)^{Q_8}) = 3(C_4, \zeta_4)^{B_{24}} - (C_2, \epsilon)^{B_{24}}.$$

If $\psi_n : C_n \longrightarrow Sp(1)$ satisfies $c(\psi_n) = \zeta_n \oplus \overline{\zeta_n}$ then, from 3.6,

$$c_{+,B_{24}}((B_{24}, \Psi_{24})^{B_{24}}) + c_{+,B_{24}}((C_4, \psi_4)^{B_{24}}) - c_{+,B_{24}}((C_6, \psi_6)^{B_{24}}) = \mathrm{Ind}_{Q_8}^{B_{24}}(c_{+,Q_8}(Q_8, \Psi)^{Q_8})$$

and therefore

$$\beta_{24} = (B_{24}, \Psi_{24})^{B_{24}} + (C_4, \psi_4)^{B_{24}} - (C_6, \psi_6)^{B_{24}} - (Q_8, \Psi)^{B_{24}} \in Ker(c_{+,B_{24}}).$$

Now consider $B_{48} =< x, y, c, d \mid \ldots >$ as in 3.2. This case is a little more delicate because there are two conjugacy classes of C_4, namely $C_4 =< d >$ and $C_4' =< y > \subset N =< x, y, c > \lhd B_{48}$. We have

$$c_{+,B_{48}}((B_{48}, \Psi_{48})^{B_{48}}) = (C_8, \zeta_8)^{B_{48}} + (C_6, \zeta_6)^{B_{48}} + (C_4, \zeta_4)^{B_{48}} - (C_2, \epsilon)^{B_{48}}.$$

Also $Q_{16} =< xd, d >$ and $C_8 =< xd >$ so that

$$c_{+,B_{48}}((Q_{16}, \Psi_{16})^{B_{48}}) = (C_8, \zeta_8)^{B_{48}} + (C_4, \zeta_4)^{B_{48}} + (C_4', \zeta_4)^{B_{48}} - (C_2, \epsilon)^{B_{48}}.$$

Furthermore $Q_{12} =< c, d >$ so that

$$c_{+,B_{48}}((Q_{12}, \Psi_{12})^{B_{48}}) = (C_6, \zeta_6)^{B_{48}} + 2(C_4, \zeta_4)^{B_{48}} - (C_2, \epsilon)^{B_{48}}.$$

Also we have $Q_8' =< x, y >$ and $Q_8 =< d, y >$ so that

$$c_{+,B_{48}}((Q_8', \Psi_8)^{B_{48}}) = 3(C_4', \zeta_4)^{B_{48}} - (C_2, \epsilon)^{B_{48}}$$

and

$$c_{+,B_{48}}((Q_8, \Psi_8)^{B_{48}}) = 2(C_4, \zeta_4)^{B_{48}} + (C_4', \zeta_4)^{B_{48}} - (C_2, \epsilon)^{B_{48}}.$$

Therefore

$$\beta_{48} = (B_{48}, \Psi_{48})^{B_{48}} - (Q_{16}, \Psi_{16})^{B_{48}} - (Q_{12}, \Psi_{12})^{B_{48}} - (Q_8', \Psi_8)^{B_{48}} \in Ker(c_{+,B_{48}}).$$

Next consider $B_{120} =< r, s, t \mid r^2 = r^3 = t^5 = rst >$. The relations

$$c_{+,B_{120}}((B_{120}, \Psi_{120})^{B_{120}}) = (C_{10}, \zeta_{10})^{B_{120}} + (C_6, \zeta_6)^{B_{120}} + (C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}},$$

$$c_{+,B_{120}}((Q_{20} =< r, t >, \Psi_{20})^{B_{120}}) = (C_{10}, \zeta_{10})^{B_{120}} + 2(C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}},$$

$$c_{+,B_{120}}((Q_{12} =< r, s >, \Psi_{12})^{B_{120}}) = (C_6, \zeta_6)^{B_{120}} + 2(C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}},$$

$$c_{+,B_{120}}((Q_8, \Psi_8)^{B_{120}}) = 3(C_4, \zeta_4)^{B_{120}} - (C_2, \epsilon)^{B_{120}}$$

imply that

$$\beta_{120} = (B_{120}, \Psi_{120})^{B_{120}} - (Q_{20}, \Psi_{20})^{B_{120}} - (Q_{12}, \Psi_{12})^{B_{120}} + (Q_8, \Psi_8)^{B_{120}} \in Ker(c_{+,B_{120}}).$$

The elements of $Ker(c_{+,G})$ of Type β are defined to be those of the form $\mathrm{Ind}_H^G(\pi^*(\beta_n)$ where $\pi : H \longrightarrow B_n$ is a surjective homomorphism and $n = 24, 48$ or 120.

Type σ:

Elements, $\Sigma \in Ker(c_{+,G})$, of Type σ are defined to be those which satisfy a relation of the form

$$2\Sigma = \sum_\alpha \tau_G(H_\alpha, \psi_\alpha).$$

Here are two examples of elements of Type σ.

i) Let

$$Q_{4n} =< x, y \,|x^n = y^2, yxy^{-1} = x^{-1}, y^4 = 1 >$$

denote the generalised quaternion group of order $4n$ and let z have order two. Set $G = Q_{4n} \times < z >$ so that G contains four copies of Q_{4n} given by $Q_1 =< x, y >$, $Q_2 =< xz, y >$, $Q_3 =< x, yz >$ and $Q_4 =< xz, yz >$. Each of these subgroups has a homomorphism, $\Psi : Q_v \longrightarrow \mathrm{Sp}(1)$, sending the generator xz^s to $e^{\pi i/n}$ and yz^t to j for appropriate s, t. Setting

$$\begin{aligned}
\Sigma_n &= (Q_1, \Psi)^G + (Q_2, \Psi)^G + (Q_3, \Psi)^G + (Q_4, \Psi)^G \\
&+2(< y^2 >, \psi_2)^G - (< x >, \psi_{2n})^G - (< y >, \psi_4)^G - (< xy >, \psi_4)^G \\
&-(< xz >, \psi_{2n})^G - (< yz >, \psi_4)^G - (< xyz >, \psi_4)^G
\end{aligned}$$

we find that $c_+(\Sigma_n) = 0$ and

$$2\Sigma_n = \tau_G(Q_1, \Psi) + \tau_G(Q_2, \Psi) + \tau_G(Q_3, \Psi) + \tau_G(Q_4, \Psi).$$

ii) If z, w have order two set $G = C_2 \ltimes (Q_8 \times < z >) \times < w >$ where $Q_8 =< x, y >$ and the generator, λ, of the left-hand C_2 acts by $\lambda(x) = xz, \lambda(y) = yz$. Then

$$\begin{aligned}
\Sigma' &= (< x, y >, \Psi)^G + (< xz, y >, \Psi)^G + (< y^2 >, \psi_2)^G - (< x >, \psi_4)^G \\
&-(< y >, \psi_4)^G - (< xy >, \psi_4)^G - (< xyz >, \psi_4)^G + (< xw, yw >, \Psi)^G \\
&+(< xwz, yw >, \Psi)^G + (< y^2 >, \psi_2)^G - (< xw >, \psi_4)^G - (< yw >, \psi_4)^G
\end{aligned}$$

satisfies $c_+(\Sigma') = 0$ and

$$\begin{aligned}
2\Sigma' &= \tau_G(< x, y >, \Psi) + \tau_G(< xz, y >, \Psi) \\
&+\tau_G(< xw, yw >, \Psi) + \tau_G(< xzw, yw >, \Psi).
\end{aligned}$$

Proposition 3.8

In the notation of 3.7, $Ker(c_{+,G})$ is generated by elements of Types τ, β and σ.

Proof

Suppose that

$$x = \sum_i n_i (H_i, \psi_i)^G \in Ker(c_{+,G}) : RSp_+(G) \longrightarrow R_+(G)).$$

For all the terms with $n_i \neq 0$ we may subtract a $\mathbf{Z}$-linear combination of the $\tau_G(H, \psi)$'s and $\mathrm{Ind}_H^G(\pi^*(\beta_n))$'s to ensure that either $\psi_i(H_i) \subset Sp(1)$ is abelian or $n_i = 1$ and $\psi_i(H_i)$ is isomorphic to a generalised quaternion group. Under these circumstances write

$$x = \sum_{\psi_i(H_i) \ non-abelian} (H_i, \psi_i)^G + \sum_{\psi_j(H_j) \ abelian} (H_i, \psi_i)^G$$

then

$$2x - \sum_{\psi_i(H_i) \ non-abelian} \tau_G(H_i, \psi_i) = \sum_m a_m (H_m, \psi_m)^G \in Ker(c_{+,G})$$

with every image, $\psi_m(H_m)$ abelian in the right-hand sum. Hence

$$2x - \sum_{\psi_i(H_i) \ non-abelian} \tau_G(H_i, \psi_i) = 0,$$

by Lemma 3.5, and x is of Type σ. ■

3.9 *A symplectic induction formula*

Let $\mathbf{Q}R^{sp}(G) := R^{sp}(G) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q}R_+^{sp}(G) := R_+^{sp}(G) \otimes_{\mathbf{Z}} \mathbf{Q}$. All homomorphims on $R^{sp}(G)$ and $R_+^{sp}(G)$, especially Res_J^G and Ind_J^G, extend in a natural way to homomorphisms between these $\mathbf{Q}$-vectorspaces. Define the map

$$a_G^{sp} : \mathbf{Q}R^{sp}(G) \to \mathbf{Q}R_+^{sp}(G)$$

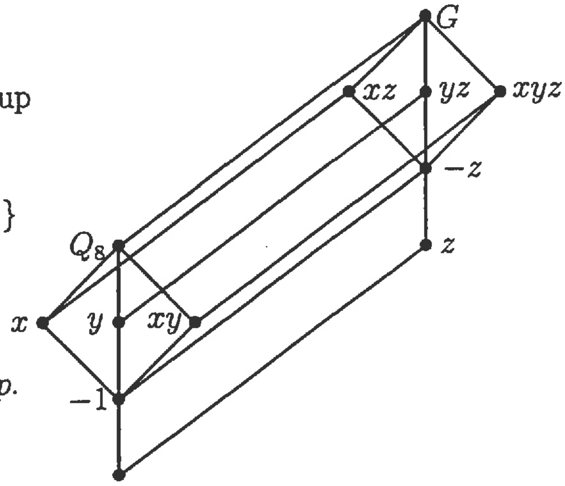
by the formula

$$a_G^{sp}(\rho) = \sum_{(H_0, \Psi_0) < .. < (H_r, \Psi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}_{H_r}^G(\rho), \Psi_r)(H_0, \Psi_0)^G$$

with $m(\theta, \Psi) = \frac{\langle c(\theta); c(\Psi) \rangle_H}{\langle c(\Psi); c(\Psi) \rangle_H}$ for $\theta, \Psi \in R^{sp}(H)$, $H \leq G$.
This is a homomorphism since m is linear in the first argument as both, the restriction map and the scalar product on characters, are linear.

3.10 *Example:* $G = C_p \times Q_8$

Let p be an odd prime and let G denote the group $C_p \times Q_8 =< x, y, z | x^4 = y^4 = z^p = 1,$ $zx = xz, zy = yz, x^2 = y^2, yx = x^3y >$. So $G = \{x^i y^j z^k | 0 \leq i \leq 3, 0 \leq j \leq 1, 0 \leq k \leq p-1\}$ is a group of order $8p$.
To shorten notation set $-1 := x^2$.
The lattice of subgroups of G is as pictured.



Let ζ denote a primitive root of unity of order p.

The character table of G is given by

	1	-1	$\pm x$	$\pm y$	$\pm xy$	z	$-z$	$\pm xz$	$\pm yz$	$\pm xyz$	..	$\pm xyz^{p-1}$
$\mathbb{1}$	1	1	1	1	1	1	1	1	1	1	..	1
ε_x	1	1	1	-1	-1	1	1	1	-1	-1	..	-1
ε_y	1	1	-1	1	-1	1	1	-1	1	-1	..	-1
ε_{xy}	1	1	-1	-1	1	1	1	-1	-1	1	..	1
$\zeta\varepsilon_x$	1	1	1	-1	-1	ζ	ζ	ζ	$-\zeta$	$-\zeta$	..	$-\zeta^{-1}$
$\vdots$	$\vdots$										$\vdots$	
$\zeta^{p-1}\varepsilon_{xy}$	1	1	-1	-1	1	ζ^{-1}	ζ^{-1}	$-\zeta^{-1}$	$-\zeta^{-1}$	ζ^{-1}	..	ζ
χ_0	2	-2	0	0	0	2	-2	0	0	0	..	0
χ_1	2	-2	0	0	0	2ζ	-2ζ	0	0	0	..	0
χ_2	2	-2	0	0	0	$2\zeta^2$	$-2\zeta^2$	0	0	0	..	0
$\vdots$	$\vdots$										$\vdots$	
χ_{p-1}	2	-2	0	0	0	$2\zeta^{-1}$	$-2\zeta^{-1}$	0	0	0	..	0

We calculate the explicit symplectic induction formula for the symplectic irreducible character $\chi_1 + \chi_{p-1}$. In fact, this is the complexified character of the symplectic representation given by $\rho : G \to \mathrm{Sp}(2)$ sending x to $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, y to $\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$ and z to $\begin{pmatrix} \frac{\zeta+\zeta^{-1}}{2} & i\frac{\zeta-\zeta^{-1}}{2} \\ -i\frac{\zeta-\zeta^{-1}}{2} & \frac{\zeta+\zeta^{-1}}{2} \end{pmatrix}$. Furthermore ρ is irreducible because χ_1 is not real-valued and hence does not arrise from a symplectic representation.

To calculate the formula one has to study the restriction of ρ on the subgroups H of G. In the following table we give all the nonzero multiplicities $m = m(\mathrm{Res}^G_H(\rho), \Psi)$:

$H, c(\Psi)$	$\langle xz\rangle, i\zeta + \overline{i\zeta}$	$\langle xz\rangle, -i\zeta + \overline{-i\zeta}$	$\langle -z\rangle, -\zeta + \overline{-\zeta}$	$\langle z\rangle, \zeta + \overline{\zeta}$	Q_8, χ	$\langle x\rangle, i + \overline{i}$	$\langle -1\rangle, 2\varepsilon$	$1, 2\mathbb{1}$
m	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{4}{2}$	$\frac{4}{2}$	$\frac{2}{1}$	$\frac{4}{2}$	$\frac{8}{4}$	$\frac{8}{4}$

Here $\pm i$ (resp. $\pm\zeta$) denotes a complex linear representation sending the generator to $\pm i$ (resp. $\pm\zeta$), χ the faithful irreducible representation on Q_8 and ε the faithful linear representation on a group of order 2. In the list we have taken $\langle xz\rangle$ resp. $\langle x\rangle$ as representatives for the groups of order $4p$ resp. 4, as the other groups behave in the same way. Observe furthermore that the represention $i\zeta{+}\overline{i\zeta}$ and $\bar{i}\zeta{+}\overline{i\zeta}$ on $\langle xz\rangle$ are conjugate by G. Now

$$\begin{aligned}
a_G^{sp}(\rho) \;=\;& \frac{1}{2}(\frac{2}{2})((\langle xz\rangle, i\zeta{+}\overline{i\zeta})^G + \frac{1}{2}(\frac{2}{2})((\langle xz\rangle, \text{-}i\zeta{+}\overline{\text{-}i\zeta})^G + \frac{1}{2}(\frac{2}{2})((\langle yz\rangle, i\zeta{+}\overline{i\zeta})^G + \\
& \frac{1}{2}((\langle yz\rangle, \text{-}i\zeta{+}\overline{\text{-}i\zeta})^G\frac{2}{2} + \frac{1}{2}(\frac{2}{2})((\langle xyz\rangle, i\zeta{+}\overline{i\zeta})^G + \frac{1}{2}(\frac{2}{2})((\langle xyz\rangle, \text{-}i\zeta{+}\overline{\text{-}i\zeta})^G + \\
& \frac{1}{4}(\frac{4}{2} - (3\frac{2}{2} + 3\frac{2}{2}))((\langle\text{-}z\rangle, \text{-}\zeta{+}\overline{\text{-}\zeta})^G + \frac{1}{8}(\frac{4}{2} - (3\frac{2}{2} + 3\frac{2}{2} + \frac{4}{2}) + (3\frac{2}{2} + 3\frac{2}{2}))((\langle z\rangle, \zeta{+}\overline{\zeta})^G + \\
& \frac{1}{p}(\frac{2}{1})(Q_8, \chi)^G + \frac{1}{2p}(\frac{4}{2} - (\frac{2}{2} + \frac{2}{2} + \frac{2}{1}))((\langle x\rangle, i{+}\bar{i})^G + \\
& \frac{1}{2p}(\frac{4}{2} - (\frac{2}{2} + \frac{2}{2} + \frac{2}{1}))((\langle y\rangle, i{+}\bar{i})^G + \frac{1}{2p}(\frac{4}{2} - (\frac{2}{2} + \frac{2}{2} + \frac{2}{1}))((\langle xy\rangle, i{+}\bar{i})^G + \\
& \frac{1}{4p}(\frac{8}{4} - (3\frac{2}{2} + 3\frac{2}{2} + \frac{4}{2} + \frac{2}{1} + 3\frac{4}{2}) + (3\cdot 2\frac{2}{2} + 3\cdot 2\frac{2}{2} + 3\frac{2}{1}))((\langle\text{-}1\rangle, 2\varepsilon)^G + \\
& \frac{1}{8p}(\frac{8}{4} - (3\frac{2}{2} + 3\frac{2}{2} + \frac{4}{2} + \frac{4}{2} + \frac{2}{1} + 3\frac{4}{2} + \frac{8}{4}) + (3{\cdot}4\frac{2}{2} + 3{\cdot}4\frac{2}{2} + 2\frac{4}{2} + 4\frac{2}{1} + 3\frac{4}{2}) + \\
& -(3^2\frac{2}{2} + 3^2\frac{2}{2} + 3\frac{2}{1}))((\langle 1\rangle, 2\mathbb{1})^G \\
=\;& ((\langle xz\rangle, i\zeta{+}\overline{i\zeta})^G + ((\langle yz\rangle, i\zeta{+}\overline{i\zeta})^G + ((\langle xyz\rangle, i\zeta{+}\overline{i\zeta})^G - ((\langle\text{-}z\rangle, \text{-}\zeta{+}\overline{\text{-}\zeta})^G + \\
& \frac{2}{p}(Q_8, \chi)^G - \frac{1}{p}((\langle x\rangle, i{+}\bar{i})^G - \frac{1}{p}((\langle y\rangle, i{+}\bar{i})^G - \frac{1}{p}((\langle xy\rangle, i{+}\bar{i})^G + \frac{1}{p}((\langle\text{-}1\rangle, 2\varepsilon)^G
\end{aligned}$$

As this example shows, the symplectic induction formula may have non-integral coefficients.

