Cosimplicial objects and little n-cubes. I.

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1 Introduction.

The little n-cubes operad $\mathcal{C}_n$ was introduced by Boardman and Vogt in [6] (except that they used the terminology of theories rather than that of operads) as a tool for understanding n-fold loop spaces. They showed that for any topological space Z the n-fold loop space $\Omega^n Z$ has an action of $\mathcal{C}_n$. In the other direction, May showed in [17] that if Y is a space with an action of $\mathcal{C}_n$ then there exists a space Z such that the group completion of Y is weakly equivalent to $\Omega^n Z$.

In the 30 years since [17] the operad $\mathcal{C}_n$ has played an important role in both unstable and stable homotopy theory. More recently, it has also been of importance (especially when $n = 2$) in quantum algebra and other areas related to mathematical physics (see, for example, [9], [13], [14], [22], [23]).

In the known applications, $\mathcal{C}_n$ can be replaced by any operad weakly equivalent to it; such operads are called E_n operads.

There is a highly developed technology that provides sufficient conditions for a space to have an action by an E_∞ operad (see [1], for example) or an E_1 operad ([17], [24]). Much less is known about actions of E_n operads for $1 < n < \infty$.

In this paper we consider the important special situation where the space (or spectrum) Y on which we want an E_n operad to act is obtained by totalization from a cosimplicial space (resp., spectrum) $X^\bullet$. We construct an E_n operad $\mathcal{D}_n$ and we show (Theorem 9.1) that if $X^\bullet$ has a certain kind of combinatorial structure (we call it a Ξ^n-structure) then $\mathcal{D}_n$ acts on $\mathrm{Tot}(X^\bullet)$.

The converse of Theorem 9.1 is not true: a $\mathcal{D}_n$-action on $\mathrm{Tot}(X^\bullet)$ does not have to come from a Ξ^n-structure on $X^\bullet$. However, in a future paper we will show that Tot induces a Quillen equivalence between the category of cosimplicial spaces with Ξ^n-structure and the category of spaces with $\mathcal{D}_n$-action; from this it will follow that if $\mathcal{D}_n$ acts on a space Y then

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there is a cosimplicial space $X^\bullet$ with a Ξ^n structure such that $\mathrm{Tot}(X^\bullet)$ is weakly equivalent to Y as a $\mathcal{D}_n$-space.

The $n = 2$ case of Theorem 9.1 was proved in [18] (by a more complicated method than in the present paper), and it has had useful applications, notably

- a proof ([18]) that the topological Hochschild cohomology spectrum of an A_∞ ring spectrum R has an action of $\mathcal{D}_2$ (this is the topological analog of Deligne's Hochschild cohomology conjecture [8]), and

- a proof by Dev Sinha that the space of knots in $\mathbb{R}\times I$ (that is, the space of embeddings of I in $\mathbb{R}\times I$ which take the two endpoints of I to the two different boundary components of $\mathbb{R} \times I$, with fixed tangent vectors at the endpoints) is a 2-fold loop space (see [20, page 25]).

The methods we use in this paper are quite general and apply to other categories of cosimplicial objects besides the categories of cosimplicial spaces and spectra. In the sequel to this paper [19] we will apply our methods to the category of cosimplicial chain complexes.

Remark 1.1. Throughout this paper we will use the following conventions for cosimplicial spaces.

(a) We define Δ to be the category of nonempty finite totally ordered sets (this is equivalent to the category usually called Δ). We write $[m]$ for the finite totally ordered set $\{0, \ldots, m\}$.

(b) A cosimplicial space $X^\bullet$ is a functor from Δ to spaces. If S is a nonempty finite totally ordered set we write X^S for the value of $X^\bullet$ at S, except that we write X^m instead of $X^{[m]}$.

Here is an outline of the paper.

As an introduction to the ideas we begin in Sections 2 and 3 with the $n = 1$ (that is, the A_∞) case. In Section 2 we recall the monoidal structure $\square$ on the category of cosimplicial spaces due to Batanin [2]. In Section 3 we give a very simple proof of the fact (first shown in [3] and [18]) that if $X^\bullet$ is a monoid with respect to $\square$ then $\mathrm{Tot}(X^\bullet)$ is an A_∞ space; the proof is based on an idea due to Beilinson ([11, Section 2]). We also give (in Remark 3.3) an explicit description of the combinatorial structure on $X^\bullet$ that constitutes a $\square$-monoid structure.

Our treatment of the E_∞ case is precisely parallel, with a symmetric monoidal structure $\boxtimes$ in place of the monoidal structure $\square$ (but for technical reasons we need to use augmented cosimplicial spaces instead of cosimplicial spaces; see Remark 6.10). As a prelude Section 4 introduces the categorical idea of "strict functor-operad" which is a technically convenient reformulation of the concept of symmetric monoidal structure. This section also introduces the general concept of functor-operad, which is needed in later sections as a way of interpolating between monoidal and symmetric monoidal structures. (The definition of functor-operad was discovered independently, in a different context, by Batanin [4]). In Section 5 we digress to offer motivation for the definition of $\boxtimes$; the definition itself, and the verification that $\boxtimes$ is indeed a symmetric monoidal structure, is given in Section 6. The main result in Section 7 (Theorem 7.1) is that if $X^\bullet$ is a commutative monoid with respect to $\boxtimes$ then $\mathrm{Tot}(X^\bullet)$ is an

E_∞ space. Section 7 also gives an explicit description of the combinatorial structure on $X^\bullet$ that constitutes a commutative $\boxtimes$-monoid structure.

In Section 8 we define for each n a functor-operad Ξ^n; the special case $n = \infty$ is the symmetric monoidal structure $\boxtimes$, and the special case $n = 1$ is essentially the same (see Remark 8.7) as the monoidal structure $\square$. (In the special case $n = 2$, a construction isomorphic to Ξ^2 was discovered independently by Tamarkin in unpublished work.) In Section 9 we use the functor-operad Ξ^n to construct an ordinary (topological) operad $\mathcal{D}_n$. The main theorem in Section 9 (Theorem 9.1) says that $\mathcal{D}_n$ is weakly equivalent to $\mathcal{C}_n$ and that if $X^\bullet$ is a Ξ^n-algebra then $\mathcal{D}_n$ acts on $\mathrm{Tot}(X^\bullet)$. Section 9 also gives an explicit description of the combinatorial structure on $X^\bullet$ that constitutes an action of Ξ^n on $X^\bullet$.

