Cubical homotopy theory: a beginning

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Introduction

This paper displays a closed model structure for the category of cubical sets and shows that the resulting homotopy category is equivalent to the ordinary homotopy category for topological spaces. The main results are Theorem 19, which gives the model structure, and Theorem 29 and Corollary 30 which together imply the equivalence of homotopy categories.

The cofibrations and weak equivalences for the theory are what one might expect, namely levelwise inclusions and maps which induce weak equivalences of topological spaces respectively. The closed model structure is relatively easy to derive, once one gets away from the preconception that fibrations should be defined by analogy with Kan fibrations. A fibration is defined to be a map which has the right lifting property with respect to all trivial cofibrations. The verification of the closed model axioms is essentially formal, and is displayed here (see also [4]) as a consequence of standard tricks from localization theory having to do with a bounded cofibration condition for countable complexes. The equivalence of the homotopy category of cubical complexes with the ordinary homotopy category is much more interesting, and follows from the assertion that the cubical singular functor satisfies excision in a non-abelian sense.

There is an underlying category of models, namely the box category $\square$, which is used to define cubical sets in the same way that the category of ordinal numbers defines simplicial sets. This means that a cubical set X is defined as a contravariant functor $X : \square^{op} \to \mathbf{Set}$ on the box category, taking values in the category of sets. The box category and its basic properties are the subject of the first section of this paper, while the first properties of cubical sets are described in the second section. The closed model structure is derived in Section 3, and appears as Theorem 19.

The assertion that the homotopy categories of cubical sets and topological spaces (or simplicial sets) are equivalent involves the final three sections of this paper.

One needs a good subdivision operator. There is certainly an obvious subdivision of an n-cube, which is just a product of barycentric subdivisions of

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intervals. The subdivision $\operatorname{sd} X$ of a cubical set X is constructed from this naive subdivision of the n-cube in the end, but a functorial description of the subdivision of cubes is required to make it work. This is done by showing that the naive subdivision is isomophic to a cubical complex associated to the poset of non-degenerate cells of the n-cube.

The catch is that the standard, easy relationship between posets and simplicial complexes is lost in the cubical setting. Cubical complexes (meaning subcomplexes of standard n-cubes) have posets of non-degenerate cells which have extra structure, formalized here as cubical posets. Furthermore, the cubical nerves of these posets are too big to be useful, but cubical posets have "minimal" cubical nerves which are cubical complexes. The naive subdivision of the n-cube is isomorphic to the minimal cubical nerve of the poset of non-degenerate cells of the cube, and the latter is the functorial construction on cubical complexes which gives rise to the subdivision operator for all cubical sets. These ideas are the subject of Sections 4 and 5 of this paper.

Despite the apparent conceptual pain of the construction of the cubical subdivision functor, the functor itself is much better behaved that the subdivision functor for simplicial sets, in that there is a canonical map $\gamma : \operatorname{sd} X \to X$ as for simplicial sets, but there is also a natural homeomorphism $h : |\operatorname{sd} X| \cong |X|$ of the associated topological spaces, and a natural homotopy $h \simeq |\gamma|$. The naturality of both the map h and the homotopy effectively does away with the necessity for showing that every cubical set can be refined by a cubical complex; this is quite unlike the corresponding situation for simplicial sets, where one needs to show that every simplicial set can be refined by a simplicial complex, via double subdivision. The proof of the cubical excision theorem (Theorem 27) makes direct use of these constructions, and then the comparison of homotopy categories (Theorem 29, Corollary 30) follows relatively quickly. These results are proved in Section 6.

I should say that none of this went exactly according to plan. The idea at the outset (and this view has been generally held) was that one should be able to develop the homotopy theory of cubical sets by analogy with the homotopy theory of simplicial sets. Unfortunately for that point of view (see Remark 8), the topological realization functor does not preserve products, even up to weak equivalence, and this has the ultimate effect of breaking the analogue of the theory of minimal fibrations. In fact, the standard n-cells $\square^n$ are not even contractible within the category of cubical sets even though their realizations are hypercubes. This phenomenon can be partially fixed by adding the Brown-Higgins connections [3] as an auxilliary set of degeneracies; this works for a long time (there's even a closed model structure), but then one sees finally that connections do not respect the subdivision operator. Connections are important and one can do a lot with them, but it appears that they will have to be addressed within the homotopy theory of cubical complexes from a more subtle point of view.

There is a a theory of combinatorial fibrations, which is defined by obvious analogy with the theory of Kan fibrations of simplicial sets, but is not displayed here. The analogy goes far enough to produce a decently behaved theory of combinatorial homotopy groups. There is a Milnor theorem which asserts that the canonical map $X \to S|X|$ induces an isomorphism between the combinatorial homotopy groups of a fibrant cubical set X and the homotopy groups of the associated space $|X|$, but one would like to have this statement hold more generally for all combinatorially fibrant objects. Its proof should have something to do with a cubical approximation theorem, meaning a suitable analogue of simplicial approximation. Cubical approximation has not been proved, and it may not yet even have a suitable expression — it appears to be one of the "hard" things that will be possible to properly state and prove only once substantial portions of the rest of the homotopy theory of cubical sets are properly developed.

There's a final punch line: one can go back and develop the homotopy theory of simplicial sets by analogy with the results given here. The closed model structure for simplicial sets is much easier to derive from this point of view, and one can prove an exact analogue of the excision result given here for simplicial sets once one understands (and successfully proves) that simplicial approximation is really about showing that an arbitrary simplicial set can be refined by a simplicial complex up to weak equivalence (see [5], [6]) — this is a much more delicate statement than the approximation technique that is used here. This collection of ideas will be the subject of a future paper.

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Contents

1 The box category

Write $\underline{\mathrm{n}} = \{1, 2, \ldots, n\}$, and let $\mathbf{1}^n$ be the n-fold product of copies of the category $\mathbf{1}$ defined by the ordinal number $\mathbf{1} = \{0, 1\}$ of the same name. Write $\mathbf{1}^0$ for the category consisting of one object and one morphism.

A face functor $(d, \epsilon_i) : \mathbf{1}^m \to \mathbf{1}^n$ is defined by an ordered inclusion $d : \underline{\mathrm{m}} \to \underline{\mathrm{n}}$ and a set of elements $\epsilon_i \in \{0, 1\}$, $i \in \underline{\mathrm{n}} - \underline{\mathrm{m}}$. The corresponding functor is specified by the diagrams

$$\begin{array}{ccc} \mathbf{1}^m & \xrightarrow{(d,\epsilon_i)} & \mathbf{1}^n \\ & {\scriptstyle d_i} \searrow & \downarrow {\scriptstyle pr_i} \\ & & \mathbf{1} \end{array}$$

where d_i is the projection $pr_{d^{-1}(i)}$ if i is in the image of d, and d_i is the constant functor at ϵ_i for $i \in \underline{\mathrm{n}} - \underline{\mathrm{m}}$.

A degeneracy functor $s = s_d : \mathbf{1}^n \to \mathbf{1}^k$ is specified by an ordered inclusion $d : \underline{\mathrm{k}} \to \underline{\mathrm{n}}$. In effect, the diagram

$$\begin{array}{ccc} \mathbf{1}^r & \xrightarrow{s_d} & \mathbf{1}^k \\ & {\scriptstyle pr_{d(i)}} \searrow & \downarrow {\scriptstyle pr_i} \\ & & \mathbf{1} \end{array}$$

is required to commute.

There is an isomorphism of posets

$$\Omega_n : \mathbf{1}^n \xrightarrow{\cong} \mathcal{P}(\underline{\mathrm{n}})$$

which is defined by associating to the n-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ the subset

$$\Omega_n(\epsilon) = \{i \mid \epsilon_i = 1\}$$

of the set $\underline{\mathrm{n}} = \{1, \ldots, n\}$.

Suppose that $(d, \epsilon_i) : \mathbf{1}^m \to \mathbf{1}^n$ is a face functor, and consider the composite poset morphism

$$\mathbf{1}^n \xrightarrow{(d,\epsilon_i)} \mathbf{1}^m \xrightarrow{\Omega_m} \mathcal{P}(\underline{\mathrm{n}}).$$

Suppose that $A = \Omega_m(d, \epsilon_i)(0, \ldots, 0)$ and that $B = \Omega_m(d, \epsilon_i)(1, \ldots, 1)$. Write $[A, B]$ for the subposet of $\mathcal{P}(\underline{\mathrm{m}})$ consisting of all subsets C such that $A \subset C \subset B$. The poset $[A, B]$ is often called the *interval* between A and B. Then one can show that there is a commutative diagram of poset morphisms

$$\begin{array}{ccc} \mathbf{1}^n & \xrightarrow{(d,\epsilon_i)} & \mathbf{1}^m \\ {\scriptstyle \Omega_n} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \Omega_m} \\ \mathcal{P}(\underline{\mathrm{n}}) & \xrightarrow[d_*]{} & \mathcal{P}(\underline{\mathrm{m}}) \end{array}$$

where the poset morphism d_* is defined by $C \mapsto d(C) \cup B$. Note that the odered inclusion $d : \underline{\mathrm{n}} \to \underline{\mathrm{m}}$ determines a bijection $\underline{\mathrm{n}} \cong B - A$, and that d_* induces a poset isomorphism $\mathcal{P}(\underline{\mathrm{n}}) \cong [A, B]$.

An ordered inclusion $d : \underline{\mathrm{k}} \to \underline{\mathrm{n}}$ can be identified with a subset $A \subset \underline{\mathrm{n}}$ of order k in the obvious way, and any degeneracy $s_d : \mathbf{1}^n \to \mathbf{1}^k$ sits in a commutative diagram

$$
\begin{array}{ccc}
\mathbf{1}^n & \xrightarrow{s_d} & \mathbf{1}^k \\
{\scriptstyle \Omega_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Omega_k} \\
\mathcal{P}(\underline{\mathrm{n}}) & & \mathcal{P}(\underline{\mathrm{k}}) \\
& \searrow & \downarrow{\scriptstyle \cong} \\
& & \mathcal{P}(A)
\end{array}
$$

where the indicated isomorphism is determined by a canonical order preserving bijection $\underline{\mathrm{k}} \cong A$ and the morphism $\mathcal{P}(\underline{\mathrm{n}}) \to \mathcal{P}(A)$ is defined by $C \mapsto C \cap A$.

In the definition of both face and degeneracy functors, the ordered inclusions can be replaced by choices of subsets. In effect, a subset of $\underline{\mathrm{n}}$ having k-elements determines a unique ordered inclusion $\underline{\mathrm{k}} \subset \underline{\mathrm{n}}$.

Consider the composite functor

$$\mathbf{1}^m \xrightarrow{(d,\epsilon_i)} \mathbf{1}^n \xrightarrow{s} \mathbf{1}^k$$

There is a pullback diagram of order preserving functions

$$
\begin{array}{ccc}
\underline{\mathrm{m}} & \xrightarrow{d} & \underline{\mathrm{n}} \\
{\scriptstyle s'}\uparrow & & \uparrow{\scriptstyle s} \\
\underline{\mathrm{r}} & \xrightarrow[d']{} & \underline{\mathrm{k}}
\end{array}
$$

and there is a corresponding commutative diagram of face and degeneracy functors

$$
\begin{array}{ccc}
\mathbf{1}^m & \xrightarrow{(d,\epsilon_i)} & \mathbf{1}^n \\
\downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\
\mathbf{1}^r & \xrightarrow[(d',\epsilon_{s(i)})]{} & \mathbf{1}^k
\end{array}
\tag{1}
$$

The sets of face and degeneracy functors are each closed under composition, and degeneracy functors can be "moved past" face functors according to the recipe specified above.

We shall write $d = (d, \epsilon_i)$ for face functors in the following, except in places where the ambiguity could cause confusion.

Lemma 1. *Suppose given a commutative diagram*

$$\begin{array}{ccc} \mathbf{1}^m & \xrightarrow{s} & \mathbf{1}^n \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle d} \\ \mathbf{1}^{n'} & \xrightarrow[d']{} & \mathbf{1}^k \end{array}$$

composed of face functors d, d' and degeneracies s, s'. Then $d = d'$ and $s = s'$.

Proof. There is a face functor $\tilde{d}$ which is a section of s. Write $\theta = s'\tilde{d} : \mathbf{1}^n \to \mathbf{1}^{n'}$. The functor d' has a left inverse given by a degeneracy, and is therefore a monomorphism. Then

$$d'\theta = d's'\tilde{d} = ds\tilde{d} = d,$$

while

$$d'\theta s = ds = d's'$$

so that $\theta s = s'$. The functor θ is also the unique functor which makes the diagram

$$\begin{array}{ccc} \mathbf{1}^m & \xrightarrow{s} & \mathbf{1}^n \\ {\scriptstyle s'}\downarrow & \swarrow{\scriptstyle \theta} & \downarrow{\scriptstyle d} \\ \mathbf{1}^{n'} & \xrightarrow[d']{} & \mathbf{1}^k \end{array}$$

commute. There is similarly a uniquely determined functor $\theta' : \mathbf{1}^{n'} \to \mathbf{1}^n$ which makes the diagram

$$\begin{array}{ccc} \mathbf{1}^m & \xrightarrow{s} & \mathbf{1}^n \\ {\scriptstyle s'}\downarrow & \nearrow{\scriptstyle \theta'} & \downarrow{\scriptstyle d} \\ \mathbf{1}^{n'} & \xrightarrow[d']{} & \mathbf{1}^k \end{array}$$

commute. It follows that the functor θ is an isomorphism of categories. In particular, $n = n'$.

