Generalised sheaf cohomology theories

J.F. Jardine*

October 14, 2002

Introduction

This paper is an expanded version of notes for a set of lectures given at the Isaac Newton Institute for Mathematical Sciences during a NATO ASI Workshop entitled "Homotopy Theory of Geometric Categories" on September 23 and 24, 2002. This workshop was part of a program entitled *New Contexts in Stable Homotopy Theory* that was held at the Institute during the fall of 2002.

The idea was to present some of the basic features of the homotopy theory of simplicial presheaves and the stable homotopy theory of presheaves of spectra, and then display their use in applications. A general outline of these theories forms the subject of Sections 1 and 2 of this paper.

There has been some renewed interest in equivariant stable categories for profinite groups of late, and the main features of that theory have been descibed here in Sections 3 and 4. I wanted to stress the calculational aspects of that theory as well as display its main features. This is done in the course of presenting an outline of the proof of Thomason's descent theorem for Bott periodic algebraic K-theory, which appears in Section 5.

The outline of the Thomason result which is presented here is a stripped down version of the proof appearing in [19], with all of the hard bits (ie. the coherence issues) carefully swept under the rug. Also, the proof works as stated only for good schemes and at good primes. The other cases, which are much more complicated to discuss, have been treated in detail elsewhere, particularly in Thomason's original paper [38] and the Thomason-Trobaugh paper [39]. One should also look at the commentary given by Mitchell in [30].

I would like to thank the über-organizers of the Workshop and the *New Contexts* program at the Institute for giving me the opportunity to speak. I would also like to take this opportunity to thank the Newton Institute for its hospitality and support.

*This research was partially supported by NSERC.

Contents

1 Simplicial presheaves

In all that follows, $\mathcal{C}$ will be a small Grothendieck site, meaning a category having a set of objects and equipped with some notion of covering. Standard examples include the following:

1) The category $op|_X$ of open subsets of a topological space X has as objects all inclusions $U \subset X$ of X, with coverings given by open coverings in the traditional sense.

2) The category underlying the Zariski site $Zar|_S$ of a scheme S has objects consisting of all open subschemes $U \subset S$ of the scheme S. A covering family is a collection of open subschemes $U_i \subset U$ such that $\cup U_i = U$. This site is really just the site of open subsets of the topological space underlying the scheme S, but we give it a special name.

3) The category underlying the étale site $et|_S$ of a scheme S consists of all étale maps $U \to X$, and has covering families consisting of sets of morphisms $\phi_i : U_i \to U$ over X such that $\cup \phi_i(U_i) = U$.

4) The Nisnevich site $Nis|_S$ of a scheme S has the same underlying category as the étale site, but a covering of U/S is an étale covering family ϕ_i : $U_i \to U$ such that every scheme morphism $Sp(F) \to U$ defined on some field F lifts to the total space of the scheme homomorphism $\sqcup U_i \to U$ which is defined by all of the maps ϕ_i. There are fewer Nisnevich covers than étale covers. Note, however, that that every Zariski cover $U_i \subset U$ is a Nisnevich cover, so that the Nisnevich topology is finer than the Zariski topology, but coarser than the étale topology.

5) All of the algebraic geometric sites described above have "big" brothers, namely $(Sch|_S)_{Zar}$ $(Sch|_S)_{et}$ and $(Sch|_S)_{Nis}$ respectively. These are the so-called big sites for the respective topologies. The underlying categories consist, in all cases of S-schemes $Y \to S$ which are locally of finite type, and secretly have some fixed infinite cardinal bounding their sets of points. The covering families have the same definitions as for the corresponding versions above. One can impose extra structure on the schemes

over S: most commonly one decorates the category $Sm|_S$ of smooth S-schemes with any of the above topologies, so that one has the the smooth sites $(Sm|_S)_{Zar}$, $(Sm|_S)_{et}$ and $(Sm|_S)_{Nis}$. The smooth Nisnevich site $(Sm|_S)_{Nis}$ is the basis for the standard description of motivic homotopy theory over S.

6) Any small category I gives rise to a Grothendieck site, with the so-called "chaotic" topology, which really is no topology at all: the covering families are precisely the identity maps $a \to a$ in the category I.

Most of the examples of Grothendieck sites listed above arise in algebraic geometry. The displayed list is is by no means complete: there are, for example, the various flavours of flat topologies, Voevodsky's h topologies, and so on. Grothendieck sites abound in nature.

A presheaf on a site $\mathcal{C}$ is a contravariant functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ taking values in the set category. Contravariant functors on $\mathcal{C}$ taking values in categories $\mathcal{D}$ are said to be presheaves in $\mathcal{D}$, or presheaves of objects in $\mathcal{D}$. For example, a presheaf of abelian groups is a contravariant functor $A : \mathcal{C}^{op} \to \mathbf{Ab}$ taking values in the category of abelian groups. A presheaf of simplicial sets, or a simplicial presheaf, is a contravariant functor $X : \mathcal{C}^{op} \to \mathbf{S}$ taking values in the category $\mathbf{S}$ of simplicial sets.

The presheaves on $\mathcal{C}$ taking values in $\mathcal{D}$ are the objects of a category, for which the morphisms are the natural transformations. From this point of view, a simplicial presheaf is a simplicial object in the category of presheaves of sets. A morphism $f : X \to Y$ of simplicial presheaves consists of simplicial set maps $f : X(U) \to Y(U)$, one for each object U of $\mathcal{C}$, which are natural with respect to the morphisms of $\mathcal{C}$ in the obvious sense. I write $s\,\mathrm{Pre}(\mathcal{C})$ for the category of simplicial presheaves on the site $\mathcal{C}$.

Here are some basic examples of simplicial presheaves:

1) Every simplicial set Y determines a constant simplicial presheaf Γ^*Y defined by $\Gamma^*Y(U) = Y$, with all morphisms of $\mathcal{C}$ sent to the identity on Y. One often dispenses with the Γ^* and just writes Y for the constant simplicial presheaf associated to Y.

2) Any object $A \in \mathcal{C}$ represents a presheaf $U \mapsto \hom(U, A) = A(U)$. Again, objects in $\mathcal{C}$ and the presheaves that they represent are often confused notationally. Any simplicial object X in $\mathcal{C}$ represents a simplicial presheaf (of the same name), by exactly the same process.

2a) A group object G of $\mathcal{C}$ represents a presheaf of groups $G(U) = \hom(U, G)$. The associated classifying simplicial presheaf BG is defined in sections by $BG(U) = B\hom(U, G)$.

2b) Suppose that $W \to V$ is a covering map of V in $\mathcal{C}$. The corresponding Čech object is the simplicial presheaf arising from the iterated fibre products

$$W, W \times_V W, W \times_V W \times_V W, \dots$$

in the presheaf category, and all of the associated projections and diagonals relating them.

It's more conceptually satisfying to think of this object as the nerve of a groupoid: if $f : X \to Y$ is a plain old function, then there is a groupoid whose objects are the elements of X, and we say that there is a (unique) morphism $x \to y$ if $f(x) = f(y)$ in Y. The nerve of this groupoid has X as its set of vertices and has all iterated cartesian products $X \times_Y \cdots \times_Y X$ as simplices. Apply this construction to all functions $W(U) \to V(U)$ in all sections and you get the Čech object associated to the covering.

2c) Suppose that L is a finite Galois extension of a field k with Galois group G. Then the Čech object associated to the étale covering $Sp(L) \to Sp(k)$ on any good site of k-schemes consists of the iterated pullbacks

$$Sp(L), Sp(L) \times_{Sp(k)} Sp(L), \ldots$$

which can be identified with the Borel construction $EG \times_G Sp(L)$, just by the identification $L \otimes_k L \cong \prod_G L$ given by Galois theory. The notation $EG \times_G Sp(L)$ also stands for the nerve of the translation category for the action of G on $Sp(L)$, and the isomorphism between the Čech object associated to the cover $Sp(L) \to Sp(k)$ and the Borel construction $EG \times_G Sp(L)$ is induced by an isomorphism of presheaves of groupoids.

3) Suppose that A is a presheaf of abelian groups, and write $K(A, n) = \overline{W}(A[-n])$. Here, $A[-n]$ means the presheaf of chain complexes which consists of a copy of A concentrated in degree n, and then we obtain $K(A, n)$ by applying the Eilenberg-Mac Lane $\overline{W}$ construction to obtain a simplicial abelian group in each section. The Eilenberg-Mac Lane construction is natural so this works, and $K(A, n)(U) = K(A(U), n)$ is the standard construction of an Eilenberg-Mac Lane space. The simplicial presheaf $K(A, n)$ is usually called an Eilenberg-Mac Lane object associated to the abelian presheaf A. Of course, there is one of these for each $n \geq 0$.

Of course there are such things as sheaves, which are presheaves which satisfy a patching property defined by the topology on $\mathcal{C}$. Most often one is dealing with covering families $U_i \to U$ in sites (as in algebraic geometry) where pullbacks are defined. In such a context, one says that a presheaf F is a sheaf if the diagrams

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

are equalizers for all covering families $U_i \to U$ of all objects U of $\mathcal{C}$. The category $\mathrm{Shv}(\mathcal{C})$ is the full subcategory of the category of presheaves whose objects are the sheaves; in other words one constructs the sheaf category by taking all natural transformations between sheaves. The inclusion of the sheaf category in the category of presheaves has a left adjoint $F \mapsto \tilde{F}$, called the associated sheaf functor. The canonical map $\eta : F \to \tilde{F}$ arising from the adjunction is called the associated sheaf map. The associated sheaf is constructed by formally adjoining (twice) all solutions of patching problems in F via certain filtered colimit diagrams; most people find this construction obtuse, and it will not be repeated here. The important point to remember is that, since $\tilde{F}$ is constructed from F by a filtered colimit construction, the functor $F \mapsto \tilde{F}$ is exact in the sense that it preserves all finite limits up to isomorphism.

I generally like to look at [33] for the basics about sheaves on Grothendieck sites, but your mileage may vary; the presentation in [26] is not quite as severe.

The basic idea behind the homotopy theory of simplicial presheaves on a site $\mathcal{C}$ is that it is determined by the topology of $\mathcal{C}$. Simplicial presheaves $X : \mathcal{C}^{op} \to \mathbf{S}$ are just diagrams of simplicial sets, and the option of choosing the chaotic topology (or rather, no topology at all) produces one of the standard diagram-theoretic homotopy theories. All other topologies on $\mathcal{C}$ are finer than the chaotic topology, and all produce different homotopy theories.

Another observation is that, while it is true that different topologies determine different homotopy theories for simplicial presheaves on a fixed category $\mathcal{C}$, the sheaves and simplicial sheaves on $\mathcal{C}$ are somehow beside the point, except that they give the means of specifying weak equivalences.