The next two sections contain material which is used in the proofs of Theorems 7.1 and 9.1 and may also be of independent interest. Let Y_k^0 denote the 0-th space of the cosimplicial space $(\Delta^\bullet)^{\boxtimes k}$. In Section 10 we prove that $(\Delta^\bullet)^{\boxtimes k}$ is isomorphic as a cosimplicial space to the Cartesian product $\Delta^\bullet \times Y_k^0$. We also show that Y_k^0 has a canonical cell structure and that it is contractible (which completes the proof of Theorem 7.1). In Section 11 we show that $\Xi_k^n(\Delta^\bullet, \ldots, \Delta^\bullet)$ is isomorphic as a cosimplicial space to the Cartesian product of its 0-th space with $\Delta^\bullet$; this is used in the proof of Theorem 9.1 (the $n = 2$ case is the "fiberwise prismatic subdivision" used in [18]).

In Section 12 we use a technique of Clemens Berger [5] to show that the operad $\mathcal{D}_n$ defined in Section 9 is weakly equivalent to the little n-cubes operad $\mathcal{C}_n$; this completes the proof of Theorem 9.1. The basic idea is to generalize the fact that two spaces are weakly equivalent if they have contractible open covers with the same nerve.

2 A monoidal structure on the category of cosimplicial spaces.

We begin with some motivation. We are concerned with the question of when Tot of a cosimplicial space has an A_∞ structure. This question is formally analogous to the question of when the normalization of a cosimplicial abelian group is a DGA (we will explore this analogy further in [19]). There is a simple, and basic, example of a cosimplicial abelian group whose normalization is a DGA: if W is a space we define $S^\bullet W = \mathrm{Map}_{\mathbb{Z}}(S_\bullet W, \mathbb{Z})$, where $S_\bullet W$ is the singular complex of W and Map is set maps. The normalization of $S^\bullet W$ has a DGA-structure induced by the cup-product on $S^\bullet W$, defined as usual for $x \in S^p W$ and $y \in S^q W$ by

$$(x \smile y)(\sigma) = x(\sigma(0, \ldots, p)) \cdot y(\sigma(p, \ldots, p+q))$$

Here $\sigma \in S_{p+q}W$, $\cdot$ is multiplication in $\mathbb{Z}$, and $\sigma(0, \ldots, p)$ (resp., $\sigma(p, \ldots, p+q)$) is the restriction of σ to the subsimplex of Δ^{p+q} spanned by the vertices $0, \ldots, p$ (resp., $p, \ldots, p+q$). The cup-product on $S^\bullet W$ is related to the coface and codegeneracy operations by the following formulas:

$$d^i(x \smile y) = \begin{cases} d^i x \smile y & \text{if } i \le p \\ x \smile d^{i-p} y & \text{if } i > p \end{cases} \tag{2.1}$$

$$d^{p+1}x \smile y = x \smile d^0 y \tag{2.2}$$

$$(2.3) \qquad s^i(x \smile y) = \begin{cases} s^i x \smile y & \text{if } i \le p-1 \\ x \smile s^{i-p} y & \text{if } i \ge p \end{cases}$$

Now let us return to the category of cosimplicial spaces. Formulas (2.1), (2.2) and (2.3) motivate the following definition.

Definition 2.1. Let $X^\bullet$ and $Y^\bullet$ be cosimplicial spaces. $X^\bullet \Box Y^\bullet$ is the cosimplicial space whose m-th space is

$$\left(\coprod_{p+q=m} X^p \times Y^q \right) / \sim$$

(where $\sim$ is the equivalence relation generated by $(x, d^0 y) \sim (d^{|x|+1} x, y)$). The cosimplicial operators are given by

$$d^i(x,y) = \begin{cases} (d^i x, y) \text{ if } i \le |x| \\ (x, d^{i-|x|} y) \text{ if } i > |x| \end{cases}$$

and

$$s^i(x,y) = \begin{cases} (s^i x, y) \text{ if } i \le |x|-1 \\ (x, s^{i-|x|} y) \text{ if } i \ge |x| \end{cases}$$

We leave it to the reader to check that the cosimplicial identities are satisfied and that the following holds.

Proposition 2.2. $\Box$ *is a monoidal structure for the category of cosimplicial spaces, with unit the constant cosimplicial space that has a point in every degree.*

There is also a monoidal structure $\Box$ for cosimplicial spectra: one simply replaces the Cartesian products in Definition 2.1 by smash products.

We conclude this section with some observations about Kan extensions. This material will not be needed logically for the rest of the paper, but it provides useful motivation for the constructions in Sections 6 and 8.

Recall the conventions in Remark 1.1.

If $S_1, S_2, \dots, S_k$ are finite totally ordered sets there is a unique total order on $S_1 \coprod \cdots \coprod S_k$ for which the inclusion maps into the coproduct are order-preserving and every element of S_i is less than every element of S_j for $i < j$. Let

$$\Phi : \Delta^{\times k} \to \Delta$$

be the functor which takes $(S_1, \dots, S_k)$ to $S_1 \coprod \cdots \coprod S_k$ with this total order.

The following fact was first noticed by Cordier and Porter (unpublished).

Proposition 2.3. *Let $X_1^\bullet, \dots, X_k^\bullet$ be cosimplicial spaces and let $X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet$ denote the composite*

$$\Delta^{\times k} \xrightarrow{X_1^\bullet \times \dots \times X_k^\bullet} \mathrm{Top} \times \cdots \times \mathrm{Top} \xrightarrow{\times} \mathrm{Top}.$$

Then $X_1^\bullet \Box \cdots \Box X_k^\bullet$ is naturally isomorphic to the left Kan extension $\mathrm{Lan}_\Phi(X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet)$.

Before giving the proof we mention an important consequence.

Remark 2.4. Let Φ^* be the functor from cosimplicial spaces to bicosimplicial spaces defined by

$$(\Phi^*(X^\bullet))^{S,T} = X^{S\coprod T}$$

It is a general fact about Kan extensions [16, beginning of Section X.3] that Lan_Φ is the left adjoint of Φ^*. This implies that there is a natural 1-1 correspondence between maps

$$\alpha : X^\bullet \Box Y^\bullet \to Z^\bullet$$

and consistent collections of maps

$$\tilde{\alpha}_{S,T} : X^S \times Y^T \to Z^{S\coprod T}$$

where "consistent" means that every pair of ordered maps $f : S \to S'$, $g : T \to T'$ induces a commutative diagram

$$\begin{array}{ccc} X^S \times Y^T & \xrightarrow{\tilde{\alpha}_{S,T}} & Z^{S\coprod T} \\ {\scriptstyle f_*\times g_*}\downarrow & & \downarrow{\scriptstyle (f\coprod g)_*} \\ X^{S'} \times Y^{T'} & \xrightarrow{\tilde{\alpha}_{S',T'}} & Z^{S'\coprod T'} \end{array}$$

Proof of Proposition 2.3. For simplicity we assume $k = 2$.

For $m \geq 0$ let $[m]$ denote the set $\{0, \ldots, m\}$. Every object in Δ is canonically isomorphic to one of the form $[m]$ so it suffices to show that the two functors in question are naturally isomorphic on the full subcategory of Δ with these objects.