The functor θ has a factorization

$$\begin{array}{ccc} \mathbf{1}^n & \xrightarrow[\cong]{\theta} & \mathbf{1}^n \\ & {\scriptstyle p}\searrow \qquad \nearrow{\scriptstyle \mu} & \\ & \mathbf{1}^r & \end{array}$$

where p is a degeneracy functor and μ is a face functor. Then p is a monomorphism as well as an epimorphism. If $r < n$ then

$$p(\epsilon_1, \dots, \overset{i}{0}, \dots, \epsilon_n) = p(\epsilon_1, \dots, \overset{i}{1}, \dots, \epsilon_n)$$

for $i \notin \underline{r}$ and p cannot be a monomorphism. It follows that $r = n$ and $p = 1$, since there is only one order-preserving monomorphism $\underline{n} \to \underline{n}$. It also follows that $\mu = 1$, and hence that $\theta = 1$. □

The box category $\square$ is the subcategory of the category of small categories which is generated by the face and degeneracy functors. Its objects consist of the categories $\mathbf{1}^k$, $k \geq 0$, and it follows from Lemma 1 that a morphism $\theta : \mathbf{1}^n \to \mathbf{1}^m$ in $\square$ can be uniquely written as a composite

$$\begin{array}{ccc} \mathbf{1}^n & \xrightarrow{\theta} & \mathbf{1}^m \\ & {\scriptstyle s} \searrow \quad \nearrow {\scriptstyle d} & \\ & \mathbf{1}^k & \end{array}$$

where s is a degeneracy functor and d is a face functor. Morphisms in the box category are also called cubical functors.

The pair (i, ϵ) consisting of $i \in \underline{n}$ and $\epsilon \in \{0, 1\}$ determines a unique face functor $d^{(i,\epsilon)} : \mathbf{1}^{n-1} \to \mathbf{1}^n$, defined by

$$d^{(i,\epsilon)}(\gamma_1, \dots, \gamma_{n-1}) = (\gamma_1, \dots, \overset{i}{\epsilon}, \dots, \gamma_{n-1}).$$

Suppose that $i < j$. Then there is a commutative diagram of face functors

$$\begin{array}{ccc} \mathbf{1}^{n-2} & \xrightarrow{d^{(i,\epsilon_1)}} & \mathbf{1}^{n-1} \\ {\scriptstyle d^{(j-1,\epsilon_2)}} \downarrow & & \downarrow {\scriptstyle d^{(j,\epsilon_2)}} \\ \mathbf{1}^{n-1} & \xrightarrow[d^{(i,\epsilon_1)}]{} & \mathbf{1}^n \end{array} \tag{2}$$

if $n \geq 2$. If $i = j$ there is a diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathbf{1}^{n-1} \\ \downarrow & & \downarrow {\scriptstyle d^{(i,1)}} \\ \mathbf{1}^{n-1} & \xrightarrow[d^{(i,0)}]{} & \mathbf{1}^{n-1} \end{array} \tag{3}$$

The degeneracy functor $s^j : \mathbf{1}^n \to \mathbf{1}^{n-1}$ is the projection which forgets the j^{th} factor, so that

$$s^j(\gamma_1, \dots, \gamma_n) = (\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n)$$

Write $s^1 : \mathbf{1} \to \mathbf{1}^0$ for the obvious map to the terminal object $\mathbf{1}^0$ in the box category $\square$.

Then there are relations

$$s^j s^i = s^i s^{j+1}, \quad \text{if } i \leq j. \tag{4}$$

Similarly,

$$s^j d^{(j,\epsilon)} = 1, \tag{5}$$

and there are commutative diagrams

$$\begin{array}{ccc} \mathbf{1}^n & \xrightarrow{d^{(i,\epsilon)}} & \mathbf{1}^{n+1} \\ {\scriptstyle s^{j-1}}\downarrow & & \downarrow{\scriptstyle s^j} \\ \mathbf{1}^{n-1} & \xrightarrow[d^{(i,\epsilon)}]{} & \mathbf{1}^n \end{array} \qquad \text{if } i < j \tag{6}$$

and

$$\begin{array}{ccc} \mathbf{1}^n & \xrightarrow{d^{(i+1,\epsilon)}} & \mathbf{1}^{n+1} \\ {\scriptstyle s^{j}}\downarrow & & \downarrow{\scriptstyle s^j} \\ \mathbf{1}^{n-1} & \xrightarrow[d^{(i,\epsilon)}]{} & \mathbf{1}^n \end{array} \qquad \text{if } i \geq j. \tag{7}$$

The projections

$$(\epsilon_1, \dots, \epsilon_{n+k}) \overset{pr_L}{\mapsto} (\epsilon_1, \dots, \epsilon_n)$$

and

$$(\epsilon_1, \dots, \epsilon_{n+k}) \overset{pr_R}{\mapsto} (\epsilon_{n+1}, \dots, \epsilon_{n+k})$$

are degeneracy functors. Thus, any morphism $\theta : \mathbf{1}^r \to \mathbf{1}^{n+k}$ is uniquely determined by the composites $pr_L\theta$ and $pr_R\theta$. That said, $\mathbf{1}^{n+k}$ is not the categorical product of $\mathbf{1}^n$ and $\mathbf{1}^k$ in the box category $\square$: one sees this by observing that the diagonal functor $\Delta : \mathbf{1} \to \mathbf{1}^2$ is not a face functor.

What can be said along these lines is the following:

Lemma 2. *The diagrams (2), (6) and (7) are pullbacks in the box category.*

Proof. A box morphism $\alpha : \mathbf{1}^r \to \mathbf{1}^n$ factors through the face $d^{(i,\epsilon)} : \mathbf{1}^{n-1} \to \mathbf{1}^n$ if and only if the images $\alpha(x) = (\alpha_1(x), \dots, \alpha_n(x))$ have the form $\alpha_i(x) = \epsilon$ for all $x \in \mathbf{1}^r$. $\square$

A poset morphism $\gamma : \mathcal{P}(\underline{n}) \to \mathcal{P}(\underline{m})$ is said to be cubical if the morphism $\gamma_* : \mathbf{1}^n \to \mathbf{1}^m$ defined by the diagram

$$\begin{array}{ccc} \mathbf{1}^n & \xrightarrow{\gamma_*} & \mathbf{1}^m \\ {\scriptstyle \Omega_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Omega_m} \\ \mathcal{P}(\underline{n}) & \xrightarrow[\gamma]{} & \mathcal{P}(\underline{m}) \end{array}$$

is cubical in the sense that it is a morphism of the box category $\square$.

Observe that there is a poset isomorphism $\theta_F : \mathcal{P}(F) \xrightarrow{\cong} \mathcal{P}(F)^{op}$ defined by $B \mapsto B^c$.

Suppose that the face functor $d : \mathcal{P}(\underline{k}) \to \mathcal{P}(\underline{n})$ is defined by the interval $[A, B] \subset \mathcal{P}(\underline{n})$, so that there is an ordered set isomorphism $\underline{k} \cong B - A$ which defines an ordered inclusion $d : \underline{\mathrm{k}} \to \underline{\mathrm{n}}$, and the functor $d : \mathcal{P}(\underline{\mathrm{k}}) \to \mathcal{P}(\underline{\mathrm{n}})$ is defined by $C \mapsto A \cup d(C)$. In particular d factors canonically as the composite

$$\mathcal{P}(\underline{\mathrm{k}}) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(\underline{\mathrm{n}})$$

where the displayed isomorphism is induced by the ordered set isomorphism $\underline{\mathrm{k}} \cong B - A$.

There is a commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}(\underline{\mathrm{k}})^{op} & \xrightarrow{\cong} & [A, B]^{op} & \longrightarrow & \mathcal{P}(\underline{\mathrm{n}})^{op} \\
{\scriptstyle \theta}\uparrow{\scriptstyle \cong} & & & & {\scriptstyle \cong}\uparrow{\scriptstyle \theta} \\
\mathcal{P}(\underline{\mathrm{k}}) & \xrightarrow[\cong]{} & [B^c, A^c] & \longrightarrow & \mathcal{P}(\underline{\mathrm{n}})
\end{array}$$

where the morphisms along the top are induced by the factorization of the original poset morphism d, and the isomorphism $\mathcal{P}(\underline{\mathrm{k}}) \cong [B^c, A^c]$ arises from the identity $A^c - B^c = B - A$ in $\mathcal{P}(\underline{\mathrm{n}})$. The point in checking the commutativity of this diagram is that, for any $C \subset \underline{\mathrm{k}}$, we have $(A \cup d(C^c))^c = A^c \cap d(C^c)^c$. Also $d(C^c) \sqcup d(C) = B - A$ so that $d(C^c)^c = A \sqcup (d(C) \sqcup B^c)$. Thus, $d(C^c)^c \cap A^c = d(C) \cup B^c$.

Suppose that the subset A of $\underline{\mathrm{n}}$ defines an ordered inclusion $A : \underline{\mathrm{k}} \to \underline{\mathrm{n}}$, which in turn induces a degeneracy functor $s : \mathcal{P}(\underline{\mathrm{n}}) \to \mathcal{P}(A) \cong \mathcal{P}(\underline{\mathrm{k}})$ given by $C \mapsto C \cap A$. Then the following diagram of functors commutes

$$\begin{array}{ccc}
\mathcal{P}(\underline{\mathrm{n}})^{op} & \xrightarrow{s^{op}} & \mathcal{P}(A)^{op} \\
{\scriptstyle \theta}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \theta} \\
\mathcal{P}(\underline{\mathrm{n}}) & \xrightarrow[s]{} & \mathcal{P}(A)
\end{array}$$

The point is that the complement of $C \cap A$ in A is the intersection $C^c \cap A$.

We have proved the following:

Lemma 3. *Suppose that the poset morphism $\omega : \mathcal{P}(\underline{n}) \to \mathcal{P}(\underline{m})$ is cubical, and let $\omega_* : \mathcal{P}(\underline{n}) \to \mathcal{P}(\underline{m})$ be defined by the requirement that the diagram*

$$\begin{array}{ccc}
\mathcal{P}(\underline{n})^{op} & \xrightarrow{\omega^{op}} & \mathcal{P}(\underline{m})^{op} \\
{\scriptstyle \theta}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \theta} \\
\mathcal{P}(\underline{n}) & \xrightarrow[\omega_*]{} & \mathcal{P}(\underline{m})
\end{array}$$

commutes. Then the functor ω_* is cubical.

2 Cubical sets

A *cubical set* X is a contravariant set-valued functor $X : \square^{op} \to \mathbf{Set}$. Write $X_n = X(\mathbf{1}^n)$, and call this set the set of *n-cells* of X. A *morphism* $f : X \to Y$

of cubical sets is a natural transformation of functors, and we have a category $c\mathbf{Set}$ of cubical sets.

The *standard n-cell* $\square^n$ is the contravariant functor on the box category $\square$ which is represented by $\mathbf{1}^n$. Thus, $\square^n$ has m-cells given by

$$\square^n_m = \hom_{\square}(\mathbf{1}^m, \mathbf{1}^n).$$

There is a *cell category* $\square \downarrow X$ for a cubical set X which is defined by analogy with the simplex category of a simplicial set. Then objects of $\square \downarrow X$ are the morphisms $\sigma : \square^n \to X$ (equivalently n-cells of X, as n varies), and a morphism is a commutative triangle of cubical set morphisms

$$\begin{array}{ccc} \square^n & & \\ \downarrow & \searrow & \\ & & X \\ & \nearrow & \\ \square^m & & \end{array}$$

There is a covariant simplicial set-valued functor $\square \to \mathbf{S}$

$$\mathbf{1}^n \mapsto B(\mathbf{1}^n) = (\Delta^1)^{\times n}$$

which is defined by the categorical nerve construction. This functor can be used to define a *cubical singular functor* $S : \mathbf{S} \to c\mathbf{Set}$, where

$$S(Y)_n = \hom_{\mathbf{S}}((\Delta^1)^{\times n}, Y).$$

This functor has a left adjoint (called *realization* or *triangulation*) $X \mapsto |X|$, where

$$|X| = \varinjlim_{\square^n \to X} (\Delta^1)^{\times n}.$$

Here, the colimit is indexed by members of the cell category $\square \downarrow X$ for X.

There are similarly defined realization and singular functors

$$|\ | : c\mathbf{Set} \leftrightarrows \mathbf{Top} : S$$

relating cubical sets and topological spaces, and of course realization is left adjoint to the singular functor in that context as well.

Remark 4. There is no notational distinction between the singular functors defined on topological spaces and simplicial sets, and no distinction between the corresponding realization functors. We shall rely on the context to tell them apart.

Example 5. Suppose that $\mathcal{C}$ is a small category. The *cubical nerve* $B_{\square}(\mathcal{C})$ is the cubical set whose n-cells are all functors of the form $\mathbf{1}^n \to \mathcal{C}$, and whose structure maps $B_{\square}(\mathcal{C})_n \to B_{\square}(\mathcal{C})_m$ are induced by precomposition with box category morphisms $\mathbf{1}^m \to \mathbf{1}^n$. Observe that there is a natural isomorphism

$$B_{\square}(\mathcal{C}) \cong S(B\mathcal{C}),$$

where $B\mathcal{C}$ is the standard nerve for the category $\mathcal{C}$ in the category of simplicial sets.