The slick way of defining weak equivalences (which as far as I know is due to Joyal [25]) starts with looking at weak equivalences of simplicial sets a little differently. A simplicial set X has homotopy groups $\pi_n(X, x)$, one for each vertex $x \in X_0$. Collecting all of the terminal maps $\pi_n(X, x) \to *$ taking values in the one point set together (one for each $x \in X_0$) produces a function

$$\bigsqcup_{x \in X_0} \pi_n(X, x) \to \bigsqcup_{x \in X_0} *$$

produces a group object $\pi_n X \to X_0$ over X_0 in the category of sets, for every $n \geq 1$. If $f : X \to Y$ is a simplicial set map, then the induced group maps $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ can be bundled together to form a morphism of group objects which is compatible with the vertex function $f : X_0 \to Y_0$ in the sense that the following diagram of functions commutes:

$$\begin{array}{ccc} \pi_n X & \xrightarrow{f_*} & \pi_n Y \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[f]{} & Y_0 \end{array} \tag{1}$$

The one sees that a map $f : X \to Y$ is a weak equivalence of simplicial sets in the usual sense if and only if

1) the induced function $f_* : \pi_0 X \to \pi_0 Y$ is a bijection, and

2) the square (1) is a pullback for all $n \geq 1$.

Here of course, $\pi_0 X$ is the set of path components of X, which can be defined to be the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X$$

of the face maps $d_0, d_1 : X_1 \to X_0$.

These constructions are completely natural, so it makes sense to talk about the homotopy group objects $\pi_n X \to X_0$ over the presheaf of vertices X_0 of a simplicial presheaf X. The presheaf of path components $\pi_0 X$ is also defined by a coequalizer of the two face maps $d_0, d_1 : X_1 \to X_0$, but this time in the presheaf category. Both constructions specialize to the corresponding thing for simplicial sets in all sections.

Definition 1. A morphism $f : X \to Y$ of simplicial presheaves on a Grothendieck site $\mathcal{C}$ is said to be a (local) weak equivalence if the following hold:

1) the induced map of sheaves $f_* : \tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and

2) the diagram of sheaf morphisms

$$\begin{array}{ccc} \tilde{\pi}_n X & \xrightarrow{f_*} & \tilde{\pi}_n Y \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \xrightarrow[f_*]{} & \tilde{Y}_0 \end{array}$$

is a pullback in the sheaf category for all $n \geq 1$.

Here, $\tilde{\pi}_n X$ is the sheaf associated to the presheaf $\pi_n X$, for all $n \geq 0$.

In the presence of stalks, this definition is equivalent (this is an exercise) to saying that a map $f : X \to Y$ is a local weak equivalence if and only if all induced maps $f_* : X_x \to Y_x$ of stalks are weak equivalences of ordinary simplicial sets. There is another more exotic definition which involve Boolean localization, which amounts to taking a fat "point" in some category of diagrams [18]. Alternatively, one can say that a map of simplicial presheaves is a local weak equivalence if it induces an isomorphism in all possible sheaves of homotopy groups, for all local choices of base points: see the definition of "combinatorial" weak equivalence in [13, p.48] — it's the same thing.

Example 2. The associated sheaf map $\eta : X \to \tilde{X}$ is a local weak equivalence of simplicial sheaves, since it is locally the identity map. This is why one tends not to single out simplicial sheaves as separate objects of study even thought there is a perfectly good historical reason for doing so in Joyal's work [25].

Definition 3. A map $i : A \to B$ of simplicial presheaves is said to be a cofibration if the induced functions $i : A_n(U) \to B_n(U)$ are one to one, for all $n \geq 0$ and objects U of $\mathcal{C}$.

Equivalently, a map $i : A \to B$ is a cofibration if it is a monomorphism in the category of simplicial presheaves on $\mathcal{C}$.

Definition 4. A map $p : X \to Y$ of simplicial presheaves is said to be a global fibration if it has the right lifting property with respect to all maps of simplicial presheaves which are simultaneously cofibrations and local weak equivalences. A simplicial presheaf X is said to be globally fibrant if the unique map $X \to *$ to the terminal simplicial presheaf is a global fibration.

The definition of global fibration means that if $p : X \to Y$ sits inside a solid arrow commutative diagram of simplicial presheaf morphisms

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration and a local weak equivalence, then the dotted arrow exists making the diagram commute.

These days (this first appeared in [1]), people sometimes use the term "injective fibration" for global fibration. The use of the term "global fibration" in [13] follows the usage of Brown and Gersten [3].

Now here's the result [13], [18] that gives the homotopy theory of simplicial presheaves:

Theorem 5. *Suppose that $\mathcal{C}$ is a small Grothendieck site. Then with the definitions of local weak equivalences, cofibrations and global fibrations given above, the category $s\,\mathrm{Pre}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category.*

The original result of this type is the corresponding statement about simplicial sheaves, which is due to Joyal [25]. Joyal's result is a consequence of Theorem 5 (see [13]), essentially on account of the fact that the canonical map $\eta : X \to \tilde{X}$ relating a simplicial presheaf X and its associated simplicial sheaf is a local weak equivalence.

The adjective "simplicial" in the statement of the theorem means that there is a well-behaved notion of function space $\mathbf{hom}(X, Y)$ for simplicial presheaves X and Y. Explicitly, the n-simplices of simplicial set $\mathbf{hom}(X, Y)$ are the simplicial presheaf maps $X \times \Delta^n \to Y$, where we have followed practice of identifying the standard n-simplex Δ^n with its associated constant simplicial presheaf. The term "proper" means that local weak equivalences are preserved by pullback along global fibrations and pushout along cofibrations; this is a property that is inherited from simplicial sets, in that it can be checked stalkwise or with a Boolean localization argument.

As usual, the function space construction only has homotopical content when the simplicial presheaf Y is globally fibrant; this is enough, since all simplicial presheaves are cofibrant. When Y is globally fibrant, the usual closed simplicial model category tricks imply that there is a natural bijection

$$\pi_0 \mathbf{hom}(X, Y) \cong [X, Y]$$

relating the set of path components of the simplicial set $\mathbf{hom}(X,Y)$ with the set of morphisms $[X,Y]$ in the homotopy category $\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}))$ associated to the category of simplicial presheaves on $\mathcal{C}$.

Suppose that K is a simplicial set, and that Y is a globally fibrant simplicial presheaf. Then there is a natural isomorphism

$$\mathbf{hom}(\Gamma^* K, Y) \cong \mathbf{hom}(K, \Gamma_* Y),$$

where the functor $\Gamma_* Y$ (called global sections of Y) is the right adjoint of the constant presheaf functor, and is thus defined by

$$\Gamma_* Y = \varprojlim_{U \in \mathcal{C}} Y(U).$$

It follows that there are induced bijections

$$[\Gamma^* K, Y]_{s\,\mathrm{Pre}(\mathcal{C})} \cong [K, \Gamma_* Y]_{\mathbf{S}}$$

relating morphisms in the homotopy categories (suitably labelled) if Y is globally fibrant.

If the site $\mathcal{C}$ happens to have a terminal object t (as is almost always the case with the sites arising in algebraic geometry), then $\Gamma_* Y = Y(t)$ by formal nonsense. In that case, one uses the homotopy category of pointed simplicial presheaves (which exists formally, in the presence of Theorem 5) to see that there is an isomorphism

$$[S^n, Y]_{s\,\mathrm{Pre}(\mathcal{C})_*} \cong \pi_n Y(t).$$

In other words, the homotopy groups of global sections of globally fibrant objects are isomorphic to groups of morphisms in the homotopy category of simplicial presheaves, and are thus determined by the topology on the site $\mathcal{C}$.

Suppose that A is an abelian sheaf. Then there is a sequence of natural isomorphisms

$$\begin{aligned}
[*, K(A,n)] &\cong [\tilde{\mathbb{Z}}*, K(A,n)] \ \text{(simplicial abelian sheaves)} \\
&\cong [\tilde{\mathbb{Z}}[0], A[-n]] \ \text{(sheaves of chain complexes)} \\
&\cong \mathrm{Ext}^n_{\mathcal{C}}(\tilde{\mathbb{Z}}, A) \\
&\cong H^n(\mathcal{C}, A).
\end{aligned} \tag{2}$$

Some comments:

1) The first of these isomorphisms relates morphisms $[*, K(A,n)]$ in the homotopy category of simplicial presheaves with morphisms in the corresponding homotopy category of simplicial abelian presheaves (or sheaves) (see [12],[24]); it is an adjointness relation that follows from the fact that the free simplicial abelian presheaf functor $X \to \mathbb{Z}X$ preserves local equivalences. This is trivial to verify in the presence of stalks, but more generally used to be known as the Illusie conjecture; the result has been known for a long time, and was one of the early applications of the Boolean localization [41].

2) The second isomorphism in the list is a consequence of the Dold-Kan correspondence, which in this case identifies morphisms in the homotopy category of simplicial abelian presheaves with morphisms in the derived category.

3) The identification of $\mathrm{Ext}^n(\tilde{\mathbb{Z}}, A)$ with morphisms $[\tilde{\mathbb{Z}}[0], A[-n]]$ is often taken to be a definition of the Ext group these days. Actually proving that it coincides with a more traditional description of Ext seems to require the use of hypercovers [12].

We have seen, in other words that the standard description of the n^{th} cohomology group $H^n(\mathcal{C}, A)$ of the site $\mathcal{C}$ with coefficients in the abelian sheaf A coincides with a group of morphisms in the homotopy category of simplicial presheaves. Thus, for example, if S is a scheme and μ_m is the étale sheaf of m^{th} roots of unity on S, then there is an isomorphism

$$[*, K(\mu_m, n)]_{et|_S} \cong H^n_{et}(S, \mu_m)$$

relating the étale cohomology of S with coefficients in μ_m to morphisms in the homotopy category of simplicial presheaves on the étale site $et|_S$ for S. The identification of sheaf cohomology with morphisms in simplicial presheaf homotopy categories is wildly general: it applies universally.

There is also a convenient way, with these techniques, to capture sheaf cohomology within standard homotopy theory. We need a definition first:

Definition 6. Suppose that X is a simplicial presheaf. A *globally fibrant model* for X is a local weak equivalence $j : X \to Z$ where Z is globally fibrant.

Here's an easily proved lemma, with far-reaching consequences:

Lemma 7. *Suppose that $f : X \to Y$ is a local weak equivalence of globally fibrant simplicial presheaves. Then f is a simplicial homotopy equivalence, and all induced maps in sections $f : X(U) \to Y(U)$, $U \in \mathcal{C}$, are simplicial homotopy equivalences.*

Proof. The map f is a weak equivalence of objects which are both cofibrant and fibrant, so it is a homotopy equivalence according to standard closed model category tricks. The simplicial presheaf $X \times \Delta^1$ is a cylinder object for a simplicial presheaf X, so we can assume that the homotopy equivalence is simplicial. If the homotopy equivalence is globally simplicial then it is simplicial in each section. □

Corollary 8. *Suppose that $f : X \to Z$ and $f' : X \to Z'$ are globally fibrant models for a fixed simplicial presheaf X. Then Z and Z' are simplicially homotopy equivalent.*

In other words, any two choices of globally fibrant models for a fixed simplicial presheaf X are homotopy equivalent in all sections. One often write $j : X \to GX$ for a choice of globally fibrant model for X: such things always exist by the closed model axioms, and j could be a trivial cofibration (ie. a cofibration and a local weak equivalence) if one likes. Some culture: the "G" stands for "global", but it can also stand for "Godement", in cases where Godement resolution theory works [13], [38], [32].