Fix m. According to [16, Equation (10) on page 240], the left Kan extension, evaluated at $[m]$, can be calculated as follows. Let $\mathcal{C}$ be the category of objects Φ-over $[m]$: an object in $\mathcal{C}$ is a pair consisting of an object $([n], [n'])$ in $\Delta \times \Delta$ and a morphism $\Phi([n], [n']) \to [m]$ in Δ; a morphism in $\mathcal{C}$ is a morphism (f, g) in $\Delta \times \Delta$ making the evident triangle commute. Then $\mathrm{Lan}_\Phi(X_1^\bullet \bar{\times} X_2^\bullet)$ is the colimit of the composite

$$\mathcal{C} \to \Delta \times \Delta \xrightarrow{X_1^\bullet \bar{\times} X_2^\bullet} \mathrm{Top}$$

where the first map is the evident forgetful functor. Now let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ consisting of pairs $(([p], [q]), f : [p + q + 1] \to [m])$ where f is a surjection, $p + q$ is either $m - 1$ or m, and if $p + q = m$ then $f(p) = f(p + 1)$. Note that an object in $\mathcal{C}'$ is determined by p and q; to simplify the notation we denote the object by (p, q). If $p + q = m - 1$ there is exactly one morphism from the object (p, q) to $(p + 1, q)$ and exactly one from (p, q) to $(p, q + 1)$, and there are no other non-identity morphisms in $\mathcal{C}'$. The map from (p, q) to $(p + 1, q)$ is the last coface map on $[p]$ and the identity on $[q]$, while the map from (p, q) to $(p, q + 1)$ is the identity on $[p]$ and the zeroth coface map on $[q]$. From this it is clear that $X_1^\bullet \Box X_2^\bullet$ is the colimit of the composite

$$\mathcal{C}' \subset \mathcal{C} \to \Delta \times \Delta \xrightarrow{X_1^\bullet \bar{\times} X_2^\bullet} \mathrm{Top}$$

In particular there is a natural map

$$X_1^\bullet \Box X_2^\bullet \to \mathrm{Lan}_\Phi(X_1^\bullet \bar{\times} X_2^\bullet).$$

By [16, Section 9.3], this will be an isomorphism if $\mathcal{C}'$ is cofinal in $\mathcal{C}$, that is, if for each $c \in \mathcal{C}$ the under-category $c \downarrow \mathcal{C}'$ is connected. The under-categories can be described explicitly: each is a nonempty full subcategory of $\mathcal{C}'$ with set of objects of the form

$$\{(p, q) \mid p \geq p_0, q \geq q_0\}$$

for some p_0 and q_0; clearly all such categories are connected.

□

3 A sufficient condition for $\mathrm{Tot}(X^\bullet)$ to be an A_∞ space.

In this section we will prove

Theorem 3.1. *If $X^\bullet$ is a monoid with respect to $\square$ then $\mathrm{Tot}(X^\bullet)$ has an A_∞ structure.*

Remark 3.2. (a) Previous proofs of Theorem 3.1 were given by Batanin [3, Theorems 5.1 and 5.2] and by us [18, Theorem 2.4].

(b) The theorem and its proof are also valid for cosimplicial spectra.

Remark 3.3. Definition 2.1 implies that $X^\bullet$ is a monoid with respect to $\square$ if and only if there are maps

$$\smile : X^p \times X^q \to X^{p+q}$$

for all $p, q \geq 0$ satisfying equations (2.1), (2.2), (2.3), the associativity condition

$$(x \smile y) \smile z = z \smile (y \smile z) \tag{3.1}$$

and the unit condition: there is an element $e \in X^0$ such that

$$x \smile e = e \smile x = x \tag{3.2}$$

for all x.

The rest of this section is devoted to the proof of Theorem 3.1.

Let $\Delta^\bullet$ denote the cosimplicial space whose m-th space is the simplex Δ^m, with the usual cofaces and codegeneracies. By definition, $\mathrm{Tot}(X^\bullet)$ is $\mathrm{Hom}(\Delta^\bullet, X^\bullet)$ (where Hom denotes the space of cosimplicial maps).

Definition 3.4. (a) For each $k \geq 0$ let $\mathcal{A}(k)$ be the space $\mathrm{Tot}((\Delta^\bullet)^{\square k})$.

(b) If $f \in \mathcal{A}(k)$ and $g_i \in \mathcal{A}(j_i)$ for $1 \leq i \leq k$ define $\gamma(f, g_1, \ldots, g_k) \in \mathcal{A}(j_1 + \cdots + j_k)$ to be the composite

$$\Delta^\bullet \xrightarrow{f} (\Delta^\bullet)^{\square k} \xrightarrow{g_1 \square \cdots \square g_k} (\Delta^\bullet)^{\square (j_1 + \cdots + j_k)}$$

Theorem 3.1 is an immediate consequence of our next result.

Proposition 3.5. *(a) Let $\mathcal{A}$ be the sequence of spaces $\mathcal{A}(k)$, $k \geq 0$, with the operations*

$$\gamma : \mathcal{A}(k) \times \mathcal{A}(j_1) \times \cdots \times \mathcal{A}(j_k) \to \mathcal{A}(j_1 + \cdots + j_k)$$

defined above. Then $\mathcal{A}$ is an operad.

(b) If $X^\bullet$ is a monoid with respect to $\square$ then $\mathcal{A}$ acts on $\mathrm{Tot}(X^\bullet)$.

(c) $\mathcal{A}$ is an A_∞ operad.

Proof. Part (a) is clear.

For part (b), given $f \in \mathcal{A}(k)$ and $x_1, \ldots, x_k \in \mathrm{Tot}(X^\bullet)$ define $f(x_1, \ldots, x_k) \in \mathrm{Tot}(X^\bullet)$ to be the composite

$$\Delta^\bullet \xrightarrow{f} (\Delta^\bullet)^{\square k} \xrightarrow{x_1 \square \cdots \square x_k} (X^\bullet)^{\square k} \xrightarrow{\mu} X^\bullet$$

where μ is the monoidal structure map of $X^\bullet$. This construction gives maps

$$\mathcal{A}(k) \times (\mathrm{Tot}(X^\bullet))^k \to \mathrm{Tot}(X^\bullet)$$

which fit together to give an action of $\mathcal{A}$ on $\mathrm{Tot}(X^\bullet)$.

For part (c) we need to show that $\mathcal{A}(0)$ is a point (which is obvious) and that each space $\mathcal{A}(k)$ is contractible. First consider the case $k = 1$. If f and g are two cosimplicial maps from $\Delta^\bullet$ to $\Delta^\bullet$, then $tf + (1-t)g$ will again be a cosimplicial map for each $0 \leq t \leq 1$ (because the cosimplicial structure maps of $\Delta^\bullet$ are affine) and so we can use the straight-line homotopy to contract $\mathrm{Hom}(\Delta^\bullet, \Delta^\bullet)$ to a point. The case $k \geq 2$ is now immediate from Lemma 3.6 below. $\square$

Remark on the proof of part (c). It might seem more natural to prove part (c) by observing that $(\Delta^\bullet)^{\square k}$ is weakly equivalent to a point in the model category structure defined by Bousfield and Kan [7, Section X.5]. The difficulty with this is that (because not all cosimplicial spaces are fibrant) $\mathrm{Hom}(\Delta^\bullet, -)$ doesn't preserve weak equivalences.