In a cubical set X, write $d_{(i,\epsilon)}$ for the function $X_n \to X_{n-1}$ which is induced by the functor $d^{(i,\epsilon)}$, and call this function a face map. Similarly, the degeneracies $s_j : X_n \to X_{n+1}$ are the functions which are induced by the functors $s^j : \mathbf{1}^{n+1} \to \mathbf{1}^n$. Say that a cell $\sigma \in X_n$ is degenerate if it is the image of some s_j, and is non-degenerate otherwise.

Define the *n-skeleton* $\operatorname{sk}_n X$ for a cubical set X to be the subcomplex which is generated by the k-cells X_k for $0 \leq k \leq n$.

Lemma 6. *A map $f : \operatorname{sk}_n X \to Y$ of cubical sets is completely determined by the restrictions $f : X_k \to Y_k$ for $0 \leq k \leq n$,*

Proof. We want to show that the maps $f : X_k \to Y_k$ extend uniquely to a morphism $f_* : sk_n X \to Y$. Suppose that $z \in \operatorname{sk}_n X_{n+1}$. Then z is degenerate, so that $z = s_i x$ for some $x \in X_n$, and it must be that $f_*(z) = s_i f(x)$ if the extension exists. Suppose that z is degenerate in two ways, so that also $z = s_j y$ for some $i < j$ and $y \in X_n$. Then

$$x = d_{(i,0)} s_i x = d_{(i,0)} s_j y = s_{j-1} d_{(i,0)} y,$$

while

$$s_j s_i (d_{(i,0)} y) = s_i s_{j-1} (d_{(i,0)} y) = s_i x = s_j y.$$

All degeneracies are injective, so that $y = s_i d_{(i,0)} y$, and

$$s_i f(x) = s_i s_{j-1} d_{(i,0)} f(y) = s_j s_i d_{(i,0)} f(y) = s_j f(y).$$

Inductively, the map $f_* : \operatorname{sk}_n(X)_r \to Y_r$ for $r = k$ is completely determined by the maps for $r < k$ in the same way. ☐

It follows that there are pushout diagrams

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} \partial\square^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in NX_n} \square^n & \longrightarrow & \operatorname{sk}_n X \end{array}$$

where NX_n denotes the non-degenerate part of X_n, and $\partial\square^n = \operatorname{sk}_{n-1} \square^n$. In other words, there is a good notion of skeletal decomposition for cubical sets.

The object $\partial\square^n$ is the subcomplex of the standard n-cell which is generated by the faces $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$. It follows from the fact that the diagram (2) is a pullback in the box category that there is a coequalizer

$$\bigsqcup_{\substack{(\epsilon_1,\epsilon_2) \\ 0 \leq i < j \leq n}} \square^{n-2} \rightrightarrows \bigsqcup_{(i,\epsilon)} \square^{n-1} \to \partial\square^n$$

where $\epsilon_i \in \{0, 1\}$.

Example 7. The cubical set $\sqcap^n_{(\epsilon,i)}$ is the subobject of $\square^n$ which is generated by all faces $d^{(j,\gamma)} : \square^{n-1} \subset \square^n$ except for $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$. From the diagram (2), it again follows that there is a coequalizer diagram

$$\bigsqcup \square^{n-2} \rightrightarrows \bigsqcup_{(j,\gamma)\neq(i,\epsilon)} \square^{n-1} \to \sqcap^n_{(\epsilon,i)}$$

where the first disjoint union is indexed over all pairs (j_1, γ_1), (j_2, γ_2) with $0 \leq j_1 < j_2 \leq n$ and $(j_k, \gamma_k) \neq (i, \epsilon)$, $k = 1, 2$.

Remark 8. Apply this sequence of ideas to the product of standard 1-cells $\square^1 \times \square^1$. Every n-cell in this product is degenerate for $n > 2$ by a simple combinatorial argument, while there is a single non-degenerate 2-cell given by the isomorphism of categories $\mathbf{1}^2 \to \mathbf{1} \times \mathbf{1}$ (NB: this is a product of box category morphisms, namely the product of left and right projections, but the isomorphism does not define $\mathbf{1}^2$ as a categorical product in the box category — see the discussion of 1-skeleta below). It follows that there is a pushout of cubical complexes

$$\begin{array}{ccc} \partial\square^2 & \longrightarrow & \mathrm{sk}_1(\square^1 \times \square^1) \\ \downarrow & & \downarrow \\ \square^2 & \longrightarrow & \square^1 \times \square^1 \end{array}$$

and hence a pushout of simplicial sets

$$\begin{array}{ccc} |\partial\square^2| & \longrightarrow & |\mathrm{sk}_1(\square^1 \times \square^1)| \\ \downarrow & & \downarrow \\ |\square^2| & \longrightarrow & |\square^1 \times \square^1| \end{array}$$

The cell category $\square \downarrow \square^n$ has a terminal object given by the identity functor on $\mathbf{1}^n$, so that there is an isomorphism

$$|\square^n| \cong (\Delta^1)^{\times n}.$$

At the same time, the definitions are rigged so that $|\partial\square^n|$ coincides with the geometric boundary of $(\Delta^1)^{\times n}$. The skeleton $\mathrm{sk}_1(\square^1 \times \square^1)$ has a 1-cell $\Delta : \mathbf{1} \to \mathbf{1}^2$ in addition to those coming from $\partial\square^2$. It follows that

$$|\,\mathrm{sk}_1(\square^1 \times \square^1)| \cong \mathrm{sk}_1(\Delta^1 \times \Delta^1).$$

It follows that $|\square^1 \times \square^1|$ has the homotopy type of the simplicial circle S^1.

The problem with realizations of products as displayed in Remark 8 can be fixed (following Kan [10]) as follows. The object $\mathbf{1}^{n+m}$ is not the product $\mathbf{1}^n \times \mathbf{1}^m$ in the box category, but there is nevertheless a functor $\tilde{\times} : \square \times \square \to \square$ which is defined on objects by

$$\mathbf{1}^n \tilde{\times} \mathbf{1}^m = \mathbf{1}^{n+m},$$

and is defined on morphisms by $\theta \tilde{\times} \gamma = \theta \times \gamma$.

If X and Y are cubical sets, define

$$X \otimes Y = \varinjlim_{\sigma:\Box^n \to X,\ \tau:\Box^m \to Y} \Box^{n+m}$$

Here, if the morphisms $\theta : \mathbf{1}^n \to \mathbf{1}^r$ and $\gamma : \mathbf{1}^m \to \mathbf{1}^s$ define morphisms $\theta : \sigma \to \sigma'$ and $\gamma : \tau \to \tau'$ in the box categories for X and Y respectively, then the corresponding map $\mathbf{1}^{n+m} \to \mathbf{1}^{r+s}$ is induced by $\theta \tilde{\times} \gamma$.

Note that there are isomorphisms

$$\Box^n \otimes \Box^m \cong \Box^{n+m}.$$

It follows that the functor $Y \mapsto Y \otimes \Box^n$ has a right adjoint $Z \mapsto Z^{(n)}$, where $Z^{(n)}_r = Z_{r+n}$ and has cubical structure map $\gamma^* : Z^{(n)}_r \to Z^{(n)}_s$ defined by $(\gamma \tilde{\times} 1)^* : Z_{r+n} \to Z_{s+n}$. In particular, there is an isomorphism

$$Y \otimes \Box^n \cong \varinjlim_{\Box^m \to Y} \Box^{m+n}.$$

The cubical function complex $\mathbf{hom}(Y, Z)$ for cubical sets Y and Z is the cubical set defined by

$$\mathbf{hom}(Y, Z)_n = \hom(Y \otimes \Box^n, Z).$$

There is a natural bijection

$$\hom(X, \mathbf{hom}(Y, Z)) \cong \hom(X \otimes Y, Z),$$

which is a consequence of the identifications

$$\hom(\Box^n, \mathbf{hom}(Y, Z)) = \hom(Y \otimes \Box^n, Z)$$

and the isomorphism

$$Y \otimes X = \varinjlim_{\Box^m \to Y,\ \Box^n \to X} \Box^{m+n} \cong \varinjlim_{\Box^n \to X} Y \otimes \Box^n.$$

There are identifications

$$\begin{array}{ccc} \Box^{n-1} \otimes \Box^m & \xrightarrow{d^{(i,\epsilon)} \otimes 1} & \Box^n \otimes \Box^m \\ \cong \downarrow & & \downarrow \cong \\ \Box^{n+m-1} & \xrightarrow[d^{(i,\epsilon)}]{} & \Box^{n+m} \end{array}$$

and

$$\begin{array}{ccc} \Box^{n} \otimes \Box^{m-1} & \xrightarrow{1 \otimes d^{(j,\epsilon)}} & \Box^n \otimes \Box^m \\ \cong \downarrow & & \downarrow \cong \\ \Box^{n+m-1} & \xrightarrow[d^{(n+j,\epsilon)}]{} & \Box^{n+m} \end{array}$$

The functor $K \mapsto K \otimes \Box^n$ has a right adjoint and therefore preserves coequalizers. Thus, if $K \subset \Box^n$ is the subcomplex which is generated by some list of faces $d^{(i,\epsilon)} : \Box^{n-1} \to \Box^n$, the $K \otimes \Box^m$ is isomorphic to the subcomplex of $\Box^{n+m}$ which is generated by the list of faces $d^{(i,\epsilon)} : \Box^{n+m-1} \to \Box^{n+m}$. Similarly, if $L \subset \Box^m$ is the subcomplex generated by faces $d^{(j,\epsilon)} : \Box^{m-1} \to \Box^m$, then $\Box^n \otimes L$ is isomorphic to the subcomplex of $\Box^{n+m}$ which is generated by the list of faces $d^{(n+j,\epsilon)} : \Box^{n+m-1} \to \Box^{n+m}$.

It follows that the induced maps $\partial\Box^n \otimes \Box^m \to \Box^n \otimes \Box^m$ and $\Box^n \otimes \partial\Box^m \to \Box^n \otimes \Box^m$ are monomorphisms of cubical sets. This implies that there are isomorphisms

$$\begin{aligned}
&(\partial\Box^n \otimes \Box^m) \cup (\Box^n \otimes \partial\Box^m) \cong \partial\Box^{n+m} \\
&(\sqcap^n_{(i,\epsilon)} \otimes \Box^m) \cup (\Box^n \otimes \partial\Box^m) \cong \sqcap^{n+m}_{(i,\epsilon)} \\
&(\partial\Box^n \otimes \Box^m) \cup (\Box^n \otimes \sqcap^m_{i,\epsilon)}) \cong \sqcap^{n+m}_{n+i,\epsilon}.
\end{aligned}$$

More generally, the functors $X \mapsto X \otimes \Box^n$ and $Y \mapsto \Box^n \otimes Y$ preserve monomorphisms of cubical sets.

There are isomorphisms

$$\begin{aligned}
|X \otimes Y| &\cong \varinjlim_{\Box^n \to X,\ \Box^m \to Y} |\Box^{n+m}| \\
&\cong \varinjlim_{\Box^n \to X,\ \Box^m \to Y} |\Box^n| \times |\Box^m| \\
&\cong |X| \times |Y|.
\end{aligned}$$

In particular, there is an isomorphism of simplicial sets.

$$|\Box^n| \cong |\Box^1|^{\times n}$$

For any $i \in \underline{n}$ there is a permutation $\theta \in \Sigma^n$ such that $\theta(i) = 0$. Using θ to permute factors therefore induces a diagram

$$\begin{array}{ccc}
|\sqcap^n_{(i,\epsilon)}| & \longrightarrow & |\Box^n| \\
{\scriptstyle\theta_*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\theta_*} \\
|\sqcap^n_{(0,\epsilon)}| & \longrightarrow & |\Box^n|
\end{array} \qquad (8)$$

The relations

$$\sqcap^n_{(0,\epsilon)} \cong (\Box^0 \otimes \Box^{n-1}) \cup (\Box^1 \otimes \partial\Box^{n-1}) \subset \Box^1 \otimes \Box^{n-1} \cong \Box^n.$$

imply that the simplicial set inclusion $|\sqcap^n_{(0,\epsilon)}| \subset |\Box^n|$ can be identified up to isomorphism with the inclusion

$$(|\Box^0| \times |\Box^{n-1}|) \cup (|\Box^1| \times |\partial\Box^{n-1}|) \subset |\Box^1| \times |\Box^{n-1}|,$$

and is therefore an anodyne extension. It follows from (8) that all induced inclusions $|\sqcap^n_{(i,\epsilon)}| \subset |\Box^n|$ are anodyne extensions of simplicial sets.

Lemma 9. *Suppose that K and L are cubical sets. Then the function $K_k \times L_l \to (K \otimes L)_{k+l}$ defined by sending the pair (σ, τ) to the cell $\sigma \otimes \tau : \square^k \otimes \square^l \to K \otimes L$ is an injection. If $k = l = 0$ this function is a bijection.*

Proof. The map $\square^n_0 \times \square^m_0 \to (\square^n \otimes \square^m)_0$ is plainly a bijection, on account of the canonical isomorphism $\square^n \otimes \square^m \cong \square^{m+n}$. The map $K_0 \times \square^m_0 \to (K \otimes \square^m)_0$ is a bijection, since this map is a colimit of maps $\square^n_0 \times \square^m_0 \to (\square^n \otimes \square^m)_0$, indexed over the cells $\square^n \to K$ of K. The map $K_0 \times L_0 \to (K \otimes L)_0$ is a colimit of maps $K_0 \times \square^m_0 \to (K \otimes \square^m)_0$, indexed over the cells $\square^m \to L$ of L, and is therefore a bijection.

