A polynomial bound on the diameter of the transportation polytope

L. STOUGIE *

October 25, 2002

Abstract

We prove that the combinatorial diameter of the skeleton of the polytope of feasible solutions to any $m \times n$ transportation problem is $O(\max\{n, m\}^3)$.

The transportation problem (TP) is a classic problem in operations research. The problem was posed for the first time by Hitchcock in 1941 [8] and independently by Koopmans in 1947 [11]. The problem appears in any standard introductory course on operations research.

The $m \times n$ TP has m supply points and n demand points. Each supply point i holds r_i units, and each demand point j wants c_j units. A solution to the problem can be written as a $m \times n$ matrix X with entries decision variables x_{ij} having value equal to the number of units transported from supply point i to demand point j. Without loss of generality we assume that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$. The objective is to minimize total transportation costs $\sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij}$, where t_{ij} is the unit transportation cost from supply point i to demand point j. The set of feasible solutions of TP, the transportation polytope, is described by

$$\sum_{j=1}^{n} x_{ij} = r_i, \qquad i = 1, 2, \ldots, m;$$

$$\sum_{i=1}^{m} x_{ij} = c_j, \qquad j = 1, 2, \ldots, n;$$

$$x_{ij} \geq 0, \qquad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots n.$$

The 1-skeleton (edge graph) of this polytope is defined as the graph with vertices the vertices of the polytope and edges its 1-dimensional faces. In 1957 W.M. Hirsch stated his famous conjecture (cf. [4]) saying that any polytope in $\mathbb{R}^d$ defined by n inequalities has diameter $n - d$. So far the best bound for any polytope is $O(n^{\log d + 1})$ [9]. Any strongly polynomial bound is still lacking. Such

bounds have been proved for some special classes of polytopes (for examples see [12]). Among those are some special classes of transportation polytopes [2] [1] and the the polytope of the dual of TP [1].

A strongly polynomial time algorithm for TP follows directly from [13]. It is not a primal simplex type algorithm. In fact the existence of a strongly polynomial primal simplex algorithm is unknown. The first strongly polynomial bound on the diameter of the transportation polytope was given by Dyer and Frieze [6]. Actually, they prove a bound on the diameter of any polytope $\{x \in \mathbb{R}^n | Ax \leq b, b \in \mathbb{R}^m\}$ where A is a totally unimodular matrix. The proof is complicated, using the probabilistic method on the outcome of a randomized algorithm. Moreover, the bound is huge ($O(m^{16}n^3(\ln(mn))^3)$ assuming $m \leq n$). We will give a simple proof of a $O(\max\{n^3, m^3\})$ bound. As will be seen, the bound is on the diameter of the transportation polytope with only a restricted class of edges. The proof is constructive: it gives an algorithm that describes how to go from any vertex to any other vertex on the transportation polytope in $O(\max\{n^3, m^3\})$ steps along the edges.

We first review some known facts about the transportation polytope [10].

Lemma 1 *The transportation polytope has dimension $nm - m - n + 1$.*
Any vertex corresponds to a basic feasible solution and each such a solution has at most $m + n - 1$ variables with non-zero values.
Any non-degenerate basic feasible solution has exactly $m + n - 1$ non-zero variables.

Thus, according to the Hirsch conjecture the diameter is bounded by $m + n - 1$.

For any feasible solution X of TP, create the bipartite graph $G(X)$ by defining a vertex for each row and column and the edges $E(X) = \{(i,j)|x_{ij} > 0, i = 1, \ldots, m, \ j = 1, \ldots, n\}$. The following facts are found in [10].

Lemma 2 *Any feasible solution X of TP is a basic feasible solution if and only if $G(X)$ is a spanning forest.*
X is a non-degenerate basic feasible solution if and only if $G(X)$ is a spanning tree.
$G(X)$ has $q + 1$ components if and only if X is a basic feasible solution of degeneracy q, i.e., a solution with $m + n - 1 - q$ non-zero variables.

In any introduction to operations research the so-called North-West Corner rule is given as a way to find a starting solution for a simplex algorithm for TP (see e.g. [7]). This rule starts by giving the highest possible value to x_{11}, and at each step when a highest possible value is given to entry x_{ij} it moves on to $x_{i,j+1}$ in case x_{ij} filled column j, or to $x_{i+1,j}$ in case x_{ij} filled row i, and proceeds until x_{mn} has received a value.

We call a basic feasible solution a North-West Corner rule solution (NWC-solution) if the solution can be obtained by applying the North-West Corner rule to some permutation of the rows and columns. We say that the row- and column permutation *genrates* the NWC-solution. Notice that any row- and column permutation generates a unique NWC-solution. The reverse is not true:

any NWC-solution can be generated by different row- and column permutations (indeed by exponentially many).

