

POWERS AND PRODUCTS OF REGENERATIVE PHENOMENA

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Summary The simultaneous occurrence of two independent regenerative phenomena defines a third, whose p -function is the product of the first two. Thus integer powers of p -functions are p -functions. The corresponding result for non-integer powers (with exponent $\alpha > 1$) was proved in 1995 for discrete time, and for standard continuous time, phenomena. There are still open questions, notably whether the class of Markov p -functions is closed under non-integer powers. These questions are addressed by means of a new technique which relates the atoms of the canonical measure to ‘kinks’ in the p -function. This provides new information even for products of p -functions.

Keywords Regenerative phenomena, p -functions, non-integer powers, kink analysis.

1. Multiplicative theory of p -functions

A *regenerative phenomenon* (Kingman, 1972a) is a random process

$$Z = (Z(t), t > 0) \tag{1.1}$$

taking only the values 0 and 1 and such that, whenever

$$0 = t_0 < t_1 < t_2 < \dots < t_n, \tag{1.2}$$

$$\mathbb{P}\{Z(t_1) = Z(t_2) = \dots = Z(t_n) = 1\} = \prod_{r=1}^n p(t_r - t_{r-1}). \tag{1.3}$$

Here p is a function on $(0, \infty)$ called the *p -function* of Z which determines the finite-dimensional distributions of Z by a simple inclusion-exclusion argument.

In particular, the first occurrence probability

$$\begin{aligned} \mathbb{P}\{Z(t_1) = \dots = Z(t_{n-1}) = 0, Z(t_n) = 1\} \\ = \mathbb{E}\{(1 - Z(t_1))(1 - Z(t_2)) \dots (1 - Z(t_{n-1})) Z(t_n)\}. \end{aligned}$$

Expanding out the product gives a sum of products of $Z(t_r)$, each of which has expectation given by (1.3) for a subsequence of (1.2). Thus

$$\mathbb{P}\{Z(t_1) = \dots = Z(t_{n-1}) = 0, Z(t_n) = 1\} = F(t_1, t_2, \dots, t_n; p), \tag{1.4}$$

where each F is a polynomial in values of p :

$$\begin{aligned} F(t_1; p) &= p(t_1), \\ F(t_1, t_2; p) &= p(t_2) - p(t_1)p(t_2 - t_1), \\ F(t_1, t_2, t_3; p) &= p(t_3) - p(t_1)p(t_3 - t_1) - p(t_2)p(t_3 - t_2) \\ &\quad + p(t_1)p(t_2 - t_1)p(t_3 - t_2) \\ &\quad \dots \end{aligned} \tag{1.5}$$

A necessary and sufficient condition for a real-valued function on $(0, \infty)$ to be a p -function is that, for any sequence (1.2),

$$F(t_1, t_2, \dots, t_n; p) \geq 0 \tag{1.6}$$

and

$$\sum_{r=1}^n F(t_1, t_2, \dots, t_r; p) \leq 1. \quad (1.7)$$

The most important classes of p -functions are \mathcal{P} , the class of *standard* p -functions satisfying

$$\lim_{t \rightarrow 0} p(t) = 1, \quad (1.8)$$

and \mathcal{R} , the class of renewal sequences which are in effect p -functions which are non-zero only when t is an integer.

Although (1.6) and (1.7) give a complete criterion for a function to be a p -function, they can be difficult to apply. For instance, it is very hard to use them to show that the product of two p -functions is a p -function. Yet this is obvious probabilistically, since if Z_1 and Z_2 are independent regenerative phenomena with p -functions p_1 and p_2 , then (1.3) is satisfied by

$$Z(t) = Z_1(t)Z_2(t), \quad (1.9)$$

with

$$p(t) = p_1(t)p_2(t). \quad (1.10)$$

This simple observation was the starting point of the rich Kendall-Davidson theory of delphic semigroups (Kendall and Harding, 1973).