We know that the functor $K \mapsto K \otimes L$ preserves monics, and that there is a canonical isomorphism

$$c : K \xrightarrow{\cong} K \otimes \square^0.$$

Suppose that $\sigma_1, \sigma_2 : \square^k \to K$ and $\tau_1, \tau_2 : \square^l \to L$ are cells of K and L, respectively, such that $\sigma_1 \otimes \tau_1 = \sigma_2 \otimes \tau_2$. There are commutative diagrams

$$\begin{array}{ccc}
\square^k & \xrightarrow{\sigma_i} & K \\
{\scriptstyle c}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle c} \\
\square^k \otimes \square^0 & \xrightarrow{\sigma_i \otimes 1} & K \otimes \square^0 \\
{\scriptstyle 1 \otimes 0}\downarrow & & \downarrow{\scriptstyle 1 \otimes \tau_i(0)} \\
\square^k \otimes \square^l & \xrightarrow[\sigma_1 \otimes \tau_1]{} & K \otimes L
\end{array}$$

Here 0 denotes the vertex $(0, \dots, 0)$ of $\square^l$.

Note that $\sigma_1(0) \otimes \tau_1(0) = \sigma_2(0) \otimes \tau_2(0)$, so that $\sigma_1(0) = \sigma_2(0)$ and $\tau_1(0) = \tau_2(0)$. Observe also that the maps $1 \otimes \tau_1(0) = 1 \otimes \tau_2(0)$ are monomorphisms. It follows that there is a monomorphism $\alpha = (1 \otimes \tau_i(0))c$ such that

$$\alpha\sigma_1 = (\sigma_1 \otimes \tau_1)(1 \otimes 0)c = (\sigma_2 \otimes \tau_2)(1 \otimes 0)c = \alpha\sigma_2,$$

so that $\sigma_1 = \sigma_2$. Similarly $\tau_1 = \tau_2$. $\square$

Write NK_n for the set of non-degenerate cells of a cubical set K in degree n.

Corollary 10. *The map $K_k \times L_l \to (K \otimes L)_{l+k}$ restricts to an injection $NK_k \times NL_l \to N(K \otimes L)_{k+l}$.*

Proof. Take $(\sigma, \tau) \in K_k \times L_l$. Any degeneracy functor $s : \square^k \otimes \square^l \to \square^n$ can be written as

$$\begin{array}{ccc}
\square^k \otimes \square^l & \xrightarrow{s_1 \otimes s_2} & \square^{n_1} \otimes \square^{n_2} \\
 & {\scriptstyle s}\searrow & \downarrow{\scriptstyle \cong} \\
 & & \square^n
\end{array}$$

where s_i is either a degeneracy functor or an identity for $i = 1, 2$ and at least one of the s_i is not the identity. There are face functors $d_i : \square^{n_1} \to \square^k$ and $d_2 : \square^{n_2} \to \square^l$ such that $s_i d_i = 1$. It follows that $\sigma \otimes \tau = s_1 d_1 \sigma \otimes s_2 d_2 \tau$, and hence that $\sigma = s_1 d_1 \sigma$ and $\tau = s_2 d_2 \tau$. Thus if $\sigma \otimes \tau$ is degenerate then one of the cells σ and τ must be degenerate. In particular, there is an induced function $NK_k \times NL_l \to N(K \otimes L)_{k+l}$. This function is the restriction of an injection, and is therefore injective. $\square$

Observe as well that the induced function

$$\bigsqcup_{0 \leq k \leq n} (NK_k \times NL_{n-k}) \to N(K \otimes L)_n$$

is surjective. In effect, the corresponding function

$$\bigsqcup_{0 \leq k \leq n} (K_k \times L_{n-k}) \to (K \otimes L)_n$$

is surjective,

The ideas in the proof of Lemma 6 can also be used to show the following:

Lemma 11. *Suppose that x and y are degenerate n-cells of a cubical set X which have the same boundary in the sense that $d_{(i,\epsilon)} x = d_{(i,\epsilon)} y$ for all i and ϵ. Then $x = y$.*

Proof. Suppose that $x = s_i u$ and $y = s_j v$ for some $i < j$. Then

$$u = d_{(i,0)} s_i u = d_{(i,0)} s_j v = s_{j-1} d_{(i,0)} v,$$

while

$$s_i u = s_i s_{j-1} d_{(i,0)} v = s_j s_i d_{(i,0)} v.$$

Then

$$d_{(j,0)} s_i u = d_{(j,0)} s_j v,$$

so that

$$s_i d_{(i,0)} v = v.$$

It follows that

$$s_i u = s_j s_i d_{(i,0)} v = s_j v.$$

so that $x = y$. $\square$

Lemma 12. *Suppose that $x, y : \square^n \to X$ are n-cells of a cubical set X such that the induced simplicial set maps $x_*; y_* : |\square^n| \to |X|$ coincide. Then $x = y$.*

Proof. The inclusion $\mathrm{sk}_n X \subset X$ induces a monomorphism $|\mathrm{sk}_n X| \to |X|$, so that we can assume that $X = \mathrm{sk}_n X$. We may further suppose that X is generated by the subcomplex $\mathrm{sk}_{n-1} X$ together with the n-cells x and y.

The proof is by induction on n. The assumption that $x_* = y_*$ therefore guarantees that x and y have the same boundary in the sense that $d_{(i,\epsilon)} x =$

$d_{(i,\epsilon)}y$ for all i and ϵ. Thus if x and y are both degenerate, then $x = y$ by Lemma 11.

Suppose that y is non-degenerate, and write X_0 for the smallest subcomplex of X containing $\operatorname{sk}_{n-1} X$ and x. Write $i : X_0 \to X$ for the inclusion of the subcomplex X_0 in X.

If $x \neq y$, then y is not in X_0. Also, the intersection $\langle y \rangle \cap X_0 = \operatorname{sk}_{n-1}\langle y \rangle$, where $\langle y \rangle$ denotes the subcomplex of X which is generated by y. This means that there is a pushout diagram

$$\begin{array}{ccc} \partial\Box^n & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \Box^n & \xrightarrow[y]{} & X \end{array}$$

The assumption that $x_* = y_*$ implies that the dotted arrow lifting exists in the solid arrow pushout diagram

$$\begin{array}{ccc} |\partial\Box^n| & \longrightarrow & |X_0| \\ \downarrow & \overset{x_*}{\nearrow} & \downarrow{\scriptstyle i_*} \\ |\Box^n| & \xrightarrow[y_*]{} & |X| \end{array}$$

making it commute. The map i_* is an inclusion which is not surjective, since the solid arrow diagram is a pushout. But the existence of the dotted arrow forces i_* to be surjective. This is a contradiction, so $x = y$. $\Box$

Corollary 13. *Suppose that $f : X \to Y$ is a map of cubical sets such that the induced simplicial set map $f_* : |X| \to |Y|$ is a monomorphism. Then f is a monomorphism of cubical sets.*

Proposition 14. *Suppose that $f : X \to Y$ is a map of cubical sets such that the induced simplicial set map $f_* : |X| \to |Y|$ is an isomorphism. Then f is an isomorphism of cubical sets.*

Proof. The map f is a monomorphism of cubical sets by Corollary 13. If f is not surjective, there is a non-degenerate cell $x : \Box^n \to Y$ of smallest dimension which is not in X. It follows that f is a composite of monomorphisms

$$X \xrightarrow{f_0} X_0 \xrightarrow{f_1} Y$$

where X_0 is obtained from X by attaching the n-cell x in the sense that there is a pushout diagram

$$\begin{array}{ccc} \partial\Box^n & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f_0} \\ \Box^n & \xrightarrow[x]{} & X_0 \end{array}$$

The triangulation functor $X \mapsto |X|$ preserves monomorphisms and pushouts so that the induced map $f_* : |X| \to |Y|$ is a composite of monomorphisms $f_{1*}f_{0*}$, and there is a pushout diagram

$$\begin{array}{ccc} |\partial\square^n| & \longrightarrow & |X| \\ \downarrow & & \downarrow{\scriptstyle f_{0*}} \\ |\square^n| & \xrightarrow[x_*]{} & |X_0| \end{array}$$

of simplicial set maps. Then the monomorphism $|\partial\square^n| \to |\square^n|$ is not surjective, so that f_{0*} is not surjective, and so f_* is not surjective. This is a contradiction, so that f must be a surjective map of cubical sets. $\square$

3 The closed model structure

The purpose of this section is to display a closed model structure for the category of cubical sets. The homotopy category associated to this model structure will later be shown to be equivalent to the standard homotopy category of topological spaces.

Basically, if you want to show that a particular category has a closed model structure, you must define three classes of morphisms in that category, namely weak equivalences, cofibrations and fibrations, and then show that they satisfy the five Quillen closed model axioms **CM1** through **CM5**. The axiom **CM1** is a completeness axiom which says that certain limits and colimits exist. The weak equivalence axiom **CM2** says that if any two of the composable maps f and g and their composite fg are weak equivalences, then so is the third. The retract axiom **CM4** says that all of the three defined classes of maps are closed under retraction. Finally the factorization axiom **CM5** says that any morphism in the category can be factored as a composite of a fibration with a trivial cofibration, and as a composite of a trivial fibration and a cofibration. Here "trivial" has the standard meaning: a trivial fibration is a morphism which is both a fibration and a weak equivalence, and a trivial cofibration is a map which is both a cofibration and a weak equivalence.

A map $f : X \to Y$ of cubical sets is said to be a *weak equivalence* if the induced map $f_* : |X| \to |Y|$ is a weak equivalence of topological spaces (or of simplicial sets). A *cofibration* $i : A \to B$ of cubical sets is a levelwise inclusion. A map $p : Z \to W$ of cubical complexes is said to be a *fibration* if it has the right lifting property with respect to all maps which are both cofibrations and weak equivalences.

The category of cubical sets certainly has all limits and colimits, so the the axiom **CM1** is satisfied. The weak equivalence axiom **CM2** is a consequence of the corresponding statement for topological spaces, and the retraction axiom **CM3** is a trivial consequence of the definitions.

For the factorization axiom, we need to show two things:

Lemma 15. *A map $p : X \to Y$ is a map which has the right lifting property with respect to all inclusions $\partial\Box^n \to \Box^n$. Then p is a fibration and a weak equivalence.*

Proof. If p has the right lifting property with respect to all inclusions $\partial\Box^n \subset \Box^n$ then p has the right lifting property with respect to all inclusions, and is therefore a fibration.

In fact, the map p is a homotopy equivalence of cubical sets, by the standard argument: the map p has a section $s : Y \to X$ since there is a commutative diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \big\downarrow & \overset{r}{\nearrow} & \big\downarrow p \\ Y & \xrightarrow[1]{} & Y \end{array}$$

and then $rp \simeq 1$ because there is a commutative diagram

$$\begin{array}{ccc} X \otimes \partial\Box^1 & \xrightarrow{(rp,1)} & X \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ X \otimes \Box^1 & \xrightarrow[pc_X]{} & Y \end{array}$$

where $c_X : X \otimes \Box^1 \to X$ is the constant homotopy at the identity on X. It follows that the induced map $p_* : |X| \to |Y|$ is a homotopy equivalence of simplicial sets. $\Box$

Lemma 16. *There is a set $\mathcal{A}$ of trivial cofibrations $A \subset B$ such that a map $p : X \to Y$ is a fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{A}$.*

Lemma 16 is a formal consequence of Lemma 17, in that Lemma 17 implies that the set $\mathcal{A}$ of trivial cofibrations of countable cubical sets does the job.

Lemma 17. *Suppose that A is a countable cubical set, and that there is a diagram*

$$\begin{array}{ccc} & & X \\ & & \big\downarrow i \\ A & \longrightarrow & Y \end{array}$$

of cubical set maps in which i is a trivial cofibration. Then there is a countable subcomplex $D \subset Y$ such that $A \to Y$ factors through D, and such that the map $D \cap Y \to D$ is a trivial cofibration.

Proof. We can assume that A is a connected subcomplex of Y.

The homotopy groups $\pi_i(|A|)$ are countable, since countable simplicial sets have countable homotopy groups (any countable simplicial set has a countable fibrant model, by the way that the small object argument works).

Suppose that x is a vertex of $A = B_0$. Then there is a finite connected subcomplex $L_x \subset Y$ such that $|L_x|$ contains a homotopy $x \to i(y)$ where y is a vertex of X. Write $C_1 = A \cup (\bigcup_x L_x)$. Suppose that w, z are vertices of $C_1 \cap X$ which are homotopic in C_1. Then there is a finite connected subcomplex $K_{w,z} \subset X$ such that $w \simeq z$ in $|K_{w,z}|$. Let $B_1 = C_1 \cup (\bigcup_{w,z} K_{w,z})$. Then every vertex of A is homotopic to a vertex of $C_1 \cap X$ inside $|C_1|$, and any two vertices $z, w \in C_1 \cap X$ which are homotopic in $|C_1|$ are also homotopic in $B_1 \cap X$. Observe also that the maps $B_0 \subset C_1 \subset B_1$ are π_0 isomorphisms.