Lemma 3 *A basic feasible solution X of TP is a NWC-solution $\Leftrightarrow$ each of the components of $G(X)$ is a caterpillar graph, i.e., a tree with a central (backbone) path and all vertices not on this path directly connected to it by an edge.*

PROOF. $\Rightarrow$: Suppose this is not true. Then there must exist a vertex in $G(X)$ that is end vertex of at least 3 edge disjoint paths of length 2. Suppose without loss of generality that row-vertex i is such a vertex, and $G(X)$ contains the paths (i, j_1, i_1), (i, j_2, i_2), and (i, j_3, i_3) given in the order in which j_1, j_2 and j_3 appear in the column permutation generating NWC-solution X. In constructing the NWC-solution all entries in column j_1 receive their value before those in column j_2, which receive theirs before those in column j_3. Thus, i_2 must be "south" of i. Since $x_{i,j_3} > 0$, x_{i_2,j_2} is non-zero only if x_{i,j_2} was given a value that neither filled column j_2 nor row i, contradicting the NWC-rule.
$\Leftarrow$: To facilitate the exposition we assume that $G(X)$ is a spanning tree. In the degenerate case the arguments below should be applied to any of the trees spanning the components, and consider them in arbitrary but prefixed order.

Take the backbone path of $G(X)$ (or any if there is a choice). Without loss of generality suppose this path starts with a row-vertex i_1 and ends with a column-vertex j_K and, following it, it reads $(i_1, j_1, i_2, j_2, \ldots, i_K, j_K)$.

Let $d(v)$ be the degree of vertex v. Any vertex not on the path has degree 1. For each row-vertex i_{k+1}, $k = 1, \ldots, K-1$ denote its adjacent column-vertices not on the backbone path in arbitrary order $j_{k_1}, \ldots, j_{k_{d(i_{k+1})-2}}$. (Clearly there are no such vertices if $d(i_{k+1}) = 2$.) In a similar way we denote the row-vertices not on the backbone path and adjacent to column-vertex j_k by $i_{k_1}, \ldots, i_{k_{d(j_k)-2}}$, $k = 1, \ldots, K-1$.

Now we are ready to construct a permutation ρ of the rows and γ of the columns.

$$
\begin{aligned}
\rho(i_1) &= 1, \ \gamma(j_1) = 1, \\
\rho(i_k) &= \rho(i_{k-1}) + d(j_{k-1}) - 1, & k &= 2, 3, \ldots, K, \\
\gamma(j_k) &= \gamma(j_{k-1}) + d(i_k) - 1, & k &= 2, 3, \ldots, K, \\
\rho(i_{k_h}) &= \rho(i_k) + h, & h &= 1, \ldots, d(j_k) - 2, \ k = 1, \ldots, K-1, \\
\gamma(j_{k_h}) &= \rho(j_k) + h, & h &= 1, \ldots, d(i_{k+1}) - 2, \ k = 1, \ldots, K-1.
\end{aligned}
$$

It is easily verified that ρ and γ are indeed permutations of the rows and the columns, respectively. Since given this permutation the NWC-solution is unique it must be X. $\square$

In the sequel we use the term *pivot step* from a solution to another solution to indicate an exchange on a 2×2-submatrix of X containing 3 non-zero entries and 1 zero entry. The zero entry is increased at the expense of two of the

non-zero entries until one of the two (or, in case of degeneracy, both) becomes zero. Notice that this is a restriction on the choice of edges of the transportation polytope. However, as we will see, restricting to these class of moves is sufficient for obtaining the desired result.

Lemma 4 *From any non-NWC-solution it requires at most $n + m - 1$ pivot steps and rearranging of rows and columns to reach a NWC-solution.*

PROOF. Let X be a non-NWC-solution and $G(X)$ the corresponding graph. Again for convenience we assume that $G(X)$ is a spanning tree. Let P be the longest path of $G(X)$. We call the vertices at distance at least 2 from the longest path non-caterpillar vertices, and the others caterpillar vertices. By Lemma 3, $G(X)$ contains at least one non-caterpillar vertex at distance exactly 2 from P. Take any vertex at distance 2 from P, u say. Suppose, without loss of generality, that the path from u to P is given by (u, v, i), ending in row-vertex i on P (whence u is a row-vertex and v a column vertex). Write P as (P_1, i, j, P_2), with P_1 the part of P until i and P_2 the part of P from j.

Insert edge (u, j) into $G(X)$, creating the circuit $(u, j), (j, i), (i, v), (v, u)$. This corresponds to increasing entry x_{uj} at the expense of decreasing either x_{ij} or x_{uv} to 0, yielding the new basic feasible solution X'. In the former case $G(X') = (G(X) \cup (u, j)) \setminus (i, j)$ has longest path (P_1, i, v, u, j, P_2). The number of non-caterpillar vertices has decreased by at least 1. If (u, v, i) was part of the longer path (Q, u, v, i), then the vertex on Q adjacent to u becomes a caterpillar vertex as well.