In particular, for any p -function p ,

$$p^\alpha(t) = \{p(t)\}^\alpha \quad (1.11)$$

defines a p -function p^α for any positive integer α . This suggested the natural question (in the discussion of (Kingman, 1966)) whether p^α is a p -function if $\alpha > 1$ is not an integer. It was a difficult question because there seemed to be no probabilistic interpretation of p^α , and it was only resolved thirty years later (Kingman, 1996) by direct use of (1.6) and (1.7). And even that proof left a gap, since it only proved that the two classes \mathcal{R} and \mathcal{P} are closed under

$$p \longmapsto p^\alpha \quad (\alpha > 1), \quad (1.12)$$

and there are many p -functions that fall outside these classes.

An important subclass of \mathcal{P} is \mathcal{PM} , which consists of those standard p -functions which can come from continuous time Markov chains. David Williams has asked whether \mathcal{PM} is closed under the map (1.12). This is still an open question, but the technique to be described below makes it very likely that the answer is positive.

2. A general proof of the power property

The purpose of this section is to give a proof that p^α is a p -function for $\alpha > 1$, a proof that applies to any p -function p , and not only to those in \mathcal{R} and \mathcal{P} which were dealt with in Kingman (1996).

It is well known (Kingman, 1972*b*) that it is often possible to avoid (1.7), since it follows from (1.6) in the presence of simpler restrictions on p . A function which satisfies (1.6) but not necessarily (1.7) is called a *semi- p -function*. Then it is known that a semi- p -function satisfying (1.8) is a p -function if and only if it is bounded, and there is a similar result for renewal sequences. Both these results are special cases of the following result.

Theorem 1 **A semi- p -function is a p -function if and only if it is bounded.**

Proof The ‘only if’ assertion is immediate, since (1.6) and (1.7) with $n = 1$ show that any p -function satisfies

$$0 \leq p(t) \leq 1 \tag{2.1}$$

for all $t > 0$. To prove the converse, we use the ‘first passage’ formula

$$p(t_n) = \sum_{r=1}^{n-1} F(t_1, t_2, \dots, t_r; p) p(t_n - t_r) + F(t_1, t_2, \dots, t_n; p), \tag{2.2}$$

which is valid for any sequence (1.2) and any function p . It is important to note that, although (2.2) has a simple probabilistic meaning when p is a p -function, it is true as an identity in the values of p . It can indeed be used by recursion on n to *define* the polynomials F .

Let p be any semi- p -function, so that

$$f_r = F(t_1, t_2, \dots, t_r; p) \geq 0 \tag{2.3}$$

for $r = 1, 2, \dots, n$, and write

$$f = f_1 + f_2 + \dots + f_n. \tag{2.4}$$

We prove by induction on $a = a_1 + a_2 + \dots + a_n$ that, for non-negative integers a_r ,

$$p(a_1 t_1 + a_2 t_2 + \dots + a_n t_n) \geq \frac{a!}{a_1! a_2! \dots a_n!} f_1^{a_1} f_2^{a_2} \dots f_n^{a_n}. \tag{2.5}$$

This is trivially true if $a = 1$; suppose it true for $1, 2, \dots, a - 1$. Replace the sequence t_1, \dots, t_n in (2.2) by

$$0 < t_1 < t_2 < \dots < t_n < T = a_1 t_1 + \dots + a_n t_n.$$

Then (2.2) and (2.5) for $a - 1$ imply that

$$p(T) \geq \sum_{r=1}^n f_r \frac{(a-1)!}{a_1! \dots (a_r-1)! \dots a_n!} f_1^{a_1} \dots f_r^{a_r-1} \dots f_n^{a_n},$$

which proves (2.5).

If the semi- p -function p is bounded above by M (say), (2.5) shows that

$$a! f_1^{a_1} \dots f_n^{a_n} \leq M a_1! \dots a_n! \tag{2.6}$$

for all a_r . Take logarithms and set $a_r = [N f_r]$ as the integer $N \rightarrow \infty$. Then Stirling's formula shows that (with simple modifications if $f_r = 0$)

$$\begin{aligned} \log a_r! &= N f_r (\log N f_r - 1) + o(N), \\ \log a! &= N f (\log N f - 1) + o(N), \end{aligned}$$

and (2.6) simplifies to

$$N f \log f + o(N) \leq 0.$$

This shows that $f \leq 1$ and therefore p satisfies (1.7) and is a p -function.