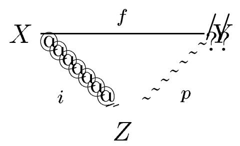
Repeat this process countably many times to find a sequence

$$A = B_0 \subset C_1 \subset B_1 \subset C_2 \subset B_2 \subset \dots$$

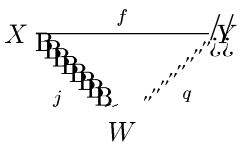
of countable subcomplexes of Y. Set $B = \bigcup B_i$. Then B is a countable subcomplex of Y such that $\pi_0(B \cap X) \cong \pi_0(B) \cong \pi_0(A) = *$.

Pick $x \in B \cap X$. The same argument (which does not disturb the connectivity) can now be repeated for the countable list of elements in all higher homotopy groups $\pi_q(|B|, x)$, to produce the desired countable subcomplex $D \subset Y$. $\square$

In the presence of Lemma 16, a standard transfinite small object argument produces a factorization



with p a fibration and i a trivial cofibration for any map $f : X \to Y$ of cubical sets. A completely standard small object argument, together with Lemma 15, shows that any map $f : X \to Y$ has a factorization

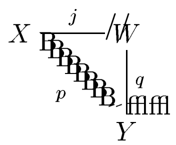


with j a cofibration and q a trivial fibration. Lemmas 15 and 16 therefore imply the factorization axiom **CM5**.

Lemma 15 has a converse, with a formal proof:

Lemma 18. *Every trivial fibration $p : X \to Y$ has the right lifting property with respect to all inclusions $\partial\square^n \subset \square^n$.*

Proof. Find a factorization



where j is a cofibration and the fibration q has the right lifting property with respect to all $\partial\Box^n \subset \Box^n$. Then q is a trivial fibration by Lemma 15, so that j is a trivial cofibration. The lifting r exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle r} & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[q]{} & Y \end{array}$$

It follows that p is a retract of q, and so p has the desired lifting property. □

The axiom **CM4** follows. We have proved the following:

Theorem 19. *With the definitions of weak equivalence, cofibration and fibration given above, the category* $c\mathbf{Set}$ *of cubical sets satisfies the axioms for a closed model category.*

The cubical set category is a closed *cubical* model category, in the sense that if $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration, then the induced map of cubical function complexes

$$(i^*, p_*) : \mathbf{hom}(B, X) \to \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

is a fibration which is also a weak equivalence of cubical sets if either i or p is a weak equivalence. This is a consequence of the observation that if $j : C \to D$ is a second cofibration of cubical sets, then the induced map

$$B \otimes C \cup_{A \otimes C} A \otimes D \to B \otimes D$$

is a cofibration which is a weak equivalence if either i or j is a weak equivalence. In effect, the triangulation functor reflects cofibrations by Corollary 13, and reflects weak equivalences by definition.

It is also clear that the class of weak equivalences is stable under pushout along cofibrations. This is half of the assertion that the model structure for cubical sets is proper. The other half of the properness assertion, namely that weak equivalences are stable under pullback along fibrations, remains to be verified.

4 Cubical posets

Recall that an interval $[A, B] \subset Q$ in a poset Q is a subposet consisting of all objects C such that $A \leq C \leq B$.

Say that a poset P is cubical if the following hold:

1) there is a fixed poset isomorphism $f : \mathbf{1}^k \xrightarrow{\cong} [A, B]$ for all non-empty intervals $[A, B]$ of P

2) any inclusion $i : [A,B] \subset [C,D]$ of non-empty intervals induces a box category morphism $i_* : \mathbf{1}^k \to \mathbf{1}^l$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{1}^k & \xrightarrow{i_*} & \mathbf{1}^l \\ f \downarrow \cong & & \cong \downarrow f \\ [A,B] & \xrightarrow[i]{} & [C,D] \end{array}$$

3) intervals are closed under intersection in P.

In the presence of the poset isomorphism $f : \mathbf{1}^k \cong [A,B]$, say that k is the dimension of $[A,B]$. We shall say that the isomorphisms f are *parameterizations* of the intervals; they are a necessary part of the structure.

The power set poset $\mathbf{1}^n \cong \mathcal{P}(\underline{\mathrm{n}})$ is a standard example. In that case, any non-empty interval $[A,B] \subset \mathcal{P}(\underline{\mathrm{n}})$ determines a unique ordered set isomorphism $d : \underline{\mathrm{k}} \xrightarrow{\cong} B - A \subset \underline{\mathrm{n}}$ which then determines a parameterization

$$\mathbf{1}^k \xrightarrow{\Omega_k} \mathcal{P}(\underline{\mathrm{k}}) \xrightarrow{d_*} [\emptyset, B-A] \cong [A,B]$$

This is the *standard parameterization* of an interval $[A,B] \subset \mathcal{P}(\underline{\mathrm{n}})$, and will always be used. Note the equality

$$[A_1, B_1] \cap [A_2, B_2] = [A_1 \cup A_2, B_1 \cap B_2]$$

so that the set of intervals of $\mathcal{P}(\underline{\mathrm{n}})$ is closed under intersection.

In some sense, the conditions 1)–3) together mean that a cubical poset P has a covering by power sets.

Cubical posets P have "minimal" cubical nerves $B_m P$. The easiest way to define $B_m P$ as a cubical set is to decree that

$$B_m P = \varinjlim_{[A,B]} \square^k,$$

where the colimit is indexed over the poset of non-empty intervals in P and the indicated colimit is for the functor $[A,B] \mapsto \square^k$, where $f : \mathbf{1}^k \to [A,B]$ is the poset isomorphism required by the structure. In particular, the poset of intervals in $\mathbf{1}^n \cong \mathcal{P}(\underline{\mathrm{n}})$ has a unique maximal element $[\emptyset, \underline{\mathrm{n}}]$, so that there is a canonical isomorphis

$$B_m \mathcal{P}(\underline{\mathrm{n}}) \cong \square^n.$$

Alternatively, it is easily seen that there is a coequalizer

$$\bigsqcup_{[A,B]\cap[C,D]\neq\emptyset} \square^r \rightrightarrows \bigsqcup_{[A,B]\neq\emptyset} \square^k \to B_m P.$$

Here, r is the dimension of the intersection $[A,B] \cap [C,D]$ and k is the dimension of $[A,B]$.

The intervals

$$\mathbf{1}^k \xrightarrow{\cong} [A,B] \subset P$$

in a cubical poset P determine cells $\sigma_{[A,B]} : \Box^k \to B_\Box P$ of the cubical nerve $B_\Box P$. The construction of these cells respects inclusion of intervals, and therefore determines a canonical natural map

$$\eta_P : B_m P \to B_\Box P.$$

An important special case of this construction is the standard map $\eta : \Box^n \to B_\Box(\mathbf{1}^n)$ which associates to an m-cell (ie. a box category morphism) $\theta : \mathbf{1}^m \to \mathbf{1}^n$ the corresponding functor $\theta : \mathbf{1}^m \to \mathbf{1}^n$ — in other words η forgets the fact that the functor θ is a box category morphism. The map $\eta : \Box^n \to B_\Box(\mathbf{1}^n)$ is plainly a monomorphism (this is the first step to a general story: all cells $\sigma_{[A,B]} : \Box^k \to B_\Box P$ are monomorphisms). It is also easy to see that any face map $d : \Box^k \to \Box^n$ determines a pullback diagram

$$\begin{array}{ccc} \Box^k & \xrightarrow{\eta} & B_\Box(\mathbf{1}^k) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_*} \\ \Box^n & \xrightarrow[\eta]{} & B_\Box(\mathbf{1}^n) \end{array} \tag{9}$$

is a pullback diagram in the category of cubical sets. In effect, if $\gamma : \mathbf{1}^s \to \mathbf{1}^k$ is a functor such that the composite $d\gamma$ is a cubical functor, then there is a degeneracy functor $s : \mathbf{1}^n \to \mathbf{1}^k$ such that $sd = 1$ and so $\gamma = sd\gamma$ is a cubical functor.

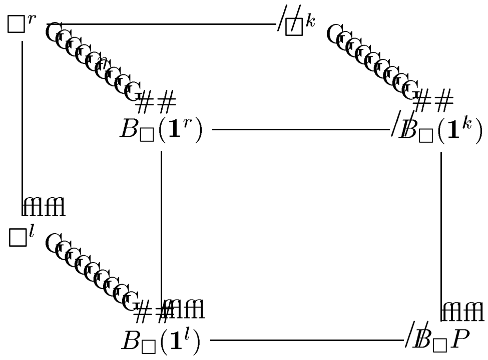
Now suppose that

$$\begin{array}{ccc} [E,F] & \longrightarrow & [A,B] \\ \downarrow & & \downarrow \\ [C,D] & \longrightarrow & P \end{array}$$

is a pullback diagram of non-empty intervals in a cubical poset P, so that $[E,F] = [A,B] \cap [C,D]$, and let

$$\begin{array}{ccc} \Box^r & \longrightarrow & \Box^k \\ \downarrow & & \downarrow \\ \Box^l & \longrightarrow & B_\Box P \end{array} \tag{10}$$

be the corresponding diagram of cells. Then the diagram (10) factors as a diagram



where all of the indicated square are pullbacks, and the map η is a monomorphism. It follows that the diagram (10) is a pullback in cubical sets.

The subobject X of $B_\square P$ which is generated by the intervals $\sigma_{[A,B]} : \square^k \to B_\square P$ is in fact covered by those intervals since (10) is a pullback, and it follows that there is a coequalizer

$$\bigsqcup_{[A,B]\cap[C,D]\neq\emptyset} \square^r \rightrightarrows \bigsqcup_{[A,B]\neq\emptyset} \square^k \to X.$$

The map $\eta_P : B_m P \to B_\square P$ factors through an isomorphism $B_m P \cong X$ by comparison of coequalizers, so that η_P is a monomorphism. We have proved

Lemma 20. *Suppose that P is a cubical poset. Then the canonical map $\eta_P : B_m P \to B_\square P$ is a monomorphism, so that $B_m P$ can be identified with the subobject of the cubical nerve $B_\square P$ which is generated by non-empty intervals.*

Lemma 21. *Suppose that P is a cubical poset. Then there is an isomorphism of simplicial sets*

$$|B_m P| \cong BP.$$

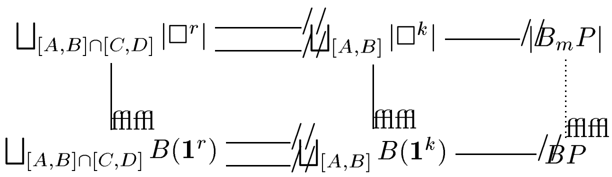
Proof. The intervals cover P, so there is a coequalizer diagram of simplicial sets

$$\bigsqcup_{[A,B]\cap[C,D]\neq\emptyset} B(\mathbf{1}^r) \rightrightarrows \bigsqcup_{[A,B]\neq\emptyset} B(\mathbf{1}^k) \to BP.$$

There are natural canonical isomorphisms

$$|\square^n| \xrightarrow{\cong} B(\mathbf{1}^n)$$

which together induce a comparison of coequalizer diagrams



so that the induced dotted arrow is an isomorphism. ∎

One can show that the isomorphism $|B_m P| \xrightarrow{\cong} BP$ coincides with the composite

$$|B_m P| \xrightarrow{i_*} |B_\square P| \cong |S(BP)| \xrightarrow{\epsilon} BP,$$

where S denotes the cubical singular functor $S : \mathbf{S} \to c\mathbf{Sets}$.

A cubical poset morphism $g : P \to Q$ is a poset morphism which respects the cubical structure of intervals in the sense that in all diagrams

$$\begin{array}{ccccc}
\mathbf{1}^k & \xrightarrow[\cong]{f} & [A,B] & \xrightarrow{i} & P \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\
\mathbf{1}^l & \xrightarrow[f]{\cong} & [g(A), g(B)] & \xrightarrow[i]{} & Q
\end{array}$$

the uniquely determined functor $g_* : \mathbf{1}^k \to \mathbf{1}^l$ is a cubical functor. All cubical functors $\theta : \mathbf{1}^n \to \mathbf{1}^m$ are cubical poset morphisms.

Suppose that P and Q are cubical posets, and consider the product poset $P \times Q$. Any interval $[(A_1, A_2), (B_1, B_2)]$ has the form

$$[(A_1, A_2), (B_1, B_2)] \cong [A_1, B_1] \times [A_2, B_2],$$

and so the parameterizations $\mathbf{1}^r \xrightarrow{\cong} [A_1, B_1]$ and $\mathbf{1}^s \xrightarrow{\cong} [A_2, B_2]$ together induce a parameterization

$$\mathbf{1}^{r+s} \xrightarrow{\cong} [(A_1, A_2), (B_1, B_2)].$$

It's plain from these identifications that any inclusion of intervals in $P \times Q$ is a cubical morphism, and of course intervals in $P \times Q$ are closed under intersection. In particular, the product poset $P \times Q$ is a cubical poset. It is also clear that the projections $P \times Q \to P$ and $P \times Q \to Q$ are cubical poset morphisms.