3.11 *Example:* $G = C_{2^n} \times Q_8$

Let $n \in \mathbb{N}$, $Z =< z >$ the cyclic group of order 2^n and $Q_8 =< x, y >$ the quaternion group of order 8, as before. Let $G_n = Z \times Q_8$ be the direct product of these groups and $\rho_n : G \to \mathrm{Sp}(2)$ the symplectic representation given by sending x to $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, y to $\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$ and z to $\begin{pmatrix} \frac{\zeta+\zeta^{-1}}{2} & i\frac{\zeta-\zeta^{-1}}{2} \\ -i\frac{\zeta-\zeta^{-1}}{2} & \frac{\zeta+\zeta^{-1}}{2} \end{pmatrix}$, ζ a primitive 2^n-th root of unity. This is an irreducible representation for $n \geq 2$, and for $n = 1$ it is twice the one-dimensional symplectic representation Ψ defined by $x \mapsto i$, $y \mapsto j$ and $z \mapsto -1$.
Now let $n \geq 2$ and $G = G_n$, and let H_t be the subgroup generated by x,y and $z^{2^{n-t}}$, $0 \leq t \leq n$. Notice that $H_1 \cong G_1$ is normal in G_n with cyclic factor group, all the intermediate groups are the groups H_t ($1 \leq t \leq n-1$), being isomorphic to G_t. Thus $\mathrm{Res}^{G_n}_{H_t}(\rho)$ stays irreducible for $t \geq 2$, while $\mathrm{Res}^{G}_{H_1}(\rho) = 2\Psi$. Hence the coefficient of $(H_1, \Psi)^G$ in $a_G^{sp}(\rho)$ is $\frac{8\cdot 2}{8\cdot 2^n} \cdot \frac{2}{1} = \frac{1}{2^{n-2}}$.
As this example shows, the denominators of the coefficients in the symplectic induction formula may contain arbitrarily large 2-powers.

Proposition 3.12

The homomorphisms a_G^{sp} *is natural with respect to restriction so that, if* $J \leq G$,

$$a_J^{sp} \circ \mathrm{Res}_J^G = \mathrm{Res}_J^G \circ a_G^{sp} : \mathbf{Q}R^{sp}(G) \to \mathbf{Q}R_+^{sp}(J)\,.$$

Proof

This is a consequence of an adjointness property of certain functors proved in [Bo3, Prop.1.4.1(ii)], applied to $p_H \quad \mathbf{Q}R^{sp}(H) \to \mathbf{Q}L^{sp}(H)$ sending $\rho : H \to \mathrm{Sp}(n)$ to $\sum_{\Psi:H\to \mathrm{Sp}(1)} m(\rho,\Psi)\Psi$. In fact $(p_H)_{H\leq G}$ is natural with respect to conjugation, and hence yields a restriction functor which composed with the Mackey functor given in [Bo3, 1.3.2] turns out to be the homomorphism sending $\rho : G \to \mathrm{Sp}(n)$ to

$$\begin{aligned}
\rho &\mapsto \sum_{K\leq H\leq G} \frac{|K|}{|G|}\mu(K:H)(K, \mathrm{Res}_K^H(\sum_{\Psi:H\to \mathrm{Sp}(1)} m(\mathrm{Res}_H^G(\rho),\Psi)\Psi))^G \\
&= \sum_{K\leq H\leq G} \frac{|K|}{|G|}\mu(K:H)(K, \sum_{\Psi:H\to \mathrm{Sp}(1)} m(\mathrm{Res}_H^G(\rho),\Psi)\mathrm{Res}_K^H(\Psi))^G \\
&= \sum_{K\leq H\leq G} \frac{|K|}{|G|}\mu(K:H) \sum_{\Psi:H\to \mathrm{Sp}(1)} m(\mathrm{Res}_H^G(\rho),\Psi)(K, \mathrm{Res}_K^H(\Psi))^G \\
&= \sum_{(H_0,\Psi_0)\leq(H,\Psi)\leq G} \frac{|H_0|}{|G|}\mu(H_0:H)m(\mathrm{Res}_H^G(\rho),\Psi)(H_0,\Psi_0)^G \\
&= a_G^{sp}(\rho)\,.
\end{aligned}$$

Thus a_G is a restriction functor in the sense of [Bo3], and in particular it is natural with respect to restriction. ∎

Proposition 3.13

Let $\rho : G \to \mathrm{Sp}(1)$ *then*

$$a_G^{sp}(\rho) = (G,\rho)^G\,.$$

Proof

Since (G,ρ) is the only element in $(G,\rho)^G$ and $m(\rho,\rho) = 1$, the coefficient of $(G,\rho)^G$ in $a_G^{sp}(\rho)$ is 1. Now let $(H,\Psi) < (G,\rho)$. Only those elements may give other nontrivial contributions to $a_G^{sp}(\rho)$. Since $\mathrm{Res}_{H_r}^G(\rho) = \Psi_r$ for $(H_r,\Psi_r) < (G,\rho)$, the multiplicities turn out to be 1. Thus we have to show that

$$\sum_{\substack{(H,\Psi)<(H_1,\Psi_1)<..<(H_r,\Psi_r)\\(H_r,\Psi_r)\leq(G,\rho)}} (-1)^r = 0\,.$$

Consider the set, $\mathcal{R}$, of chains the sum runs over. Let $\mathcal{P} < \mathcal{R}$ denote the subset of those chains which will not end in (G,ρ). Then

$$(\,(H,\Psi) < \ldots < (H_r,\Psi_r)\,) \mapsto (\,(H,\Psi) < \ldots < (H_r,\Psi_r) < (G,\rho)\,)$$

gives a bijection $\mathcal{P} \to \mathcal{R}\setminus\mathcal{P}$, where chains of length r are in correspondence to chains of length $r+1$. So the terms cancel in pairs, and indeed the sum above equals 0. ∎

Proposition 3.14
The homomorphism $a_G^{sp} : \mathbf{Q}R^{sp}(G) \to \mathbf{Q}R_+^{sp}(G)$ is the only homomorphism being natural with respect to restriction and satisfies Prop. 3.13 when ρ is one-dimensional.

The proof is similar to the one of theorem 2.2.15 in [SnEBI].

Proposition 3.15
The homomorphisms a_G^{sp} is natural with respect to inflation so that, for $N \triangleleft G$,

$$a_G^{sp} \circ \mathrm{Infl}_{G/N}^G = \mathrm{Infl}_{G/N}^G \circ a_{G/N}^{sp} : \mathbf{Q}R^{sp}(G/N) \to \mathbf{Q}R_+^{sp}(G)$$

Proof
We have to show that the coefficients of a basis element $(H,\Psi)^G$ in $a_G^{sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho}))$ and $\mathrm{Infl}_{G/N}^G(a_{G/N}^{sp}(\overline{\rho}))$ for $\rho : G \to \mathrm{Sp}(n)$ with $N \le \ker\rho$ coincide. Since all maps are morphisms, we can assume ρ being irreducible. In case $\rho : G \to \mathrm{Sp}(1)$ the statement follows directly from 3.13.
Now we assume $n \ge 2$ and argue using induction on $|G|$. If $H = G$, the formula 3.9 tells that the coefficient of $(G,\Psi)^G$ in $a_G^{sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho}))$ is zero, as $\rho = \mathrm{Infl}_{G/N}^G(\overline{\rho})$ is irreducible, and on the other hand the coefficient of $(G,\Psi)^G$ in $\mathrm{Infl}_{G/N}^G(a_{G/N}^{sp}(\overline{\rho}))$ also vanishes, because

$$\mathrm{Infl}_{G/N}^G\Big(\sum_{(HN/N,\overline{\Psi})^{G/N}} C_{(HN/N,\overline{\Psi})^{G/N}}(HN/N,\overline{\Psi})^{G/N}\Big)$$

contributes to this coefficient only from the base elements with $HN = G$, and for those the coefficient $C_{(G/N,\overline{\Psi})^{G/N}} = 0$ since $\overline{\Psi}$ is irreducible. Now we suppose $H < G$ and use induction on $[G:H]$. The coefficient of $(H,\Psi)^H$ in $\mathrm{Res}_H^G(\sum_{(H_0,\Psi_0)^G} C_{(H_0,\Psi_0)^G}(H_0,\Psi_0)^G)$ is determined by $C_{(H_0,\Psi_0)^G}$ with $(H,\Psi)^G \le (H_0,\Psi_0)^G$. Since by induction on $[G:H]$ we know that the coefficients coincide for $(H,\Psi)^G < (H_0,\Psi_0)^G$, it suffices to proof that the coefficients of $(H,\Psi)^H$ in $\mathrm{Res}_H^G(a_G^{sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho})))$ and $\mathrm{Res}_H^G(\mathrm{Infl}_{G/N}^G(a_{G/N}^{sp}(\overline{\rho})))$ are the same. But in fact

$$\begin{aligned}
\mathrm{Res}_H^G(a_G^{sp}(\mathrm{Infl}_{G/N}^G(\overline{\rho}))) &= a_H^{sp}(\mathrm{Res}_H^G(\mathrm{Infl}_{G/N}^G(\overline{\rho})))\\
&= a_H^{sp}(\mathrm{Infl}_{H/(H\cap N)}^H(\iota_{HN/N}^{H/N\cap H}(\mathrm{Res}_{HN/N}^{G/N}(\overline{\rho}))))\\
&= \mathrm{Infl}_{H/(H\cap N)}^H(a_{H/(H\cap N)}^{sp}(\iota_{HN/N}^{H/N\cap H}(\mathrm{Res}_{HN/N}^{G/N}(\overline{\rho}))))\\
&= \mathrm{Infl}_{H/(H\cap N)}^H(\iota_{HN/N}^{H/N\cap H}(a_{HN/N}^{sp}(\mathrm{Res}_{HN/N}^{G/N}(\overline{\rho}))))\\
&= \mathrm{Infl}_{H/(H\cap N)}^H(\iota_{HN/N}^{H/N\cap H}(\mathrm{Res}_{HN/N}^{G/N}(a_{G/N}^{sp}(\overline{\rho}))))\\
&= \mathrm{Res}_H^G(\mathrm{Infl}_{G/N}^G(a_{G/N}^{sp}(\overline{\rho}))) .
\end{aligned}$$

Here $\iota_{HN/N}^{H/(N\cap H)}$ denotes the canonical isomorphism induced by the canonical group isomorphism $HN/N \cong H/(N\cap H)$, and we have induced the hypotheses on G/N which is of smaller order than G because the statement is trivial for $N = 1$. ■

Definition 3.16 Let $\rho : G \longrightarrow \mathrm{Sp}(n)$ be a symplectic representation of a finite group, G. The *centre of* ρ is the maximal subgroup, $Z(\rho)$, such that $\mathrm{Res}^G_{Z(\Psi)}(\rho) = n\chi$ for some homomorphism of the form $\chi : Z(\rho) \longrightarrow \{\pm 1\} \subset \mathrm{Sp}(1)$. Since $\{\pm 1\}$ is central of $\mathrm{Sp}(1)$ it is easy to see that such a maximal $Z(\rho)$ exists and is unique (cf. [SnEBI] Corollary 2.2.40).

Proposition 3.17
In the notation of Definition 3.16, the coefficient of $(H, \Psi)^G$ in $a^{sp}_G(\rho)$ is zero unless $(Z(\rho), \chi) \leq (H, \Psi)$.

Proof
Since a^{sp}_G commutes with inflation by 3.15 we may assume that ρ is injective.
Recall the formula for $a^{sp}_G(\rho)$

$$a^{sp}_G(\rho) = \sum_{(H_0,\Psi_0)<..<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} \frac{\left\langle c(\mathrm{Res}^G_{H_r}(\rho)); c(\Psi_r)\right\rangle_{H_r}}{\langle c(\Psi_r); c(\Psi_r)\rangle_{H_r}} (H_0, \Psi_0)^G \in \mathbf{Q}R^{sp}_+(G).$$

If H contains $Z(\rho)$ then $(Z(\rho), \chi) \leq (H, \Psi)$ because $\mathrm{Res}^G_{Z(\rho)}(\rho) = n\chi$. Therefore we must show that the coefficient of $(H, \Psi)^G$ is zero when H does not contain $Z(\rho)$. In this case $Z(\rho)$ is not trivial and we may choose $g \in Z(\rho)$ such that $\chi(g) = -1$, since ρ (and hence χ) is injective on $Z(\rho)$.

If $g \notin \widetilde{H}$ and $\widetilde{\Psi} : \widetilde{H} \longrightarrow \mathrm{Sp}(1)$ is a homomorphism then there exists a unique extension, $\widehat{\Psi}$, of $\widetilde{\Psi}$ to $\widehat{H} =< \widetilde{H}, g >$ such that $\widehat{\Psi}(g) = \chi(g) = -1$. Now consider the set, $\mathcal{R}$, of chains $(H, \Psi) < (H_1, \Psi_1) < .. < (H_r, \Psi_r)$ appearing in the formula with $\left\langle c(\mathrm{Res}^G_{H_r}(\rho)); c(\Psi_r)\right\rangle_{H_r}$ non-zero. Let $\mathcal{P} \subset \mathcal{R}$ denote the subset consisting of those chains for which no $H_i = \hat{H}_{i-1}$ and $H_1 \neq \hat{H}$. For each chain in $\mathcal{P}$ there exists a smallest integer, j, such that $g \notin H_{j-1}$ but $g \in H_j$. If there is no such H_j we set $j = r + 1$. For each such chain we have $(H_{j-1}, \Psi_{j-1}) < (\widehat{H}_{j-1}, \widehat{\Psi}_{j-1}) < (H_j, \Psi_j)$ or $(H_r, \Psi_r) < (\hat{H}_r, \widehat{\Psi}_r)$ if $j = r + 1$. Furthermore, when $j = r+1$ the multiplicities of $\widehat{\Psi}_r$ and Ψ_r in $\mathrm{Res}^G_{Z(\rho)}(\rho)$ are equal and hence non-zero.