Theorem 2 **If p is a semi- p -function and $\alpha > 1$, then p^α is a semi- p -function. If p is a p -function, then p^α is a p -function.**

Proof The second assertion follows from the first by Theorem 1. Thus we only have to prove, if a function p satisfies (1.6) for all sequences (1.2), then so does p^α for all $\alpha > 1$. We shall in fact prove, by induction on n , the stronger result that, *if p satisfies (1.6) for all sequences of length at most n , then so does p^α for all $\alpha > 1$.*

This assertion is trivially true for $n = 1, 2$. Suppose it is true for $1, 2, \dots, n - 1$, and let p be any function satisfying (1.6) for every sequence of length at most n . Then p^α satisfies (1.6) for every sequence of length less than n , and our task is to prove that

$$F(t_1, t_2, \dots, t_n; p^\alpha) \geq 0. \tag{2.7}$$

Suppose first that $p(t_n) = 0$. Applying (2.2) with the 2-element sequence $0 < t_r < t_n$ we have $p(t_r)p(t_n - t_r) = 0$, and so $p^\alpha(t_r)p^\alpha(t_n - t_r) = 0$. But

$$p^\alpha(t_r) \geq F(t_1, \dots, t_r; p^\alpha) \geq 0,$$

so that

$$F(t_1, \dots, t_r; p^\alpha) p^\alpha(t_n - t_r) = 0$$

for $r = 1, 2, \dots, n - 1$. Equation (2.2) then shows that (2.7) is true with equality.

In what follows we therefore assume that

$$p(t_n) > 0. \quad (2.8)$$

Use the conventions

$$p(0) = 1, \quad t_0 = 0, \quad (2.9)$$

and write

$$f_{ij}(\alpha) = F(t_{i+1} - t_i, t_{i+2} - t_i, \dots, t_j - t_i; p^\alpha) \quad (2.10)$$

for $0 \leq i < j \leq n$, so that by the inductive hypothesis

$$f_{ij}(\alpha) \geq 0 \quad (\alpha > 1) \quad (2.11)$$

except perhaps for $i = 0, j = n$. Equation (2.2) applied to the sequence $t_{i+1} - t_i < t_{i+2} - t_i < \dots < t_j - t_i$ shows that, for $0 \leq i < j \leq n$,

$$p^\alpha(t_j - t_i) = \sum_{k=i+1}^j f_{ik}(\alpha) p^\alpha(t_j - t_k). \quad (2.12)$$

In matrix notation

$$P_\alpha = I + F_\alpha P_\alpha, \quad (2.13)$$

where P_α and F_α are square matrices of order $(n + 1)$, the (i, j) th element of P_α being

$$p^\alpha(t_j - t_i)$$

if $0 \leq i \leq j \leq n$ and 0 otherwise, and the (i, j) th element of F_α being $f_{ij}(\alpha)$ if $0 \leq i < j \leq n$ and 0 otherwise.

Thus

$$F_\alpha = I - P_\alpha^{-1}, \quad P_\alpha = I + P_\alpha F_\alpha, \quad (2.14)$$

and using primes to denote differentiation with respect to α ,

$$P'_\alpha = F_\alpha P'_\alpha + F'_\alpha P_\alpha,$$

so that

$$\begin{aligned} F'_\alpha &= (I - F_\alpha) P'_\alpha P_\alpha^{-1} = (I - F_\alpha) P'_\alpha (I - F_\alpha) \\ &= P'_\alpha - F_\alpha P'_\alpha - P'_\alpha F_\alpha + F_\alpha P'_\alpha F_\alpha. \end{aligned}$$