It is now a consequence of the identifications (2) that if A is an abelian sheaf on a site $\mathcal{C}$ having terminal object t, and $GK(A,n)$ is a globally fibrant model for the Eilenberg-Mac Lane object $K(A,n)$ then there are isomorphisms

$$\pi_i GK(A,n)(t) \cong \begin{cases} H^{n-i}(\mathcal{C}, A) & \text{if } 0 \leq i \leq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The moral is that sheaf cohomology with coefficients in A can be recovered from the spaces of global sections $GK(A,n)(t)$ of the globally fibrant objects $GK(A,n)$.

Remark 9. It causes no pain at all to see that if B is an abelian presheaf and $GK(B,n)$ is a globally fibrant model for $K(B,n)$, then there are isomorphisms

$$\pi_i GK(B,n)(t) \cong \begin{cases} H^{n-i}(\mathcal{C}, \tilde{B}) & \text{if } 0 \leq i \leq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The point is that there is a local weak equivalence $K(B,n) \to K(\tilde{B},n)$ which induces a homotopy equivalence $GK(B,n) \to GK(\tilde{B},n)$, so that the map $GK(B,n)(t) \to GK(\tilde{B},n)(t)$ in global sections is also a homotopy equivalence.

Suppose that G is a sheaf of groups on the site $\mathcal{C}$, and let BG be its associated classifying simplicial sheaf. There is a non-abelian analogue of the cohomology identifications (2), in that there is a bijection

$$[*, BG] \cong \{\text{isomorphism classes of } G\text{-torsor over the point } *\}. \tag{3}$$

The thing on the right is one of the standard descriptions of the classical non-abelian invariant $H^1(\mathcal{C}, G)$. The result itself is a hypercover argument (a *hypercover* is a map which is both a "local fibration" and a local weak equivalences in the modern world [12] — this concept will not be explained here) which makes use of the fact that the fundamental groupoid functor preserves local weak equivalences [16]. Insofar as the category of G-torsors "is" the stack associated to the sheaf of groups G, this result gave the first indication that simplicial presheaf homotopy theory had something to do with stacks [22].

Here is how it came up in a first application: with the identification (3) in hand, it is clear that if k is a field of characteristic not 2 and O_n is the algebraic group of automorphisms for the trivial form of rank n, then there is an identification of the set $[*, BO_n]$ of the set of morphisms in the homotopy category of simpicial presheaves for the étale topology on $Sp(k)$ with the set of isomorphism classes of non-degenerate symmetric bilinear forms of rank n over k. This identification gives rise to a theory of characteristic classes for quadratic forms over k in the mod 2 Galois cohomology of k as follows:

1) there is a ring isomorphism

$$H^*_{et}(BO_n, \mathbb{Z}/2) \cong A[HW_1, \ldots, HW_2]$$

where $A = H^*_{et}(k, \mathbb{Z}/2)$ is the mod 2 Galois cohomology of k and the degree of the polynomial generator HW_i is i.

2) Every form α of rank n determines a morphism $[\alpha] : * \to BO_n$ in the homotopy category of simplicial presheaves for the étale topology on $Sp(k)$, and hence determines a map

$$\alpha^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{et}(k, \mathbb{Z}/2)$$

taking values in the mod 2 Galois cohomology of k. The generators HW_i get mapped to elements $HW_i(\alpha) \in H^i_{et}(k, \mathbb{Z}/2)$ which are called the higher Hasse-Witt invariants of the form α.

It is easy to see that $HW_1(\alpha)$ is induced by the determinant $O_n \to \mathbb{Z}/2$, and that $HW_2(\alpha)$ coincides with the classical Hasse-Witt invariant. The higher Hasse-Witt invariants were originally defined by Delzant, but not in this form.

The foregoing will make more sense in the presence of the definition of the cohomology of a simplicial presheaf. Explicitly, if X is a simplicial presheaf on $\mathcal{C}$ and A is an abelian presheaf on that site, then one defines the cohomology group $H^n(X, A)$ by setting

$$H^n(X, A) = [X, K(A, n)].$$

This definition specializes to all geometric cohomology theories of schemes (use the constant simplicial presheaf associated to a scheme) and simplicial schemes (use the simplicial presheaf represented by the simplicial scheme) [13]. In particular, in the example above,

$$H^n_{et}(BO_n, \mathbb{Z}/2) \cong [BO_n, K(\mathbb{Z}/2)]$$

where the morphisms are in the homotopy category of simplicial presheaves on the étale site for k.

The definition of the *cohomology groups* must be held in stark contrast to the *homology sheaves* of a simplicial presheaf X: if A is a presheaf of abelian groups on $\mathcal{C}$, then one defines the n^{th} homology sheaf $H_n(X, A)$ by

$$H_n(X, A) = \tilde{H}_n(\mathbb{Z}X \otimes A),$$

where the object on the right is the sheaf associated to the n^{th} homology sheaf of the simplicial abelian presheaf $\mathbb{Z}X \otimes A$.

Homology sheaves and cohomology groups are related by a universal coefficients spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p(H_q(X, \mathbb{Z}), A) \Rightarrow H^{p+q}(X, A).$$

It follows from a standard comparison argument that any map $f : X \to Y$ which induces an isomorphism in all homology sheaves must also induce an isomorphism cohomology groups. Similar statements apply to ℓ-torsion coefficients, where ℓ is a prime: if f induces an isomorphism of sheaves $H_*(X, \mathbb{Z}/\ell) \cong H_*(Y, \mathbb{Z}/\ell)$ then f induces an isomorphism in all cohomology groups with ℓ-torsion coefficients.

Example 10. Suppose that k is an algebraically closed field of characteristic not equal to ℓ, where ℓ is some prime number. The general linear presheaf of groups Gl is defined by $Gl = \varinjlim Gl_n$ in the presheaf category on the smooth étale site $(Sm|_k)_{et}$. One interpretation of the Gabber rigidity theorem says that the adjunction map $\epsilon : \Gamma^* BGl(k) \to BGl$ of simplicial presheaves induces an isomorphism

$$H_*(\Gamma^* BGl(k), \mathbb{Z}/\ell) \cong H_*(BGl, \mathbb{Z}/\ell)$$

in all mod ℓ homology sheaves for the étale topology. It follows that the map ϵ induces cohomology isomorphisms

$$H^*_{et}(BGl, \mathbb{Z}/\ell) \cong H^*_{et}(\Gamma^* BGl(k), \mathbb{Z}/\ell).$$

On the other hand, the algebraically closed field k has trivial étale cohomology groups, so that there are isomorphisms

$$H^*_{et}(\Gamma^* BGl(k), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell).$$

We have therefore identified the mod ℓ cohomology of the discrete group $Gl(k)$ with étale cohomology $H^*_{et}(BGl, \mathbb{Z}/\ell)$, which is well known to be a polynomial ring $\mathbb{Z}/\ell[c_1, c_2, \dots]$ in Chern classes which is invariant of the underlying algebraically closed field. Suslin's calculation

$$K_n(k, \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell & \text{if } n = 2i,\ i \geq 0, \\ 0 & \text{if } n = 2i+1,\ i \geq 0, \end{cases}$$

follow pretty quickly.

This last example was one of the early calculational successes of the homotopy theory of simplicial presheaves. It also illustrates an idea, namely rigidity, which has proven to be quite robust:

1) There are comparison maps $\epsilon : \Gamma^* BG(k) \to BG$ associated to any reductive algebraic group over G, and the generalized isomorphism conjecture of Friedlander and Milnor says that the discrete and étale cohomology with mod ℓ coefficients for the classifying spaces of such groups G should be an isomorphism. The rigidity program for proving the conjecture is staring at you: show that the map $\Gamma^* BG(k) \to BG$ induces an isomorphism in mod ℓ étale homology sheaves (it's just too bad that it hasn't worked yet outside of stable cases).

2) A rigidity argument like the one displayed above is at the heart of the proof of the Suslin-Voevodsky theorem [37] which asserts that the singular cohomology of a scheme coincides with its étale or qfh cohomology if the coefficient sheaves are finite and constant.

2 Presheaves of spectra

Suppose that A is a presheaf of abelian groups on $\mathcal{C}$. The Eilenberg-Mac Lane objects $K(A,n)$ naturally organize themselves into a presheaf of spectra, which will be called $H(A)$. Specifically, this structure includes the full sequence of pointed simplicial presheaves

$$K(A,0), K(A,1), K(A,2), \dots$$

and maps of pointed simplicial presheaves $S^1 \wedge K(A,n) \to K(A,n+1)$ which are induced by shifting the underlying chain complexes, as usual. Here, S^1 is the simplicial circle $\Delta^1/\partial\Delta^1$, identified with a constant simplicial presheaf — Voevodsky denotes this object by S^1_s [40].

More generally, a presheaf of spectra X on the site $\mathcal{C}$ consists of pointed simplicial presheaves X^n, $n \geq 0$, and maps of pointed simplicial presheaves $\sigma : S^1 \wedge X^n \to X^{n+1}$ which are sometimes called bonding maps. Really what we're doing here by initiating the study of these objects is taking the Bousfield-Friedlander model for the stable category [2], and adapting it to the presheaf context: a spectrum, for us, will be just a sequence of pointed simplicial sets and bonding maps displayed according to the recipe above, and the category of (Bousfield-Friedlander) spectra will be denoted by **Spt**. From this point of view, a presheaf of spectra X on the site $\mathcal{C}$ is a functor $X : \mathcal{C}^{op} \to \mathbf{Spt}$.

A map $f : X \to Y$ of presheaves of spectra is the obvious thing; it consists of maps $f : X^n \to Y^n$, $n \geq 0$, of pointed simplicial presheaves which respect structure in the obvious sense that the diagrams

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle S^1 \wedge \sigma}\downarrow & & \downarrow{\scriptstyle f} \\ S^1 \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute. Alternatively, f is just a natural transformation between the functors $X, Y : \mathcal{C}^{op} \to \mathbf{Spt}$. I shall write $\mathbf{Spt}(\mathcal{C})$ for the category of presheaves of spectra on the site $\mathcal{C}$.

There are adjoint functors

$$\mathbf{Spt} \underset{\Gamma_*}{\overset{\Gamma^*}{\rightleftarrows}} \mathbf{Spt}(\mathcal{C})$$

which are induced by the global sections functor Γ_* and the constant presheaf functor Γ^*, as the notations suggest. As in the case of simplicial presheaves, one often confuses a spectrum with its associated constant presheaf of spectra. Thus, for example, one writes $S = \Gamma^* S$ for the constant object associated to the sphere spectrum S. This presheaf of spectra, in all sections, consists of the pointed simplicial sets

$$S^0, S^1, S^1 \wedge S^1, \dots$$

with bonding maps given by identities.

As a matter of convenience, one writes $S^n = S^1 \wedge \cdots \wedge S^1$ (n-fold smash); it is standard to call this object the simplicial n-sphere.

The weak equivalences that define the stable homotopy theory of presheaves are much easier to define than weak equivalences of simplicial presheaves, essentially because there is no ambiguity about the choice of base point. Explicitly, a presheaf of spectra X has presheaves $\pi_n X$, $n \in \mathbb{Z}$, of stable homotopy groups, and these presheaves have associated sheaves $\tilde{\pi}_n X$ of stable homotopy groups.