Associating to each chain in $\mathcal{P}$ the unique chain obtained by interpolating $(\widehat{H}_{j-1}, \widehat{\Psi}_{j-1})$ gives a multiplicity-preserving bijection between chains of length r in $\mathcal{P}$ and length $r + 1$ in $\mathcal{R} \backslash \mathcal{P}$. This bijection shows that the terms in the coefficient of $(H, \psi)^G$ cancel in pairs, as required. ■

Proposition 3.18
The homomorphisms a_G and a^{sp}_G are connected via complexification, that is

$$a_G \circ c_G = c_{+,G} \circ a^{sp}_G : \mathbf{Q}R^{sp}(G) \to \mathbf{Q}R_+(G)$$

Proof

Let $\rho : G \to \mathrm{Sp}(n)$ and let $(J,\phi)^G$ be a base element of $\mathbf{Q}R_+(G)$. We have to show that the coefficients of $(J,\phi)^G$ in $a_G(c(\rho))$ and $c_+(a_G^{sp}(\rho))$ coincide. The first one is easy to express, namely

$$\frac{|J|}{|G|}\sum_{(J_0,\phi_0)\in(J,\phi)^G}\;\sum_{(J_0,\phi_0)<\dots<(J_r,\phi_r)}(-1)^r\left\langle \mathrm{Res}^G_{J_r}(c(\rho));\phi_r\right\rangle_{J_r}.$$

Now we calculate the coefficient, denoted C, of $(J,\phi)^G$ in

$$c_+(a_G^{sp}(\rho)) = \sum_{(H_0,\Psi_0)<\dots<(H_t,\Psi_t)}(-1)^t\frac{|H_0|}{|G|}m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)\,c_+((H_0,\Psi_0)^G).$$

Only those summands will have a contribution to C which have a non-zero term $(J,\phi)^G$ in $c_+((H_0,\Psi_0)^G)$. If $c(\Psi_0)=\phi_0+\overline{\phi_0}$, then $c_+((H_0,\Psi_0)^G) = (H_0,\phi_0)^G + (H_0,\overline{\phi_0})^G$, and this can only contributes to C, if $(H_0,\phi_0)^G = (J,\phi)^G$ or $(H_0,\overline{\phi_0})^G = (J,\phi)^G$. If $c(\Psi_0)=\psi_0$ is irreducible, then

$$c_+((H_0,\Psi_0)^G) = \mathrm{Ind}^G_{H_0}(a_{H_0}(\psi_0)) = \sum_{(J_0,\phi_0)<\dots<(J_t,\phi_t)}(-1)^t\frac{|J_0|}{|H_0|}\left\langle \mathrm{Res}^{H_0}_{J_t}(\psi_0);\phi_t\right\rangle_{J_t}(J_0,\phi_0)^G,$$

and this contributes

$$\frac{|J|}{|H_0|}\sum_{(J_0,\phi_0)\in(J,\phi)^G}\;\sum_{\substack{(J_0,\phi_0)<\dots<(J_t,\phi_t)\\ J_t\le H_0}}(-1)^t\left\langle \mathrm{Res}^{H_0}_{J_t}(\psi_0);\phi_t\right\rangle_{J_t}.$$

Therefore, C can be expressed as

$$\begin{aligned}
C \;=\; & \sum_{\substack{(H_0,\Psi_0)<\dots<(H_t,\Psi_t)\\ c(\Psi_0)=\phi_0+\overline{\phi_0},\,(H_0,\phi_0)\,\mathrm{xor}\,(H_0,\overline{\phi_0})\in(J,\phi)^G}}(-1)^t\frac{|H_0|}{|G|}m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)\;+\\
& \sum_{\substack{(H_0,\Psi_0)<\dots<(H_t,\Psi_t)\\ c(\Psi_0)=\phi_0+\overline{\phi_0},\,(H_0,\phi_0)\,\mathrm{and}\,(H_0,\overline{\phi_0})\in(J,\phi)^G}}(-1)^t\frac{|H_0|}{|G|}2m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)\;+\\
& \sum_{\substack{(H_0,\Psi_0)<\dots<(H_t,\Psi_t)\\ c(\Psi_0)=\psi_0\ \mathrm{irred.}}}(-1)^t\frac{|H_0|}{|G|}m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)\left(\frac{|J|}{|H_0|}\sum_{\substack{(J_0,\phi_0)<\dots<(J_r,\phi_r)\\ (J_0,\phi_0)\in(J,\phi)^G,\,J_r<H_0}}(-1)^r\left\langle \mathrm{Res}^{H_0}_{J_r}(\psi_0);\phi_r\right\rangle_{J_r}\right)
\end{aligned}$$

Note that in each summand the factor $\frac{|J|}{|G|}$ appears and thus can be factored out. We expand the three subsums of C according to the type of decomposition of $c(\Psi_i)$, namely $c(\Psi_i)=\phi_i+\overline{\phi_i}$ or $c(\Psi_i)=\psi_i$ irreducible. The first subsum

$$\sum_{\substack{(H_0,\Psi_0),\,c(\Psi_0)=\phi_0+\overline{\phi_0}\\ (H_0,\phi_0)\in(J,\phi)^G,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}}\;\sum_{(H_0,\Psi_0)<\dots<(H_t,\Psi_t)}(-1)^t m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

splits into $C_1 + C_2$ with

$$C_1 = \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)\lessdot..\lessdot(H_t,\Psi_t) \\ c(\Psi_t)=\phi_t+\overline{\phi_t}}} (-1)^t m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

$$= \sum_{\substack{(H_0,\phi_0)\in(J,\phi)^G \\ (H_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{(H_0,\phi_0)\lessdot..\lessdot(H_t,\phi_t)} (-1)^t \left\langle \mathrm{Res}^G_{H_t}(c(\rho));\phi_t \right\rangle_{H_t}$$

because $\langle c(\Psi_t);c(\Psi_t)\rangle_{H_t} = 2$ and $\left\langle \mathrm{Res}^G_{H_t}(c(\rho));\phi_r \right\rangle_{H_t} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho));\overline{\phi_r} \right\rangle_{H_t}$, and

$$C_2 = \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)\lessdot..\lessdot(H_t,\Psi_t) \\ c(\Psi_t)=\psi_t}} (-1)^t m(\mathrm{Res}^G_{H_t}(\rho),\Psi_r)$$

$$= \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)\lessdot..\lessdot(H_r,\Psi_r) \\ c(\Psi_r)=\phi_r+\overline{\phi_r}}} \sum_{\substack{(H_{r+1},\Psi_{r+1})\lessdot..\lessdot(H_t,\Psi_t) \\ (H_r,\Psi_r)\lessdot(H_{r+1},\Psi_{r+1})\,,\,c(\Psi_{r+1})=\psi_{r+1}}} (-1)^t m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

$$= \sum_{\substack{(J_0,\phi_0)\lessdot..\lessdot(J_r,\phi_r) \\ (J_0,\phi_0)\in(J,\phi)^G\,,\,(J_0,\overline{\phi_0})\notin(J,\phi)^G}} \sum_{\substack{(H_{r+1},\Psi_{r+1})\lessdot..\lessdot(H_t,\Psi_t) \\ J_r\lessdot H_{r+1}\,,\,c(\Psi_{r+1})=\psi_{r+1}}} (-1)^t \left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1});\phi_r \right\rangle_{J_r} m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

The second subsum

$$\sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G \text{ and } (H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{(H_0,\Psi_0)\lessdot..\lessdot(H_t,\Psi_t)} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

splits into $C_3 + C_4$ with

$$C_3 = \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)\lessdot..\lessdot(H_t,\Psi_t) \\ c(\Psi_t)=\phi_t+\overline{\phi_t}}} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

$$= \sum_{\substack{(H_0,\phi_0)\in(J,\phi)^G \\ \text{with } (H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{(H_0,\phi_0)\lessdot..\lessdot(H_t,\phi_t)} (-1)^t \left\langle \mathrm{Res}^G_{H_t}(c(\rho));\phi_t \right\rangle_{H_t},$$

and

$$C_4 = \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)\lessdot..\lessdot(H_t,\Psi_t) \\ c(\Psi_t)=\psi_t}} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_r)$$

$$= \sum_{\substack{(H_0,\Psi_0)\,,\,c(\Psi_0)=\phi_0+\overline{\phi_0} \\ (H_0,\phi_0)\in(J,\phi)^G\,,\,(H_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0)\lessdot..\lessdot(H_r,\Psi_r) \\ c(\Psi_r)=\phi_r+\overline{\phi_r}}} \sum_{\substack{(H_{r+1},\Psi_{r+1})\lessdot..\lessdot(H_t,\Psi_t) \\ (H_r,\Psi_r)\lessdot(H_{r+1},\Psi_{r+1})\,,\,c(\Psi_{r+1})=\psi_{r+1}}} (-1)^t 2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

$$= \sum_{\substack{(J_0,\phi_0)\lessdot..\lessdot(J_r,\phi_r) \\ (J_0,\phi_0)\in(J,\phi)^G \text{ and } (J_0,\overline{\phi_0})\in(J,\phi)^G}} \sum_{\substack{(H_{r+1},\Psi_{r+1})\lessdot..\lessdot(H_t,\Psi_t) \\ J_r\lessdot H_{r+1}\,,\,c(\Psi_{r+1})=\psi_{r+1}}} (-1)^t \left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1});\phi_r \right\rangle_{J_r} m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t)$$

In fact, for the calculation of C_3 we notice that if $\phi_0 = \overline{\phi_0}$ then either in case $\phi_t = \overline{\phi_t}$ the factor 2 cancels because of

$$2\cdot m(\mathrm{Res}^G_{H_t}(\rho),\Psi_t) = 2\frac{\left\langle \mathrm{Res}^G_{H_t}(c(\rho));2\phi_t \right\rangle_{H_t}}{\langle 2\phi_t;2\phi_t\rangle_{H_t}} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho));\phi_t \right\rangle_{H_t},$$

or in case $\phi_t \neq \overline{\phi_t}$ each chain $(H_0, \Psi_0) \lessdot .. \lessdot (H_t, \Psi_t)$ yields two chains $(H_0, \phi_0) \lessdot .. \lessdot (H_t, \phi_t)$ and

$$m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t) = \frac{\left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t + \overline{\phi_t} \right\rangle_{H_t}}{\left\langle \phi_t + \overline{\phi_t}; \phi_t + \overline{\phi_t} \right\rangle_{H_t}} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t},$$

and that if $\phi_0 \neq \overline{\phi_0}$ then the factor 2 vanishes because each chain $(H_0, \Psi_0) \lessdot .. \lessdot (H_t, \Psi_t)$ yields two chains $(H_0, \phi_0) \lessdot .. \lessdot (H_t, \phi_t)$ starting in either (H_0, ϕ_0) or $(H_0, \overline{\phi_0})$, and again

$$m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t) = \frac{\left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t + \overline{\phi_t} \right\rangle_{H_t}}{\left\langle \phi_t + \overline{\phi_t}; \phi_t + \overline{\phi_t} \right\rangle_{H_t}} = \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t}.$$

For the calculation of C_4 the same arguments hold for the subchains $(H_0, \Psi_0) \lessdot .. \lessdot (H_r, \Psi_r)$, where in case $\phi_r = \overline{\phi_r}$ we have $\left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1}); \phi_r \right\rangle_{J_r} = 2$.
Finally $C_2 + C_4$ add up to

$$\sum_{\substack{(J_0,\phi_0) \lessdot .. \lessdot (J_r,\phi_r) \\ (J_0,\phi_0) \in (J,\phi)^G}} \sum_{\substack{(H_0,\Psi_0) \lessdot .. \lessdot (H_t,\Psi_t) \\ J_r \lessdot H_0,\, c(\Psi_0) = \psi_{r+1}}} (-1)^{r+t-1} \left\langle \mathrm{Res}^{H_{r+1}}_{J_r}(\psi_{r+1}); \phi_r \right\rangle_{J_r} m(\mathrm{Res}^G_{H_t}(\rho), \Psi_t)$$

which now turns out to be the negative of the third subsum above, and $C_1 + C_3$ add up to

$$\sum_{(H_0,\phi_0) \in (J,\phi)^G} \sum_{(H_0,\phi_0) \lessdot .. \lessdot (H_t,\phi_t)} (-1)^t \left\langle \mathrm{Res}^G_{H_t}(c(\rho)); \phi_t \right\rangle_{H_t},$$

and this indeed coincides with the coefficient of $(J, \phi)^G$ in $a_G(c(\rho))$. ■

3.19 *Example 3.10:* $G = C_p \times Q_8$

We will apply the complexification maps c_+ to the formula above:

$$\begin{aligned} c_+(a_G{}^{sp}(\rho)) &= (\langle xz\rangle, i\zeta)^G + (\langle xz\rangle, \overline{i\zeta})^G + (\langle yz\rangle, i\zeta)^G + (\langle yz\rangle, \overline{i\zeta})^G + (\langle xyz\rangle, i\zeta)^G + \\ &\quad (\langle xyz\rangle, \overline{i\zeta})^G - (\langle\text{-}z\rangle, \text{-}\zeta)^G - (\langle\text{-}z\rangle, \overline{\text{-}\zeta})^G + \frac{2}{p}((\langle x\rangle, i)^G + (\langle y\rangle, i)^G + \\ &\quad (\langle xy\rangle, i)^G - (\langle\text{-}1\rangle, \varepsilon)^G) - \frac{1}{p}(\langle x\rangle, i)^G - \frac{1}{p}(\langle x\rangle, \overline{i})^G - \frac{1}{p}(\langle y\rangle, i)^G - \frac{1}{p}(\langle y\rangle, \overline{i})^G - \\ &\quad \frac{1}{p}(\langle xy\rangle, i)^G + \frac{1}{p}(\langle xy\rangle, \overline{i})^G + \frac{2}{p}(\langle\text{-}1\rangle, \varepsilon)^G \\ &= (\langle xz\rangle, i\zeta)^G + (\langle xz\rangle, \overline{i\zeta})^G + (\langle yz\rangle, i\zeta)^G + (\langle yz\rangle, \overline{i\zeta})^G + (\langle xyz\rangle, i\zeta)^G + \\ &\quad (\langle xyz\rangle, \overline{i\zeta})^G -\!-(\langle\text{-}z\rangle, \text{-}\zeta)^G - (\langle\text{-}z\rangle, \overline{\text{-}\zeta})^G \\ &= a_G(\chi_1) + a_G(\chi_{p-1}) = a_G(c(\rho)) \end{aligned}$$

Theorem 3.20
The map a_G^{sp} *induced an induction formula, that is*

$$b_G^{sp} a_G^{sp} = \mathrm{id} : R^{sp}(G) \to R^{sp}(G) .$$

Proof

By 3.1 and 3.18 the diagram

$$\begin{array}{ccccc} \mathbf{Q}R^{sp}(G) & \xrightarrow{a_G^{sp}} & \mathbf{Q}R_+^{sp}(G) & \xrightarrow{b_G^{sp}} & \mathbf{Q}R^{sp}(G) \\ c\downarrow & & c_+\downarrow & & c\downarrow \\ \mathbf{Q}R(G) & \xrightarrow{a_G} & \mathbf{Q}R_+(G) & \xrightarrow{b_G} & \mathbf{Q}R(G) \end{array}$$

is commutative. Hence, for $\rho : G \to \mathrm{Sp}(n)$, we use 2.1 to calculate

$$c(b_G^{sp}(a_G^{sp}(\rho))) = b_G(a_G(c(\rho))) = c(\rho) \ ,$$

and since c is injective, we conclude $\rho = b_G^{sp}(a_G^{sp}(\rho))$. ■

Theorem 3.21

Let $K/\mathbf{Q}_p$, (p odd), and let L/K be a finite, totally ramified Galois extension with group G. Then, for $\rho : G \to \mathrm{Sp}(n)$, $a_G^{sp}(\rho) \in R_+^{sp}(G)$.

Proof

First we observe that the structure of G is restricted; G is a semidirect product of a p-group by a cyclic group of order prime to p. Especially the 2-Sylow-subgroup of G has to be cyclic. Hence, if $\rho : G \to \mathrm{Sp}(n)$ is irreducible, then $c(\rho)$ will not be irreducible. In fact, if $c(\rho)$ would be irreducible, its character would be real-valued and therefore of Schur index 2 (over $\mathbf{R}$). By the Brauer-Witt theorem (CR74.38) there would be a $(\mathbf{R}, 2)$-elementary subgroup H of G loaded with an irreducible character of Schur index 2. But H, a semidirect product of an odd cyclic normal subgroup with a 2-group, which is cyclic by itself, does not admit such a character.
Thus $c(\rho) = \theta + \overline{\theta}$ for some irreducible unitary representation $\theta : G \to \mathrm{U}(n)$. Since

$$a_G^{sp}(\rho) = \sum_{(H,\Psi)^G} \alpha_{(H,\Psi)^G}(\rho)\ (H,\Psi)^G \text{ with}$$

$$\alpha_{(H,\Psi)^G}(\rho) = \frac{|H|}{|G|} \sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r)\\ (H_0,\Psi_0)\in(H,\Psi)^G}} (-1)^r m(\mathrm{Res}^G_{H_r}\rho, \Psi_r) \ ,$$

we have to show that $\alpha_{(H,\Psi)^G}(\rho) \in \mathbf{Z}$. This we do by showing that these coefficients equal to coefficients of the canonical unitary induction formula of $c(\rho)$ or θ, which by Boltje's results are known to be integral.
Case 1: $c(\Psi) = \lambda + \overline{\lambda}$ with $\lambda \neq \overline{\lambda}$.
In this case all one-dimension representations Ψ_i split into $c(P_i) = \lambda_i + \overline{\lambda_i}$ with $\lambda_i \neq \overline{\lambda_i}$.
Hence $m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r) = \left\langle \theta + \overline{\theta}; \lambda_r \right\rangle_{H_r}$.
If $(H,\lambda)^G \neq (H,\overline{\lambda})^G$, we receive

$$\alpha_{(H,\Psi)^G}(\rho) = \frac{|H|}{|G|} \sum_{\substack{(H_0,\lambda_0)<..<(H_r,\lambda_r)\\ (H_0,\lambda_0)\in(H,\lambda)^G}} (-1)^r \left\langle \theta + \overline{\theta}; \lambda_r \right\rangle_{H_r} = \alpha_{(H,\lambda)^G}(\theta + \overline{\theta}) \ ,$$

the coefficient of $(H,\lambda)^G$ in $a_G(\theta+\overline{\theta})$.
If $(H,\lambda)^G = (H,\overline{\lambda})^G$, each symplectic chain gives two unitary chains. Thus the same equation shows that $\alpha_{(H,\Psi)^G}(\rho) = \frac{1}{2}\alpha_{(H,\lambda)^G}(\theta+\overline{\theta})$. But as

$$\alpha_{(H,\lambda)^G}(\theta+\overline{\theta}) = \alpha_{(H,\lambda)^G}(\theta) + \alpha_{(H,\lambda)^G}(\overline{\theta}) = \alpha_{(H,\lambda)^G}(\theta) + \alpha_{(H,\overline{\lambda})^G}(\overline{\theta}) = 2\alpha_{(H,\lambda)^G}(\theta) ,$$

we conclude $\alpha_{(H,\Psi)^G}(\rho) = \alpha_{(H,\lambda)^G}(\theta) \in \mathbf{Z}$.
Case 2: $c(\Psi) = 2\lambda$.
We show that $\alpha_{(H,\Psi)^G}(\rho) = \alpha_{(H,\lambda)^G}(\theta) \in \mathbf{Z}$. Therefore we take a chain

$$(*) \quad (H_0,\Psi_0) \lessdot .. \lessdot (H_r,\Psi_r)$$

and study the corresponding multiplicity. Let $c(\Psi_r) = \lambda_r + \overline{\lambda_r}$.
If $\lambda_r = \overline{\lambda_r}$ then $m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r) = \frac{1}{2}\mathrm{Res}^G_{H_r}(\theta+\overline{\theta})\lambda_r H_r = \mathrm{Res}^G_{H_r}(\theta)\lambda_r H_r$, and there is exactly one unitary chain derived from $(*)$, namely $(H_0,\lambda_0) \lessdot .. \lessdot (H_r,\lambda_r)$. On the other side, if $\lambda_r \neq \overline{\lambda_r}$ then $m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r) = \frac{1}{2}\mathrm{Res}^G_{H_r}(\theta+\overline{\theta})\lambda_r H_r$ and $(*)$ affords the two unitary chains $(H_0,\lambda_0) \lessdot .. \lessdot (H_r,\lambda_r)$ and $(H_0,\lambda_0) \lessdot .. \lessdot (H_r,\overline{\lambda_r})$. Since $\mathrm{Res}^G_{H_r}(\overline{\theta})\lambda_r H_r = \mathrm{Res}^G_{H_r}(\theta)\overline{\lambda_r} H_r$ we conclude

$$\sum_{\substack{(H_0,\Psi_0)\lessdot .. \lessdot (H_r,\Psi_r) \\ (H_0,\Psi_0)\in (H,\Psi)^G}} (-1)^r m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r) \quad = \sum_{\substack{(H_0,\lambda_0)\lessdot .. \lessdot (H_r,\lambda_r) \\ (H_0,\lambda_0)\in (H,\lambda)^G}} (-1)^r m(\mathrm{Res}^G_{H_r}(\theta),\lambda_r) \ ,$$

and this finishes the proof. ∎

4 Symplectic Adams operations

4.1 We may represent Sp(1) ≅ SU(2) by matrices in U(2) of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where a and b are complex numbers which satisfy the relation $|a|^2+|b|^2 = 1$. The standard maximal torus in Sp(1) is the circle corresponding to $a = e^{i\theta}, b = 0$. Writing S^1 for this circle, the normaliser is given by

$$N_{\mathrm{Sp}(1)}S^1 = \{\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \mid |a| = 1 = |b|\}.$$