Recall that the minimal nerve $B_m P$ of a cubical poset P is defined by the identification

$$B_m P = \varinjlim_{[A,B]} \square^k$$

where the limit is indexed over the intervals $[A, B] \subset P$ and k is the dimension of $[A, B]$. Write $[A, B] : \square^k \to B_m P$ for the canonical cubical set map corresponding to the interval $[A, B]$. It follows that there is an isomorphism

$$B_m P \otimes B_m Q \cong \varinjlim_{[A_1,B_1],[A_2,B_2]} \square^r \otimes \square^s,$$

where $[A_1, B_1]$ and $[A_2, B_2]$ vary over the intervals with corresponding dimensions r, s of P and Q respectively. The composites

$$\square^r \otimes \square^s \cong \square^{r+s} \xrightarrow{[(A_1,A_2),(B_1,B_2)]} B_m(P \times Q)$$

determine a cubical set map

$$\nu : B_m P \otimes B_m Q \to B_m(P \times Q) \tag{11}$$

The construction can plainly be reversed, and it follows that ν is an isomorphism. The isomorphism ν is natural with respect to cubical poset morphisms in both variables.

5 Cubical subdivision

Write $N\square^n$ for the poset of non-degenerate cells in the cubical complex $\square^n$. Observe that an object σ of $N\square^n$ can be identified with a coface $(d, \epsilon) : \mathbf{1}^k \to \mathbf{1}^n$, and hence with an interval $[A, B] \subset \mathcal{P}(\underline{n})$. Here, A is identified with the image of $(0, \dots, 0)$ under (d, ϵ), while B is the image of $(1, \dots, 1)$.

Write $N\mathcal{P}(\underline{n})$ for the poset of intervals in $\mathcal{P}(\underline{n})$. We have just displayed a poset isomorphism

$$N\square^n \cong N\mathcal{P}(\underline{n}).$$

Under this identification, a face relation $\tau \leq \sigma$ between non-degenerate cells corresponds to an inclusion of intervals $[C, D] \subset [A, B]$, where $A \subset C \subset D \subset B$. The corresponding interval $[[C, D], [A, B]]$ in the poset $N\square^n$ can be identified up to isomorphism with the product poset $[A, C]^{op} \times [D, B]$ (with $C \subset D$), via the map $(E, F) \mapsto [E, F]$. There is a parameterization

$$\mathbf{1}^s \times \mathbf{1}^t \to [C^c, A^c] \times [D, B] \xrightarrow{\theta \times 1} [A, C]^{op} \times [D, B] \cong [[C, D], [A, B]]$$

which arises from the standard parameterizations for $[C^c, A^c]$ and $[D, B]$ and the canonical isomorphism $\theta : [C^c, A^c] \to [A, C]^{op}$

An intersection

$$[[C_1, D_1], [A_1, B_1]] \cap [[C_2, D_2], [A_2, B_2]]$$

of intervals in $N\square^n$ consists of intervals $[E, F]$ such that

$$E \in [A_1, C_1] \cap [A_2, C_2] = [A_1 \cup A_2, C_1 \cap C_2]$$

and

$$F \in [D_1, B_1] \cap [D_2, B_2] = [D_1 \cup D_2, B_1 \cap B_2].$$

It follows that the displayed intersection is equal to the interval

$$[[C_1 \cap C_2, D_1 \cup D_2], [A_1 \cup A_2, B_1 \cap B_2]].$$

This interval can be empty, of course.

Any cubical morphism $\theta : \mathcal{P}(\underline{n}) \to \mathcal{P}(\underline{m})$ restricts to cubical morphisms $\theta : [E, F] \to [\theta(E), \theta(F)]$. There is a commutative diagram

$$\begin{array}{ccc} [[C, D], [A, B]] & \xrightarrow{\theta_*} & [[\theta(C), \theta(D)], [\theta(A), \theta(B)]] \\ \cong \uparrow & & \uparrow \cong \\ [A, C]^{op} \times [D, B] & \xrightarrow[\theta^{op} \times \theta]{} & [\theta(A), \theta(C)]^{op} \times [\theta(D), \theta(B)] \end{array} \tag{12}$$

The same observation applies to inclusions

$$[[C_1, D_1], [A_1, B_1]] \subset [[C_2, D_2], [A_2, B_2]]$$

of intervals in $N\Box^n$: such a map coincides up to isomorphism with a product

$$[A_1, C_1]^{op} \times [D_1, B_1] \to [A_2, C_2]^{op} \times [D_2, B_2]$$

of inclusions of intervals.

It follows in particular that the poset $N\Box^n$ has a cubical structure, and we define the cubical set $\operatorname{sd}\Box^n$ by

$$\operatorname{sd}\Box^n = B_m N\Box^n.$$

We now know as well that any cubical set map $\theta : \Box^n \to \Box^m$ induces a morphism of cubical posets $\theta : N\Box^n \to N\Box^m$, and hence functorially determines a cubical set map $\theta_* : \operatorname{sd}\Box^n \to \operatorname{sd}\Box^m$. Note finally that the assignment $[A, B] \mapsto B$ defines a cubical poset map $\gamma : N\Box^n \to \mathcal{P}(\underline{n})$ which respects all cubical structure maps $\theta : \Box^n \to \Box^m$ in the sense that all diagrams of poset maps

$$\begin{array}{ccc} N\Box^n & \xrightarrow{\theta_*} & N\Box^m \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathcal{P}(\underline{n}) & \xrightarrow[\theta_*]{} & \mathcal{P}(\underline{m}) \end{array}$$

commute. It follows that there are cubical set maps $\gamma : \operatorname{sd}\Box^n \to \Box^n$ which respect all cubical set maps $\Box^n \to \Box^m$.

The subdivision $\operatorname{sd} X$ of a cubical set X is defined by

$$\operatorname{sd} X = \varinjlim_{\Box^n \to X} \operatorname{sd}\Box^n.$$

This construction is functorial in X, and there is a natural transformation

$$\gamma : \operatorname{sd} X \to X$$

which is induced by the maps $\gamma : \operatorname{sd}\Box^n \to \Box^n$.

Suppose in general that P is a cubical poset, and that $Q \subset P$ is a subposet which is closed under taking subobjects in the sense that if $A \leq B$ and $B \in Q$ then $A \in Q$. Then the induced poset morphism

$$[A, B]_Q \subset [A, B]_P$$

is an isomorphism if $[A, B]_Q$ is non-empty. It follows that Q is a cubical poset, and the inclusion $Q \subset P$ is a morphism of cubical posets.

Example 22. Suppose that K is a *cubical complex* in the sense that $K \subset \Box^n$ for some $\Box^n$, and let NK denote the poset of non-degenerate cells in K. Then as a subposet of $N\Box^n$, NK is closed under taking subobjects, and is therefore a cubical poset.

Suppose that $K \subset \square^n$ is a cubical complex. Then the intersection of any two non-degenerate cells $\sigma : \square^k \subset K$ and $\tau : \square^m \subset K$ is again a non-degenerate cell $\sigma \cap \tau : \square^r \subset K$, simply because this is true in $\square^n$. It follows that there is a coequalizer

$$\bigsqcup_{\sigma\cap\tau} \square^r \rightrightarrows \bigsqcup_{\sigma} \square^k \to K$$

which is determined by the covering $\{\sigma : \square^n \subset K\}$ arising from the collection of non-degenerate cells. The functor $K \mapsto \operatorname{sd} K$ plainly has a right adjoint, and therefore preserves colimits, so that the picture

$$\bigsqcup_{\sigma\cap\tau} \operatorname{sd} \square^r \rightrightarrows \bigsqcup_{\sigma} \operatorname{sd} \square^k \to \operatorname{sd} K$$

is a coequalizer.

There is a comparison of fork diagrams

$$\begin{array}{ccccc}
\bigsqcup_{\sigma\cap\tau} \operatorname{sd} \square^r & \rightrightarrows & \bigsqcup_{\sigma} \operatorname{sd} \square^k & \to & \operatorname{sd} K \\
\cong \downarrow & & \downarrow \cong & & \downarrow \zeta \\
\bigsqcup_{\sigma\cap\tau} B_m N\square^r & \rightrightarrows & \bigsqcup_{\sigma} B_m N\square^k & \to & B_m NK
\end{array}$$

which becomes a comparison of coequalizers in simplicial sets after triangulating. In effect, the poset NK is covered by the posets $N\square^k$ corresponding to non-degenerate cells $\sigma : \square^k \to K$, so that the fork

$$\bigsqcup_{\sigma\cap\tau} BN\square^r \rightrightarrows \bigsqcup_{\sigma} BN\square^k \to BNK$$

is a coequalizer of simplicial sets. Now use Lemma 21.

It follows from Proposition 14 that the induced map $\zeta : \operatorname{sd} K \to B_m NK$ is an isomorphism. There is a cubical set monomomorphism $B_m NK \subset B_\square NK$. It follows that the cubical subdivision functor preserves monomorphisms between cubical complexes, and this in turn implies the following:

Lemma 23. *The functor $X \mapsto \operatorname{sd} X$ preserves monomorphisms of cubical sets.*

Proof. Use a relative skeletal decomposition for a monomorphisms $i : X \to Y$, in conjunction with the fact that all induced maps $\operatorname{sd} \partial\square^n \subset \operatorname{sd} \square^n$ are monomorphisms. $\square$

We have also proved

Lemma 24. *Suppose that $K \subset \square^n$ is a cubical complex. Then there is an isomorphism*

$$\zeta : \operatorname{sd} K \xrightarrow{\cong} B_m NK.$$

There is a poset isomorphism

$$\mathcal{P}(\underline{\mathrm{n}}) \xrightarrow{\cong} \mathcal{P}(\underline{1})^{\times n} \tag{13}$$

where the composite

$$\mathcal{P}(\underline{\mathrm{n}}) \xrightarrow{\cong} \mathcal{P}(\underline{1})^{\times n} \xrightarrow{pr_i} \mathcal{P}(\underline{1})$$

with the i^{th} projection functor pr_i coincides with the degeneracy functor $s_i : \mathcal{P}(\underline{\mathrm{n}}) \to \mathcal{P}(\underline{1})$ which is defined by intersection with the subset $\{i\}$. In other words,

$$s_i(A) = \begin{cases} \emptyset & \text{if } i \notin A, \\ \underline{1} = \{1\} & \text{if } i \in A. \end{cases}$$

This is on account of the identification

$$\underline{\mathrm{n}} \cong \underline{1} \sqcup \cdots \sqcup \underline{1}.$$

The poset isomorphism (13) induces a cubical poset isomorphism

$$N\mathcal{P}(\underline{\mathrm{n}}) \xrightarrow{\cong} N\mathcal{P}(\underline{1})^{\times n}$$

of the corresponding posets of intervals. Any interval $[A, B]$ of dimension n in $\mathcal{P}(\underline{\mathrm{m}})$ induces a cubical morphism $[A, B] : \mathcal{P}(\underline{\mathrm{n}}) \to \mathcal{P}(\underline{\mathrm{m}})$ in the usual way, and there is a commutative diagram

$$\begin{array}{ccc} N\mathcal{P}(\underline{\mathrm{n}}) & \xrightarrow{[A,B]_*} & N\mathcal{P}(\underline{\mathrm{m}}) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ N\mathcal{P}(\underline{1})^{\times n} & \xrightarrow[{[A,B]_*}]{} & N\mathcal{P}(\underline{1})^{\times m} \end{array}$$

To describe the bottom horizontal map, write $d : \underline{\mathrm{n}} \cong B - A \subset \underline{\mathrm{m}}$ for the unique ordered monomorphism associated to the interval $[A, B]$. Then the composite

$$N\mathcal{P}(\underline{1})^{\times n} \xrightarrow{[A,B]_*} N\mathcal{P}(\underline{1})^{\times n} \xrightarrow{pr_i} N\mathcal{P}(\underline{1})$$

factors through the object $[\emptyset, \emptyset]$ if $i \notin B$, factors through the object $[\underline{1}, \underline{1}]$ if $i \in A$ and is the projection

$$N\mathcal{P}(\underline{1})^{\times n} \xrightarrow{pr_{d^{-1}(i)}} N\mathcal{P}(\underline{1})$$

if $i \in B - A$.

Suppose that the subset $A \subset \underline{\mathrm{n}}$ determines an ordered set monomorphism $d : \underline{\mathrm{r}} \to \underline{\mathrm{n}}$ via the composite

$$\underline{\mathrm{r}} \cong A \subset \underline{\mathrm{n}}$$

in the usual way. Then restriction along d (aka. intersection with A) induces a cubical morphism $d^* : \mathcal{P}(\underline{\mathrm{n}}) \to \mathcal{P}(\underline{\mathrm{r}})$ in the usual way, and this morphism induces a cubical poset morphism $d^* : N\mathcal{P}(\underline{n}) \to N\mathcal{P}(\underline{r})$ on the corresponding posets of intervals. There is a corresponding commutative diagram

$$\begin{array}{ccc} N\mathcal{P}(\underline{n}) & \xrightarrow{d^*} & N\mathcal{P}(\underline{r}) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ N\mathcal{P}(\underline{1})^{\times n} & \xrightarrow[d^*]{} & N\mathcal{P}(\underline{1})^{\times r} \end{array}$$

and each composite

$$N\mathcal{P}(\underline{1})^{\times n} \xrightarrow{d^*} N\mathcal{P}(\underline{1})^{\times r} \xrightarrow{pr_i} N\mathcal{P}(\underline{1})$$

coincides with the projection $pr_{d(i)} : N\mathcal{P}(\underline{1})^{\times n} \to N\mathcal{P}(\underline{1})$.