Definition 11. A map $f : X \to Y$ of presheaves of spectra on $\mathcal{C}$ is a (local) stable equivalence if it induces isomorphisms $\tilde{\pi}_* X \cong \tilde{\pi} Y$ in all associated sheaves of stable homotopy groups.

Definition 12. A cofibration is a map $i : A \to B$ of $\mathbf{Spt}(\mathcal{C})$ such that

1) the map $i : A^0 \to B^0$ is an inclusion (aka. cofibration) of simplicial presheaves, and

2) all maps
$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}, \ \ n \geq 0,$$
are cofibrations of simplicial presheaves.

Definition 13. A global (or stable) fibration is a map $p : X \to Y$ of presheaves of spectra on $\mathcal{C}$ which has the right lifting property with respect to all maps which are simultaneously cofibrations and local stable equivalences.

Now here's the first main result [14]:

Theorem 14. *With the definition of local stable equivalence, cofibration and global fibration given above, the category* $\mathbf{Spt}$($\mathcal{C}$ *of presheaves of spectra on a Grothendieck site* $\mathcal{C}$ *satisfies the axioms for a proper closed simplicial model category.*

The associated homotopy category $\mathrm{Ho}(\mathbf{Spt}(\mathcal{C}))$ is the stable homotopy category associated to the site $\mathcal{C}$ and its underlying topology.

The statement of the Theorem implies the existence of a well-behaved function space $\mathbf{hom}(X, Y)$ for presheaves of spectra X and Y. This is the usual thing: its set $\mathbf{hom}(X, Y)_n$ of n-simplices is the set of all maps $X \wedge \Delta^n_+ \to Y$ of presheaves of spectra, where Δ^n_+ is the simplcial set Δ^n with a disjoint base point attached.

There are some things that you just get for free out of this result, along with some knowledge of the ordinary stable homotopy category, since all maps $X \to Y$ of presheaves of spectra which induce stable equivalences $X(U) \to Y(U)$ in each section must also be local stable equivalences. Here are some of the more striking examples:

Example 15. Suppose given a finite list of presheaves of spectra X_i, $1 \leq i \leq n$. Then the canonical map
$$\bigvee_{i=1}^{n} X_i \to \prod_{i=1}^{n} X_i$$

is a local stable equivalence, because the corresponding property holds for ordinary spectra. Among other things, this means that the stable category $\mathrm{Ho}(\mathbf{Spt}(\mathcal{C}))$ has an additive structure.

Example 16. Fibre and cofibre sequences coincide in the stable category of presheaves of spectra, just as fibre and cofibre sequences coincide in the ordinary stable category. The point is that the proof of the result in the ordinary stable category involves natural constructions, and hence passes to the category of presheaves of spectra.

The proof of Theorem 14 involves a stabilization construction which is very similar to the one met in real life [2], except that one has to be careful to use globally fibrant models on the simplicial presheaf level. More explicitly, if X is a presheaf of spectra, then one first constructs a level fibrant model $j : X \to X_f$; this map consists of inductively constructed globally fibrant models $X^n \to X_f^n$ in all levels. Then there is a presheaf of spectra QX_f with a natural map $\eta : X_f \to QX_f$ such that the simplicial presheaf QX_f^n is the filtered colimit of the inductive system

$$X_f^n \to \Omega X_f^{n+1} \to \Omega^2 X_f^{n+2} \to$$

arising from the adjoints of the bonding maps for X. The indicated loop spaces make homotopical sense ($\Omega Y = \mathbf{hom}_*(S^1, Y)$, in general and as usual) because all of the simplicial presheaves Y_f^n are globally fibrant. Finally (this is important, and is a standard source of errors) a filtered colimit of globally fibrant objects might not be globally fibrant, so we have to take a level fibrant model of the presheaf of spectra QX_f. The stabilization of X is then the composite

$$X \to X_f \to QX_f \to (QX_f)_f.$$

One can assume that this map is natural in X, because the small object constructions by which we construct globally fibrant models are all natural.

In fact, the composite map $X \to (QX_f)_f$ is a fibrant model for X: it induces an isomorphism in stable homotopy groups (this is obvious, because we're stabilizing almost in a standard way in sections), and the object $(QX_f)_f$ is globally fibrant in the sense of Theorem 14.

The observation that this thing is globally fibrant is an outcome of the proof of Theorem 14, which follows a formal script as outlined by Bousfield and Friedlander. One of the outcomes of that proof is a formal recognition principle for global fibrations of presheaves of spectra:

Lemma 17. *A map $p : X \to Y$ is a global fibration of presheaves of spectra if and only if it satisfies the following two properties:*

1) *all constituent maps $p : X^n \to Y^n$, $n \geq 0$, are global fibrations of simplicial presheaves, and*

2) all diagrams of simplicial presheaf maps

$$\begin{array}{ccc} X^n & \longrightarrow & (QX^n_f)_f \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y^n & \longrightarrow & (QY^n_f)_f \end{array}$$

are homotopy cartesian diagrams in the category of simplicial presheaves.

It's an easy exercise to draw the following consequence:

Corollary 18. *A presheaf of spectra X is globally fibrant if and only if all level objects X^n are globally fibrant simplicial presheaves and all adjoint bonding maps $X^n \to \Omega X^{n+1}$ are local weak equivalences.*

In other words, a globally fibrant presheaf of spectra is an Ω-spectrum object for the category of presheaves of spectra on the site $\mathcal{C}$. It is in particular a presheaf of Ω-spectra in the usual sense, but it has more structure arising from the topology on $\mathcal{C}$. Globally fibrant presheaves of spectra are really the central mystery of the subject, as they encode the notion of descent.

Definition 19. A presheaf of spectra X on the site $\mathcal{C}$ satisfies descent if some (and hence any) globally fibrant model $j : X \to Z$ induces stable equivalences $j : X(U) \to Z(U)$, $U \in \mathcal{C}$, of ordinary spectra in all sections.

This is just an example of a very general notion: we can talk , for example, about simplicial presheaves that satisfy descent, or presheaves of groupoids G such that BG satisfies descent. The modern definition of stack is a presheaf of groupoids that satisfies descent in this sense.

We care about presheaves of spectra X that satisfy descent because we can often explicitly calculate their presheaves of stable homotopy groups π_*X, meaning that we can compute the groups $\pi_*X(U)$ in all sections. This is usually done with cohomological techniques that depend on phenomena such as the following example:

Example 20. Suppose that A is an abelian sheaf on the étale site $et|_S$ for some scheme S. Let $H(A)$ be the corresponding Eilenberg-Mac Lane presheaf of spectra, and take a globally fibrant model $j : H(A) \to GH(A)$. In fact, to make the construction, it is enough to take a level fibrant model $H(A)_f$ for $H(A)$ since we're starting with a presheaf of Ω-spectra.

By definition,

$$\pi_i GH(A)(S) = \varinjlim_n \pi_{i+n} GK(A,n)(S),$$

and we know from before that there are isomorphisms

$$\pi_j GK(A,n)(S) \cong \begin{cases} H^{n-j}_{et}(S,A) & \text{if } 0 \leq j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

Then the object $GH(A)(S)$ is an Ω-spectrum by construction, so that there are isomorphisms

$$\pi_i GH(A)(S) \cong \begin{cases} H_{et}^{-i}(S,A) & \text{if } i \leq 0, \\ 0 & \text{if } i > 0. \end{cases}$$

I chose to work on an étale site to make the example more real, but this calculation works in complete generality: if B is an abelian presheaf on a site $\mathcal{C}$ and $GH(B)$ is a globally fibrant model for the corresponding Eilenberg-Mac Lane presheaf of spectra, then there are isomorphisms

$$\pi_i \Gamma_* GH(B) \cong \begin{cases} H^{-i}(\mathcal{C},\tilde{B}) & \text{if } i \leq 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The calculations just made are the basis of the construction of the descent spectral sequence for the stable homotopy groups $\pi_* GF$ of a globally fibrant model GF of a presheaf of spectra F.

Again we'll introduce some assumptions to make the construction a little more real: suppose that F is a presheaf of connective spectra on the étale site $et|_S$ of a decent scheme S. The connectivity assumption means that $\tilde{\pi}_i F = 0$ for $i < 0$, or that the stable homotopy group sheaves of F vanish in negative degrees. Ordinary spectra E have natural Postnikov towers $P_* E$ [19], and so the presheaf of spectra F has a Postnikov tower

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \downarrow & & \downarrow \\ & & P_2F & \xrightarrow{\simeq} & GP_2F \\ & \nearrow & \downarrow & & \downarrow \\ & & P_1F & \xrightarrow{\simeq} & GP_1F \\ & \nearrow & \downarrow & & \downarrow \\ F & \longrightarrow & P_0F & \xrightarrow[\simeq]{} & GP_0F \end{array}$$

The Postnikov tower splits off stable homotopy groups in the usual sense, and we have taken globally fibrant models $P_nF \to GP_nF$ of all of the P_nF in such a way that the map $GP_{n+1}F \to GP_nF$ are stable fibrations having fibres of the form $GK(\pi_{n+1}F, n+1)$. It's not hard to see that the inverse limit

$$\varprojlim_n GP_nF$$

is globally fibrant. It, however, more difficult to conclude that the map

$$F \to \varprojlim_n GP_nF$$

is a local stable equivalence; this usually requires an assumption of a uniform bound on étale cohomological dimension in all sections, but this is often met in decent geometric examples provided that the sheaves of stable homotopy groups are torsion sheaves of some kind. If that works, the inverse limit of the tower of fibrations

$$\cdots \to GP_2F(S) \to GP_1F(S) \to GP_0F(S)$$

is stably equivalent to $GF(S)$, and a suitably re-indexed Bousfield-Kan spectral sequence [38], [19] has the form

$$E_2^{s,t} = H^s_{et}(S, \tilde{\pi}_t F) \Rightarrow \pi_{t-s} GF(S). \tag{4}$$

The convergence has to be taken with a grain of salt: typically (again) a uniform bound on étale cohomological dimension with respect to the sheaves $\tilde{\pi}_* F$ is required to make it work. The spectral sequence (4) is called the *descent spectral sequence* for F, or for GF. One often sees it referred to as the étale cohomological descent spectral sequence, or as the topological descent spectral sequence.

Example 21. Suppose that K/ℓ is the mod ℓ K-theory presheaf of spectra on the étale site $et|_S$, where ℓ is a prime which is distinct from the residue characteristics of S. Suppose that S is otherwise well behaved as a geometric object [38], and in particular has finite Krull dimension d. Take a stably fibrant model $j : K/\ell \to GK/\ell$. Then $GL/\ell(S)$ is a variant of the étale K-theory spectrum of S, and has descent spectral sequence

$$E_2^{s,t} = H^s_{et}(S, \tilde{\pi}_t K/\ell) \Rightarrow \pi_{t-s} GK/\ell(S).$$

The étale sheaves $\tilde{\pi}_* K/\ell$ are known, by Suslin's calculation of the mod ℓ K-theory of algebraically closed fields: the sheaf $\tilde{\pi}_{2t} K/\ell$ is the twist $\mu_l^{\otimes t}$ in even positive degrees, and is 0 elsewhere.