Hence $N_{\mathrm{Sp}(1)}S^1 =< S^1, w >$ where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider a finite subgroup, $H \subset N_{\mathrm{Sp}(1)}S^1$, then H may be conjugated by an element of $N_{\mathrm{Sp}(1)}S^1$ so that $H = H \cap S^1$ or $H =< H \cap S^1, w >$, such a subgroup H will temporarily be called *standard*. Write $C_m \subset S^1$ for the cyclic subgroup of order m and $Q_{4m} =< C_m, w >$. These are all the standard finite subgroups, $H \subset N_{\mathrm{Sp}(1)}S^1$. The inclusion, $i : H \subset \mathrm{Sp}(1)$, of a finite, standard subgroup yields a one-dimensional symplectic representation which is determined by the Sp(1)-conjugacy class of i. However, if $H = H \bigcap S^1$ any automorphism of H induced by conjugation in Sp(1) may also be realised by conjugation in $N_{\mathrm{Sp}(1)}S^1$. The same is true for any standard subgroup of the form $H =< H \cap S^1, w >$ of order strictly larger than eight. The cyclic group C_4 of order four in S^1 may, however, be conjugated within Sp(1), onto $< w >= Q_4$. In fact, if $\Lambda(X)$ is given by

$$\Lambda(X) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} X \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

then

$$\Lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = w, \quad \Lambda(w) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Also Λ induces an automorphism of the standard subgroup isomorphic to Q_8, the quaternion group of order eight.

Up to conjugation in $N_{\mathrm{Sp}(1)}S^1$ these are the only Sp(1)-conjugation automorphisms between standard subgroups which are not induced by conjugation in $N_{\mathrm{Sp}(1)}S^1$.

Let p be an odd prime. Then we may define a homomorphism

$$\Psi^p : N_{\mathrm{Sp}(1)}S^1 \longrightarrow N_{\mathrm{Sp}(1)}S^1$$

by the formulae

$$\Psi^p \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} a^p & 0 \\ 0 & \overline{a}^p \end{pmatrix}, \quad \Psi^p \begin{pmatrix} 0 & b \\ -\overline{b} & 0 \end{pmatrix} = (-1)^{(p-1)/2} \begin{pmatrix} 0 & b^p \\ -\overline{b}^p & 0 \end{pmatrix}.$$

Proposition 4.2

Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Let p be an odd prime. Then there is a natural homomorphism

$$\Psi^p : R_+^{sp}(G) \longrightarrow R_+^{sp}(G)$$

given by $\Psi^p((H,\psi)^G) = (H, \Psi^p \cdot \psi)^G$ when $\psi(H) \subset N_{\mathrm{Sp}(1)}S^1$ is standard in the sense of §4.1.

Proof

The hypothesis on G is inherited by H and by $\psi(H)$. This means that $\psi(H)$ is solvable and not isomorphic to the binary tetrahedral group of order twenty-four, which implies that we may conjugate in Sp(1) to get make $\psi(H)$ standard. The element, $(H,\psi)^G$, depends only on the $G-\mathrm{Sp}(1)$-conjugacy class of (H,ψ). Varying (H,ψ) by $G-N_{\mathrm{Sp}(1)}S^1$-conjugation does not alter $(H, \Psi^p \cdot \psi)^G$. By the discussion of §4.1, this means that Ψ^p is well-defined on $(H,\psi)^G$ except possibly if $\psi(H) = C_4, Q_4, Q_8$. However it is easily verified that the Sp(1)-conjugation, Λ, commutes with Ψ^p in these exceptional cases, which completes the proof. ■

Proposition 4.3

As in ([SnEBI] p.109), define $\Psi^p : R_+(G) \longrightarrow R_+(G)$ by $\Psi^p((H,\phi)^G) = (H,\phi^p)^G$. Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Let p be an odd prime. Then

$$\Psi^p \cdot c_+ = c_+ \cdot \Psi^p : R_+^{sp}(G) \longrightarrow R_+(G).$$

Proof

If $\psi(H)$ is abelian and standard with $c(\psi) = \phi \oplus \overline{\phi}$ then $c(\Psi^p \cdot \psi) = \phi^p \oplus \overline{\phi}^p$ as complex representations of H so that

$$\Psi^p(c_+(H,\psi)^G) = (H,\phi^p)^G + (H,\overline{\phi}^p)^G = c_+(\Psi^p(H,\psi)^G).$$

Otherwise, being standard, $\psi(H)$ is Q_{4n} for some $n \geq 1$ and the result follows from the formulae of Proposition 2.6 for $a_{Q_{4n}}(Q_{4n},\psi)^{Q_{4n}})$. $a_{Q_{4n}}(Q_{4n},\psi)^{Q_{4n}}$. More precisely, from

Proposition 2.6 , we have

$$a_{Q_{4n}}(c(\Psi)) = \begin{cases} (<x>,\phi_x)^{Q_{4n}} + (<y>,\rho_y)^{Q_{4n}} + (<y>,\overline{\rho}_y)^{Q_{4n}} - (<y^2>,\chi)^{Q_{4n}} \\ \qquad if\ n\ is\ odd, \\ (<x>,\phi_x)^{Q_{4n}} + (<y>,\rho_y)^{Q_{4n}} + (<xy>,\rho_{xy})^{Q_{4n}} - (<y^2>,\chi)^{Q_{4n}} \\ \qquad if\ n\ is\ even \end{cases}$$

where $\rho_x(x) = \xi_{2n}$. Therefore, if n is odd, we have

$$\Psi^p(c_+(H,\psi)^G)$$

$$= \Psi^p((<x>,\phi_x)^G + (<y>,\rho_y)^G + (<y>,\overline{\rho}_y)^G - (<y^2>,\chi)^G)$$

$$= ((<x>,\phi_x^p)^G + (<y>,\rho_y^p)^G + (<y>,\overline{\rho}_y^p)^G - (<y^2>,\chi^p)^G)$$

On the other hand, if the symplectic representation $\psi : H \longrightarrow \mathrm{Sp}(1)$ satisfies $c(\psi) = \mathrm{Ind}_{C_{2n}}^{Q_{4n}}(\phi)$ then $c(\Psi^p(\psi)) = \mathrm{Ind}_{C_{2n}}^{Q_{4n}}(\phi^p)$ so that

$$c_+(\Psi^p(H,\psi)^G))$$

$$= ((<x>,\phi_x^p)^G + (<y>,\rho_y^p)^G + (<y>,\overline{\rho}_y^p)^G - (<y^2>,\chi^p)^G)$$

$$= \Psi^p(c_+(H,\psi)^G).$$

The case when n is even is similar. ■

Corollary 4.4 *Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Let p be an odd prime. Then the composition*

$$R^{sp}(G) \xrightarrow{a_G^{sp}} R_+^{sp}(G) \otimes \mathbf{Q} \xrightarrow{\Psi^p \otimes 1} R_+^{sp}(G) \otimes \mathbf{Q} \xrightarrow{b_G \otimes 1} R^{sp}(G) \otimes \mathbf{Q}$$

sends z to $\psi^p(z) \otimes 1$, where ψ^p is the usual Adams operation.

Proof

It suffices to show that $c(b_G \otimes 1(\Psi^p \otimes 1(a_G^{sp}(z))))$ is equal to $\psi^p(c(z)) \otimes 1$. However

$$\begin{aligned} c(b_G \otimes 1(\Psi^p \otimes 1(a_G^{sp}(z)))) &= b_G \otimes 1(c_+(\Psi^p \otimes 1(a_G^{sp}(z)))) \\ &= b_G \otimes 1(\Psi^p \otimes 1(c_+(a_G^{sp}(z)))) \\ &= b_G \otimes 1(\Psi^p \otimes 1(a_G(c(z)))) \\ &= \psi^p(c(z)) \otimes 1 \end{aligned}$$

by ([SnEBI] Theorem 4.1.6). ■

Remark 4.5 *Orthogonal representations and Ψ^2*

We continue to assume that G is a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic.

Now we turn to the orthogonal group, O(2), whose maximal torus is the circle,

$$\mathrm{SO}(2) = \{ \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \mid \theta \in \mathbf{R} \}.$$

This is normal is O(2) which may be written as a semi-direct product, $\mathbf{Z}/2 \ltimes S^1$, given in terms of generators and relations as

$$\mathrm{O}(2) = \{ \tau, e^{i\theta} \ (\theta \in \mathbf{R}) \mid \tau^2 = 1, \ \tau e^{i\theta} \tau = e^{-i\theta} \}.$$

The formulae

$$\Psi^2(w) = \tau, \ \Psi^2(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}) = e^{2i\theta}$$

defines a homomorphism

$$\Psi^2 : N_{\mathrm{Sp}(1)} S^1 \longrightarrow \mathrm{O}(2)$$

since $\Psi^2(w^2) = \tau^2 = 1 = \Psi^2(-I)$ and $\tau e^{2i\theta} \tau = e^{-2i\theta}$.

However

$$\Psi^2(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}) = -1, \ \Psi^2(w) = \tau$$

which are two elements of order two which are not conjugate in O(2). This means that we cannot define a homomorphism Ψ^2 by the formula of Proposition 4.2, in the light of the discussion of §4.1 of Sp(1)-conjugacy of standard subgroups. The difficulty occurs with the standard subgroups C_4, Q_4, Q_8. The following result is the best we can do.

Proposition 4.6

Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Then there is a homomorphism

$$\Psi^2 : R_+^{sp}(G) \longrightarrow R_+^{o}(G)$$

given by $\Psi^2((H,\psi)^G) = (H, \Psi^2 \cdot \psi)^G$ when $\psi(H) \subset N_{\mathrm{Sp}(1)} S^1$ is standard and different from C_4, Q_4, Q_8. When $\psi(H) \subset N_{\mathrm{Sp}(1)} S^1$ is standard and H is one of C_4, Q_4, Q_8 set

$$\Psi^2((H,\psi)^G) = (H, \Psi^2 \cdot \psi)^G + (H, \Psi^2 \cdot \psi')^G$$

where

$$\psi' = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad \psi = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

5 Symplectic Local Root Numbers

5.1 *Local root numbers*

Now suppose that L/K is a Galois extension of p-adic local fields with group $G(L/K)$. An important invariant of a finite dimensional, complex representation ρ of $G(L/K)$ is the local root number $W_K(\rho)$, which is a complex number of unit norm ([SnGMS] §§1.4.10-1.4.14). When ρ is one-dimensional $W_K(\rho)$ is given by a Gauss sum/Artin conductor formula which extends uniquely to an exponential homomorphism on the representation ring $R(G(L/K))$ of the form

$$W_K : R(G(L/K)) \longrightarrow S^1 = \{z \in \mathbf{C}^* \mid |z| = 1\}$$

which satisfies the following properties:

(i) If $K \subset L \subset N$ is a chain of finite Galois extensions and $G(N/K) \longrightarrow G(L/K)$ is the canonical map then

$$W_K(\mathrm{Infl}_{G(L/K)}^{G(N/K)}(\rho)) = W_K(\rho).$$

(ii) If F is an intermediate field of L/K and $\rho : G(L/F) \longrightarrow \mathrm{GL}(V)$ is a representation then

$$W_K(\mathrm{Ind}_{G(L/F)}^{G(L/K)}(\rho - dim(\rho))) = W_F(\rho).$$

Note that $W_K(1) = 1$.

When ρ is the complexification of an orthogonal (i.e. real) representation, $\rho = c(\rho_1)$, then we have a formula of Deligne ([De]; see also [Sn5], [Sn7] Theorem 2.26 p.270)

$$W_K(\rho) = SW_2(\rho_1) \cdot W_K(det(\rho_1)).$$

Here $SW_2(\rho_1) \in H^2(K; \mathbf{Z}/2) \cong \{\pm 1\}$ is the second Stiefel-Whitney class of ρ_1 and $W_K(det(\rho_1))$ is a fourth root of unity given by the quadratic Gauss sum/Artin conductor formula, since $det(\rho_1)$ is a one-dimensional representation given by a quadratic character. In particular, this formula applies to the case of permuation representations $\rho = \mathrm{Ind}_{G(L/F)}^{G(L/K)}(1)$.

The case when ρ is the underlying complex representation of a symplectic (i.e. quaternionic) representation, $\rho = c(\rho_2)$, is particularly important in number theory (for example, see [Fr] and [SnGMS]). In this case $W_K(\rho) \in \{\pm 1\}$. On the other hand, the authors know of no formula for symplectic root numbers in general. When ρ_2 is one-dimensional of the form $\rho_2 : G(L/K) \longrightarrow \mathrm{Sp}(1)$ and K has odd residue characteristic the results of [PR1], [PR2] amount to a formula for $W_K(\rho)$.

Let $G(L/K)$ denote the Galois group of a finite extension of local fields of odd residue characteristic. Suppose that $\rho : G(L/K) \longrightarrow \mathrm{Sp}(n)$ is a symplectic representation and that, in $R_+^{sp}(G(L/K)) \otimes_{\mathbf{Z}} \mathbf{Q}$,

$$a_{G(L/K)}^{sp}(\rho) = \sum_{(G(L/F),\Psi)^{G(L/K)}} n_{(G(L/F),\Psi)^{G(L/K)}} \cdot (G(L/F), \Psi)^{G(L/K)}$$

is the symplectic Explicit Brauer Induction formula of 3.9. If each of the rational numbers $n_{(G(L/F),\Psi)^{G(L/K)}}$ actually lies in the 2-adic integers then $W_K(\rho)$ would be given by the formula

$$\begin{aligned} W_K(\rho) &= W_K(\rho - n) = \prod_{(G(L/F),\Psi)^{G(L/K)}} W_K(\mathrm{Ind}_{G(L/F)}^{G(L/K)}(\Psi - 1))^{n_{(G(L/F),\Psi)^{G(L/K)}}} \\ &= \prod_{(G(L/F),\Psi)^{G(L/K)}} W_F(\Psi)^{n_{(G(L/F),\Psi)^{G(L/K)}}}, \end{aligned}$$

which makes sense because $W_F(\Psi) \in \{\pm 1\}$ and $n_{(G(L/F),\Psi)^{G(L/K)}} \in \mathbf{Z}_2$.

The above formula for local symplectic roots numbers is the motivation for the following integrality conjecture.

Conjecture 5.2 Let G be a finite solvable group which is isomorphic to the group of a Galois extension of local fields of odd residue characteristic. Then, in §3.9,

$$a^{sp}_{G(L/K)}(\rho) \in R^{sp}_{+}(G(L/K)) \otimes_{\mathbf{Z}} \mathbf{Z}_2.$$

Remark 5.3 *Evidence for Conjecture 5.2*

We have explained the motivation for Conjecture 5.2 in 5.1. Here are two pieces of evidence in its favour.

(i) In 3.11 we gave an example of a symplectic representation ρ_n of $G = Q_8 \times C_{2^n}$ for which $a^{sp}_G(\rho_n)$ was not 2-adically integral.

When can $G = Q_8 \times C_{2^n}$ occur as the Galois group of an extension of p-adic local fields?

If $p = 2$ one can take $L/\mathbf{Q}_2$ as in Case B or Case C of [HSvT]. Namely either

$$L = \mathbf{Q}_2(\sqrt{2}, \sqrt{3})(\alpha_{\pm})$$

where $\alpha^2_{\pm} = \pm(\sqrt{6}/6)(1 + \sqrt{2})(\sqrt{2} + \sqrt{3})$ or

$$L = \mathbf{Q}_2(\sqrt{10}, \sqrt{3})(\alpha_{\pm})$$

where $\alpha^2_{\pm} = \pm(1 + \sqrt{3} + \sqrt{10}/10 + \sqrt{30}/10)$. In all these four cases $\mathbf{Q}_2^{nr}$, the maximal unramified extension of $\mathbf{Q}_2$, satisfies $\mathbf{Q}_2^{nr} \bigcap L = \mathbf{Q}_2$. Therefore if we take $K/\mathbf{Q}_2$ to be the unique unramified extension of degree 2^n then $LK/\mathbf{Q}_2$ is Galois with group $G(LK/\mathbf{Q}_2) \cong Q_8 \times C_{2^n}$.