The cubical poset morphisms $\gamma_n : N\mathcal{P}(\underline{n}) \to N\mathcal{P}(\underline{1})$ respect the cubical structure functors s_i. It follows that there is a commutative diagram

$$\begin{array}{ccc} N\mathcal{P}(\underline{n}) & \xrightarrow{\cong} & N\mathcal{P}(\underline{1})^{\times n} \\ {\scriptstyle\gamma_n}\downarrow & & \downarrow{\scriptstyle(\gamma_1)^{\times n}} \\ \mathcal{P}(\underline{n}) & \xrightarrow[\cong]{} & \mathcal{P}(\underline{1})^{\times n} \end{array}$$

Finally, we know that the minimal nerve construction B_m takes products to $\otimes$ products, and that $\operatorname{sd}\square^n = B_m N\mathcal{P}(\underline{n})$, while $B_m\mathcal{P}(\underline{n}) = \square^n$. It follows that there are isomorphisms

$$\operatorname{sd}\square^n \xrightarrow{\cong} (\operatorname{sd}\square^1)^{\otimes n}$$

which respect the cubical structure functors, and that there are commutative diagrams

$$\begin{array}{ccc} \operatorname{sd}\square^n & \xrightarrow{\cong} & (\operatorname{sd}\square^1)^{\otimes n} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma^{\otimes n}} \\ \square^n & \xrightarrow[\cong]{} & (\square^1)^{\otimes n} \end{array}$$

There is a homemorphism $h : |\operatorname{sd}\square^1| \to |\square^1|$ which is defined by sending the vertex $[\emptyset,\emptyset]$ to 0, the vertex $[\underline{1},\underline{1}]$ to 1, and the vertex $[\emptyset,\underline{1}]$ to 1/2, and then extending linearly. The map $\gamma_* : |\operatorname{sd}\square^1| \to |\square^1|$ is the affine map which sends $[\emptyset,\emptyset]$ to 0 and the other two vertices to 1. There is plainly a convex homotopy $H : h \to \gamma_*$. Since it's convex, and h and γ_* have the same effect on the vertices $[\emptyset,\emptyset]$ and $[\underline{1},\underline{1}]$, the homotopy H is constant on the images of these vertices.

Topological realization takes $\otimes$ products to products, so there are commutative diagrams

$$\begin{array}{ccc} |\operatorname{sd}\square^n| & \xrightarrow{\cong} & |\operatorname{sd}\square^1|^{\times n} \\ {\scriptstyle\gamma_*}\downarrow & & \downarrow{\scriptstyle\gamma_*^{\times n}} \\ |\square^n| & \xrightarrow[\cong]{} & |\square^1|^{\times n} \end{array}$$

Any interval $[A,B]$ of dimension n in $\mathcal{P}(\underline{m})$ induces a diagram

$$\begin{array}{ccc} |\operatorname{sd}\Box^n| & \xrightarrow{[A,B]_*} & |\operatorname{sd}\Box^m| \\ \cong \downarrow & & \downarrow \cong \\ |\operatorname{sd}\Box^1|^{\times n} & \xrightarrow[{[A,B]_*}]{} & |\operatorname{sd}\Box^1|^{\times m} \end{array}$$

in which the map $[A,B]_* : |\operatorname{sd}\Box^1|^{\times n} \to |\operatorname{sd}\Box^1|^{\times m}$ is defined by the composites

$$|\operatorname{sd}\Box^1|^{\times n} \xrightarrow{[A,B]_*} |\operatorname{sd}\Box^1|^{\times m} \xrightarrow{pr_i} |\operatorname{sd}\Box^1|$$

where $pr_i[A,B]_*$ factors through the vertex $[\emptyset,\emptyset]$ if $i \notin B$, factors through $[\underline{1},\underline{1}]$ if $i \in A$ and coincides with the projection $pr_{d^{-1}(i)} : |\operatorname{sd}\Box^1|^{\times n} \to |\operatorname{sd}\Box^1|$ if $i \in B - A$. Again, d is the unique ordered monomorphism $\underline{n} \cong B - A \subset \underline{m}$ which is determined by the interval $[A,B]$.

Similarly, if $A \subset \underline{n}$ of order r determines the ordered set monomorphism $d : \underline{r} \to \underline{n}$ in the usual way, then there is a commutative diagram

$$\begin{array}{ccc} |\operatorname{sd}\Box^n| & \xrightarrow{d^*} & |\operatorname{sd}\Box^r| \\ \cong \downarrow & & \downarrow \cong \\ |\operatorname{sd}\Box^1|^{\times n} & \xrightarrow[d^*]{} & |\operatorname{sd}\Box^1|^{\times r} \end{array}$$

where the composite

$$|\operatorname{sd}\Box^1|^{\times n} \xrightarrow{d^*} |\operatorname{sd}\Box^1|^{\times r} \xrightarrow{pr_i} |\operatorname{sd}\Box^1|$$

is the projection $pr_{d(i)}$.

It follows that the product homeomorphisms $h^{\times n} : |\operatorname{sd}\Box^1|^{\times n} \to |\Box^1|$ determine homeomorphisms $h_n : |\operatorname{sd}\Box^n| \to |\Box^n|$ which commute with all maps induced by cubical set maps $\theta : \Box^n \to \Box^m$ in the sense that the diagrams

$$\begin{array}{ccc} |\operatorname{sd}\Box^n| & \xrightarrow{\theta_*} & |\operatorname{sd}\Box^m| \\ h_n \downarrow \cong & & \cong \downarrow h_m \\ |\Box^n| & \xrightarrow[\theta_*]{} & |\Box^m| \end{array}$$

commute. The homotopies

$$H_* : |\operatorname{sd}\Box^1|^{\times n} \times |\Box^1| \to |\Box^1|^{\times n}$$

which are defined by

$$H_{n*}(t_1, \dots, t_n, s) = (H(t_1,s), \dots, H(t_n,s))$$

induce homotopies

$$H'_n : |\operatorname{sd}\square^n| \times |\square^1| \to |\square^n|$$

from $h_n \to \gamma_{n*}$ which respect cubical set maps $\theta : \square^n \to \square^m$ in the sense that the diagrams

$$\begin{array}{ccc} |\operatorname{sd}\square^n| \times |\square^1| & \xrightarrow{\theta_* \times 1} & |\operatorname{sd}\square^m| \times |\square^1| \\ {\scriptstyle H'_n}\downarrow & & \downarrow{\scriptstyle H'_m} \\ |\square^n| & \xrightarrow[\theta_*]{} & |\square^m| \end{array}$$

commute.

We have assembled a proof of the following

Theorem 25. *There is a homeomorphism $h : |\operatorname{sd}X| \to |X|$ which is natuaral in cubical sets X, and a natural homotopy $H : |\operatorname{sd}X| \times |\square^1| \to |X|$ from h to γ_*.*

6 Cubical excision

Lemma 26. *Suppose that U_1 and U_2 are open subsets of a topological space Y such that $Y = U_1 \cup U_2$. Suppose given a commutative diagram of pointed cubical set maps*

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & S(U_1) \cup S(U_2) \\ {\scriptstyle i}\downarrow & & \downarrow \\ L & \xrightarrow[\beta]{} & S(Y) \end{array}$$

where i is an inclusion of finite cubical sets. Then for some n the composite diagram

$$\begin{array}{ccccc} \operatorname{sd}^n K & \xrightarrow{\gamma^n} & K & \xrightarrow{\alpha} & S(U_1) \cup S(U_2) \\ {\scriptstyle i_*}\downarrow & & & & \downarrow \\ \operatorname{sd}^n L & \xrightarrow[\gamma^n]{} & L & \xrightarrow[\beta]{} & S(Y) \end{array}$$

is pointed homotopic to a diagram

$$\begin{array}{ccc} \operatorname{sd}^n K & \longrightarrow & S(U_1) \cup S(U_2) \\ {\scriptstyle i_*}\downarrow & \nearrow & \downarrow \\ \operatorname{sd}^n L & \longrightarrow & S(Y) \end{array}$$

admitting the indicated lifting.

Proof. Suppose that $K \subset K' \subset L$. We begin by showing inductively that there is an N such that the composite

$$\mathrm{sd}^N K' \xrightarrow{\eta} S|\mathrm{sd}^N K'| \xrightarrow{Sh^N} S|K'| \xrightarrow{S(\beta')_*} SY$$

factors uniquely through a map $\tilde{\beta'} : \mathrm{sd}^N K' \to S(U_1) \cup S(U_2)$, where β' is the composite $K' \subset L \xrightarrow{\beta} S(Y)$, and $\beta'_* : |K'| \to Y$ is the adjoint of β'.

Note that a map $f : K'' \to S(Y)$ lifts to $S(U_1) \cup S(U_2)$ if and only if for every cell $\sigma : \square^k \to K''$ the adjoint $(f\sigma)_* : |\square^k| \to Y$ of the composite $f\sigma$ lifts to U_1 or U_2.

Suppose that the composite

$$\mathrm{sd}^n K' \xrightarrow{\eta} S|\mathrm{sd}^n K'| \xrightarrow{Sh^n} S|K'| \xrightarrow{S\beta'_*} S(Y)$$

lifts to $S(U_1) \cup S(U_2)$. Then I claim that the composite

$$\mathrm{sd}^{n+1} K' \xrightarrow{\eta} S|\mathrm{sd}^{n+1} K'| \xrightarrow{Sh^{n+1}} S|K'| \xrightarrow{S\beta'_*} S(Y) \tag{14}$$

lifts to $S(U_1) \cup S(U_2)$.

In effect, suppose that $\sigma : \square^s \to \mathrm{sd}^{n+1} K'$ is a cell of $\mathrm{sd}^{n+1} K'$. Then the realization $\sigma_* : |\square^s| \to |\mathrm{sd}^{n+1} K'|$ is carried on a cell $\tau : \square^r \to \mathrm{sd}^n K'$ in the sense that there is a commutative diagram

$$\begin{array}{ccc} |\square^s| & \xrightarrow{f} & |\square^r| \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow{\scriptstyle \tau_*} \\ |\mathrm{sd}^{n+1} K'| & \xrightarrow[h]{\cong} & |\mathrm{sd}^n K'| \end{array}$$

The cell τ lifts to $S(U_1) \cup S(U_2)$ by assumption, so that its adjoint $\tau_* : |\square^r| \to Y$ factors through either U_1 or U_2. The adjoint σ_* of σ is the composite

$$|\square^s| \xrightarrow{f} |\square^r| \xrightarrow{\tau_*} Y,$$

so that σ_* factors through either U_1 or U_2. It follows that the composite (14) factors through $S(U_1) \cup S(U_2)$.

Suppose that $L' \subset L$ is obtained from K' by attaching a cell, so that there is a diagram

$$\begin{array}{ccc} \partial\square^r & \longrightarrow & K' \\ \downarrow & & \downarrow \\ \square^r & \longrightarrow & L' \end{array}$$

Suppose further that there is some n such that the composite

$$\mathrm{sd}^n K' \xrightarrow{\eta} S|\mathrm{sd}^n K'| \xrightarrow{Sh^n} S|K'| \xrightarrow{S(\beta_{K'})_*} SY$$

lifts to $S(U_1) \cup S(U_2)$, where $\beta_{K'}$ is the composite $K' \subset L \xrightarrow{\beta} S(Y)$. There is a number m such that the composite

$$\operatorname{sd}^m \square^r \xrightarrow{\eta} S|\operatorname{sd}^m \square^r| \xrightarrow{h^m} S|\square^r| \to S|L'| \xrightarrow{S(\beta_{L'*})} S(Y)$$

lifts to $S(U_1) \cup S(U_2)$ by a standard Lebesgue number argument.

Now consider the diagram

$$\begin{array}{ccc} S|\partial\square^r| & \longrightarrow & S|K'| \\ \big\downarrow & & \big\downarrow {\scriptstyle S\beta_{K'*}} \\ S|\square^r| & \underset{S\beta_{\square^r *}}{\longrightarrow} & S(Y) \end{array} \tag{15}$$

Then there is a number N such that after refinement along the maps $S(h^N)\eta$ the diagram (15) lifts to a commutative diagram

$$\begin{array}{ccc} \operatorname{sd}^N \partial\square^r & \longrightarrow & \operatorname{sd}^N K' \\ \big\downarrow & & \big\downarrow \\ \operatorname{sd}^N \square^r & \longrightarrow & S(U_1) \cup S(U_2) \end{array}$$

The subdivision functor preserves pushouts, so there is a uniquely determined lift $\operatorname{sd}^N L' \to S(U_1) \cup S(U_2)$ of the composite

$$\operatorname{sd}^N L' \xrightarrow{\eta} S|\operatorname{sd}^N L'| \xrightarrow{Sh^N} S|L'| \xrightarrow{S\beta_{L'*}} S(Y).$$

Thus, we can suppose that we've found the requisite number N. The composite

$$\operatorname{sd}^N K' \xrightarrow{\eta} S|\operatorname{sd}^N K'| \xrightarrow{Sh^N} S|K'| \xrightarrow{S\beta_{K'*}} S(Y).$$

is naturally homotopic to the composite

$$\operatorname{sd}^N K' \xrightarrow{\gamma^N} K' \subset L \xrightarrow{\beta} S(Y)$$

for all complexes K' between K and L.