The Lichtenbaum-Quillen Conjecture can be viewed as the assertion that the canonical homomorphism

$$\pi_i K/\ell(S) \to \pi_i GK/\ell(S)$$

is an isomorphism for $i \geq d-1$ for such schemes S.

In the case where S is defined over a field F containing a primitive ℓ^{th} of unity ζ, "the" Bott element is easily described as an element of $\pi_2 K/\ell(F)$ that maps to ζ under the surjection

$$\pi_2 K/\ell(F) \to \mathrm{Tor}(\mathbb{Z}/\ell, F^*)$$

(The notation is a bit pedantic: the group $\pi_2 K/\ell(F)$ is otherwise known as $K_2(F, \mathbb{Z}/\ell)$ and $\pi_1 K(F) = K_1(F) \cong F^*$.) In more general cases, some care has to be taken — see the "Gang of Four" paper [5].

One can show that the groups $\pi_i GK/\ell(S)$ are Bott periodic for $i \geq -1$ [19]. Inverting multiplication by the Bott element β gives a spectrum $GK/\ell(S)(1/\beta)$, which is stably equivalent to the étale K-theory spectrum of Dwyer and Friedlander [4]. The canonical map $GK/\ell(S) \to GK/\ell(S)(1/\beta)$ induces an isomorphism in stable homotopy groups in degrees $i \geq -1$, under the standard assumptions. One might also want to be careful about the coefficients and insist that $\ell \neq 2$; these issues have been treated in some detail in the literature [38].

Étale K-theory is an example of a generalized étale cohomology theory, or more broadly, of a a generalized sheaf cohomology theory. Quite generally, if F is a presheaf of spectra on a site $\mathcal{C}$, take a globally fibrant model $j : F \to GF$ and define

$$\mathbb{H}^i(\mathcal{C}, F) = \pi_{-i}\Gamma_* GF, \ \ i \in \mathbb{Z}.$$

All sheaf cohomology theories are examples of generalized sheaf cohomology theories: in the present notation, we have already seen that there is a natural isomorphism

$$\mathbb{H}^i(\mathcal{C}, H(A)) \cong H^i(\mathcal{C}, \tilde{A}).$$

for all abelian presheaves A. This isomorphism also explains the sign change in the defining index: the idea is to make the notation for the generalized theories compatible with the notation for ordinary sheaf cohomology.

Other examples of these theories abound, particularly in algebraic K-theory, where there is a flavour of K-theory for each of the standard geometric topologies, and interesting descent theorems (or conjectures) which relate these theories to algebraic K-theory itself. In particular, we have

1) Zariski K-theory arising from the Zariski topology, which coincides with ordinary K-theory for regular schemes by the Brown-Gersten descent theorem [3], and

2) Nisnevich K-theory arising from the Nisnevich (or cd) topology, which coincides with the K-theory of regular schemes by the Nisnevich descent theorem [32] (an unstable version of Nisnevich descent is the starting point for motivic homotopy theory [31]).

Finally, in this language, Thomason's descent theorem [38], [19] asserts that the Bott periodic K-theory presheaf of spectra $K/\ell(1/\beta)$ satisfies descent for the étale topology in the same range of examples (including the primes ℓ) for which the Lichtenbaum-Quillen conjecture is believed to hold. In other words, if you take the mod ℓ K-theory presheaf of spectra K/ℓ and formally invert multiplication by the Bott element, you end up constructing a model for étale K-theory. The overall moral is that descent is everywhere.

I want to close this section by mentioning some other general developments that could not be treated at length here:

1) There is a homotopy theory (ie. model structure) for the category $\mathbf{Spt}^{\Sigma}(\mathcal{C})$ of presheaves of symmetric spectra on an arbitrary small site $\mathcal{C}$, such that the associated homotopy category is equivalent to the stable category $\mathrm{Ho}(\mathbf{Spt}(\mathcal{C}))$ of presheaves of spectra on $\mathcal{C}$ [20]. The point of the

construction, as for the case of ordinary symmetric spectra [11], is that the category of presheaves of symmetric spectra has an internal symmetric monoidal smash product.

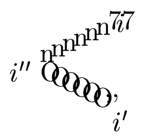
2) Suppose that S is a scheme of finite Krull dimension, and let $(Sm|_S)_{Nis}$ denote the site of smooth S-schemes, equipped with the Nisnevich topology. One can construct the unstable motivic homotopy category [31] from the category $s\,\mathrm{Pre}((Sm|_S)_{Nis})$ by formally contracting the affine line $\mathbb{A}^1_S$ over S [9]. One can then go on to formally invert smashing with an object T which is compact in a suitable sense (eg. $T = S^1, \mathbf{G}_m, \mathbf{P}^1$) to produce a motivic stable category $\mathbf{Spt}_T((Sm|_S)_{Nis})$ of T-spectra [21]. In the case $T = \mathbb{P}^1$, we obtain the motivic stable category of Morel and Voevodsky. There is a corresponding category $\mathbf{Spt}^{\Sigma}_T((Sm|_S)_{Nis})$ of symmetric T-spectra which is defined by analogy with the constructions in [11] and [20]. This category is a model for the motivic stable category of T-spectra if the cyclic permutation $(1,2,3)$ acts trivially on the three-fold smash $T^{\wedge 3}$; examples include $T = S^1, \mathbb{P}^1$. Again, this category of symmetric spectrum objects has an internal symmetric monoidal smash product, so there is a good theory of smash products for the motivic stable category.

3 Profinite groups

For our purposes, a *profinite group* $G = \{G_i\}$ is a finite group-valued functor $G : I \to \mathbf{Grp}$ with $i \mapsto G_i$, which is defined on a small left filtered category I and such that all morphisms $i \to j$ of I are mapped to *surjective* group homomorphisms $\pi : G_i \to G_j$.

I'll recall what it means for the index category I to be left filtered. There are two conditions:

1) any two objects i, i' of I have a "common lower bound", meaning that there is an object i'' and morphisms



2) for any two morphisms $\alpha, \alpha' : i \to i'$ there is a morphism $e : i'' \to i$ such that $\alpha \cdot e = \alpha' \cdot e$

As a result (and we will use this all the time) colimits of contravariant functors defined on I are filtered in the usual sense.

Although it's a little crime to do so, I'm going to confuse notations by writing $G = \varprojlim_i G_i$. The main reason for assuming that the all transition homomorphisms $G_i \to G_j$ in the profinite group G is that all induced homomorphisms $G \to G_i$ are surjective. I shall write H_i for the kernel of this homomorphism, which can be viewed either as a subgroup of the inverse limit or a profinite group defined on the left filtered category $I \downarrow i$ whose objects are all morphisms $j \to i$ in I.

Example 22. The central examples for us will be the Galois group G of Galois field extensions F/K of a fixed field K. By this we really mean the collection of all Galois groups $G(L/K)$ of the finite Galois extensions L within F. Keep in mind that nobody has said anything about F being separably closed.

A *discrete G-set* X (for a profinite group G) is a set equipped with G-action $G \times X \to X$ which factors through an action by one of the quotients G_i in the sense that there is a commutative diagram

$$\begin{array}{ccc} G \times X & & \\ \downarrow & \searrow & \\ G_i \times X & \longrightarrow & X \end{array}$$

There is a corresponding category of such gadgets, consists of the discrete G-sets and all G-equivariant maps beteen them; this category will be denoted by $G - \mathbf{Set}_d$.

A *finite discrete G-set* is a discrete G-set X which (you guessed it) happens to be finite. We write $G - \mathbf{Set}_{df}$ for the full subcategory of $G - \mathbf{Set}_d$ on such finite objects. It is not hard at all to see that a finite discrete G-set X can be identified up to equivariant isomorphism with a finite disjoint union of the form

$$\bigsqcup G_i/N_i.$$

where N_i is a subgroup (not necessarily normal) of the corresponding group G_i. Observe that the one point set $*$ with trivial G-action is a member of this category, and is terminal.

The category of finite discrete G-sets has the structure of a Grothendieck site, where the covering families $\{\phi_i : X_i \to Y\}$ are finite lists of G-equivariant maps $\phi_i : X_i \to X$ such that the induced morphisms $\sqcup X_i \to Y$ are surjective.

A sheaf $F : (G - \mathbf{Set}_{df})^{op} \to \mathbf{Set}$ is a contravariant functor (or presheaf) such that the induced map

$$F(\sqcup G_i/N_i) \to \prod F(G_i)^{N_i}$$

Well, a sheaf had better take finite disjoint unions to finite products, and the indentification

$$F(G_i/N_i) \cong F(G_i)^{N_i}$$

is a consequence of the fact that the covering morphism $G_i \to G_i/N_i$ determines a coequalizer

$$G_i \times N_i \cong G_i \times_{G_i/N_i} G_i \rightrightarrows G_i \to G_i/N_i$$

in the category of finite discrete G-sets.

Write $\mathbf{Shv}(G-\mathbf{Set}_{df})$ for the category of sheaves on the site of finite discrete G-modules. There is an equivalence of categories

$$\mathbf{Shv}(G-\mathbf{Set}_{df}) \underset{R}{\overset{L}{\rightleftarrows}} G-\mathbf{Mod}_d$$

The notation $G-\mathbf{Mod}_d$ refers to Serre's category of discrete G-modules: these are sets equipped with a G-action $G \times X \to X$ such that X is a filtered colimit of fixed points in the sense that

$$X = \varinjlim_i X^{H_i}.$$

Recall that H_i is defined to the kernel of the group homomorphism $G \to G_i$. The functors L and R are easy to describe: if F is a sheaf then

$$LF = \varinjlim_i F(G_i),$$

while

$$RX = \hom_G(\ , X)$$

is the functor represented on the category of G-sets by X.

Example 23. Every discrete G-set X is a discrete G-module, since it's fixed by some H_i. The object therefore represents a sheaf $X = \mathbf{hom}(\ , X)$ on the site $G-\mathbf{Set}_{df}$ of finite discrete G-modules. In particular, every object of $G-\mathbf{Set}_{df}$ represents a sheaf.

Here are some remarks:

1) The topos $G-\mathbf{Mod}_d$ of discrete G-modules has enough points: there is a functor

$$u^* : G - \mathbf{Mod}_d \to \mathbf{Set}$$

which is defined by forgetting the group structure, so that u^*X is the set underlying X. Colimits and finite limits are formed in the category of discrete G-modules as they are in the set category, so it's easy to see that the functor u^* is faithful and exact. That's fine, but the wierdness here is that there's only one stalk.