On the other hand, if p is odd and F is a p-adic local field then $F^* \otimes \mathbf{Z}/2$ has four elements. Hence if L/F is Galois with $G(L/F) \cong Q_8$ then $N_{L/F}(L^*)$ is equal to the squares in F^*. If E/L is such that E/F is Galois with group $Q_8 \times C_{2^n}$ then there is an intermediate field M/F with $F^*/N_{M/F}(M^*) \cong C_2 \times C_2 \times C_2$ but then the surjection F^* onto $C_2 \times C_2 \times C_2$ must factor through $F^* \otimes \mathbf{Z}/2$ which has only four elements. Thus the expected counterexample to Conjecture 5.1 coming from 3.11 cannot exist when p is odd.

(ii) The prototypical integrality argument for an Explicit Brauer Induction formula is due to Boltje ([SnEBI] Theorem 2.3.43). The symplectic modification of that argument is rather more involved but can be used to establish 2-adic integrality in almost all cases.

In addition, 2-adic integrality holds under the conditions of Theorem 3.21.

6 Induction formula for orthogonal representations

6.1 *Orthogonal representations*

Let G be a finite group. Let $R^o(G)$ denote the Grothendieck group of even-dimensional $\mathbf{R}G$-modules. By definition, this is the quotient group of the free abelian group $\mathcal{F}^o(G)$ over the isomorphism classes of the category of even-dimensional $\mathbf{R}G$-modules, factored out by the subgroup generated by expressions coming from short exact sequences. This gives a canonical surjective morphism

$$\kappa_G : \mathcal{F}^o(G) \to R^o(G) .$$

We identify $R^o(G)$ (resp. $\mathcal{F}^o(G)$) with the Grothendieck group of (resp. free abelian group generated by) equivalent classes of even-dimensional orthogonal representations

$$\rho : G \longrightarrow \mathrm{O}(2n) := \mathrm{O}(2n, \mathbf{R})$$

for some $n \in \mathbf{N}$, and with the group of (resp. free abelian group generated by) $\mathbf{R}$-characters on G. Endowed with the standard maps

$$\mathrm{Res}^G_J : R^o(G) \to R^o(J) \quad \Big(\text{resp. } \mathrm{Res}^G_J : \mathcal{F}^o(G) \to \mathcal{F}^o(J)\Big)$$

and

$$\mathrm{Ind}^G_J : R^o(J) \to R^o(G) \quad \Big(\text{resp. } \mathrm{Ind}^G_J : \mathcal{F}^o(J) \to \mathcal{F}^o(G)\Big)$$

for $J \leq G$, this defines a Mackey functor structure on $H \mapsto R^o(H)$. But $H \mapsto \mathcal{F}^o(H)$ is not a Mackey functor because the Mackey formula does not hold.

Deligne [Ma] has shown that every representation $\rho : G \to \mathrm{O}(2n)$ is a $\mathbf{Z}$-linear combination of two-dimensional orthogonal representations on subgroups induced to G. So there exist $H_i \leq G$, $\Psi_i : H_i \to \mathrm{O}(2)$ and $n_i \in \mathbf{Z}$ such that

$$\rho = \sum_i n_i \mathrm{Ind}^G_{H_i}(\Psi_i) .$$

Thus let $L^o(G)$ (resp. $\mathcal{T}^o(G)$) denote the subgroup in $R^o(G)$ (resp. $\mathcal{F}^o(G)$) generated by the classes of two-dimensional orthogonal $\mathbf{R}H$-modules, that is the $\mathrm{O}(2)$-conjugacy classes of homomorphisms

$$\Psi : H \to \mathrm{O}(2) .$$

These two groups are canonically isomorphic via κ_G.

6.2 *The $_+$-construction*

Let $R^o_+(G)$ denote the Mackey functor obtained by the $_+$-construction on $H \mapsto L^o(H)$. More precisely, let (H, Ψ) be a pair consisting of a subgroup $H \leq G$ and the equivalent class of an orthogonal representation $\Psi : H \to \mathrm{O}(2)$, and let $\mathcal{M}^o(G)$ be the set of all those pairs. There is an obvious action of G on $\mathcal{M}^o(G)$. Let $(H, \Psi)^G$ denote the G-orbit of (H, Ψ) in $\mathcal{M}^o(G)$, and let $\mathcal{M}^o(G)/_G$ denote the set of those orbits. Then $R^o_+(G)$ is defined as the free abelian group generated by the elements of $\mathcal{M}^o(G)/_G$. Indeed $H \mapsto R^o_+(H)$ is the Mackey functor induced by $H \mapsto L^o(H)$, and this comes with homomorphisms

$$\mathrm{Res}_J^G : R_+^o(G) \longrightarrow R_+^o(J)$$

and

$$\mathrm{Ind}_J^G : R_+^o(J) \longrightarrow R_+^o(G)$$

for $J \leq G$. For $N \lhd G$ we have the inflation map

$$\mathrm{Infl}_{G/N}^G : R_+^o(G/N) \rightarrow R_+^o(G)$$

defined by $\mathrm{Infl}_{G/N}^G((HN/N, \overline{\Psi})^G) = (HN, \Psi)^G$ for $\Psi : HN \rightarrow \mathrm{O}(2)$ with $N \leq \ker \Psi$.

Let $b_G^o : R_+^o(G) \rightarrow R^o(G)$ be the homomorphism defined by

$$b_G^o : (H, \Psi)^G \mapsto \mathrm{Ind}_H^G(\Psi) .$$

This map behaves naturally with respect to restriction, induction and inflation.

6.3 *Complexification*

Let c denote the natural homomorphism

$$c = c_G : R^o(G) \longrightarrow R(G)$$

given by embedding $\mathbf{R}$ into $\mathbf{C}$, that is by tensoring with $\mathbf{C}$, so that for $\rho : G \rightarrow \mathrm{O}(2n)$,

$$c(\rho) : G \ni g \mapsto \rho(g) \in \mathrm{O}(2n) \subseteq \mathrm{U}(2n) .$$

For $\Psi : G \rightarrow \mathrm{O}(2)$ we will have to distinguish between the behaviours of $c(\Psi)$. Either $c(\Psi)$ stays irreducible or it splits into a sum of two one-dimensional unitary representations. If $c(\Psi) = \psi$ is irreducible, we will indicate this by writing ψ instead of Ψ. If $c(\Psi) = \lambda + \overline{\lambda}$ with $\lambda : G \rightarrow \mathrm{U}(1)$ not real-valued and $\overline{\lambda}$ the complex conjugated character, we will write $\lambda + \overline{\lambda}$. If $c(\Psi) = 2\phi$ is twice a linear character, we use the notation 2ϕ, and finally in case $c(\Psi)$ is the sum of two different linear characters $\phi + \phi'$ taking values in ± 1 we will indicate this by writing $\phi + \phi'$. So ϕ will always denote a one-dimensional real-valued representation, λ a one-dimensional representation which differs from its complex conjugate denoted $\overline{\lambda}$, and ψ a two-dimensional irreducible representation.

Define the homomorphism

$$c_+ = c_{+,G} : R_+^o(G) \rightarrow R_+(G)$$

by the formula

$$c_+((H, \Psi)^G) = \mathrm{Ind}_H^G(a_H(c(\Psi))) .$$

This definition does not depend on the choice of (H, Ψ) in $(H, \Psi)^G$. Since we will have to apply this formula we will give it in detail:

$$c_+((H,\Psi)^G) = \begin{cases} 2 \cdot (H, \phi)^G & \text{if } c(\Psi) = 2\phi \\ (H,\phi)^G + (H,\phi')^G & \text{if } c(\Psi) = \phi + \phi' \\ (H,\lambda)^G + (H,\overline{\lambda})^G & \text{if } c(\Psi) = \lambda + \overline{\lambda} \end{cases}$$

in case $c(\Psi)$ is not irreducible, and in case $c(\Psi) = \psi$ is irreducible we choose a representative (H, Ψ) of $(H, \Psi)^G$ and define

$$c_+((H,\Psi)^G) = \begin{array}{l} \sum\limits_{\substack{(K,\phi)<..<(K_s,\phi_s) \\ \text{in } \mathcal{M}(H)}} (-1)^s \frac{|K|}{|H|} \left\langle \mathrm{Res}^H_{K_s}(\psi); \phi_s \right\rangle (K,\phi)^G + \\ \sum\limits_{\substack{(K,\phi)<..<(K_s,\lambda_s) \\ \text{in } \mathcal{M}(H)}} (-1)^s \frac{|K|}{|H|} \left\langle \mathrm{Res}^H_{K_s}(\psi); \lambda_s \right\rangle (K,\phi)^G + \\ \sum\limits_{\substack{(K,\lambda)<..<(K_s,\lambda_s) \\ \text{in } \mathcal{M}(H)}} (-1)^s \frac{|K|}{|H|} \left\langle \mathrm{Res}^H_{K_s}(\psi); \lambda_s \right\rangle (K,\lambda)^G \end{array}$$

Observe that b_G and b^o_G are naturally connected via complexification, which means that $b_G \circ c_+ = c_+ \circ b^o_G : R^o_+(G) \to R(G)$.

6.4 *An orthogonal induction formula*

Let $\mathbf{Q}R^o(G) = R^o(G) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q}R^o_+(G) = R^o_+(G) \otimes_{\mathbf{Z}} \mathbf{Q}$. All homomorphims on $R^o(G)$ and $R^o_+(G)$, especially Res^G_J and Ind^J_G, extend in a natural way to homomorphisms between these $\mathbf{Q}$-vectorspaces.

The homomorphism $a^o_G : \mathcal{F}^o(G) \to \mathbf{Q}R^o_+(G)$ is defined by mapping an orthogonal representation $\rho : G \to \mathrm{O}(2n)$ to

$$a^o_G(\rho) = \sum_{(H_0,\Psi_0)<..<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho), \Psi_r)(H_0, \Psi_0)^G$$

with multiplicity m given by the formula

$$m(\theta,\Psi) := \begin{cases} \langle c(\theta); \psi \rangle_H & \text{if } c(\Psi) = \psi \\ \langle c(\theta); \lambda \rangle_H & \text{if } c(\Psi) = \lambda + \overline{\lambda} \\ [\langle c(\theta); \phi \rangle_H / 2] & \text{if } c(\Psi) = 2\phi \\ 1 & \text{if } c(\Psi) = \phi + \phi',\ \langle c(\theta); \phi \rangle_H \text{ odd and } \langle c(\theta); \phi' \rangle_H \text{ odd} \\ 0 & \text{else} \end{cases},$$

for $\theta : H \to \mathrm{O}(2n)$, $\Psi : H \to \mathrm{O}(2)$, where $[x]$ denotes the integral part of a rational number x. Notice that $\langle c(\theta); \lambda \rangle_H = \left\langle c(\theta); \overline{\lambda} \right\rangle_H$.

Note that $m(\Psi, \Psi) = 1$ in all cases.

The following examples will show that in general this homomorphism doesn't factor through $R^o(G)$ and does not take values in $R^o_+(G)$. But first we give an analogue of 3.13.

Proposition 6.5

Let $\rho : G \to \mathrm{O}(2)$ *then*

$$a^o_G(\rho) = (G, \rho)^G .$$

Proof

Since (G,ρ) is the only element in $(G,\rho)^G$ and $m(\rho,\rho)=1$, the coefficient of $(G,\rho)^G$ in $a_G^o(\rho)$ is 1. Now let $(H,\Psi) < (G,\rho)$. Only those elements may give other nontrivial contributions to $a_G^o(\rho)$. Since $\mathrm{Res}^G_{H_r}(\rho) = \Psi_r$ for $(H_r,\Psi_r) < (G,\rho)$, the multiplicities turn out to be 1. Thus we have to show that

$$\sum_{\substack{(H,\Psi)<(H_1,\Psi_1)<..<(H_r,\Psi_r)\\ (H_r,\Psi_r)\le(G,\rho)}} (-1)^r = 0\,.$$

Consider the set, $\mathcal{R}$, of chains the sum runs over. Let $\mathcal{P} < \mathcal{R}$ denote the subset of those chains which will not end in (G,ρ). Then

$$(\,(H,\Psi) < \ldots < (H_r,\Psi_r)\,) \;\mapsto\; (\,(H,\Psi) < \ldots < (H_r,\Psi_r) < (G,\rho)\,)$$

gives a bijection $\mathcal{P} \to \mathcal{R}\setminus\mathcal{P}$, where chains of length r are in correspondence to chains of length $r+1$. So the terms cancel in pairs, and indeed the sum above equals 0. ■

6.6 *Example1:* $G = C_2 \times C_2$

Let $G = \langle A, B | A^2 = B^2 = AB^2 = 1\rangle$ be the Kleinian 4-Group. Let $\mathbb{1}$ denote the trivial character on G and ε_X the linear character on G with kernel $< X >$ for $X = A, B, AB$. By 6.5

$$a_G^o(\mathbb{1}+\varepsilon_A) + a_G^o(\varepsilon_B+\varepsilon_{AB}) = (G,\mathbb{1}+\varepsilon_A)^G + (G,\varepsilon_B+\varepsilon_{AB})^G\,.$$

But for the regular $\rho = \mathbb{1}+\varepsilon_A+\varepsilon_B+\varepsilon_{AB}$ an easy but lengthy calculation gives

$$\begin{aligned} a_G^o(\rho) \;=\;& (G,\mathbb{1}+\varepsilon_A)^G + (G,\mathbb{1}+\varepsilon_B)^G + (G,\mathbb{1}+\varepsilon_{AB})^G + \\ & (G,\varepsilon_A+\varepsilon_B)^G + (G,\varepsilon_A+\varepsilon_{AB})^G + (G,\varepsilon_B+\varepsilon_{AB})^G - 2(\langle A\rangle,\mathbb{1}+(-\mathbb{1}))^G - \\ & 2(\langle B\rangle,\mathbb{1}+(-\mathbb{1}))^G - 2(\langle AB\rangle,\mathbb{1}+(-\mathbb{1}))^G + 2(\langle 1\rangle,2\mathbb{1})^G \end{aligned}$$

Since the element $(\mathbb{1}+\varepsilon_A)+(\varepsilon_B+\varepsilon_{AB}) - (\mathbb{1}+\varepsilon_A+\varepsilon_B+\varepsilon_{AB}) \in \mathcal{F}^o(G)$ will not be killed by a_G^o, a_G^o does not factor through $R^o(G)$.
The situation is the same after applying c because

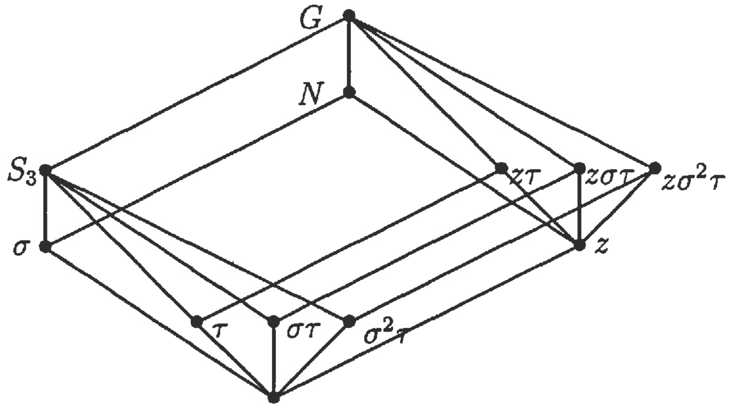
$$a_G(c(\rho)) = (G,\mathbb{1})^G + (G,\varepsilon_A)^G + (G,\varepsilon_B)^G + (G,\varepsilon_{AB})^G\,,$$

while

$$\begin{aligned} c_+(a_G^o(\rho)) \;=\;& 3(G,\mathbb{1})^G + 3(G,\varepsilon_A)^G + 3(G,\varepsilon_B)^G + 3(G,\varepsilon_{AB})^G + \\ & \sum_{X\neq 1}\left(-2(\langle X\rangle,\mathbb{1})^G - 2(\langle X\rangle,(-\mathbb{1}))^G\right) + 4(\langle 1\rangle,\mathbb{1})^G\,. \end{aligned}$$

6.7 *Example2:* $G = C_p \times S_3$

Let $p \geq 5$ be an odd prime and let G be the direct product of a cyclic group of order p and the symmetric group on three letters, so $G = < z, \sigma, \tau | z^p = \sigma^3 = \tau^2 = 1,$ $z\sigma = \sigma z, z\tau = \tau z, \tau\sigma = \sigma^2\tau >$. The lattice of subgroups of G is as pictured.



Let ζ denote a primitive p-th root of unity. The table of irreducible complex characters is giben by

	1	σ, σ^2	$\tau, \sigma\tau, \sigma^2\tau$	z	$z\sigma, z\sigma^2$	$z\tau, z\sigma\tau, z\sigma^2\tau$	z^2	..
$\mathbb{1}$	1	1	1	1	1	1	1	..
ε	1	1	-1	1	1	-1	1	..
ζ	1	1	1	ζ	ζ	ζ	ζ^2	..
$\vdots$	$\vdots$							$\vdots$
$\varepsilon\zeta^{p-1}$	1	1	-1	ζ^{-1}	ζ^{-1}	$-\zeta^{-1}$	ζ^{-2}	..
χ	2	-2	0	2	-2	0	2	..
$\chi\zeta$	2	-2	0	2ζ	-2ζ	0	$2\zeta^2$	..
$\vdots$	$\vdots$							$\vdots$
$\chi\zeta^{p-1}$	2	-2	0	$2\zeta^{-1}$	$-2\zeta^{-1}$	0	$2\zeta^{-2}$	..