The map $\beta : K \to S(Y)$ already lifts to $S(U_1) \cup S(U_2)$ so that there is a commutative diagram

$$\begin{array}{ccccccc} \operatorname{sd}^N \square^k & \xrightarrow{\eta} & S|\operatorname{sd}^N \square^k| & \underset{\cong}{\xrightarrow{Sh^N}} & S|\square^k| & \xrightarrow{S(\tau)} & S(U_1) \cup S(U_2) \\ {\scriptstyle \sigma_*}\big\downarrow & & {\scriptstyle \sigma_*}\big\downarrow & & {\scriptstyle \sigma_*}\big\downarrow & & \big\downarrow \\ \operatorname{sd}^N K & \underset{\eta}{\longrightarrow} & S|\operatorname{sd}^N K| & \underset{Sh^N}{\xrightarrow{\cong}} & S|K| & \underset{S(\alpha_*)}{\longrightarrow} & S(Y) \end{array} \tag{16}$$

for all cells $\sigma : \square^k \to K$, where the composite

$$|\square^k| \xrightarrow{\sigma_*} |K| \xrightarrow{\alpha_*} Y$$

factors through some map $\tau : |\Box^k| \to U_i$. It follows that the restriction to $\mathrm{sd}^N \Box^k$ of the homotopy $S(\alpha_*)S(h^N)\eta \simeq \alpha\gamma^N$ stays inside $S(U_1) \cup S(U_2)$. This is true for all σ, so that the homotopy of the lifting $\mathrm{sd}^N K \to S(U_1) \cup S(U_2)$ with the composite

$$\mathrm{sd}^N K \xrightarrow{\gamma^N} K \xrightarrow{\alpha} S(U_1) \cup S(U_2)$$

stays inside $S(U_1) \cup S(U_2)$.

It also follows from the commutativity of diagram (16) that the homotopy of diagrams preserves base points: in particular, take the cell $\sigma : \Box^k \to K$ to be the base point $\Box^0 \to K$. $\Box$

Theorem 27 (cubical excision). *Suppose that Y is covered by open subsets U_1 and U_2. Then the induced map of cubical sets $i : S(U_1) \cup S(U_2) \subset S(Y)$ is a weak equivalence.*

Proof. Suppose that X is a pointed cubical set. The category $\mathcal{F}_*(X)$ of pointed finite cubical subsets $K \subset X$ has all finite limits is plainly filtered, and there is an isomorphism

$$\pi_q|X| \cong \varinjlim_{K \in \mathcal{F}_*(X)} \pi_q|K|.$$

Suppose that $[\alpha] \in \pi_q(|S(Y)|, x)$ is carried on a finite subcomplex $\omega : K \subset S(Y)$ in the sense that $[\alpha] = \omega_*[\alpha']$ for some $[\alpha'] \in \pi_q|K|$. There is an $N \geq 0$ such that the diagram

$$\begin{array}{ccccc}
\mathrm{sd}^N \Box^0 & \xrightarrow[\cong]{\gamma^N} & \Box^0 & \xrightarrow{x} & S(U_1) \cup S(U_2) \\
\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle i} \\
\mathrm{sd}^N K & \xrightarrow[\gamma^N]{} & K & \xrightarrow[\omega]{} & S(Y)
\end{array}$$

is pointed homotopic to a diagram

$$\begin{array}{ccc}
\mathrm{sd}^N \Box^0 & \xrightarrow{x} & S(U_1) \cup S(U_2) \\
\downarrow & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle i} \\
\mathrm{sd}^N K & \longrightarrow & S(Y)
\end{array}$$

in which the indicated lift σ exists. But γ^N is a weak equivalence, so that $[\alpha'] = \gamma^N_*[\alpha'']$ for some α''. But then $[\alpha] = \omega_*\gamma^N_*[\alpha''] = i_*\sigma_*[\alpha'']$ so that i_* is surjective on homotopy groups.

Suppose that $[\beta] \in \pi_q|S(U_1) \cup S(U_2)|$ is carried on the subcomplex $K \subset S(U_1) \cup S(U_2)$ and suppose that $i_*[\beta] = 0$. Then there is a commutative diagram of cubical set inclusions

$$\begin{array}{ccc} K & \xrightarrow{i_1} & S(U_1) \cup S(U_2) \\ {\scriptstyle j}\downarrow & & \downarrow \\ L & \xrightarrow[i_2]{} & S(Y) \end{array}$$

such that $[\beta] \mapsto 0$ in $\pi_q|L|$. There is an $N \geq 0$ such that the composite diagram

$$\begin{array}{ccccc} \operatorname{sd}^N K & \xrightarrow{\gamma^n} & K & \xrightarrow{i_1} & S(U_1) \cup S(U_2) \\ {\scriptstyle j_*}\downarrow & & & & \downarrow{\scriptstyle i} \\ \operatorname{sd}^N L & \xrightarrow[\gamma^n]{} & L & \xrightarrow[i_2]{} & S(Y) \end{array}$$

is pointed homotopic to a diagram

$$\begin{array}{ccc} \operatorname{sd}^N K & \xrightarrow{i_1'} & S(U_1) \cup S(U_2) \\ {\scriptstyle j_*}\downarrow & \overset{\tau}{\nearrow} & \downarrow{\scriptstyle i} \\ \operatorname{sd}^N L & \xrightarrow[i_2']{} & S(Y) \end{array}$$

in which the indicated lifting exists. Again, the maps γ^n are weak equivalences, so that $[\beta] = \gamma_*^n[\beta']$ for some $[\beta'] \in \pi_q|\operatorname{sd}^N K|$ and

$$i_{1*}[\beta] = i_{1*}\gamma_*^N[\beta'] = i_{1*}'[\beta'] = \tau_* j_*[\beta'].$$

Finally, $\gamma_*^N j_*[\beta'] = j_*[\beta] = 0$ so that $j_*[\beta'] = 0$ in $\pi_q|\operatorname{sd}^N L|$ and so $i_{1*}[\beta] = 0$ in $\pi_q|S(U_1) \cup S(U_2)|$. □

The category c**Sets** of cubical sets is a category of cofibrant objects for a homotopy theory, for which the cofibrations are inclusions of cubical sets and the weak equivalences are those maps $f : X \to Y$ which induce weak equivalences $f_* : |X| \to |Y|$ of CW-complexes. As such, it has most of the usual formal calculus of homotopy cocartesian diagrams (specifically II.8.5 and II.8.8 of [8]).

Corollary 28. *Suppose that the diagram*

$$\begin{array}{ccc} \bigsqcup_i \partial\Delta^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bigsqcup_i \Delta^n & \longrightarrow & Y \end{array}$$

is a pushout in the category of simplicial sets. Then the diagram of cubical set morphisms

$$\begin{array}{ccc} \bigsqcup_i S|\partial\Delta^n| & \longrightarrow & S|X| \\ \downarrow & & \downarrow \\ \bigsqcup_i S|\Delta^n| & \longrightarrow & S|Y| \end{array}$$

is homotopy cocartesian.

Proof. The usual classical arguments say that one can find an open subset $U \subset |Y|$ such that $|X| \subset U$ and this inclusion is a homotopy equivalence. The set U is constructed by fattening up all $|\partial\Delta^n|$ to an open subset U_i of $|\Delta^n|$ (by radial projection) such that $|\partial\Delta^n| \subset U_i$ is a homotopy equivalence. We can therefore assume that the inclusion

$$\bigsqcup_i |\partial\Delta^n| \subset (\bigsqcup_i |\Delta^n|) \cap U$$

is a homotopy equivalence. We can also assume that there is an open subset $V_i \subset |\Delta^n|$ such that the inclusion is a homotopy equivalence, such that $V_i \cap U_i \subset U_i$ is a homotopy equivalence, and such that $|\Delta^n| = V_i \cup U_i$. The net result is a commutative diagram

$$\begin{array}{ccccc} & & \bigsqcup_i S|\partial\Delta^n| & \longrightarrow & S|X| \\ & & {\scriptstyle\simeq}\downarrow & \mathbf{III} & \downarrow{\scriptstyle\simeq} \\ S(V \cap U) & \xrightarrow{\simeq} & S(U \cap (\bigsqcup_i |\Delta^n|)) & \longrightarrow & S(U) \\ \downarrow & \mathbf{I} & \downarrow & \mathbf{II} & \downarrow \\ S(V) & \xrightarrow[\simeq]{} & \bigsqcup_i S|\Delta^n| & \longrightarrow & S|Y| \end{array}$$

of cubical set homomorphisms in which all vertical maps are cofibrations and the labelled maps are weak equivalences. The the composite diagram $\mathbf{I} + \mathbf{II}$ is homotopy cocartesian by cubical excision (Theorem 27), so that the diagram $\mathbf{II}$ is homotopy cocartesian by the usual argument. It follows that the composite diagram $\mathbf{III} + \mathbf{II}$ is homotopy cocartesian, again by a standard argument. □

Theorem 29. *Suppose that Y is a topological space, and let $\epsilon : |S(Y)| \to Y$ be the adjunction map arising from the cubical set singular functor S and its left adjoint $|\ | : c\mathbf{Sets} \to \mathbf{Top}$. Then the map $\epsilon : |S(Y)| \to Y$ is a weak equivalence.*

Proof. The cubical singular functor $S : \mathbf{Top} \to c\mathbf{Sets}$ preserves weak equivalences. In effect, all spaces are fibrant, so the standard construction which replaces a map by a fibration can be used to show that any weak equivalence

$f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where π is a trivial fibration and j is a section of a trivial fibration $Z \to X$. It is therefore enough to show that the cubical singular functor takes trivial fibrations to weak equivalences. Finally, if $\pi : Z \to Y$ is such a trivial fibration, then it has the right lifting property with respect to all inclusions $|\partial \Box^n| \subset |\Box^n|$, so that the induced map $S(\pi) : SZ \to SY$ has the right lifting property with respect to all inclusions $\partial \Box^n \subset \Box^n$ by adjointness. We already know that this means that $S(\pi)$ is a weak equivalence (in fact, a homotopy equivalence).

The functor $Z \mapsto |S(Z)|$ therefore preserves weak equivalences. We can thus assume that $Y = |X|$ for some simplicial set X.

The functor $Z \mapsto |S(Z)|$ also preserves disjoint unions and filtered colimits of CW complexes (because the spaces $|\Box^n|$ are compact). It also preserves homotopies, and therefore preserves contractible spaces; in particular, the map $\epsilon : |S(|\Delta^n|)| \to |\Delta^n|$ is a weak equivalence for all standard simplices Δ^n.

Finally, we can induct along skeleta of simplicial sets X and suppose that the map $\epsilon : |S(|\operatorname{sk}_{n-1} X|)| \to |\operatorname{sk}_{n-1} X|$ is a weak equivalence for all simplicial sets X. But then the induced diagram

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} |S(|\partial \Delta^n|)| & \longrightarrow & |S(|\operatorname{sk}_{n-1} X|)| \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in NX_n} |S(|\Delta^n|)| & \longrightarrow & |S(|\operatorname{sk}_n X|)| \end{array}$$

is homotopy cocartesian for all simplicial sets X by Corollary 28. The various occurrences of ϵ then give a comparison of homotopy cocartesian diagrams, and the map $\epsilon : |S(|\operatorname{sk}_n X|)| \to |\operatorname{sk}_n X|$ is a weak equivalence by the gluing lemma [8, II.8.8]. $\Box$

Corollary 30. *The counit map $\eta : X \to S(|X|)$ is a weak equivalence for any cubical set X.*

Proof. The map η is a weak equivalence of cubical sets if and only if the induced map $|\eta| : |X| \to |S(|X|)|$ is a weak equivalence of topological spaces. This, however, follows from a triangle identity and Theorem 29. $\Box$

References

[1] R. Antolini, *Cubical structures, homotopy theory*, Ann. Mat. Pura Appl. **178**(4) (2000), 317–324.

[2] R. Antolini and B. Wiest, *The cubical singular complex of a topological space*, Math. Proc. Camb. Phil. Soc. **126** (1999), 149–154.

[3] R. Brown and P. Higgins, *On the algebra of cubes*, Journal Pure App. Alg. **21** (1981), 233–260.

[4] D-C. Cisinski, *Les Préfaisceaux comme modèles des types d'homotopie*, Thèse, Université Paris 7 (2002).

[5] E. Curtis, *Simplicial homotopy theory*, Advances in Math. **6**(2) (1971), 107–209.

[6] R. Fritsch and R. Piccinini, *Cellular Structures in Topology*, Cambridge Studies in Advanced Mathematics **19**, Cambridge University Press, Cambridge (1990).

[7] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, New York (1967).

[8] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics **174**, Birkhäuser, Basel-Boston-Berlin (1999).

[9] V.K.A.M. Gugenheim, *On supercomplexes*, Trans. AMS **85** (1957), 35–51.

[10] D.M. Kan, *Abstract homotopy. I*, Proc. Nat. Acad. Sci, **41** (1955), 1092–1096.

[11] D.M. Kan, *Abstract homotopy. II*, Proc. Nat. Acad. Sci, **42** (1956), 255–258.

[12] D.M. Kan, *Abstract homotopy. III*, Proc. Nat. Acad. Sci, **42** (1956), 419–421.

[13] K. Lamotke, *Semisimpliziale algebraiche Topologie*, Springer-Verlag, Berlin-Heidelberg-New York (1968).

[14] J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. of Math. **65**(2) (1957), 357–362.

[15] C.P. Rourke and B.J. Sanderson, *Δ-sets I: Homotopy theory*, Quart. J. Math. Oxford (2), **22** (1971), 321–338.

[16] B.J. Sanderson, *The simplicial extension theorem*, Math. Proc. Camb. Phil. Soc. **77** (1975), 497–498.

[17] E.C. Zeeman, *Relative simplicial approximation*, Proc. Camb. Phil. Soc. **60** (1964), 39–43.