In the latter case the longest path of $G(X') = (G(X) \cup (u, j)) \setminus (u, v)$ is again P. However, u is now connected to P by the edge (u, j) and hence has become a caterpillar vertex, decreasing the number of non-caterpillar vertices by 1.

Therefore, we need to repeat this procedure, inserting the edge connecting a vertex at distance 2 from the longest path to a vertex at distance 3 on the longest path, at most $n + m - 1$ times to arrive at a caterpillar graph, since $G(X)$ contains no more than $n + m - 1$ non-caterpillar vertices. $\square$

As a preliminary to Lemma 6 we bound the number of pivot steps required to reach a NWC-solution from a non-NWC-solution without permuting rows or columns for problem instances with two rows or two columns.

Lemma 5 *Given a basic feasible solution X of a $2 \times n$ TP ($m \times 2$ TP), at most n (m) pivot steps are required to arrive at a NWC-solution, maintaining the order of the rows and columns in which X is given.*

PROOF. We prove the statement for a $2 \times n$ TP only. The proof for $m \times 2$ TP's is analogous. Consider a $2 \times n$ TP with given basic feasible solution

$$X = \left(\begin{array}{cccc} x_{11}, & x_{12}, & \ldots, & x_{1n} \\ x_{21}, & x_{22}, & \ldots, & x_{2n} \end{array} \right).$$

Notice that there exists at most one column in X with two non-zero entries. We denote this column $k \in \{1, \ldots, n\}$ if it exists, and we use $k = 0$ to say that no

such column exists (X is degenerate). By induction on the number of columns we will prove the lemma and the claim that in case $k \in \{0, 1, n\}$ only $n-1$ pivot steps are required.

The claim and the lemma are obviously true if there is only 1 column. Suppose it is true for every $2 \times (n-1)$ TP and consider X. We distinguish three cases.

CASE 1. $x_{21} = 0$ ($\Rightarrow x_{11} > 0$) or $x_{1n} = 0$ ($\Rightarrow x_{2n} > 0$): We fix, respectively the first or the last column and create a NWC-solution on the remaining $n-1$ columns. Notice that if the conditions of the claim hold for X then they will hold again for the matrix restricted to the remaining $n-1$ columns. Therefore, the induction hypothesis proves both claim and lemma.

CASE 2. $x_{21} > 0$, $x_{1n} > 0$, and $k \in \{0, 1, n\}$, i.e., the condition of the claim holds: We do a pivot step on the 2×2-submatrix made up by columns 1 and n. After this step $x_{21} = 0$ or $x_{1n} = 0$ and we are in Case 1, having done 1 pivot step. The induction hypothesis proves the claim and hence the lemma for this case.

CASE 3. $x_{21} > 0$, $x_{1n} > 0$, and $k \in \{2, \ldots n-1\}$: We do a pivot step on the submatrix made up by columns 1 and k. After this step $x_{11} > 0$, and $x_{21} = 0$, bringing us in Case 1, or $x_{1k} = 0$, bringing us in Case 2. In the former situation the induction hypothesis proves the lemma. In the latter situation we arrive at at the conditions of the claim after one pivot step, from where we need to make only $n-1$ pivot steps. This proves the lemma. □

Consider a NWC-solution X and a row- and column permutation that generates X. An exchange of position of two consecutive rows or columns of the permutations we call a *neighbour exchange*. In general, X will not be the NWC-solution generated by the permutation obtained after a neighbour exchange.

Lemma 6 *After a neighbour exchange on a NWC-solution the number of pivot steps required to arrive at the NWC-solution generated by the permutation after the exchange is bounded by the number of non-zero entries in the two rows or columns that were exchanged.*

PROOF. We give the proof for a neighbour row-exchange. The one for neighbour column-exchanges is analogous. Let X be the solution generated by the permutation before the exchange, and let the exchange be made on the i-th and the $i+1$-th row of this permutation. Notice that X restricted to rows i and $i+1$ is a solution of the $2 \times n$ TP with row sums r_i and r_{i+1} and column sums $c'_j = x_{ij} + x_{i+1,j}$, $j = 1, \ldots, n$.

There exists at most one column of X in which both rows i and $i+1$ have a non-zero-entry. Let this be column k. Since X is a NWC-solution row i contains non-zero entries in all columns $k-l, k-l+1, \ldots, k$, and zero entries elsewhere, and row $i+1$ contains non-zero entries in all columns $k, k+1, \ldots, k+h$ and zero entries elsewhere (l and h may be 0, and, in a degenerate solution, either row i or row $i+1$ may have a 0 in column k). Hence, $c'_j = 0$, $\forall j \notin \{k-l, \ldots, k+h\}$, and X restricted to rows i and $i+1$ and to columns $k-l$ up to $k+h$ is a NWC

5

solution for the $2 \times (l + h + 1)$ TP with row sums r_i and r_{i+1} and column sums c'_j, $j = k - l, \ldots, k + h$. After reversing the rows we know from Lemma 5 that we can obtain the NWC-solution in this restricted $2 \times (l + h + 1)$ TP within at most $l + h + 1$ pivot steps.