2) For a presheaf X on the site of finite discrete G-modules, the object LX is still defined and is a discrete G-module, and the canonical map $X \to RL(X)$ can be identified with the associated sheaf map. The map itself is a little complicated to describe, but reduces in sections corresponding to G_i/N_i to the composite

$$X(G_i/N_i) \to X(G_i)^{N_i} \to \varinjlim_{j \to i} X(G_j)^{\pi^{-1}N_i} \cong \hom(G_i/N_i, LX).$$

3) There is an isomorphism
$$\Gamma_* F \cong LF^G$$
for all sheaves F. Recall that the global sections $\Gamma_* F$ of F is given by taking the inverse limit of F over the objects of the underlying site. Then one has
$$LF^G = \varinjlim F(G_i)^G = \varinjlim F(G_i)^{G_i} = F(*)$$

The topos $G - \mathbf{Mod}_d$ (aka. $\mathbf{Shv}(G - \mathbf{Set}_{df})$ is often called the *classifying topos* for the profinite group G.

A well known theorem of Giraud [26], [33] asserts that if a category satisfies a certain list of exactness properties, then it must be equivalent to the category of sheaves on some Grothendieck site. Furthermore, the site itself is defined on a generating family in a completely explicit way. The category of discrete G-modules for a profinite group G, and the identification with sheaves on the site of finite discrete G-sets is a result of the constructions of Giraud's theorem.

Giraud's theorem is a very useful tool for attaching explicit sites to toposes. It applies, in particular, to all flavours of categories of sets or spaces admitting actions by a group G (topological, discrete, sheaf theoretic, etc.) — these categories are the classifying toposes for the corresponding group objects.

4 Generalised Galois cohomology theory

We shall continue to talk about general profinite groups $G = \{G_i\}$, as in the previous section.

The covering map $G_i \to *$ determines a Čech resolution
$$\begin{array}{ccccccc} G_i & \leftleftarrows & G_i \times G_i & \leftleftarrows & G_i \times G_i \times G_i & & \cdots \\ \downarrow & & & & & & \\ * & & & & & & \end{array}$$
of the terminal object $*$ in the finite discrete G-module set $G - \mathbf{Mod}_{df}$. This simplicial object of left G-modules can be identified with the *sheaf-theoretic* Borel construction $G_i \tilde{\times}_{G_i} EG_i$ arising from the action of G_i on itself by right multiplication. The notation $G_i \tilde{\times}_{G_i} EG_i$ means that this object is the sheaf associated to the obvious presheaf $G_i \times_{G_i} EG_i$ — this distinction can be a very subtle point.

Suppose that B is an abelian presheaf on $G - \mathbf{Set}_{df}$. The cochain complex of *presheaf* maps $\hom(G_i \times_{G_i} EG_i, B)$ is a cosimplicial abelian group with n-cochains
$$\hom((G_i \times_{G_i} EG_i)_n, B) \cong \prod_{* \xleftarrow{g_1} * \cdots * \xleftarrow{g_n} *} B(G_i).$$
The simplicial presheaf maps $G_j \times_{G_j} EG_j \to G_i \times_{G_i} EG_i$ arising from the transition homomorphisms $G_j \to G_i$ induce cochain complex maps
$$\hom(G_i \times_{G_i} EG_i, B) \to \hom(G_j \times_{G_j} EG_j, B)$$

and we define the Čech cohomology groups $\check{H}^*(G,B)$ of G with coefficients in the abelian presheaf B by

$$\check{H}^*(G,B) = \varinjlim_{G_i} H^* \hom(G_i \times_{G_i} EG_i, B) \cong \varinjlim_{G_i} H^*(G_i, B(G_i))$$

If A is an abelian sheaf on $G - \mathbf{Set}_{df}$, then there are isomorphisms

$$\prod_{* \xleftarrow{g_1} * \cdots * \xleftarrow{g_n} *} A(G_i) \cong \prod_{* \xleftarrow{g_1} * \cdots * \xleftarrow{g_n} *} LA^{H_i},$$

where (from the last section) H_i is the kernel of the canonical surjection $G \to G_i$. It follows there are isomorphisms

$$\varinjlim_{i} H^* \hom(G_i \times_{G_i} EG_i, A) \cong \varinjlim_{i} H^*(G_i, LA^{H_i}) \cong H^*(G, LA)$$

In other words, the Čech cohomology groups $\check{H}^*(G,A)$ of G with coefficients in the abelian sheaf A coincide up to isomorphism with the traditional Galois cohomology groups $H^*(G,LA)$ of G with coefficients in the discrete G-module LA associated to the sheaf A.

Now here's the central fact about this construction:

Lemma 24. *Suppose that B is an abelian presheaf on the site $G - \mathbf{Set}_{df}$, and suppose that its associated sheaf $\tilde{B}$ is 0. Then $\check{H}^*(G,B) = 0$.*

Proof. A n-cochain in $\hom(G_i \times_{G_i} EG_i, B)$ is a tuple (α_σ) of elements $a_\sigma \in B(G_i)$, where the index σ corresponds to the set of n-tuples of elements in G_i. The associated sheaf $\tilde{B}$ is 0, so there is a covering $X_\sigma \to G_i$ such that $\alpha_\sigma \mapsto 0 \in B(X_\sigma)$. It follows that there is a transition homomorphism $G_{j_\sigma} \to G_i$ in the profinite group G such that $\alpha_\sigma \mapsto 0 \in B(G_{j_\sigma})$. Pick $j \geq j_\sigma$ for all σ. Then the cochain α maps to 0 in

$$\prod_{(BG_j^{op})_n} B(G_j).$$

This is true for all cochains, so the filtered colimit of complexes that defines $\check{H}^*(G,B)$ in all degrees is 0. □

Here's an easy corollary:

Corollary 25. *Suppose that B is an abelian presheaf. Then the associated sheaf map $B \to \tilde{B}$ induces an isomorphism*

$$\check{H}^*(G,B) \cong \check{H}^*(G,\tilde{B}).$$

Proof. The kernel and cokernel of the map $B \to \tilde{B}$ are abelian presheaves whose associated sheaves are 0. □

Write $H^*(G, A)$ for the sheaf cohomology of the topos $G - \mathbf{Mod}_d$ of discrete G-modules with coefficients in an abelian sheaf A. There are, of course, isomorphisms

$$H^*(G, A) \cong H^*\Gamma_* I^* \cong H^* I^*(*)$$

where $A \to I^*$ is an injective resolution of the abelian sheaf A.

The following result says that sheaf and Čech cohomology coincide:

Proposition 26. *There is an isomorphism*

$$\check{H}^*(G, A) \cong H^*(G, A)$$

which is natural in abelian sheaves A.

Proof. Let $A \to I^*$ be an injective resolution of A. Then the simplicial presheaf maps $G_i \times_{G_i} EG_i \to *$ and the injective resolution together determine natural isomorphisms

$$\check{H}^*(G, A) \overset{(A)}{\cong} \check{H}^*(G, I^*) \overset{(B)}{\cong} H^*(G, A).$$

The cohomology groups $\check{H}^*(G, I^*)$ arise from a filtered colimit of bicomplexes

$$\prod_{* \xleftarrow{g_1} * \cdots * \xleftarrow{g_n} *} I^p(G_i)$$

in an obvious way. The isomorphism labelled (A) is a consequence of Lemma 24, since the cohomology presheaves $H^* I^*$ satisfy $\tilde{H}^p I^* = 0$ for $p > 0$. The isomorphism (B) is induced by the collection of local weak equivalences

$$G_i \times_{G_i} EG_i \to *.$$

One uses the fact that all functors $\hom(\ , I^p)$ are exact on the sheaf category since the abelian sheaves I^p are injective. □

Example 27. Suppose that k is a field, and let $G = Gal(k_{sep}/k)$ be its absolute Galois group, where k_{sep} denotes the separable closure of the field k. There is a "site isomorphism"

$$et|_k \xrightarrow{\cong} G - \mathbf{Set}_{df}$$

which is defined by taking a finite étale map $U = \sqcup\ Sp(L_i) \to Sp(k)$ to the set $\hom_k(Sp(k_{sep}), U)$. The notation means that U is a finite disjoint union of spectra of separable extensions L_i/k. Any abelian sheaf A for the étale topology on $Sp(k)$ therefore corresponds uniquely to an abelian sheaf on the site $G - \mathbf{Set}_{df}$ of discrete finite G-sets for the Galois group G, and there is an isomorphism

$$H^*_{et}(k, A) \cong \check{H}^*(G, A)$$

which is induced by the site isomorphism and the identification of sheaf cohomology with Čech cohomology given by Proposition 26.

These ideas have analogs for presheaves of spectra F on the site $G - \mathbf{Set}_{df}$. There is a stable equivalence

$$\hom((G_i \times_{G_i} EG_i)_+, F) \simeq \underleftarrow{\mathrm{holim}}_{\,G_i} F(G_i),$$

so one is entitled to define the generalised Čech cohomology groups $\check{\mathbb{H}}^n(G, F)$ with coefficients in the presheaf of spectra F by

$$\check{\mathbb{H}}^n(G, F) = \varinjlim_i \pi_{-n} \underleftarrow{\mathrm{holim}}_{\,G_i} F(G_i)$$

for $n \in \mathbb{Z}$.

Here's a key point: if F is globally fibrant, then the local weak equivalence $G_i \times_{G_i} EG_i \to *$ induces a stable equivalence

$$\hom_*((G_i \times_{G_i} EG_i)_+, F) \xrightarrow{\simeq} \hom_*(S^0, F) \cong F(*).$$

In particular, there is a stable equivalence

$$\underleftarrow{\mathrm{holim}}_{\,G_i} F(G_i) \simeq F(*).$$

This is the *finite descent property* for globally fibrant presheaves of spectra on the site$G - \mathbf{Set}_{df}$.

There are two consequences:

1) There is an isomorphism

$$\check{\mathbb{H}}^*(G, F) \cong \mathbb{H}^*(G, F)$$

if F is globally fibrant. Recall that $\mathbb{H}^n(G, F) \cong \pi_n F(*)$ in this case.

2) If $E \to F$ is a globally fibrant model for a presheaf of spectra E then the induced map $\check{\mathbb{H}}^*(G, E) \to \check{\mathbb{H}}^*(G, F)$ can be identified with a morphism

$$\check{\mathbb{H}}^*(G, E) \to \mathbb{H}^*(G, E)$$

from the generalised Čech cohomology theory associated to E to the generalized sheaf cohomology theory associated to E.

Remark 28 (Warning). The map

$$\check{\mathbb{H}}^*(G, E) \to \mathbb{H}^*(G, E)$$

relating generalised Čech and generalised sheaf cohomology is *not* known to be an isomorphism in general. We have effectively seen that it is an isomorphism when E is an Eilenberg-Mac Lane spectrum object $K(A, n)$. Both constructions see fibre sequences so it follows that the map is an isomorphism if E has only finitely many non-trivial presheaves of homotopy groups ... but that's it. It is even not known that $\check{\mathbb{H}}^*(G, E) = 0$ in the presence of a local stable equivalence $E \to *$

In general, if F is a presheaf of spectra on $G-\mathbf{Set}_{df}$ and G_i is one of the quotients (components) of the profinite group G, then the right multiplication maps $\cdot g : G_i \to G_i$ by elements $g \in G_i$ determine an action

$$G_i \times F(G_i) \to F(G_i).$$

Write $g = (\cdot g)^* : F(G_i) \to F(G_i)$. The finite collection of maps $g : F(G_i) \to F(G_i)$ of spectra can be added up in the stable category to produce the *norm map* $N : F(G_i) \to F(G_i)$; this map is defined to be the "composite"

$$F(G_i) \xrightarrow{\Delta} \prod_{g\in G} F(G_i) \xleftarrow{\simeq} \bigvee_{g\in G} F(G_i) \xrightarrow{\nabla} F(G_i).$$

in the stable category. The notation here is a little bit strange, although you find it in the literature [19]: the map Δ is not a diagonal, but is rather defined by the requirement that all diagrams

$$\begin{array}{ccc} F(G_i) & \xrightarrow{\Delta} & \prod_{g\in G} F(G_i) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle pr_g} \\ & & F(G_i) \end{array}$$

commute, and is therefore multiplication by the group elements in the corresponding factors.