Lemma 3.6. *For each $k \geq 1$, $(\Delta^\bullet)^{\square k}$ is isomorphic as a cosimplicial space to $\Delta^\bullet$.*

Proof. It suffices to do the case $k = 2$; the general case follows by induction. First we define maps

$$f^m : (\Delta^\bullet \square \Delta^\bullet)^m \to \Delta^m$$

for $m \geq 0$ by

$$f^m((s_0, \ldots, s_p), (t_0, \ldots, t_q)) = (\tfrac{1}{2}s_0, \ldots, \tfrac{1}{2}(s_p + t_0), \ldots, \tfrac{1}{2}t_q)$$

These are well defined and fit together to give a cosimplicial map $f : \Delta^\bullet \square \Delta^\bullet \to \Delta^\bullet$. Next define

$$g^m : \Delta^m \to (\Delta^\bullet \square \Delta^\bullet)^m$$

as follows: given $(u_0, \ldots, u_m) \in \Delta^m$, choose the smallest p for which

$$u_0 + \cdots + u_p \geq \frac{1}{2}$$

and let

$$g^m(u_0, \ldots, u_m) = [(2u_0, \ldots, 1 - 2u_0 - \cdots - 2u_{p-1}), (2u_0 + \cdots + 2u_p - 1, 2u_{p+1}, \ldots, 2u_m)]$$

The g^m are continuous, they fit together to give a cosimplicial map $g : \Delta^\bullet \to \Delta^\bullet \Box \Delta^\bullet$, and f and g are mutually inverse. $\square$

Remark 3.7. Lemma 3.6 is due to Grayson [10, Section 4].

4 Functor-operads.

The purpose of this section is to describe a general setting in which there are analogs of Definition 3.4 and Proposition 3.5(a) and (b) (see Definition 4.3 and Propositions 4.4 and 4.6).

Given a category $\mathcal{C}$ let $\mathcal{C}^{\times k}$ denote the k-fold Cartesian product. For each permutation $\sigma \in \Sigma_k$ we define

$$\sigma_\# : \mathcal{C}^{\times k} \to \mathcal{C}^{\times k}$$

to be the functor taking $(A_1, \ldots, A_k)$ to $(A_{\sigma(1)}, \ldots, A_{\sigma(k)})$.

In order to motivate the definition of functor-operad, let us consider the situation in which $\mathcal{C}$ has a symmetric monoidal structure $\boxtimes$. For each $k \geq 0$ define a functor

$$\mathcal{F}_k : \mathcal{C}^{\times k} \to \mathcal{C}$$

by

$$\mathcal{F}_k(X_1, \ldots, X_k) = X_1 \boxtimes (X_2 \boxtimes (X_3 \boxtimes \cdots))$$

The definition of symmetric monoidal structure implies that there are canonical natural isomorphisms

$$\sigma_* : \mathcal{F}_k \to \mathcal{F}_k \circ \sigma_\#$$

and

$$\Gamma_{j_1,\ldots,j_k} : \mathcal{F}_k(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_k}) \to \mathcal{F}_{j_1 + \cdots j_k}$$

satisfying certain consistency conditions. The following definition is an abstract version of this situation, except that instead of requiring the Γ's to be natural isomorphisms we allow them merely to be natural transformations.

Definition 4.1. Let $\mathcal{C}$ be a category enriched over Top. A *functor-operad* $\mathcal{F}$ in $\mathcal{C}$ is a sequence of continuous functors $\mathcal{F}_k : \mathcal{C}^{\times k} \to \mathcal{C}$ together with

(i) for each $\sigma \in \Sigma_k$, a continuous natural isomorphism

$$\sigma_* : \mathcal{F}_k \to \mathcal{F}_k \circ \sigma_\#$$

(ii) for each choice of $j_1, \ldots, j_k \geq 0$, a continuous natural transformation

$$\Gamma_{j_1,\ldots,j_k} : \mathcal{F}_k(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_k}) \to \mathcal{F}_{j_1 + \cdots j_k}$$

such that

(a) $\mathcal{F}_1$ is the identity functor, and the natural transformations

$$\Gamma_{1,\ldots,1} : \mathcal{F}_k(\mathcal{F}_1, \ldots, \mathcal{F}_1) \to \mathcal{F}_k$$

$$\Gamma_k : \mathcal{F}_1(\mathcal{F}_k) \to \mathcal{F}_k$$

are equal to the identity.

(b) All diagrams of the following form commute:

$$\begin{array}{ccc} \mathcal{F}_k(\mathcal{F}_{j_1}(\mathcal{F}_{i_{11}}, \ldots, \mathcal{F}_{i_{1j_1}}), \ldots, \mathcal{F}_{j_k}(\mathcal{F}_{i_{k1}}, \ldots, \mathcal{F}_{i_{kj_k}})) & \xrightarrow{\Gamma} & \mathcal{F}_{j_1+\ldots+j_k}(\mathcal{F}_{i_{11}}, \ldots, \mathcal{F}_{i_{kj_k}}) \\ \downarrow{\scriptstyle \mathcal{F}_k(\Gamma,\ldots,\Gamma)} & & \downarrow{\scriptstyle \Gamma} \\ \mathcal{F}_k(\mathcal{F}_{i_{11}+\cdots+i_{1j_1}}, \ldots, \mathcal{F}_{i_{k1}+\cdots+i_{kj_k}}) & \xrightarrow{\Gamma} & \mathcal{F}_{i_{11}+\cdots+i_{kj_k}} \end{array}$$

(c) $(\sigma\tau)_* = \tau_*\sigma_*$ for all $\sigma, \tau \in \Sigma_k$

(d) Let $\tau_i \in \Sigma_{j_i}$ for $1 \leq i \leq k$ and let τ be the image of $(\tau_1, \ldots, \tau_k)$ under the map

$$\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_{j_1+\cdots+j_k}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_k(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_k}) & \xrightarrow{\Gamma} & \mathcal{F}_{j_1+\cdots+j_k} \\ \downarrow{\scriptstyle \mathcal{F}_k(\tau_{1*},\ldots,\tau_{k*})} & & \downarrow{\scriptstyle \Gamma} \\ \mathcal{F}_k(\mathcal{F}_{j_1} \circ \tau_{1\#}, \ldots, \mathcal{F}_{j_k} \circ \tau_{k\#}) & \xrightarrow{\tau_*} & \mathcal{F}_{j_1+\cdots+j_k} \circ \tau_\# \end{array}$$