We calculate $a_G^o(\rho)$ for the orthogonal representation $\rho : G \longrightarrow \mathrm{O}(4)$ defined by

$$\rho(\sigma) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & 0 & \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \; \rho(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; \rho(z) = \begin{pmatrix} \frac{\zeta+\zeta^{-1}}{2} & 0 & \frac{\zeta-\zeta^{-1}}{-2i} & 0 \\ 0 & \frac{\zeta+\zeta^{-1}}{2} & 0 & \frac{\zeta-\zeta^{-1}}{-2i} \\ \frac{\zeta-\zeta^{-1}}{2i} & 0 & \frac{\zeta+\zeta^{-1}}{2} & 0 \\ 0 & \frac{\zeta-\zeta^{-1}}{2i} & 0 & \frac{\zeta+\zeta^{-1}}{2} \end{pmatrix},$$

which is irreducible as an orthogonal representation, because $c(\rho) = \chi\zeta + \chi\zeta^{p-1}$ splits into two irreducible non-orthogonal representations.
To calculate the formula one has to know the multiplicities $m = m(\mathrm{Res}_H^G(\rho), \Psi)$ for all subgroups $H \leq G$ and all $\Psi : H \longrightarrow \mathrm{O}(2)$. This can be taken out from the following table.

$H, c(\Psi)$	$N, \zeta\zeta_3 + \overline{\zeta\zeta_3}$	$N, \zeta\zeta_3^2 + \overline{\zeta\zeta_3^2}$	$S3, \chi$	$\langle\sigma\rangle, \zeta_3 + \overline{\zeta_3}$	$\langle z\rangle, \zeta + \overline{\zeta}$
m	1	1	2	2	2

$H, c(\Psi)$	$\langle z\tau\rangle, \zeta + \overline{\zeta}$	$\langle z\tau\rangle, \varepsilon\zeta + \overline{\varepsilon\zeta}$	$\langle\tau\rangle, 2\mathbb{1}$	$\langle\tau\rangle, 2\varepsilon$	$1, 2\mathbb{1}$
m	1	1	1	1	2

Here ζ_3 denotes a fixed primitive third root of unity resp. the corresponding representation defined by $z \mapsto \zeta_3$, $\chi : S_3 \longrightarrow \mathrm{O}(2)$ the faithful irreducible representation on S_3 and ε the character on $\langle()\tau\rangle$ sending $\tau \mapsto -1$. Now

$$\begin{aligned}
a^o_G(\rho) &= \frac{1}{2}(1)(N,\zeta\zeta_3+\overline{\zeta\zeta_3})^G + \frac{1}{2}(1)(N,\zeta\zeta_3^2+\overline{\zeta\zeta_3^2})^G + \frac{1}{p}(2)(S_3,\chi)^G \\
&\quad + \frac{1}{2p}(2-(2+1+1))(\langle\sigma\rangle,\zeta_3+\overline{\zeta_3})^G + \frac{1}{3}(1)(\langle z\tau\rangle,\zeta+\overline{\zeta})^G + \frac{1}{3}(1)(\langle z\tau\rangle,\varepsilon\zeta+\overline{\varepsilon\zeta})^G \\
&\quad + \frac{1}{3}(1)(\langle z\sigma\tau\rangle,\zeta+\overline{\zeta})^G + \frac{1}{3}(1)(\langle z\sigma\tau\rangle,\varepsilon\zeta+\overline{\varepsilon\zeta})^G + \frac{1}{3}(1)(\langle z\sigma^2\tau\rangle,\zeta+\overline{\zeta})^G \\
&\quad + \frac{1}{3}(1)(\langle z\sigma^2\tau\rangle,\varepsilon\zeta+\overline{\varepsilon\zeta})^G + \frac{1}{6}(2-(3\cdot(1+1)+2))(\langle z\rangle,\zeta+\overline{\zeta})^G \\
&\quad + \frac{1}{3p}(1-1)(\langle\tau\rangle,2\mathbb{1})^G + \frac{1}{3p}(1-1)(\langle\tau\rangle,2\varepsilon)^G + \frac{1}{3p}(0-2)(\langle\tau\rangle,\mathbb{1}+\varepsilon)^G \\
&\quad + \frac{1}{3p}(1-1)(\langle\sigma\tau\rangle,2\mathbb{1})^G + \frac{1}{3p}(1-1)(\langle\sigma\tau\rangle,2\varepsilon)^G + \frac{1}{3p}(0-2)(\langle\sigma\tau\rangle,\mathbb{1}+\varepsilon)^G \\
&\quad + \frac{1}{3p}(1-1)(\langle\sigma^2\tau\rangle,2\mathbb{1})^G + \frac{1}{3p}(1-1)(\langle\sigma^2\tau\rangle,2\varepsilon)^G + \frac{1}{3p}(0-2)(\langle\sigma^2\tau\rangle,\mathbb{1}+\varepsilon)^G \\
&\quad + \frac{1}{6p}(2-(2+2+3(1+1)+(1+1)+3(1+1)+2)+(4\cdot2+2(1+1)+3\cdot2(1+1)))(\langle 1\rangle,2\mathbb{1})^G \\
&= (N,\zeta\zeta_3+\overline{\zeta\zeta_3})^G + \frac{2}{p}(S_3,\chi)^G - \frac{1}{p}(\langle\sigma\rangle,\zeta_3+\overline{\zeta_3})^G + (\langle z\tau\rangle,\zeta+\overline{\zeta})^G + (\langle z\tau\rangle,\varepsilon\zeta+\overline{\varepsilon\zeta})^G \\
&\quad - (\langle z\rangle,\zeta+\overline{\zeta})^G - \frac{2}{p}(\langle\tau\rangle,\mathbb{1}+\varepsilon)^G - \frac{1}{p}(\langle 1\rangle,2\mathbb{1})^G
\end{aligned}$$

Proposition 6.8

The homomorphisms a^o_G is natural with respect to restriction so that, if $J \le G$,

$$a^o_J \circ \mathrm{Res}^G_J = \mathrm{Res}^G_J \circ a^o_G : \mathcal{F}^o(G) \longrightarrow \mathbf{Q}R^o_+(J)\ .$$

Proof

Similar to the proof of 3.12.

Proposition 6.9

Let G be a finite group. For $\rho : G \longrightarrow \mathrm{O}(2n)$, the defect for commutativity in

$$\begin{array}{ccc}
\mathcal{F}^o(G) & \xrightarrow{\ a^o_G\ } & \mathbf{Q}R^o_+(G) \\
{\scriptstyle \mathrm{co}\kappa_G}\big\downarrow & & {\scriptstyle c_+}\big\downarrow \\
R(G) & \xrightarrow{\ a_G\ } & \mathbf{Q}R_+(G)
\end{array}$$

is given by

$$c_+(a^o_G(\rho)) - a_G(c(\rho)) = \sum_{\substack{(H_0,\phi_0)<\ldots<(H_r,\phi_r) \\ \phi_r \text{ odd}}} (-1)^r \frac{|H_0|}{|G|}(n_{H_r}(\rho)-2)(H_0,\phi_0)^G\ ,$$

where the sum runs over all chains ending in some pair (H_r, ϕ_r) with linear character ϕ_r (taking values in ± 1) such that $\left\langle \mathrm{Res}^G_{H_r}(c(\rho)); \phi_r \right\rangle_{H_r}$ is an odd number, and $n_{H_r}(\rho)$ denotes the number of such characters on H_r.

The proof is of this fact follows directly from lemma 6.11 and lemma 6.12. It is straightforward, but one has to keep book on lots of cases. Therefore we have to introduce some more notation and will prepare the proof in two lemmata.

6.10 *Notation*

Recall the convention explained in 6.3 to denote a base element $(H, \Psi)^G$ of $\mathbf{Q}R^o_+(G)$ by $(H, c(\Psi))^G$ and to use ψ, λ and ϕ to indicate the type of splitting of $c(\Psi)$.
Since we will have to compare coefficients, we use for each for each base element $(H, \Psi)^G \in \mathcal{M}^o(G)/_G$ the homomorphism $\pi_{(H,\Psi)^G} : \mathbf{Q}R^o_+(G) \to \mathbf{Q}$ defined by

$$\pi_{(H,\Psi)^G}((H', \Psi')^G) = \begin{cases} 1 & \text{if } (H, \Psi)^G = (H', \Psi')^G \\ 0 & \text{else} \end{cases}$$

To abbreviate notation we denote for $(H, \Psi) \in \mathcal{M}^o(G)$

$$\mathcal{M}^o(H, \Psi) := \{(H', \Psi') \in \mathcal{M}(G) | (H', \Psi') \leq (H, \Psi)\}$$

and

$$\mathcal{M}(H, \Psi) := \{(H', \varepsilon') \in \mathcal{M}(G) | H' \leq H, \left\langle \mathrm{Res}^H_{H'}(c(\Psi)); \varepsilon' \right\rangle_{H'} > 0\} .$$

Furthermore, for $(H, \phi) \in \mathcal{M}(G)$ and $(K, \Psi) \in \mathcal{M}^o(G)$ we will write $(H, \phi) \preceq (K, \Psi)$, if $H \leq K$ and $\left\langle \mathrm{Res}^K_H(c(\Psi)); \phi \right\rangle_H > 0$. If additionally $H < K$ we write $(H, \phi) \prec (K, \Psi)$. For $(H, \phi)^G \in \mathcal{M}(G)/_G$ and $(K, \Psi)^G \in \mathcal{M}^o(G)/_G$ we use $(H, \phi)^G \preceq (K, \Psi)^G$ (resp. $(H, \phi)^G \prec (K, \Psi)^G$) to express the fact that there exist $(H_0, \phi_0) \in (H, \phi)^G$ and $(K_0, \Psi_0) \in (K, \Psi)^G$ such that $(H_0, \phi_0) \preceq (K_0, \Psi_0)$ (resp. $(H_0, \phi_0) \prec (K_0, \Psi_0)$).
Since ρ will not be changed through our calculations, we will write briefly $\langle \chi \rangle$ instead of $\left\langle \mathrm{Res}^G_H(c(\rho)); \varepsilon \right\rangle_H$, where ε is an irreducible complex representation on a subgroup H of G.
For $\phi : H \to \{\pm 1\}$ we will say "ϕ is odd" or briefly "ϕ odd" , if $\langle \phi \rangle = \left\langle \mathrm{Res}^G_H(\rho \otimes \mathbf{C}); \phi \right\rangle_H$ is an odd number.

Lemma 6.11
Let $(H, \lambda)^G \in \mathcal{M}(G)/_G$ for some $\lambda\colon H \to \mathrm{U}(1)$ with $\lambda \neq \overline{\lambda}$. Then

$$\pi_{(H,\lambda)^G}(a_G(c(\rho))) = \pi_{(H,\lambda)^G}(c_+(a^o_G(\rho))) .$$

Proof
Let $(H, \lambda)^G \in \mathcal{M}(G)/_G$. The coefficient B of $(H, \lambda)^G$ in $a_G(c(\rho))$ is, according to 2.2,

$$B = \frac{|H|}{|G|} \sum_{\substack{(H_0,\lambda_0) < .. < (H_r,\lambda_r) \\ (H_0,\lambda_0) \in (H,\lambda)^G}} (-1)^r \langle \lambda_r \rangle .$$

The calculation of the coefficient C of $(H,\lambda)^G$ in $c_+(a^o_G(\rho))$ takes some more effort. Since c_+ is a homomorphism, it is the coefficient of $(H,\lambda)^G$ in

$$\sum_{(H_0,\Psi_0)<\cdot\cdot<(H_r,\Psi_r)} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\ c_+((H_0,\Psi_0)^G)\,.$$

But $(H,\lambda)^G$ can only have a nontrivial contributions from $c_+((\widetilde{H},\widetilde{\Psi})^G)$ if either $(\widetilde{H},\widetilde{\Psi})^G = (H,\lambda+\overline{\lambda})^G$ or $(\widetilde{H},\widetilde{\Psi})^G = (K,\psi)^G$ with $(H,\lambda+\overline{\lambda})^G < (K,\psi)^G$. So, if for $(\widetilde{H},\widetilde{\Psi}) \in \mathcal{M}^o(G)$

$$C_{\widetilde{H},\widetilde{\Psi}} := \pi_{(H,\lambda)^G}\left(\sum_{\substack{(H_0,\Psi_0)<\cdot\cdot<(H_r,\Psi_r)\\ (H_0,\Psi_0)=(\widetilde{H},\widetilde{\Psi})}} (-1)^r \frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\ c_+((\widetilde{H},\widetilde{\Psi})^G) \right)$$

then C can be expressed as

$$C = \sum_{\substack{(H_0,\lambda_0+\overline{\lambda}_0)\in\mathcal{M}^o(G)\\ (H_0,\lambda_0+\overline{\lambda}_0)^G=(H,\lambda+\overline{\lambda})^G}} C_{H_0,\lambda_0+\overline{\lambda}_0} + \sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\ (K,\psi)^G>(H,\lambda+\overline{\lambda})^G}} C_{K,\psi}.$$

We will calculate these coefficients in each case, starting with $C_{H_0,\lambda_0+\overline{\lambda}_0}$. Without loss of generality we may suppose that λ_0 is such that $(H_0,\lambda_0)^G = (H,\lambda)^G$. With this fixed λ_0 we change notation and write briefly H for H_0 and λ for λ_0. This will not cause any confusion. We decompose $C_{H,\lambda+\overline{\lambda}}$ into the sum of the coefficient of $(H,\lambda)^G$ in

$$\sum_{(H,\lambda+\overline{\lambda})<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r)} (-1)^r \frac{|H|}{|G|} \langle\lambda_r\rangle((H,\lambda)^G + (H,\overline{\lambda})^G)$$

and the coefficient of $(H,\lambda)^G$ in

$$\sum_{\substack{(H,\lambda+\overline{\lambda})<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r)<\\ <(K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s)}} (-1)^{r+s} \frac{|H|}{|G|} \langle\psi_s\rangle((H,\lambda)^G + (H,\overline{\lambda})^G)\,.$$

Any long chain $((H_0,\lambda_0+\overline{\lambda}_0)<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r)<(K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s))$ of length $r+s$ can be broken up uniquely into the *lower* chain $((H_0,\lambda_0+\overline{\lambda}_0)<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r))$ in $\mathcal{M}^o(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s))$ of length $s-1$, and on the other side any such lower and upper chain define a unique long chain. Hence the second sum turns out to be the coefficient of $(H,\lambda)^G$ in

$$\sum_{(K,\psi)>(H,\lambda+\overline{\lambda})}\ \sum_{\substack{(H,\lambda+\overline{\lambda})<\cdot\cdot<(H_r,\lambda_r+\overline{\lambda}_r)\\ \text{in } \mathcal{M}(K,\psi)}}\ \sum_{(K,\psi)<(K_1,\psi_1)<\cdot\cdot<(K_s,\psi_s)} (-1)^{r+s+1} \frac{|H|}{|G|} \langle\psi_s\rangle((H,\lambda)^G + (H,\overline{\lambda})^G)\,.$$

Now, for $(H_i,\lambda_i+\overline{\lambda}_i) > (H,\lambda+\overline{\lambda})$, we choose λ_i such that $\mathrm{Res}^{H_i}_H(\lambda_i) = \lambda$, and distinguish two cases.