If g, h are elements of G_i, then there is a commutative diagram

$$\begin{array}{ccc} F(G_i) & \xrightarrow{N} & F(G_i) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ F(G_i) & \xrightarrow[N]{} & F(G_i) \end{array}$$

in the stable category (actually much more care is required), and so the norm map has a factorization

$$\begin{array}{ccc} F(G_i) & \xrightarrow{N} & F(G_i) \\ \downarrow & & \uparrow \\ \underrightarrow{\operatorname{holim}}_{G_i} F(G_i) & \xrightarrow[N_h]{} & \underleftarrow{\operatorname{holim}}_{G_i} F(G_i) \end{array}$$

through a map

$$N_h : \underrightarrow{\operatorname{holim}}_{G_i} F(G_i) \to \underleftarrow{\operatorname{holim}}_{G_i} F(G_i)$$

called the *hypernorm*.

Recall that the map

$$F(*) \to \underleftarrow{\operatorname{holim}}_{G_i} F(G_i)$$

is a stable equivalence if F is globally fibrant; in that case the composition

$$F(G_i) \to \underrightarrow{\mathrm{holim}}_{G_i} F(G_i) \xrightarrow{N_h} \underleftarrow{\mathrm{holim}}_{G_i} F(G_i) \xleftarrow{\simeq} F(*)$$

defines an *abstract transfer map* $\tau : F(G_i) \to F(*)$. This transfer map has the usual list of good properties (including a projection formula) where it is defined.

Assumption 29. We're now going to restrict to the cases of presheaves of spectra F which are *bounded below* in the sense that the presheaves of stable homotopy groups $\pi_i F$ vanish for i below some number N, and such that the cohomological dimension of the profinite group G with respect to all sheaves $\tilde{\pi}_* F$ is bounded above by some number M. This means that $H^j(H, \tilde{\pi}_j F) = 0$ for $j > M$ for all closed subgroups H of G. Saying that H is closed means in practice that H is the pullback in G of some finite subgroup of some finite quotient G_i.

Example 30. The examples of such profinite groups to keep in mind arise from Galois groups of fields k, when the étale sheaves $\tilde{\pi}_* F$ are ℓ-torsion where ℓ is a prime not equal to either 2 or the characteristic of k, and k has finite transcendance degree over some field N containing a primitive ℓ^{th} root of unity ζ_ℓ, and such that either $cd_\ell(N) < 1$ (eg. $N = \mathbb{F}_p(\zeta_\ell)$ a finite field) or $cd_l(N(\mu_{\ell^\infty})) < 1$ (eg. $\mathbb{Q}(\zeta_\ell)$). These are special examples of fields k for which Thomason's descent theorem for Bott periodic K-theory holds (and is more easily described), and for which the Lichtenbaum-Quillen conjecture should hold.

Under these assumptions, all of the descent machinery works:

1) If F is a globally fibrant presheaf of spectra on $G - \mathbf{Set}_{df}$ the Galois cohomological descent spectral sequence

$$E_2^{s,t} = H^s(G, \tilde{\pi}_t F) \Rightarrow \pi_{t-s} F(*) = \mathbb{H}^{s-t}(G, F)$$

converges. If G is the absolute Galois group of one of the fields in the list above, then this spectral sequence has the form

$$E_2^{s,t} = H^s_{et}(k, \tilde{\pi}_t F) \Rightarrow \pi_{t-s} F(k) = \mathbb{H}^{s-t}_{et}(k, F).$$

2) Suppose that $F \to P_n F$ is a Postnikov section of F, and consider the composite

$$F \to P_n F \xrightarrow{j} GP_n F$$

where j is a globally fibrant model. The presheaf of spectra $GP_n F$ has only finitely many non-trivial presheaves of stable homotopy groups on account of the global bound on cohomological dimension, and the fibre of the composite map $F \to GP_n F$ has presheaves of stable homotopy groups which are 0 below some fixed integer, in all sections. In fact, one can show that the presheaf maps $\pi_i F \to \pi_i GP_n F$ are isomorphisms for $i < n - M$. This gives a technique for approximating the presheaves of stable homotopy groups of F on presheaves of spectra having only finitely many non-trivial presheaves of stable homotopy groups.

3) Suppose that E is a presheaf of spectra whose presheaves of stable homotopy groups $\pi_* E$ satisfy the assumptions above. The n^{th} Postnikov section $E \to P_n E$ induces a comparison diagram

$$\begin{array}{ccc} \check{\mathbb{H}}^i(G,E) & \longrightarrow & \check{\mathbb{H}}^i(G,P_nE) \\ \downarrow & & \downarrow \simeq \\ \mathbb{H}^i(G,E) & \longrightarrow & \mathbb{H}^i(G,P_nE) \end{array}$$

We have just seen that the map $\mathbb{H}^i(G,E) \to \mathbb{H}^i(G,P_nE)$ is an isomorphism for $i > M-n$. It follows that $\check{\mathbb{H}}^i(G,P_nE)$ is isomorphic to $\mathbb{H}^i(G,E)$ in the same range. (One should very quickly revert to interpreting this in terms of actual stable homotopy groups of spectra, because the sign change is much too confusing).

Now we're in position to believe a pretty good result:

Theorem 31 (Tate Theorem). *Suppose that F is a globally fibrant presheaf of spectra on the site $G-\mathbf{Set}_{df}$, and suppose that the presheaves of stable homotopy groups of F are bounded below and that G has finite cohomological dimension with respect to all sheaves $\tilde{\pi}_* F$ of F. The the hypernorm map*

$$N_h : \underrightarrow{\text{holim}}_{G_i} F(G_i) \to \underleftarrow{\text{holim}}_{G_i} F(G_i) \simeq F(*)$$

is a stable equivalence for all finite quotients G_i of G.

The Tate Theorem is due to Thomason [38], but appeared for the first time in [19] in its present form. Its proof of is an inductive argument that starts with the Tate lemma that asserts the norm map induces an isomorphism

$$H_0(G_i, A(G_i)) \cong H^0(G_i, A(G_i))$$

if A is a discrete abelian G-module of cohomological dimension 0 [34]. This gives the case of the Theorem for presheaves of spectra $GH(A)$ associated to such sheaves A. Both sides of the comparison respect fibre sequences, allowing one to prove the case corresponding to presheaves of spectra $GK(B,n)$ where G has bounded cohomological dimension with respect to the abelian sheaf $\tilde{B}$. A second induction involving fibre sequences allows one to verify the statement for all globally fibrant models GP_nF of finite Postnikov sections, and then one finishes the proof by using the approximation technique given in item 2) above.

In the cases to which it applies, the Tate Theorem give rise to the *Tate spectral sequence*

$$E_2^{p,q} = H_p(G_i, \pi_q F(G_i)) \Rightarrow \pi_{p+q} F(*)$$

Thus, for the sort of base field k listed in Example 30, if L/k is a finite Galois extension with Galois group G and F is a globally fibrant presheaf of spectra on the étale site $et|_k$ which satisfies the usual assumptions, then the Tate spectral sequence has the form

$$E_2^{p,q} = H_p(G, \pi_q F(L)) \Rightarrow \pi_{p+q} F(k) \tag{5}$$

We shall switch to this context for the remainder of this paper.

5 Thomason's descent theorem

The Tate spectral sequence (5) and low dimensional calculations are the basis for the following result of Thomason. For the statement, we need to know that if L/k is a finite Galois extension of a field k with Galois group G, then the K-theory transfer map $i : K/\ell(L) \to K/\ell(k)$ is G-equivariant for the obvious action of G on $K/\ell(L)$ and the trivial action on $K/\ell(k)$. It therefore induces a map of spectra

$$i_h : \underrightarrow{\text{holim}}_G K/\ell(L) \to K/\ell(k)$$

which is called the *hypertransfer*.

Theorem 32. *Suppose that k is a field satisfying the list of assumptions in Example 30, and suppose that there is a Galois extension N/k such that $cd_\ell(N) \leq 1$ and $cd_\ell(H) \leq 1$ where $H = Gal(N/k)$. Suppose that L/k is a finite Galois subextension of N/k with Galois group G. Then there is an element*

$$Ind(\beta) \in \pi_2 \underrightarrow{\text{holim}}_G K/\ell(L)$$

such that the homomorphism $i_{h*} : \pi_2 \underrightarrow{\text{holim}}_G K/\ell(L) \to \pi_2 K/\ell(k)$ induced by the hypertransfer maps $Ind(\beta)$ to the Bott element $\beta \in \pi_2 K/\ell(k)$.

The element $Ind(\beta)$ is called an *inductor* for the Bott element β. The punchline in the proof of Theorem 32 is the Tate Theorem for the globally fibrant model of the suspended Moore spectrum $S[2]/\ell$.

Now, in the same generality as the Theorem 32:

1) The Bott periodic K-theory presheaf of spectra $K/\ell(1/\beta)$ and its globally fibrant model $GK/\ell(1/\beta)$ are both modules over the mod ℓ K-theory presheaf K/ℓ. In particular the composite

$$K/\ell(1/\beta)(L) \wedge K/\ell(L) \xrightarrow{\cup} K/\ell(1/\beta)(L) \xrightarrow{i_*} K/\ell(1/\beta)(k)$$

has an adjoint

$$c : K/\ell(1/\beta)(L) \to \mathbf{Hom}(K/\ell(L), K/\ell(1/\beta)(k))$$

taking values in a function spectrum. Taking homotopy inverse limits for the G-action induces a map c_* which fits into the picture

$$
\begin{array}{ccc}
\underleftarrow{\operatorname{holim}}_G K/\ell(1/\beta)(L) & \xrightarrow{c_*} & \underleftarrow{\operatorname{holim}}_G \mathbf{Hom}(K/\ell(L), K/\ell(1/\beta)(k)) \\
\uparrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \cong} \\
 & & \mathbf{Hom}(\underrightarrow{\operatorname{holim}}_G K/\ell(L), K/\ell(1/\beta)(k)) \\
 & & \downarrow{\scriptstyle Ind(\beta)^*} \\
K/\ell(1/\beta)(k) & \dashrightarrow & \mathbf{Hom}(S[2], K/\ell(1/\beta)(k))
\end{array}
$$

The indicated dotted arrow map is induced by multiplication by the Bott element, by a projection formula argument. A similar picture exists for the globally fibrant model $GK/\ell(1/\beta)$, and the two may be compared.