(e) Let $\sigma \in \Sigma_k$ and let $\bar{\sigma}$ be the permutation in $\Sigma_{j_1+\ldots+j_k}$ which permutes the blocks $\{1, \ldots, j_1\}, \ldots, \{j_1 + \ldots + j_{k-1} + 1, \ldots, j_1 + \ldots + j_k\}$ in the same way that σ permutes the numbers $1, \ldots, k$. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}_n(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_k}) & \xrightarrow{\Gamma} & \mathcal{F}_{j_1+\cdots+j_k} \\ \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \bar{\sigma}_*} \\ \mathcal{F}_n(\mathcal{F}_{j_{\sigma(1)}}, \ldots, \mathcal{F}_{j_{\sigma(k)}}) \circ \bar{\sigma}_\# & \xrightarrow{\Gamma} & \mathcal{F}_{j_1+\cdots+j_k} \circ \bar{\sigma}_\# \end{array}$$

Remark 4.2. (a) Batanin [4] has independently proposed a similar but more general definition: if $\mathcal{O}$ is an operad in the category of categories Batanin defines an *internal operad* in $\mathcal{O}$ to be a collection consisting of an object a_k in $\mathcal{O}(k)$ for each $k \geq 0$ and morphisms

$$\sigma_* : a_k \to \tilde{\sigma}(a_k)$$

for each $\sigma \in \Sigma_k$ (where $\tilde{\sigma}$ denotes the action of $\sigma \in \Sigma_k$ on $\mathcal{O}(k)$) and

$$\Gamma_{j_1,\ldots,j_k} : \gamma(a_k, a_{j_1}, \ldots, a_{j_k}) \to a_{j_1+\cdots+a_{j_k}}$$

for each $j_1, \ldots, j_k \geq 0$ (where γ is the structure map of the operad $\mathcal{O}$) satisfying the analogs of properties (a)–(e) in Definition 4.1. A functor-operad in $\mathcal{C}$ is then an internal operad in the endomorphism operad of $\mathcal{C}$.

(b) If $\mathcal{B}$ is an operad in the category Top we can define a functor-operad $\mathcal{F}$ in Top by

$$\mathcal{F}_k(X_1, \ldots, X_k) = \mathcal{B}_k \times X_1 \times \cdots \times X_k$$

with the obvious structure maps.

Definition 4.3. Let $\mathcal{F}$ be a functor-operad in $\mathcal{C}$ and let A be an object of $\mathcal{C}$.

(a) Define $\mathcal{F}_A$ to be the collection of spaces

$$\mathcal{F}_A(k) = \mathrm{Hom}(A, \mathcal{F}_k(A, \ldots, A)), \quad k \geq 0.$$

(b) Give $\mathcal{F}_A(k)$ the action induced by the σ_*.

(c) Define $1 \in \mathcal{F}_A(1)$ to be the identity map of A.

(d) For each choice of $j_1, \ldots, j_k \geq 0$ define

$$\gamma : \mathcal{F}_A(k) \times \mathcal{F}_A(j_1) \times \cdots \mathcal{F}_A(j_k) \to \mathcal{F}_A(j_1 + \cdots + j_k)$$

to be the composite

$$\mathrm{Hom}(A, \mathcal{F}_k(A, \ldots, A)) \times \mathrm{Hom}(A, \mathcal{F}_{j_1}(A, \ldots, A)) \times \cdots \mathrm{Hom}(A, \mathcal{F}_{j_k}(A, \ldots, A)) \to$$
$$\mathrm{Hom}(A, \mathcal{F}_k(\mathcal{F}_{j_1}(A, \ldots, A), \ldots, \mathcal{F}_{j_k}(A, \ldots, A))) \xrightarrow{\mathrm{Hom}(A,\Gamma)} \mathrm{Hom}(A, \mathcal{F}_{j_1+\cdots+j_k}(A, \ldots, A))$$

Proposition 4.4. *These choices make $\mathcal{F}_A$ an operad.*

The proof is an easy verification.

Definition 4.5. Let $\mathcal{F}$ be a functor-operad in $\mathcal{C}$. An *algebra over* $\mathcal{F}$ is an object X of $\mathcal{C}$ together with continuous maps $\Theta_k : \mathcal{F}_k(X, \ldots, X) \to X$ for $k \geq 0$ such that

(a) Θ_1 is the identity map.

(b) The following diagram commutes for each choice of $j_1, \ldots, j_k \geq 0$

$$\begin{array}{ccc} \mathcal{F}_k(\mathcal{F}_{j_1}(X, \ldots, X), \ldots, \mathcal{F}_{j_k}(X, \ldots, X)) & \xrightarrow{\Gamma} & \mathcal{F}_{j_1+\cdots+j_k}(X, \ldots, X) \\ \downarrow{\scriptstyle \mathcal{F}_k(\Theta_{j_1}, \ldots, \Theta_{j_k})} & & \downarrow{\scriptstyle \Theta_{j_1+\cdots+j_k}} \\ \mathcal{F}_k(X, \ldots, X) & \xrightarrow{\Theta_k} & X \end{array}$$

(c) $\Theta_k \circ \sigma_* = \Theta_k$ for all $\sigma \in \Sigma_k$.

Now let A be an object of $\mathcal{C}$, let X be an algebra over $\mathcal{F}$, and for each $k \geq 0$ define

$$\theta_k : \mathcal{F}_A(k) \times \mathrm{Hom}(A, X)^k \to \mathrm{Hom}(A, X)$$

to be the composite

$$\mathrm{Hom}(A, \mathcal{F}_k(A, \ldots, A)) \times \mathrm{Hom}(A, X)^k \to \mathrm{Hom}(A, \mathcal{F}_k(X, \ldots, X)) \xrightarrow{\mathrm{Hom}(A,\Theta_k)} \mathrm{Hom}(A, X)$$

Proposition 4.6. *The maps θ_k make* $\mathrm{Hom}(A, X)$ *an algebra over the operad $\mathcal{F}_A$.*

Again, the proof is an easy verification.

Definition 4.7. A functor-operad $\mathcal{F}$ is *strict* if the natural transformations $\Gamma_{j_1,\ldots,j_k}$ are isomorphisms.

Proposition 4.8. *If $\mathcal{F}$ is a strict functor-operad, then $\mathcal{F}_2$ is a symmetric monoidal structure for $\mathcal{C}$ with identity object $\mathcal{F}_0$. The commutative monoids with respect to this structure are the same as the algebras over $\mathcal{F}$.*

Once more, the proof is an easy verification.

5 A family of operations in $S^\bullet W$

In Section 6 we will define a symmetric monoidal product $\boxtimes$ is the category of augmented cosimplicial spaces. In this section we pause to offer motivation for this definition. The results in this section are not needed logically for later sections.