If $(H,\lambda)^G \neq (H,\overline{\lambda})^G$, then every chain $((H,\lambda+\overline{\lambda})\lessdot..\lessdot(H_r,\lambda_r+\overline{\lambda}_r))$ starting in $(H,\lambda+\overline{\lambda})$ determines the unique chain $((H,\lambda)\lessdot..\lessdot(H_r,\lambda_r))$ starting in an element of $(H,\lambda)^G$, and the other way arround. Thus, in this case,

$$C_{H,\lambda+\overline{\lambda}} = \frac{|H|}{|G|}\left(\sum_{(H,\lambda)\lessdot..\lessdot(H_r,\lambda_r)} (-1)^r \langle\lambda_r\rangle + \sum_{(K,\psi)>(H,\lambda+\overline{\lambda})} \sum_{\substack{(H,\lambda)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle \right)$$

If $(H,\lambda)^G = (H,\overline{\lambda})^G$, then every chain $((H,\lambda+\overline{\lambda})\lessdot..\lessdot(H_r,\lambda_r+\overline{\lambda}_r))$ determines uniquely the two chains $((H,\lambda)\lessdot..\lessdot(H_r,\lambda_r))$ and $((H,\overline{\lambda})\lessdot..\lessdot(H_r,\overline{\lambda}_r))$ starting in elements of $(H,\lambda)^G$, and the other way arround. Therefore, in this case

$$C_{H,\lambda+\overline{\lambda}} = \frac{|H|}{|G|}\left(\sum_{(H,\lambda)\lessdot..\lessdot(H_r,\lambda_r)} (-1)^r \langle\lambda_r\rangle + \sum_{(H,\overline{\lambda})\lessdot..\lessdot(H_r,\overline{\lambda}_r)} (-1)^r \langle\lambda_r\rangle \right) +$$

$$+\frac{|H|}{|G|} \sum_{(K,\psi)>(H,\lambda+\overline{\lambda})} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \left(\sum_{\substack{(H,\lambda)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^{r+s+1}\langle\psi_s\rangle + \sum_{\substack{(H,\overline{\lambda})\lessdot..\lessdot(H_r,\overline{\lambda}_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^{r+s+1}\langle\psi_s\rangle \right)$$

Next we express $C_{K,\psi}$ with $(K,\psi)^G > (H,\lambda+\overline{\lambda})^G$ for some $(K,\psi) \in \mathcal{M}^o(G)$. By definition, the coefficient of $(H,\lambda)^G$ in $c_+((K,\psi)^G)$ is

$$\sum_{\substack{(H_0,\lambda_0)\in(H,\lambda)^G\\ (H_0,\lambda_0+\overline{\lambda}_0)\lessdot(K,\psi)}} \sum_{\substack{(H_0,\lambda_0)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r \frac{|H|}{|K|} \left\langle \mathrm{Res}^K_{H_r}(\psi); \lambda_r \right\rangle$$

Since $\left\langle \mathrm{Res}^K_{H_i}(\psi); \lambda_i \right\rangle = 1$, the coefficient $C_{K,\psi}$ of $(H,\lambda)^G$ in

$$\sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^s \frac{|K|}{|G|} \left\langle \mathrm{Res}^G_K(c(\rho)); \psi \right\rangle_K c_+((K,\psi)^G)$$

turns out to be

$$\begin{aligned} C_{K,\psi} &= \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^s \frac{|K|}{|G|} \langle\psi_s\rangle \sum_{\substack{(H_0,\lambda_0)\\ \in(H,\lambda)^G}} \sum_{\substack{(H_0,\lambda_0)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r \frac{|H|}{|K|} \\ &= \frac{|H|}{|G|} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\lambda_0)\\ \in(H,\lambda)^G}} \sum_{\substack{(H_0,\lambda_0)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^{s+r}\langle\psi_s\rangle \end{aligned}$$

Using these three expressions for $C_{(\widetilde{H},\widetilde{\Psi})^G}$, the formula for C turns into

$$C = \sum_{\substack{(H_0,\lambda_0+\overline{\lambda}_0)\\ \in(H,\lambda+\overline{\lambda})^G}} C_{H_0,\lambda_0+\overline{\lambda}_0} + \sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\ (K,\psi)^G>(H,\lambda+\overline{\lambda})^G}} c_{K,\psi}$$

$$= \frac{|H|}{|G|}\left(\sum_{(H_0,\lambda_0)\in(H,\lambda)^G} \sum_{(H_0,\lambda_0)<\cdot\cdot<(H_r,\lambda_r)} (-1)^r\langle\lambda_r\rangle + \right.$$

$$\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G>(H,\lambda+\overline{\lambda})^G}} \sum_{(K,\psi)<\cdot\cdot<(K_s,\psi_s)} \sum_{\substack{(H_0,\lambda_0)<\cdot\cdot<(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,p),(H_0,\lambda_0)^G=(H,\lambda)^G}} (-1)^{r+s+1}\langle\psi_s\rangle +$$

$$\left. \sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G>(H,\lambda+\overline{\lambda})^G}} \sum_{(K,\psi)<\cdot\cdot<(K_s,\psi_s)} \sum_{\substack{(H_0,\lambda_0)\\ \in(H,\lambda)^G}} \sum_{\substack{(H_0,\lambda_0)<\cdot\cdot<(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} (-1)^{s+r}\langle\psi_s\rangle \right)$$

The two last terms cancel out, and we finally get

$$\begin{aligned} C &= \frac{|H|}{|G|} \sum_{\substack{(H_0,\lambda_0)\in\mathcal{M}(G)\\(H_0,\lambda_0+\overline{\lambda}_0)^G=(H,\lambda+\overline{\lambda})^G}} \sum_{(H_0,\lambda_0)<\cdot\cdot<(H_r,\lambda_r)} (-1)^r\langle\lambda_r\rangle \\ &= B \end{aligned}$$

as claimed in 6.11. ■

Lemma 6.12

Let $(H,\phi)^G \in \mathcal{M}(G)/_G$ with $\phi : H \to \{\pm 1\}$ an orthogonal representation. Then

$$\pi_{(H,\phi)^G}(c_+(a^o_G(\rho)) - a_G(c(\rho))) = \sum_{\substack{(H_0,\phi_0)<\cdot\cdot<(H_r,\phi_r)\\(H_0,\phi_0)\in(H,\phi)^G,\ \phi_r \text{ odd}}} \frac{|H_0|}{|G|}(-1)^r(n_{H_r}-2)\,,$$

where n_{H_r} denote the number of elements in the set $\{\phi_r : H_r \to \{\pm1\} | \phi_r \text{ odd}\}$.

Proof

Let $(H,\phi)^G \in \mathcal{M}(G)/_G$. The coefficient B of $(H,\phi)^G$ in $a_G(c(\rho))$ is

$$B = \frac{|H|}{|G|}\left(\sum_{\substack{(H_0,\phi_0)<\cdot\cdot<(H_r,\phi_r)\\(H_0,\phi_0)\in(H,\phi)^G}} (-1)^r\langle\phi_r\rangle + \sum_{\substack{(H_0,\phi_0)<\cdot\cdot<(H_r,\lambda_r)\\(H_0,\phi_0)\in(H,\phi)^G}} (-1)^r\langle\lambda_r\rangle \right)$$

The coefficient C of $(H,\phi)^G$ in $c_+(a^o_G(\rho))$ turns out to be the sum of a lot of parial sums coming from different kinds of chains. Since c_+ is a homomorphism, C is the coefficient of $(H,\phi)^G$ in

$$\sum_{(H_0,\Psi_0)<\cdot\cdot<(H_r,\Psi_r)} (-1)^r\frac{|H_0|}{|G|} m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\ c_+((H_0,\Psi_0)^G)\,.$$

Since $(H,\phi)^G$ can only have a nontrivial contribution from elements of the form $c_+((H_0,\Psi_0)^G)$ if $(H_0,\Psi_0)^G$ is either $(H,2\phi)^G$ or $(H,\phi+\phi')^G$ or $(K,\psi)^G$ with $(H,\phi)^G \prec (K,\psi)^G$, we can expresse C as

$$C = \sum_{\substack{(H_0,2\phi_0)\in\mathcal{M}^o(G)\\(H_0,2\phi_0)^G=(H,2\phi)^G}} C_{H_0,2\phi_0} + \sum_{\substack{(H_0,\phi_0+\phi'_0)\in\mathcal{M}^o(G)\\(H_0,\phi_0+\phi'_0)^G=(H,\phi+\phi')^G}} C_{H_0,\phi_0+\phi'_0} + \sum_{\substack{(H_0,\psi_0)\in\mathcal{M}^o(G)\\(H_0,\psi_0)^G\prec(H,\phi)^G}} C_{H_0,\psi_0}$$

where the second subsum runs over all possible pairs $\phi+\phi'$, and, for $(\widetilde{H},\widetilde{\Psi})\in\mathcal{M}^o(G)$,

$$C_{\widetilde{H},\widetilde{\Psi}} := \pi_{(H,\phi)^G}\left(\sum_{\substack{(H_0,\Psi_0)<..<(H_r,\Psi_r)\\ (H_0,\Psi_0)=(\widetilde{H},\widetilde{\Psi})}} (-1)^r\frac{|H_0|}{|G|}m(\mathrm{Res}^G_{H_r}(\rho),\Psi_r)\; c_+((\widetilde{H},\widetilde{\Psi})^G)\right).$$

Now we calculate these coefficients $C_{\widetilde{H},\widetilde{\Psi}}$ in each of the three cases.

We begin with the case $(\widetilde{H},\widetilde{\Psi})^G=(\widetilde{H},2\widetilde{\phi})^G=(H,2\phi)^G$ and can, without causing confusion, change notation back to H instead of $\widetilde{H}$ and ϕ instead of $\widetilde{\phi}$. Using the explicit formula for the multiplicities, we split $C_{H,2\phi}$ into $\frac{|H|}{|G|}\sum_{i=1}^6 C_i(H,2\phi)$, where

$$\begin{aligned}
C_1(H,2\phi) &:= \sum_{(H,2\phi)<..<(H_r,2\phi_r)} (-1)^r 2\,[\langle\phi_r\rangle/2] \\
C_2(H,2\phi) &:= \sum_{\substack{(H,2\phi)<..<(H_i,2\phi_i)<(H_{i+1},\phi_{i+1}+\phi'_{i+1})<\\ <..<(H_r,\phi_r+\phi'_r),\ \phi_r,\phi'_r\ \text{odd}}} (-1)^r 2\cdot 1 \\
C_3(H,2\phi) &:= \sum_{\substack{(H,2\phi)<..<(H_i,2\phi_i)<\\ <(H_{i+1},\lambda_{i+1}+\overline{\lambda}_{i+1})<..<(H_r,\lambda_r+\lambda'_r)}} (-1)^r 2\langle\lambda_r\rangle \\
C_4(H,2\phi) &:= \sum_{\substack{(H,2\phi)<..<(H_r,2\phi_r)<\\ <(K_1,\psi_1)<..<(K_s,\psi_s)}} (-1)^{r+s} 2\langle\psi_s\rangle \\
C_5(H,2\phi) &:= \sum_{\substack{(H,2\phi)<..<(H_i,2\phi_i)<(H_{i+1},\phi_{i+1}+\phi'_{i+1})<..<\\ <(H_r,\phi_r+\phi'_r)<(K_1,\psi_1)<..<(K_s,\psi_s)}} (-1)^{r+s} 2\langle\psi_s\rangle \\
C_6(H,2\phi) &:= \sum_{\substack{(H,2\phi)<..<(H_i,2\phi_i)<(H_{i+1},\lambda_{i+1}+\overline{\lambda}_{i+1})<..<\\ <(H_r,\lambda_r+\overline{\lambda}_r)<(K_1,\psi_1)<..<(K_s,\psi_s)}} (-1)^{r+s} 2\langle\psi_s\rangle
\end{aligned}$$

Now we will modify these terms using the following observations.
1) Of course we can identify any chain of the form $((H,2\phi)<..<(H',2\phi'))$ in $\mathcal{M}^o(G)$ with the corresponding chain $((H,\phi)<..<(H',\phi'))$ in $\mathcal{M}(G)$.
2) Clearly $2\,[\langle\phi_r\rangle/2]$ equals $\langle\phi_r\rangle-1$, if ϕ_r odd, and coincides with $\langle\phi_r\rangle$ otherwise.
3) Any chain $((H,2\phi)<..<(H_r,\lambda_r+\overline{\lambda}_r))$ in $\mathcal{M}^o(G)$ corresponds uniquely to the pair of chains given by $((H,\phi)<..<(H_r,\lambda_r))$ and $((H,\phi)<..<(H_r,\overline{\lambda}_r))$. Thus, taking the sum over all chains $((H,\phi)<..<(H_r,\lambda_r))$ instead of $((H,2\phi)<..<(H_r,\lambda_r+\overline{\lambda}_r))$ gives twice as much summands, and this will take care of the factor 2.
4) A chain $((H,2\phi)<..<(H_r,2\phi_r)<(K_1,\psi_1)<..<(K_s,\psi_s))$ of length $r+s$ breaks uniquely up into the *lower* chain $((H,2\phi)<..<(H_r,2\phi_r))$ in $\mathcal{M}^o(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)<..<(K_s,\psi_s))$ of length $s-1$.
5) A chain $((H,2\phi)<..<(H_r,\phi_r+\phi'_r)<(K_1,\psi_1)<..<(K_s,\psi_s))$ of length $r+s$ breaks up uniquely into the *lower* chain $((H,2\phi)<..<(H_r,\phi_r+\phi'_r))$ in $\mathcal{M}^o(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)<..<(K_s,\psi_s))$ of length $s-1$.
6) A chain $((H,2\phi)<..<(H_r,\lambda_r+\overline{\lambda}_r)<(K_1,\psi_1)<..<(K_s,\psi_s))$ of length $r+s$ breaks up

uniquely into the *lower* chain $((H,2\phi)\lessdot..\lessdot(H_r,\lambda_r+\overline{\lambda}_r))$ in $\mathcal{M}^o(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $s-1$.

Thus we have

$$\begin{aligned}
C_1(H,2\phi) &= \sum_{(H,\phi)\lessdot..\lessdot(H_r,\phi_r)} (-1)^r\langle\phi_r\rangle - \sum_{\substack{(H,\phi)\lessdot..\lessdot(H_r,\phi_r)\\ \phi_r \text{ odd}}} (-1)^r \\
C_2(H,2\phi) &= 2 \sum_{\substack{(H,2\phi)\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} (-1)^r\cdot 1 \\
C_3(H,2\phi) &= \sum_{(H,\phi)\lessdot..\lessdot(H_r,\lambda_r)} (-1)^r\langle\lambda_r\rangle \\
C_4(H,2\phi) &= 2\sum_{(K,\psi)>(H,2\phi)} \sum_{\substack{(H,\phi)\lessdot..\lessdot(H_r,\phi_r)\\ \text{in } \mathcal{M}(K,\psi)}} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle \\
C_5(H,2\phi) &= 2\sum_{(K,\psi)>(H,2\phi)} \sum_{\substack{(H,2\phi)\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \text{in } \mathcal{M}^o(K,\psi)}} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle \\
C_6(H,2\phi) &= \sum_{(K,\psi)>(H,2\phi)} \sum_{\substack{(H,\phi)\lessdot..\lessdot(H_r,\lambda_r)\\ \text{in } \mathcal{M}(K,\psi)}} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle
\end{aligned}$$

Next we study the case $(\widetilde{H},\widetilde{\Psi})^G = (\widetilde{H},\widetilde{\phi}+\widetilde{\phi}')^G = (H,\phi+\phi')^G$ for some $\phi'\neq\phi$. Again we change notation and write H instead of $\widetilde{H}$ and ϕ,ϕ' instead of $\widetilde{\phi},\widetilde{\phi}'$. The explicit formula for the multiplicities transform $C_{H,\phi+\phi'}$ into

$$C_{H,\phi+\phi'} = \frac{|H|}{|G|} m_{H,\phi+\phi'} \left(\sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\lessdot\\ \lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s)}} (-1)^{r+s+1}\langle\psi_s\rangle + \sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} (-1)^r \right)$$

with $m_{H,\phi+\phi'}=2$, if $(H,\phi)^G=(H,\phi')^G$, and $m_{H,\phi+\phi'}=1$ otherwise.
We split any chain $((H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\lessdot(K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $r+s$ into the *lower* chain $((H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r))$ in $\mathcal{M}^o(K_1,\psi_1)$ of length r and the *upper* chain $((K_1,\psi_1)\lessdot..\lessdot(K_s,\psi_s))$ of length $s-1$. So, if

$$C_7(H,\phi+\phi') := m_{H,\phi+\phi'} \sum_{(K,\psi)>(H,\phi+\phi')} \sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \text{in } \mathcal{M}^o(K,\psi)}} \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^{r+s+1}\langle\psi_s\rangle$$

and

$$C_8(H,\phi+\phi') := m_{H,\phi+\phi'} \sum_{\substack{(H,\phi+\phi')\lessdot..\lessdot(H_r,\phi_r+\phi'_r)\\ \phi_r,\phi'_r \text{ odd}}} (-1)^r ,$$

then $C_{H,\phi+\phi'} = \frac{|H|}{|G|}\,(\,C_7(H,\phi+\phi') + C_8(H,\phi+\phi')\,)$.

Finally we expand $C_{\widetilde{H},\widetilde{\Psi}}$ in the case $(\widetilde{H},\widetilde{\Psi})^G = (K,\psi)^G \succ (H,\phi)^G$ for some $(K,\psi) \in \mathcal{M}^o(G)$.
By definition of c_+, the coefficient of $(H,\phi)^G$ in $c_+((K,\psi)^G)$ is

$$\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G \\ (H_0,\phi_0)\prec(K,\psi)}} \frac{|H_0|}{|K|} \left(\sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r \left\langle \mathrm{Res}^K_{H_r}(\psi); \phi_r \right\rangle + \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\lambda_r) \\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r \left\langle \mathrm{Res}^K_{H_r}(\psi); \lambda_r \right\rangle \right).$$

Therefore, the coefficient $C_{K,\psi}$ of $(H,\phi)^G$ in

$$\sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^s \frac{|K|}{|G|} m(\mathrm{Res}^G_K(\rho),\psi)\ (\otimes \mathbf{C}_+)((K,\psi)^G)$$

is given by

$$C_{K,\psi} = \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} (-1)^s \frac{|K|}{|G|} \langle \psi_s \rangle \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G \\ (H_0,\phi_0)\prec(K,\psi)}} \frac{|H|}{|K|} \cdot$$

$$\left(\sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r \left\langle \mathrm{Res}^K_{H_r}(\psi); \phi_r \right\rangle + \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\lambda_r) \\ \text{in } \mathcal{M}(K,\psi)}} (-1)^r \left\langle \mathrm{Res}^K_{H_r}(\psi); \lambda_r \right\rangle \right)$$

This we split into several subsums and obtain $C_{K,\psi} = \frac{|H|}{|G|} \sum_{i=4}^{7} C_i(K,\psi)$, where

$$C_4(K,\psi) := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G \\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi) \\ \mathrm{Res}^K_{H_r}(\psi)=2\phi_r}} (-1)^{r+s} \langle \psi_s \rangle \cdot 2$$

$$C_5(K,\psi) := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G \\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi) \\ \mathrm{Res}^K_{H_r}(\psi)=\phi_r+\phi'_r, \mathrm{Res}^K_{H_0}(\psi)=2\phi_0}} (-1)^{r+s} \langle \psi_s \rangle$$

$$C_6(K,\psi) := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G \\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\lambda_r) \text{in } \mathcal{M}(K,\psi) \\ \mathrm{Res}^K_{H_0}(\psi)=2\phi_0}} (-1)^{r+s} \langle \psi_s \rangle$$

$$C_7(K,\psi) := \sum_{(K,\psi)\lessdot..\lessdot(K_s,\psi_s)} \sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G \\ (H_0,\phi_0)\prec(K,\psi)}} \sum_{\substack{(H_0,\phi_0)\lessdot..\lessdot(H_r,\phi_r) \text{ in } \mathcal{M}(K,\psi) \\ \mathrm{Res}^K_{H_0}(\psi)=\phi_0+\phi'_0}} (-1)^{r+s} \langle \psi_s \rangle$$