2) Consider the resulting comparison diagram

$$
\begin{array}{ccccc}
K/\ell(1/\beta)(k) & \xrightarrow{i^*} & \underleftarrow{\operatorname{holim}}_G K/\ell(1/\beta)(L) & \longrightarrow & \Omega^2 K/\ell(1/\beta)(k) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j*} & & \downarrow{\scriptstyle j} \\
GK/\ell(1/\beta)(k) & \underset{i^*}{\xrightarrow{\simeq}} & \underleftarrow{\operatorname{holim}}_G GK/\ell(1/\beta)(L) & \longrightarrow & \Omega^2 GK/\ell(1/\beta)(k)
\end{array} \tag{6}
$$

The vertical maps arise from the choice $j : K/\ell(1/\beta) \to GK/\ell(1/\beta)$ of globally fibrant model, the maps labelled i^* are canonical maps into the respective homotopy inverse limits, and both horizontal composites are induced by multiplication by the Bott element.

Actually, this is all a bit of a lie: all of the objects in the diagram (6) are supposed to be constructed from finite Postnikov sections of the presheaf of spectra $K/\ell(1/\beta)$ so that the following "Čech descent" arguments work — the concept of approximation by finite Postnikov sections was invented to handle exactly this point (and Thomason used to say that this was the hardest thing in his proof) — but the display in front of you is already formidable enough.

3) We have a diagram (6) for all finite subextensions L/k of N/k, and the idea is that the map $j : K/\ell(1/\beta) \to GK/\ell(1/\beta)$ should induce an isomorphism in generalized Čech cohomology theory for the Galois group of N/k (this is the thing that requires a finite Postnikov section instead). Then any element $\alpha \in \pi_* GK/\ell(1/\beta)(k)$ lifts to an element of $\pi_* \underleftarrow{\operatorname{holim}}_G K/\ell(1/\beta)(L)$ *for some extension* L, so that $\beta \cup \alpha$ is in the image of $j_* : \pi_* \Omega^2 K/\ell(1/\beta)(k) \to \pi_* \Omega^2 GK/\ell(1/\beta)(k)$. Similarly, if $j_*(\gamma) = 0$ in $\pi_* GK/\ell(1/\beta)(k)$ then $i^*(\gamma) = 0$ in $\pi_2 \underleftarrow{\operatorname{holim}}_G K/\ell(1/\beta)(L)$ for some L, and so $\beta \cup \alpha = 0$. Multiplication by β has been inverted on $K/\ell(1/\beta)$ and on its globally fibrant model, so it follows that $j : K/\ell(1/\beta)(k) \to GK/\ell(1/\beta)(k)$ is a stable equivalence. This is also true if k is replaced by any of its finite

extensions inside the field N, because the same assumptions are at work. One then can replace k by N, by a continuity argument.

4) The previous step effectively finishes the case of relative Galois cohomological dimension 1. One can show that the map $j : K/\ell(1/\beta)(k) \to GK/\ell(1/\beta)(k)$ for all fields satisfying the standard assumption by inducting up through a Tate-Tsen filtration [19], [38], meaning via an induction through steps of relative Galois cohomological dimension one.

These have been, grossly speaking, the steps in proving Thomason's descent theorem for Bott periodic K-theory for good fields (and at good primes). The version of Thomason's theorem that is given in [19, Thm 7.31] asserts the following:

Theorem 33. *Suppose that ℓ is a prime such that $\ell > 3$. Suppose that X is a scheme which is separated, Noetherian and regular, of finite Krull dimension, and suppose that the ring $\Gamma(X, \mathcal{O}_X)$ of functions on X contains $1/\ell$. Suppose that each residue field $k(x)$ of X is of finite transcendance degree over some field k_x such that $cd_l(k_x) \leq 1$ or $cd_\ell(k_x(\mu_{l^\infty}) \leq 1$. Then the map*

$$K/\ell(1/\beta)(X) \to GK/\ell(1/\beta)(X)$$

is a stable equivalence.

The assumptions on the residue fields mean that the statement of Thomason's theorem holds for those fields: the requirement that the respective fields contain a primitive ℓ^{th} root of unity disappears by a trick. The important thing is the regularity assumption, which means that we are in the realm where the Nisnevich descent theorem holds.

Here's how to finish the proof. Take a globally fibrant model

$$j : K/\ell(1/\beta) \to GK/\ell(1/\beta)$$

with respect to the étale topology on smooth schemes over X. This is a map of presheaves of spectra, but now interpret it in the stable category associated to the Nisnevich topology. The presheaf of spectra $GK/\ell(1/\beta)$ is globally fibrant with respect to the Nisnevich topology, since direct image functors preserve global fibrations (this is a ubiquitous fact, which first appeared in [15]). At the same time, by working in stalks at points x on smooth schemes Y/X, one finds a commutative diagram

$$\begin{array}{ccc} K/\ell(1/\beta)(\mathcal{O}^h_{x,Y}) & \xrightarrow{j_*} & GK/\ell(1/\beta)(\mathcal{O}^h_{x,Y}) \\ \simeq \downarrow & & \downarrow \simeq \\ K/\ell(1/\beta)(k(x)) & \xrightarrow[j_*]{\simeq} & GK/\ell(1/\beta)(k(x)) \end{array}$$

where $\mathcal{O}^h_{x,Y}$ is the henselization of the local ring $\mathcal{O})_{x,Y}$ and the vertical maps are induced by the residue homomorphism $\mathcal{O}^h_{x,Y} \to k(x)$. The residue maps are both stable equivalences, essentially by Gabber rigidity, and the bottom horizontal map is a stable equivalence by the Thomason theorem for fields.

It follows that the map $j : K/\ell(1/\beta) \to GK/\ell(1/\beta)$ is a local stable equivalence and hence a globally fibrant model for the Nisnevich topology. At the same time, the Nisnevich descent theorem implies that the presheaf of spectra $K/\ell(1/\beta)$ satisfies Nisnevich descent, and is therefore sectionwise stably equivalent to any globally fibrant model for that topology. The map j is therefore a stable equivalence in all sections.

References

[1] B. Blander, *Local projective model structures on simplicial presheaves*, K-Theory **24**(3) (2001), 283–301.

[2] A.K. Bousfield and E.M. Friedlander, *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Springer Lecture Notes in Math. **658** (1978), 80–150.

[3] K.S. Brown and S.M. Gersten, *Algebraic K-theory as generalized sheaf cohomology*, Springer Lecture Notes in Math. **341** Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1973), 266–292.

[4] W. Dwyer and E. Friedlander, *Algebraic and étale K-theory*, Trans. AMS **292** (1985), 247–280.

[5] W. Dwyer, E. Friedlander, V. Snaith and R. Thomason, *Algebraic K-theory eventually surjects onto topological K-theory*, Invent. math. **66** (1982), 481–491.

[6] O. Gabber, *K-theory of Henselian local rings and Henselian pairs*, Contemp. Math. **126** (1992), 59–70.

[7] H. Gillet and R.W. Thomason, *The K-theory of strict hensel local rings and a theorem of Suslin*, J. Pure Applied Algebra **34** (1984), 241–254.

[8] P.G. Goerss, *Homotopy fixed points for Galois groups*, Contemp. Math. **181** (1995), 187–224.

[9] P.G. Goerss and J.F. Jardine, *Localization theories for simplicial presheaves*, Can. J. Math. **50(5)** (1998), 1048–1089.

[10] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, **174**, Birkhäuser, Basel-Boston-Berlin (1999).

[11] M. Hovey, B. Shipley and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.

[12] J.F. Jardine, *Simplicial objects in a Grothendieck topos*, Contemp. Math **55** (1986), 193–239.

[13] J.F. Jardine, *Simplicial presheaves*, J. Pure Applied Algebra **47** (1987), 35–87.

[14] J.F. Jardine, *Stable homotopy of simplicial presheaves*, Can. J. Math. **39** (1987), 733–747.

[15] J.F. Jardine, *The Leray spectral sequence*, J. Pure Applied Algebra **61** (1989), 189–196.

[16] J.F. Jardine, *Universal Hasse-Witt classes*, Contemporary Math. **83** (1989), 83-100.

[17] J.F. Jardine¡ *The Lichtenbaum-Quillen conjecture for fields*, Canad. Math. Bull. **36** (1993), 426–441.

[18] J.F. Jardine, *Boolean localization, in practice*, Doc. Math. **1** (1996), 245–275.

[19] J.F. Jardine, *Generalized Etale Cohomology Theories*, Progress in Math. **146**, Birkhäuser, Basel-Boston-Berlin (1997).

[20] J.F. Jardine, *Presheaves of symmetric spectra*, J. Pure App. Alg. **150** (2000) 137–154.

[21] J.F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552.

[22] J.F. Jardine, *Stacks and the homotopy theory of simplicial sheaves*, Homology, Homotopy and Applications **3**(2) (2001), 361–384.

[23] J.F. Jardine, *The separable transfer*, Preprint (2000), to appear in J. Pure Applied Algebra.

[24] J.F. Jardine, *Presheaves of chain complexes*, Preprint (2001).

[25] A. Joyal, *Letter to A. Grothendieck* (1984).

[26] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer-Verlag, Berlin-Heidelberg-New York (1992).

[27] A.S. Merkurjev and A.A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Math. USSR Izvestiya **21** (1983), 307–340.

[28] A.S. Merkurjev and A.A. Suslin, *On the norm residue homomorphism of degree 3*, Math. USSR Izvestiya **36** (1991), 349–367.

[29] J.S. Milne, *Étale Cohomology*, Princeton University Press, Princeton (1980).

[30] S. Mitchell, *Hypercohomology spectra and Thomason's descent theorem*, Algebraic K-theory, Fields Institute Communications **16**, AMS (1997), 221–278.

[31] F. Morel and V. Voevodsky, *$\mathbb{A}^1$ homotopy theory of schemes*, Publ. Math. IHES **90** (1999), 45–143.

[32] Y.A. Nisnevich, *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory*, Algebraic K-theory: Connections with Geometry and Topology, NATO ASI Series C **279**, Kluwer, Dordrecht (1989), 241–342.

[33] H. Schubert, *Categories*, Springer-Verlag, New York-Heidelberg-Berlin (1972).

[34] J-P. Serre, *Cohomologie Galoisienne*, Springer Lecture Notes in Math. **5** (4th edition), Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1973).

[35] A.A. Suslin, *On the K-theory of algebraically closed fields*, Invent. Math. **73** (1983), 241–245.

[36] A.A. Suslin, *On the K-theory of local fields*, J. Pure Applied Algebra **34** (1984), 301–318.

[37] A. Suslin and V. Voevodsky, *Singular homology of abstract algebraic varieties*, Invent. math. **123** (1996), 61–103.

[38] R.W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Scient. Éc. Norm. Sup. 4^e série **18** (1985), 437–552.

[39] R.W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Volume III, Progress in Mathematics **88**, Birkhäuser, Boston-Basel-Berlin (1990), 247–436.

[40] V. Voevodsky, *$\mathbf{A}^1$-Homotopy theory*, Doc. Math. Extra Vol. ICM **I** (1998), 579-604.

[41] D.H. Van Osdol, *Simplicial homotopy in an exact category*, Amer. J. Math. **99**(6) (1977), 1193–1204.