The definition of the monoidal product $\square$ was motivated in Section 2 by the properties of the cup product in $S^\bullet W$. The cup product is part of a larger family of operations in $S^\bullet W$ whose properties could be used as the basis for a definition of $\boxtimes$. However, this larger family is rather inconvenient to work with (because the analog of equation (2.2) for the larger family is complicated) so we will use a related family which has somewhat simpler properties.

We begin with a variant of the cup product. Given $x \in S^p W$ and $y \in S^q W$ we define

$$x \sqcup y \in S^{p+q+1} W$$

by

$$(x \sqcup y)(\sigma) = x(\sigma(0, \dots, p)) \cdot y(\sigma(p+1, \dots, p+q))$$

(note that, in contrast to the cup product, the vertex p is not repeated).

This operation is related to the coface and codegeneracy operations in $S^\bullet W$ by the following equations:

$$d^i(x \sqcup y) = \begin{cases} d^i x \sqcup y & \text{if } i \leq p+1 \\ x \sqcup d^{i-p-2} y & \text{if } i > p+1 \end{cases} \tag{5.1}$$

$$s^i(x \sqcup y) = \begin{cases} s^i x \sqcup y & \text{if } i < p \\ x \sqcup s^{i-p-1} y & \text{if } i > p \end{cases} \tag{5.2}$$

Note that there is no analog for $\sqcup$ of equation (2.2).

The operations $\smile$ and $\sqcup$ determine each other:

$$x \sqcup y = (d^{p+1} x) \smile y = x \smile d^0 y$$

$$x \smile y = s^p(x \sqcup y)$$

Now observe that equations (5.1) and (5.2) can be used as the basis for a characterization of $\square$-monoids: Remark 2.4 implies that $X^\bullet$ is a $\square$-monoid if and only if there are maps

$$\sqcup : X^p \times X^q \to X^{p+q+1}$$

satisfying (5.1), (5.2), the associativity condition

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \tag{5.3}$$

and the unit condition: there exists $e \in X^0$ with

$$s^p(x \sqcup e) = s^0(e \sqcup x) = x \tag{5.4}$$

(compare this to Remark 3.3).

In the remainder of this section we will define a family of operations in $S^\bullet W$ which generalize $\sqcup$; the definition of $\boxtimes$ in Section 6 will be suggested by the properties of this family.

Given a map $\sigma : \Delta^m \to W$ and a subset T of $\{0, \dots, m\}$ let $\sigma(T)$ denote the restriction of σ to the sub-simplex of Δ^m spanned by the vertices in T.

Recall the conventions in Remark 1.1. Given a nonempty finite totally ordered set T we define $S^T W$ to be equal to $S^m W$, where m is the cardinality of T minus 1 (so that $[m] \cong T$ as totally ordered sets).

Suppose we are given a function

$$f : T \to \{1, \dots, k\}$$

and elements $x_i \in S^{f^{-1}(1)} W$ for $1 \le i \le k$. We can define an element

$$\langle f \rangle (x_1, \dots, x_k) \in S^T W$$

by

$$\langle f \rangle (x_1, \dots, x_k)(\sigma) = x_1(\sigma(f^{-1}(1)) \cdot x_2(\sigma(f^{-1}(2)) \cdot \dots \cdot x_k(\sigma(f^{-1}(k))$$

where $\cdot$ denotes multiplication in $\mathbb{Z}$. This procedure gives a natural transformation

$$\langle f \rangle : S^{f^{-1}(1)} W \otimes \dots \otimes S^{f^{-1}(k)} W \to S^T W$$

Remark 5.1. In the special case where f is the function from $\{0, \dots, p+q+1\}$ to $\{1, 2\}$ which takes $\{0, \dots, p\}$ to 1 and $\{p+1, \dots, p+q+1\}$ to 2, we have $\langle f \rangle (x, y) = x \sqcup y$

Next we describe the relation between the operations $\langle f \rangle$ and the cosimplicial structure maps of $S^\bullet W$.

Proposition 5.2. *Let*

$$\begin{array}{ccc} T & \xrightarrow{\;h\;} & T' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \{1 \dots, k\} & \end{array}$$

be a commutative diagram, where h is a map in Δ (i.e., an order-preserving map). For each $i \in \{1 \dots, k\}$ let

$$h_i : f^{-1}(i) \to g^{-1}(i)$$

be the restriction of h.

Then the diagram

$$\begin{array}{ccc} S^{f^{-1}(1)} W \otimes \dots \otimes S^{f^{-1}(k)} W & \xrightarrow{\langle f \rangle} & S^T W \\ {\scriptstyle (h_1)_* \otimes \dots \otimes (h_k)_*} \downarrow & & \downarrow {\scriptstyle h_*} \\ S^{g^{-1}(1)} W \otimes \dots \otimes S^{g^{-1}(k)} W & \xrightarrow{\langle g \rangle} & S^{T'} W \end{array}$$

commutes.

The proof is an immediate consequence of the definitions. In the special case of Remark 5.1 we recover equations (5.1) and (5.2).

6 A symmetric monoidal structure on the category of augmented cosimplicial spaces.

¿From now on we will work with augmented cosimplicial spaces (the reason for this is given in Remark 6.10). By definition, an augmented cosimplicial space is a functor $X^\bullet$ from Δ_+ to Top, where Δ_+ is the category of finite totally ordered sets (including the empty set). We write X^S for the value of $X^\bullet$ at the finite totally ordered set S.

Our goal in this section is to construct a symmetric monoidal product $\boxtimes$ in the category of cosimplicial spaces. We do this by constructing a strict functor-operad Ξ and letting $\boxtimes = \Xi_2$; see Proposition 4.8. We will describe each Ξ_k as a Kan extension (cf. Proposition 2.3).

Definition 6.1. Let $k \geq 0$. Define $\bar{k}$ to be the set $\{1, \ldots, k\}$ when $k \geq 1$ and the empty set when $k = 0$.

Our next two definitions are motivated by Proposition 5.2.

Definition 6.2. Let $\mathfrak{Q}_k$ be the category whose objects are pairs (f, S), where S is an object of Δ_+ and f is a map of sets from S to $\bar{k}$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \bar{k} & \end{array}$$

where h is a map in Δ_+.

There is a forgetful functor $\Phi : \mathfrak{Q}_k \to \Delta_+$ which takes (f, S) to S, and a functor Ψ from $\mathfrak{Q}_k$ to the k-fold Cartesian product $(\Delta_+)^{\times k}$ which takes (f, S) to the k-tuple $(f^{-1}(1), \ldots, f^{-1}(k))$.