The $(0, n)$ element of this matrix identity is

$$\begin{aligned} f'_{0n}(\alpha) &= p^\alpha(t_n) \log p(t_n) - \sum_{r=1}^n f_{0r}(\alpha) p^\alpha(t_n - t_r) \log p(t_n - t_r) \\ &\quad - \sum_{s=0}^{n-1} p^\alpha(t_s) \log p(t_s) f_{sn}(\alpha) \\ &\quad + \sum_{1 \leq r \leq s < n} f_{0r}(\alpha) p^\alpha(t_s - t_r) \log p(t_s - t_r) f_{sn}(\alpha), \end{aligned}$$

where terms in $\log 0$ are ignored. Applying (2.2) and its dual from (2.14), this is easily thrown into the form

$$\begin{aligned} f'_{0n}(\alpha) &= f_{0n}(\alpha) \log p(t_n) \\ &\quad + \sum f_{0r}(\alpha) f_{sn}(\alpha) p^\alpha(t_s - t_r) \log \left\{ \frac{p(t_s - t_r) p(t_n)}{p(t_n - t_r) p(t_s)} \right\}, \end{aligned} \tag{2.15}$$

where the sum extends over all r, s satisfying

$$1 \leq r \leq s < n, \quad f_{0r}(\alpha) f_{sn}(\alpha) p(t_s - t_r) > 0, \tag{2.16}$$

these conditions ensuring by (2.11) that the logarithm is defined.

Equation (2.15) is a general form of equation (16) of (Kingman, 1996) and the proof now proceeds exactly as in that paper. We apply Jensen's inequality to the sum in (2.15), using the convexity of

$$\psi(x) = x \log x,$$

to show that the sum cannot be less than

$$\alpha^{-1} p(t_n)^{-\alpha} A \psi(B/A) = \alpha^{-1} p(t_n)^{-\alpha} B \log(B/A),$$

where

$$\begin{aligned}
A &= \sum f_{0r}(\alpha) f_{sn}(\alpha) p^\alpha(t_n - t_r) p^\alpha(t_s) \\
&\leq \sum_{r=1}^{n-1} f_{0r}(\alpha) p^\alpha(t_n - t_r) \sum_{s=1}^{n-1} p^\alpha(t_s) f_{sn}(\alpha) \\
&= \{p^\alpha(t_n) - f_{0n}(\alpha)\}^2
\end{aligned}$$

and

$$\begin{aligned}
B &= \sum f_{0r}(\alpha) f_{sn}(\alpha) p^\alpha(t_s - t_r) p^\alpha(t_n) \\
&= \{p^\alpha(t_n) - f_{0n}(\alpha)\} p^\alpha(t_n).
\end{aligned}$$

Thus (2.15) implies that

$$f'_{0n}(\alpha) \geq f_{0n}(\alpha) \log p(t_n) + \alpha^{-1} \{p^\alpha(t_n) - f_{0n}(\alpha)\} \log \left\{ \frac{p^\alpha(t_n)}{p^\alpha(t_n) - f_{0n}(\alpha)} \right\}, \quad (2.17)$$

which means that

$$g(\alpha) = f_{0n}(\alpha) p(t_n)^{-\alpha}$$

satisfies

$$\alpha g'(\alpha) \geq -\{1 - g(\alpha)\} \log \{1 - g(\alpha)\} \quad (2.18)$$

in $\alpha \geq 1$. This differential inequality, with $g(1) \geq 0$, is shown in (Kingman, 1996) to imply that $g(\alpha) \geq 0$ for all $\alpha > 1$, so that

$$f_{0n}(\alpha) = F(t_1, t_2, \dots, t_n; p^\alpha) \geq 0$$

as required, and the proof is complete.

3. Kink analysis for products

The class \mathcal{P} of standard p -functions is characterised by the Laplace transform identity

$$\int_0^\infty p(t) e^{-\theta t} dt = \left\{ \theta + \int_{(0, \infty]} (1 - e^{-\theta t}) \mu(dt) \right\}^{-1} \quad (3.1)$$

valid for $\theta > 0$. For every $p \in \mathcal{P}$, there is a unique measure μ on $(0, \infty]$, the *canonical measure* of p , satisfying

$$\int_{(0, \infty]} \min(t, 1) \mu(dt) < \infty \quad (3.2)$$

and (3.1). Conversely, for any μ satisfying (3.2), there is exactly one continuous function p satisfying (3.1), and it belongs to \mathcal{P} . This bijection is continuous, if \mathcal{P} is given the topology of pointwise convergence, and the set of measures satisfying (3.2) is given the vague topology that is the weakest making

$$\mu \longmapsto \int_{(0, \infty]} f(t) \min(t, 1) \mu(dt)$$

continuous for any bounded continuous f .