Bringing all these expressions together, we conclude

$$C = \sum_{\substack{(H_0,2\phi_0) \\ \in(H,2\phi)^G}} C_{H_0,2\phi_0} + \sum_{(H,\phi+\phi')^G} \sum_{\substack{(H_0,\phi_0+\phi'_0) \\ \in(H,\phi+\phi')^G}} C_{H_0,\phi_0+\phi'_0} + \sum_{\substack{(K,\psi)\in\mathcal{M}^o(G) \\ (K,\psi)^G\succ(H,\phi)^G}} C_{K,\psi}$$

$$= \frac{|H|}{|G|} \left(\sum_{i=1}^{6} \sum_{\substack{(H_0,2\phi_0) \\ \in(H,2\phi)^G}} C_i(H_0,2\phi_0) + \sum_{i=7}^{8} \sum_{(H,\phi+\phi')^G} \sum_{\substack{(H_0,\phi_0+\phi'_0) \\ \in(H,\phi+\phi')^G}} C_i(H_0,\phi_0+\phi'_0) \right.$$

$$+\sum_{i=4}^{7}\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}} C_i(K,\psi)\Bigg).$$

Some of these terms cancel against each other. The terms involving C_4 vanish, as

$$\sum_{\substack{(H_0,2\phi_0)\\\in(H,2\phi)^G}} C_4(H_0,2\phi_0)+\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}} C_4(K,\psi)=$$

$$\sum_{\substack{(H_0,2\phi_0)\\\in(H,2\phi)^G}} 2\sum_{(K,\psi)>(H_0,2\phi_0)}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\\text{in }\mathcal{M}(K,\psi)}}\sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle+$$

$$\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}}\sum_{(K,\psi)<..<(K_s,\psi_s)}\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\(H_0,\phi_0)\prec(K,\psi)}}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\text{ in }\mathcal{M}(K,\psi)\\\mathrm{Res}^K_{H_r}(\psi)=2\phi_r}} 2(-1)^{r+s}\langle\psi_s\rangle=0.$$

The C_5-terms add up to 0, because

$$\sum_{\substack{(H_0,2\phi_0)\\\in(H,2\phi)^G}} C_5(H_0,2\phi_0)+\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}} C_5(K,\psi)=$$

$$\sum_{\substack{(H_0,2\phi_0)\\\in(H,2\phi)^G}} 2\sum_{(K,\psi)>(H_0,2\phi_0)}\sum_{\substack{(H_0,2\phi_0)<..<(H_r,\phi_r+\phi'_r)\\\text{in }\mathcal{M}^o(K,\psi)}}\sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle+$$

$$\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}}\sum_{(K,\psi)<..<(K_s,\psi_s)}\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\(H_0,\phi_0)\prec(K,\psi)}}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\text{ in }\mathcal{M}(K,\psi)\\\mathrm{Res}^K_{H_r}(\psi)=\phi_r+\phi'_r,\mathrm{Res}^K_{H_0}(\psi)=2\phi_0}}(-1)^{r+s}\langle\psi_s\rangle$$

Indeed, every chain $(H_0<..<H_r)$ of subgroups in K determine exactly two chains in $\mathcal{M}(K,\psi)$, namely $((H_0,\phi_0)<..<(H_r,\phi_r))$ and $((H_0,\phi_0)<..<(H_r,\phi'_r))$.
Also the C_6-terms cancel each other, as

$$\sum_{\substack{(H_0,2\phi_0)\\\in(H,2\phi)^G}} C_6(H_0,2\phi_0)+\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}} C_6(K,\psi)=$$

$$\sum_{\substack{(H_0,2\phi_0)\\\in(H,2\phi)^G}}\sum_{(K,\psi)>(H_0,2\phi_0)}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\lambda_r)\\\text{in }\mathcal{M}^o(K,\psi)}}\sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle+$$

$$\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}}\sum_{(K,\psi)<..<(K_s,\psi_s)}\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\(H_0,\phi_0)\prec(K,\psi)}}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\lambda_r)\text{in }\mathcal{M}(K,\psi)\\\mathrm{Res}^K_{H_0}(\psi)=2\phi_0}}(-1)^{r+s}\langle\psi_s\rangle=0$$

Finally, the terms with C_7 go off, since

$$\sum_{(H,\phi+\phi')^G}\sum_{\substack{(H_0,\phi_0+\phi'_0)\\\in(H,\phi+\phi')^G}} C_7(H_0,\phi_0+\phi'_0)+\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}} C_7(K,\psi)=$$

$$\sum_{(H,\phi+\phi')^G}\sum_{\substack{(H_0,\phi_0+\phi'_0)\\\in(H,\phi+\phi')^G}} m_{H_0,\phi_0+\phi'_0}\sum_{(K,\psi)>(H_0,\phi_0+\phi'_0)}\sum_{\substack{(H_0,\phi_0+\phi'_0)<..<(H_r,\phi_r+\phi'_r)\\\text{in }\mathcal{M}^o(K,\psi)}}\sum_{(K,\psi)<..<(K_s,\psi_s)}(-1)^{r+s+1}\langle\psi_s\rangle+$$

$$\sum_{\substack{(K,\psi)\in\mathcal{M}^o(G)\\(K,\psi)^G\succ(H,\phi)^G}}\sum_{(K,\psi)<..<(K_s,\psi_s)}\sum_{\substack{(H_0,\phi_0)\in(H,\phi)^G\\(H_0,\phi_0)\prec(K,\psi)}}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\text{ in }\mathcal{M}(K,\psi)\\\mathrm{Res}^K_{H_0}(\psi)=\phi_0+\phi'_0}}(-1)^{r+s}\langle\psi_s\rangle$$

In fact this is 0, because if we have a pair $(K,\psi) > (H_0, \phi_0 + \phi_0')$ with $(H_0,\phi_0)^G = (H,\phi)^G \neq (H_0,\phi_0')^G$, the $m_{H_0,\phi_0+\phi_0'} = 1$, and taking sums over chains in $\mathcal{M}^o(K,\psi)$ starting in $(H_0,\phi_0+\phi_0')$ is the same as taking sums over chains in $\mathcal{M}(K,\psi)$ starting in (H_0,ϕ_0). Otherwise, if a pair $(K,\psi) > (H_0,\phi_0+\phi_0')$ satisfies $(H_0,\phi_0)^G = (H,\phi)^G = (H_0,\phi_0')^G$, then $m_{H_0,\phi_0+\phi_0'} = 2$, and taking sums over chains in $\mathcal{M}(K,\psi)$ starting in (H_0,ϕ_0) or (H_0,ϕ_0') is twice as much as taking sums over chains in $\mathcal{M}^o(K,\psi)$ starting in $(H_0,\phi_0+\phi_0')$.

So C reduces to

$$C = \frac{|H|}{|G|}\left(\sum_{i=1}^{3}\sum_{\substack{(H_0,2\phi_0)\\ \in(H,2\phi)^G}} C_i(H_0,2\phi_0) + \sum_{(H,\phi+\phi')^G}\sum_{\substack{(H_0,\phi_0+\phi_0')\\ \in(H,\phi+\phi')^G}} C_8(H_0,\phi_0+\phi_0')\right).$$

Now we can compare B and C, which is the coefficient of $(H,\phi)^G$ in $c_+(a_G^o(\rho)) - a_G(c(\rho)$. We notice that, taking the expressions for B and C into account,

$$\frac{|H|}{|G|}\sum_{(H_0,\phi_0)\in(H,\phi)^G}(C_1(H_0,2\phi_0)+C_3(H_0,2\phi_0)) = B + \frac{|H|}{|G|}\sum_{(H_0,\phi_0)\in(H,\phi)^G}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}}(-1)^r .$$

Thus we have to compute

$$\sum_{(H_0,\phi_0)\in(H,\phi)^G}\left(\sum_{\substack{(H_0,2\phi_0)<..<(H_r,\phi_r+\phi_r')\\ \phi_r,\phi_r' \text{ odd}}} 2(-1)^r - \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}}(-1)^r\right)$$

and

$$\sum_{(H,\phi+\phi')^G}\sum_{(H_0,\phi_0+\phi_0')\in(H,\phi+\phi')^G} C_8(H_0,\phi_0+\phi_0').$$

The first term can be simplified by the following observation. Let $(H_0,\phi_0)\in(H,\phi)^G$ fixed and, for $H_0 \leq H_r \leq G$, let $X_{H_r,\phi_0} := \{\phi_r : H_r \to \pm 1 | \mathrm{Res}^{H_r}_{H_0}(\phi_r) = \phi_0, \phi_r \text{ odd}\}$ and $n = n_{H_r,\phi_0}$ denote the number of elements in X_{H_r,ϕ_0}. Then there are precisely n elements $(H_r,\phi_r)\in\mathcal{M}(G)$ with $(H_0,\phi_0)\leq(H_r,\phi_r)$ and ϕ_r odd, and precisely $\frac{n(n-1)}{2}$ elements $(H_r,\phi_r+\phi_r')\in\mathcal{M}(G)$ with $(H_0,2\phi_0)\leq(H_r,\phi_r+\phi_r')$ and ϕ_r,ϕ_r' odd. Thus

$$\sum_{\substack{(H_0,2\phi_0)<..<(H_r,\phi_r+\phi_r')\\ \phi_r,\phi_r' \text{ odd}}} 2(-1)^r = \sum_{H_0<..<H_r} 2\frac{n_{H_r,\phi_0}(n_{H_r,\phi_0}-1)}{2}(-1)^r = \sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}}(-1)^r(n_{H_r,\phi_0}-1)$$

So the first term can be rewritten as

$$\sum_{(H_0,\phi_0)\in(H,\phi)^G}\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ \phi_r \text{ odd}}}(-1)^r(n_{H_r,\phi_0}-2)$$

Next we reduce the second expression. Let $(H_0, \phi_0) \in (H,\phi)^G$ fixed. For $H_0 \leq H_r \leq G$ let $X'_{H_r,\phi_0} := \{\phi_r : H_r \to \pm 1 | \mathrm{Res}^{H_r}_{H_0}(\phi_r) \neq \phi_0, \phi_r \text{ odd}\}$ and $n' = n'_{H_r,\phi_0}$ denote the number of elements in X'_{H_r,ϕ_0}. Now

$$\sum_{\phi'_0 \neq \phi_0} \sum_{\substack{(H_0,\phi_0+\phi'_0) \lessdot .. \lessdot (H_r,\phi_r+\phi'_r) \\ \phi_r,\phi'_r \text{ odd}}} (-1)^r = \sum_{\substack{(H_0,\phi_0) \lessdot .. \lessdot (H_r,\phi_r) \\ \phi_r \text{ odd}}} (-1)^r n'_{H_r,\phi_0} ,$$

since, for any chain $(H_0, \phi_0) \lessdot .. \lessdot (H_r, \phi_r)$ with ϕ_r odd, a chain $(H_0, \phi_0 + \phi'_0) \lessdot .. \lessdot (H_r, \phi_r + \phi'_r)$ with ϕ_r, ϕ'_r odd determines and is determined by $\phi'_r \in X'_{H_r,\phi_0}$. Furthermore, for (H_r, ϕ_r) fixed with $(H_r, \phi_r)^G \succeq (H, \phi)^G$,

$$\sum_{\substack{(H_0,\phi_0+\phi'_0) \\ (H_0,\phi_0+\phi'_0)^G \succeq (H,\phi_0)^G}} \sum_{\substack{(H_0,\phi_0+\phi'_0) \lessdot .. \lessdot (H_r,\phi_r+\phi'_r) \\ \phi_r,\phi'_r \text{ odd}}} m_{H_0,\phi_0+\phi'_0} (-1)^r = \sum_{(H_0,\phi_0) \in (H,\phi)^G} \sum_{\substack{(H_0,\phi_0) \lessdot .. \lessdot (H_r,\phi_r) \\ \phi_r \text{ odd}}} (-1)^r n'_{H_r,\phi_0} ,$$

as $m_{H_0,\phi_0+\phi'_0} = 1$ in the case $(H_0, \phi_0)^G \neq (H_0, \phi'_0)^G$, and $m_{H_0,\phi_0+\phi'_0} = 2$ in the case $(H_0, \phi_0)^G = (H_0, \phi'_0)^G$. Finally, we may take the sum over all (H_r, ϕ_r) to rewrite the second expression as

$$\sum_{(H_0,\phi_0) \in (H,\phi)^G} \sum_{\substack{(H_0,\phi_0) \lessdot .. \lessdot (H_r,\phi_r) \\ \phi_r \text{ odd}}} (-1)^r n'_{H_r,\phi_0}$$

Taking these simplifications into account, we conclude from $n_{H_r,\phi_0} + n'_{H_r,\phi_0} = n_{H_r}$, that

$$C - B = \frac{|H|}{|G|} \sum_{\substack{(H_0,\phi_0) \lessdot .. \lessdot (H_r,\phi_r) \\ (H_0,\phi_0) \in (H,\phi)^G, \phi_r \text{ odd}}} (-1)^r (n_{H_r} - 2)$$

and lemma 6.12 is proved. ■

Theorem 6.13

The map a^o_G induced an explicit induction formula, that is, for $\rho : G \to \mathrm{O}(2n)$ in $\mathcal{F}^o(G)$,

$$b^o_G(a^o_G(\rho)) = \kappa_G(\rho) \in R^o(G) .$$

Proof

We study the diagram

$$\begin{array}{ccccc} \mathcal{F}^o(G) & \xrightarrow{a^o_G} & \mathbb{Q}R^o_+(G) & \xrightarrow{b^o_G} & \mathbb{Q}R^o(G) \\ {\scriptstyle co\kappa_G}\downarrow & & {\scriptstyle c_+}\downarrow & & {\scriptstyle c}\downarrow \\ \mathbb{Q}R(G) & \xrightarrow{a_G} & \mathbb{Q}R_+(G) & \xrightarrow{b_G} & \mathbb{Q}R(G) \end{array}$$

Note that the right square is a commutative diagram, since all maps involved are homomorphisms and a_G, used in the definition of c_+, is a section for b_G. Indeed

$$b_G(c_+((H,\Psi)^G)) = b_G(\mathrm{Ind}^G_H(a_H(c(\Psi)))) = \mathrm{Ind}^G_H(b_H a_H(c(\Psi))) = \mathrm{Ind}^G_H(c(\Psi)) = c(\mathrm{Ind}^G_H(\Psi)) .$$

Since c_+ is injective, it suffices to show

$$c_+((b^o_G \circ a^o_G)(\rho)) = c(\kappa_G(\rho)) \ \ (= (b_G \circ a_G)(c \circ \kappa_G)(\rho))\ .$$

Therefore it is enough to prove that the defect on the commutativity in the left square, given by (6.9), vanishes after been hidden with b_G. In fact, for $\rho : G \longrightarrow \mathrm{O}(2n)$ and $n_H(\rho)$ as in 6.9, we calculate

$$\begin{aligned}
&b_G\,(c_+(a^o_G(\rho)) \;-\; a_G(c(\rho))) = \\
&b_G\left(\sum_{\substack{(H,\phi)\\ \phi\ \mathrm{odd}}}(n_H(\rho)-2)\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_r,\phi_r)=(H,\phi)}}(-1)^r\frac{|H_0|}{|G|}(H_0,\phi_0)^G\right) = \\
&\sum_{\substack{(H,\phi)\\ \phi\ \mathrm{odd}}}(n_H(\rho)-2)\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_r,\phi_r)=(H,\phi)}}(-1)^r\frac{|H_0|}{|G|}\mathrm{Ind}^G_{H_0}(\phi_0) = \\
&\sum_{\substack{(H,\phi)\\ \phi\ \mathrm{odd}}}(n_H(\rho)-2)\frac{|H|}{|G|}\mathrm{Ind}^G_H(\sum_{\substack{(H_0,\phi_0)<..<(H_r,\phi_r)\\ (H_r,\phi_r)=(H,\phi)}}(-1)^r\frac{|H_0|}{|H|}\mathrm{Ind}^H_{H_0}(\phi_0)) = \\
&\sum_{\substack{(H,\phi)\\ \phi\ \mathrm{odd}}}(n_H(\rho)-2)\frac{|H|}{|G|}\mathrm{Ind}^G_H(\phi\sum_{\substack{H_0<..<H_r\\ H_r=H}}(-1)^r\frac{|H_0|}{|H|}\mathrm{Ind}^H_{H_0}(\mathbb{1})) = 0\ ,
\end{aligned}$$

since firstly, for $H \leq G$ a fixed noncyclic group,

$$\sum_{H_0<..<H_r=H}(-1)^r\frac{|H_0|}{|H|}\mathrm{Ind}^H_{H_0}(\mathbb{1}) = 0\ . \qquad (1)$$

(see for example [Bo3, III.1.4]), and secondly, if $H \leq G$ has a nontrivial contribution to the sum above, then $n_H(\rho) > 0$ (so there exists an odd ϕ) and $n_H(\rho) - 2 \neq 0$ (so $n_H(\rho) \geq 4$), so that H has an elementary abelian 2 group of order at least 4 as a factor group and can not be cyclic. ■

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Institut für Mathematik
Universitätsstraße 14
D-86135 Augsburg, Germany
olaf.neisse@math.uni-augsburg.de

Faculty of Mathematical Studies
University of Southampton
Southampton, SO17 1BJ United Kingdom
V.P.Snaith@maths.soton.ac.uk