Definition 6.3. For each $k \geq 0$ define a functor Ξ_k as follows. Given augmented cosimplicial spaces $X_1^\bullet, \ldots, X_k^\bullet$, let $X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet$ denote the composite

$$(\Delta_+)^{\times k} \xrightarrow{X_1^\bullet \times \cdots \times X_k^\bullet} \text{Top} \times \cdots \times \text{Top} \xrightarrow{\times} \text{Top}.$$

We define the augmented cosimplicial space $\Xi_k(X_1^\bullet, \ldots, X_k^\bullet)$ to be the Kan extension

$$\text{Lan}_\Phi((X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet) \circ \Psi)$$

Remark 6.4. (a) Ξ_0 is the augmented cosimplicial space which takes every S to a point (because a Cartesian product indexed by the empty set is a point).

(b) The adjointness property of Lan_Φ [16, beginning of Section X.3] implies that a map $\Xi(X_1^\bullet, \ldots, X_k^\bullet) \to Y^\bullet$ is the same thing as a collection of maps

$$\langle f \rangle : X_1^{f^{-1}(1)} \times \cdots \times X_k^{f^{-1}(k)} \to Y^T,$$

one for each $f : T \to \bar{k}$, such that the analog of Proposition 5.2 is satisfied.

Our next goal is to specify the structure maps σ_* and $\Gamma_{j_1,\dots,j_k}$ of the functor-operad Ξ. For each of these we will use [16, Equation (10) on page 240] to write the relevant Kan extension as a colimit, and we will then use the following observation, whose proof is left to the reader.

Lemma 6.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $G : \mathcal{A} \to Top$ and $H : \mathcal{B} \to Top$ be functors. Each pair consisting of a functor $K : \mathcal{A} \to \mathcal{B}$ and a natural transformation $\nu : G \to H \circ K$ induces a map*

$$\operatorname{colim}_{\mathcal{A}} G \to \operatorname{colim}_{\mathcal{B}} H$$

We begin by constructing the transformation σ_*. Let $X_1^\bullet, \dots, X_k^\bullet$ be augmented cosimplicial spaces and let S be a totally ordered finite set; we want to construct

$$\sigma_* : \Xi_k(X_1^\bullet, \dots, X_k^\bullet)^S \to \Xi_k(X_{\sigma(1)}^\bullet, \dots, X_{\sigma(k)}^\bullet)^S$$

Let $\mathcal{A}_1$ be the category whose objects are the diagrams

$$\bar{k} \xleftarrow{f} T \xrightarrow{h} S$$

where T is a totally ordered finite set, f is a map of sets and h is an ordered map; we denote such a diagram by (f, h). A morphism from (f, h) to (f', h') is a commutative diagram

$$\begin{array}{ccccc} \bar{k} & \xleftarrow{f} & T & \xrightarrow{h} & S \\ \Big\downarrow{=} & & \Big\downarrow{g} & & \Big\downarrow{=} \\ \bar{k} & \xleftarrow{f'} & T' & \xrightarrow{h'} & S \end{array}$$

where g is an ordered map. Let

$$G_1 : \mathcal{A}_1 \to \text{Top}$$

be the functor which takes (f, h) to $\prod X_i^{f^{-1}(i)}$.

By [16, Equation (10) on page 240] we have

$$\Xi_k(X_1^\bullet, \dots, X_k^\bullet)^S = \operatorname{colim}_{\mathcal{A}_1} G_1 \tag{6.1}$$

Next let $\sigma \in \Sigma_k$ and let H_1 be the functor which takes (f, h) to $\prod X_{\sigma(i)}^{f^{-1}(i)}$; we have

$$\Xi_k(X_{\sigma(1)}^\bullet, \dots, X_{\sigma(k)}^\bullet)^S = \operatorname{colim}_{\mathcal{A}_1} H_1$$

Definition 6.6. The map

$$\sigma_* : \Xi_k(X_1^\bullet, \dots, X_k^\bullet)^S \to \Xi_k(X_{\sigma(1)}^\bullet, \dots, X_{\sigma(k)}^\bullet)^S$$

is induced by the functor $K_1 : \mathcal{A}_1 \to \mathcal{A}_1$ which takes (f, h) to $(\sigma^{-1} \circ f, h)$ and the natural transformation

$$\nu_1 : G_1 \to H_1 \circ K_1$$

which takes $(x_1, \dots, x_k)$ to $(x_{\sigma(1)}, \dots, x_{\sigma(k)})$.

Next let $j_1, \ldots, j_k \geq 0$, let $X_1^\bullet, \ldots, X_{j_1+\cdots+j_k}^\bullet$ be augmented cosimplicial spaces and let S be a finite totally ordered set. We want to construct

$$\Gamma_{j_1,\ldots,j_k} : \Xi_k(\Xi_{j_1}(X_1^\bullet, \ldots), \ldots, \Xi_{j_k}(\ldots, X_{j_1+\cdots+j_k}^\bullet))^S \to \Xi_{j_1+\cdots+j_k}(X_1^\bullet, \ldots, X_{j_1+\cdots+j_k}^\bullet)^S$$

First observe that

$$\Xi_k(\Xi_{j_1}(X_1^\bullet, \ldots), \ldots)^S = \operatorname{colim}_{\mathcal{A}_1} H,$$

where $\mathcal{A}_1$ is the category defined above and H is the functor which takes (f, h) to

$$\prod_{i=1}^{k} \Xi_{j_i}(X_{j_1+\cdots+j_{i-1}+1}^\bullet, \ldots, X_{j_1+\cdots+j_i}^\bullet)^{f^{-1}(i)}$$

Thus a point in $\Xi_k(\Xi_{j_1}(X_1^\bullet, \ldots), \ldots)^S$ is an equivalence class represented by a diagram

$$\bar{k} \xleftarrow{f} T \xrightarrow{h} S$$

together with points $x_i \in \Xi_{j_i}(X_{j_1+\cdots+j_{i-1}+1}^\bullet, \ldots)^{f^{-1}(i)}$ for $1 \leq i \leq k$. Similarly, each x_i is represented by a diagram

$$\bar{j}_i \xleftarrow{f_i} T_i \xrightarrow{h_i} f^{-1}(i)$$

together with a point

$$x_i' \in \prod_{p=j_1+\cdots+j_{i-1}+1}^{j_1+\cdots+j_i} X_p^{f_i^{-1}(p-j_1-\cdots-j_{i-1})}$$

We can assemble this information into a diagram

(6.2)
$$\begin{array}{ccccc}
\overline{j_1+\cdots+j_k} & \xleftarrow{\chi} & \bar{j}_1 \coprod \cdots \coprod \bar{j}_k & \xleftarrow{e} & U \\
 & & \Big\downarrow \psi & & \Big\downarrow g \\
 & & \bar{k} & \xleftarrow{f} & T \xrightarrow{h} S
\end{array}$$

and a point $x \in \prod X_p^{(\chi \circ e)^{-1}(p)}$. Here χ is the unique ordered bijection, ψ takes $\bar{j}_i$ to i, U is $\coprod T_i$, e is $\coprod f_i$, the restriction of g to T_i is h_i, and we give U the unique total order for which g and the inclusions of the T_i are ordered maps.