If standard p -functions p_1 and p_2 have canonical measures μ_1 and μ_2 , the canonical measure μ of $p = p_1 p_2$ is determined by μ_1 and μ_2 . Indeed,

$$(\mu_1, \mu_2) \longmapsto \mu \tag{3.3}$$

is a commutative, associative binary operation making the space of measures satisfying (3.1) a semigroup isomorphic to \mathcal{P} . It is however very difficult to describe (3.3) explicitly.

Kink analysis addresses these problems using Theorem 3.4 of (Kingman, 1972a), which asserts that any $p \in \mathcal{P}$ has right and left derivatives $p'(t+)$ and $p'(t-)$ at all t in $(0, \infty)$ and that

$$p'(t+) - p'(t-) = \mu\{t\}. \tag{3.4}$$

Thus an atom of μ at t corresponds to a ‘kink’ in the graph of p . Since

$$p'(t+) = p_1(t)p_2'(t+) + p_1'(t+)p_2(t)$$

and

$$p'(t-) = p_1(t)p_2'(t-) + p_1'(t-)p_2(t)$$

(3.4) implies that

$$\mu\{t\} = p_1(t)\mu_2\{t\} + p_2(t)\mu_1\{t\}. \tag{3.5}$$

Thus the atoms of μ in $(0, \infty)$ form the union of the atoms of μ_1 and of μ_2 . (This is not necessarily true at ∞ , where it is possible that $\mu_1\{\infty\} = \mu_2\{\infty\} = 0$ but that $\mu\{\infty\} > 0$.)

Now suppose that μ_1 and μ_2 are purely atomic. Then (3.5) shows that the atomic part μ_a of μ on $(0, \infty)$ is given by

$$\mu_a(dt) = p_1(t)\mu_2(dt) + p_2(t)\mu_1(dt), \tag{3.6}$$

so that

$$\mu(dt) \geq p_1(t)\mu_2(dt) + p_2(t)\mu_1(dt). \quad (3.7)$$

This inequality now extends by continuity to all μ_1 and μ_2 , since the purely atomic measures form a dense subset and all the relevant operations are continuous. Thus we have proved the following.

Theorem 3 **The canonical measure μ of the product $p = p_1p_2$ of standard p -functions with canonical measures μ_1 and μ_2 satisfies (3.7) on $(0, \infty)$. The atom of μ at any $t \in (0, \infty)$ is given by (3.5).**

The equation (3.5) and the inequality (3.7) have natural probabilistic interpretations, which can be developed into a probabilistic description of the semigroup operation (3.3). This is both complex and subtle, and space does not allow it to be described here.

4. Kink analysis for powers

The same argument, but without a probabilistic meaning, applies to the canonical measure of the power of a standard p -function.

Theorem 4 **Let $p \in \mathcal{P}$ have canonical measure μ , and let $\alpha > 1$. Then p^α belongs to \mathcal{P} , and has canonical measure μ_α (say). The atoms of μ_α in $(0, \infty)$ are at the same points as those of μ , and**

$$\mu_\alpha\{t\} = \alpha p(t)^{\alpha-1} \mu\{t\} \quad (4.1)$$

for each $t \in (0, \infty)$. If the measure ν_α is defined on $(0, \infty)$ by

$$\mu_\alpha(dt) = \alpha p(t)^{\alpha-1} \nu_\alpha(dt), \quad (4.2)$$

then ν_α increases with α in the sense that, for $1 < \alpha < \beta$,

$$\mu \leq \nu_\alpha \leq \nu_\beta. \quad (4.3)$$

Proof Since p^α has derivative $\alpha p(t)^{\alpha-1} p'$ to right and left, (3.4) shows that