Fixing X everywhere except in its i-th and $i + 1$-th row, and creating a NWC-solution on the $2 \times (l+h+1)$ TP defined above, but with the order of the two rows reversed, creates a NWC solution on the whole problem, generated by the permutation after the neighbour row-exchange. As noted above this must be the unique NWC-solution generated by the new permutation. $\qquad \square$

The following lemma is straightforward from the definition of NWC-solution.

Lemma 7 *Any NWC-solution can be reached from any other NWC-solution by a series of at most $m^2 + n^2$ neighbouring row- and column exchanges and restoring the NWC-property after each exchange.*

From Lemmas 6 and 7 we get the following corollary.

Corollary 1 *At most $O(\max\{n, m\}^3)$ pivot steps are required to get from any NWC-solution to any other NWC-solution.*

Since according to Lemma 4 we can get from any non-NWC-solution to any NWC-solution in at most $n + m$ pivot steps, the main theorem follows.

Theorem 1 *The transportation polytope has diameter $O(\max\{n, m\}^3)$.*

We notice that the bound on the diameter is an enormous improvement over the bound that was known before [6] and presented above. Moreover, the proof is rather simple and uses only basic knowledge about TP. It would be interesting to investigate if the result and its proof could help in the design of a strongly polynomial time primal simplex algorithm.

We emphasize that we do not use all the edges of the transportation polytope in establishing the bound. Hence, the bound even holds on the restricted transportation polytope in which only edges exist that correspond to 2×2-pivot steps, as defined above. Firstly, this may give some space to make shortcuts on the paths defined in this paper on the transportation polytope by using also other edges, bringing the bound on the diameter further down, in the direction of the linear bound implied by the Hirsch conjecture.

Secondly, it opens the possibility that a random walk on the vertices of the transportation polytope using only 2×2 moves mixes rapidly. The analysis of such a walk seems easier than one that allows steps along any edge of the polytope. This would be a crucial step in devising a polynomial randomized approximation scheme for counting the vertices of the transportation polytope, a $\sharp$P-complete problem [5] [1]. So far rapid mixing on the transportation polytope has been shown only for problems with a fixed number of rows or a fixed number of columns [3].

---

[1]In fact, [5] only claims NP-hardness, but the proof establishes $\sharp$P-completeness.

References

[1] M.L. Balinski, On two special classes of transportation polytopes, *Mathematical Programming Studies 1*, 1974, 43–58.

[2] E.D. Bolker, Transportation polytopes, *Journal of Combinatorial Theory B 13*, 1972, 251–262.

[3] M. Cryan, M. Dyer, H. Müller and L. Stougie, Random walks on the vertices of transportation polytopes with constant number of sources, in *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, Baltimore, Maryland, 2003.

[4] G.B. Dantzig, *Linear programming and extensions*, Princeton University Press, Princeton, 1963.

[5] M.E. Dyer, The complexity of vertex enumeration methods, *Mathematics of Operations Research 8*, 1983, 381–402.

[6] M. E. Dyer and A. M. Frieze, Random walks, totally unimodular matrices and a randomized dual simplex method, *Mathematical Programming 64*, 1994, 1–16.

[7] F.S. Hillier and G.J. Lieberman, *Introduction to operations research*, McGraw Hill, New York, 6th edition, 1995.

[8] F.L. Hitchcock, The distribution of a product from several sources to numerous localities, *Journal of Mathematical Physics 20*, 1941, 224–230.

[9] G. Kalai and D.J. Kleitman, A quasi-polynomial bound for diameter of graphs of polyhedra, *Bulletin of the American Mathematics Society 26*, 1992, 315–316.

[10] V. Klee and C. Witzgall, Facets and vertices of transporation polyhedra, in G.B. Dantzig and A.F. Veinott (eds.), *Mathematics of the decision sciences, Part 1*, American Mathematics Society, Providence, Richmond, 1968, 257–282.

[11] Tj.C. Koopmans, Optimum utilization of the transportation system, in D.H. Leavens (ed.), *The Econometric Society Meeting, Washington DC, 1947*, 1948, 136–146.

[12] A. Schrijver, Polyhedral combinatorics, in R. Graham, M. Grötschel and L. Lovász (eds.), *Handbook of combinatorics, Volume II*, North-Holland Elsevier, 1995, 1649–1704.

[13] E. Tardos, A strongly polynomial minimum cost circulation algorithm, *Combinatorica 5*, 1985, 247–256.