$$\begin{aligned}\mu_\alpha\{t\} &= \alpha p(t)^{\alpha-1} p'(t+) - \alpha p(t)^{\alpha-1} p'(t-) \\ &= \alpha p(t)^{\alpha-1} \mu\{t\},\end{aligned}$$

proving (4.1). In particular, if μ is purely atomic, the atomic part of μ_α is $\alpha p(t)^{\alpha-1} \mu(dt)$, so that

$$\mu_\alpha(dt) \geq \alpha p(t)^{\alpha-1} \mu(dt) \tag{4.4}$$

on $(0, \infty)$. As before, this inequality extends to general μ by continuity. In the notation of (4.2),

$$\nu_\alpha(dt) \geq \mu(dt),$$

so that the first inequality in (4.4) is proved.

To prove the second inequality, apply (4.4) to p^γ ($\gamma > 1$). This gives

$$\mu_{\alpha\gamma}(dt) \geq \alpha p(t)^{(\alpha-1)\gamma} \mu_\gamma(dt)$$

or

$$\alpha\gamma p(t)^{\alpha\gamma-1} \nu_{\alpha\gamma}(dt) \geq \alpha p(t)^{\alpha\gamma-\gamma} \gamma p(t)^{\gamma-1} \nu_\gamma(dt).$$

Thus we have proved that

$$\nu_{\alpha\gamma} \geq \nu_\gamma$$

for $\alpha, \gamma > 1$, which becomes the second inequality of (4.4) if we replace γ by α and α by β/α .

How does this apply to the Williams problem started at the end of Section 1? The class \mathcal{PM} of standard p -functions arising from Markov chains with countable state space is closed under products. Thus $p^\alpha \in \mathcal{PM}$ if $p \in \mathcal{PM}$ and $\alpha > 1$ is an integer. Is this true if α is not an integer?

Suppose therefore that $p \in \mathcal{PM}$. The characterisation theorem for \mathcal{PM} (Kingman, 1972a) shows that the canonical measure μ of p is absolutely continuous on $(0, \infty)$, and admits a lower semicontinuous density f which is either identically zero or satisfies

$$f(t) > 0 \ (t > 0), \quad f(t) \geq e^{-\gamma t} \ (t \geq 1) \tag{4.5}$$

for some γ . The case $f \equiv 0$ is easily dealt with, so we assume (4.5).

Theorem 4 shows that the canonical measure μ_α of p^α ($\alpha > 1$) is free of atoms in $(0, \infty)$. Indeed, more is true, as can be seen by taking β to an integer in (4.3). Then we know that p^β belongs to \mathcal{PM} , so that μ_β is absolutely continuous in $(0, \infty)$. So therefore is ν_β , and the inequality (4.3) implies that ν_α is absolutely continuous. This implies that μ_α has a density f_α on $(0, \infty)$ and, if g_α is defined by

$$f_\alpha(t) = \alpha p(t)^{\alpha-1} g_\alpha(t), \quad (4.6)$$

g_α is a density for μ_α . By (4.3), if $1 < \alpha < \beta$,

$$f(t) \leq g_\alpha(t) \leq g_\beta(t) \quad (4.7)$$

for almost all t , and a standard regularisation technique enables the densities to be chosen so that (4.7) holds for all $t \in (0, \infty)$ so that g_α is non-decreasing in α .

From (4.5), (4.6) and (4.7), $f_\alpha(t) > 0$ for all $t > 0$. For $t \geq 1$,

$$f_\alpha(t) \geq \alpha p(t)^{\alpha-1} e^{-\gamma t} e^{-\gamma(\alpha)t}$$

for some $\gamma(\alpha)$, since standard p -functions decay at most exponentially fast (Kingman, 1972a, p. 53). Thus we have almost established the necessary conditions for p^α to belong to \mathcal{PM} . The only gap is the need to prove that f_α can be taken to be lower semicontinuous when α is not an integer, and the techniques of this paper are not powerful enough to prove this. Williams's question remains open, but might perhaps be promoted to the status of a conjecture.

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