Let $\mathcal{A}_2$ be the category of diagrams of the form (6.2) for which g and h are ordered; an object of $\mathcal{A}_2$ will be denoted (e, f, g, h). What we have shown so far is that $\Xi_k(\Xi_{j_1}(X_1^\bullet, \ldots), \ldots)^S$ is the colimit over $\mathcal{A}_2$ of the functor G_2 which takes (e, f, g, h) to $\prod X_i^{(\chi \circ e)^{-1}(i)}$. Next we note that $\Xi_{j_1+\cdots+j_k}(X_1^\bullet, \ldots, X_{j_1+\cdots+j_k}^\bullet)^S$ is the colimit over the category $\mathcal{B}$ of diagrams

$$\overline{j_1+\cdots+j_k} \xleftarrow{f} T \xrightarrow{h} S$$

(denoted (f, h)) of the functor H_2 which takes (f, h) to $\prod X_i^{f^{-1}(i)}$. Now let

$$K_2 : \mathcal{A}_2 \to \mathcal{B}$$

take (e, f, g, h) to $(\chi \circ e, h \circ g)$; note that $H_2 \circ K_2 = G_2$.

Definition 6.7. The map

$$\Gamma_{j_1,\ldots,j_k} : \Xi_k(\Xi_{j_1}(X_1^\bullet,\ldots),\ldots)^S \to \Xi_{j_1+\cdots+j_k}(X_1^\bullet,\ldots,X_{j_1+\cdots+j_k}^\bullet)^S$$

is induced by the functor $K_2 : \mathcal{A}_2 \to \mathcal{B}$ and the identity natural transformation from G_2 to $H_2 \circ K_2$.

Finally, let $K_3 : \mathcal{B} \to \mathcal{A}_2$ be the functor which takes (f,h) to $(f, \psi \circ f, \mathrm{id}, h)$. Then $G_2 \circ K_3 = H_2$ and we can let $\nu_3 : H_2 \to G_2 \circ K_3$ be the identity natural transformation; the result is a natural transformation

$$\Lambda : \Xi_{j_1+\cdots+j_k}(X_1^\bullet,\ldots,X_{j_1+\cdots+j_k}^\bullet)^S \to \Xi_k(\Xi_{j_1}(X_1^\bullet,\ldots),\ldots)^S$$

With these definitions it is easy to check that σ_*, Γ and Λ are natural in S, that conditions (a)–(e) of definition 4.1 are satisfied, and that Λ is inverse to Γ. We have now shown

Theorem 6.8. *The collection Ξ_k, $k \geq 0$, with the structure maps σ_* and Γ defined above, is a strict functor-operad in the category of augmented cosimplicial spaces. In particular Ξ_2 is a symmetric monoidal product with unit Ξ_0.* □

Remark 6.9. Theorem 6.8 and its proof are valid for the category of augmented cosimplicial objects in any symmetric monoidal category $\mathcal{A}$, with Cartesian products in Top replaced by the symmetric monoidal product in $\mathcal{A}$.

Remark 6.10. All of the constructions in this section can be imitated for the category of ordinary (nonaugmented) cosimplicial spaces, provided that the maps f in the definition of $\mathcal{Q}_k$ are required to be surjective. In this setting the product Ξ_2 is still associative and commutative, but it is not unital: Ξ_0 is empty, and neither Ξ_0 nor any other cosimplicial space is an identity object for Ξ_2.

7 A sufficient condition for $\mathrm{Tot}(X^\bullet)$ to be an E_∞ space.

If $X^\bullet$ is an augmented cosimplicial space, we define $\mathrm{Tot}(X^\bullet)$ to be the usual Tot of the cosimplicial space obtained by restricting $X^\bullet$ to Δ. Equivalently, $\mathrm{Tot}(X^\bullet)$ is

$$\mathrm{Hom}(\Delta^\bullet, X^\bullet)$$

where Hom denotes maps of augmented cosimplicial spaces and we extend $\Delta^\bullet$ to an augmented cosimplicial space by setting $\Delta^\emptyset = \emptyset$.

Now apply Proposition 4.4, letting $\mathcal{F}$ be the functor operad Ξ constructed in Section 6 and A the augmented cosimplicial space $\Delta^\bullet$. This gives an operad $\mathcal{D}$ with k-th space $\mathcal{D}(k) = \mathrm{Tot}(\Xi_k(\Delta^\bullet,\ldots,\Delta^\bullet))$.

Recall that we have defined $\boxtimes$ to be Ξ_2; this is a symmetric monoidal product on the category of augmented cosimplicial spaces.

Theorem 7.1. (a) *$\mathcal{D}$ is an E_∞ operad.*

(b) *If $X^\bullet$ is a commutative monoid with respect to $\boxtimes$ (equivalently, if $X^\bullet$ is an algebra over Ξ) then $\mathcal{D}$ acts on Tot($X^\bullet$).*

Proof. Part (b) is immediate from Propositions 4.6 and 4.8. Part (a) will be proved in Section 10. $\square$

The analog of Theorem 7.1(b) is valid, with the same proof, when $X^\bullet$ is an augmented cosimplicial spectrum.

In the remainder of this section we use the adjointness property of Lan_Φ [16, beginning of Section X.3] to give an explicit characterization of commutative $\boxtimes$-monoids.

Definition 7.2. Let $X^\bullet$ be an augmented cosimplicial space. A $\langle\ \rangle$-structure on $X^\bullet$ consists of a map

$$\langle f\rangle : X^{f^{-1}(1)} \times X^{f^{-1}(2)} \to X^T$$

for each totally ordered set T and each $f : T \to \{1,2\}$.

Definition 7.3. A $\langle\ \rangle$-structure on $X^\bullet$ is *consistent* if for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & T' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \{1,2\} & \end{array}$$

the diagram

$$\begin{array}{ccc} X^{f^{-1}(1)} \times X^{f^{-1}(2)} & \xrightarrow{\langle f\rangle} & X^T \\ \downarrow{\scriptstyle (h_1)_* \times (h_2)_*} & & \downarrow{\scriptstyle h_*} \\ X^{g^{-1}(1)} \times X^{g^{-1}(2)} & \xrightarrow{\langle g\rangle} & X^{T'} \end{array}$$

commutes, where h_i is the restriction of h to $f^{-1}(i)$.

Recall the notation of Definition 6.3. Since Lan_Φ is left adjoint to Φ^*, a map $X^\bullet \boxtimes X^\bullet \to X^\bullet$ is the same thing as a natural transformation

$$(X^\bullet \bar{\times} X^\bullet) \circ \Psi \to X^\bullet \circ \Phi$$

and it's easy to check that this is the same thing as a consistent $\langle\ \rangle$-structure on $X^\bullet$. It remains to translate the commutativity, associativity and unitality conditions satisfied by a commutative $\boxtimes$-monoid into this language.

Definition 7.4. A $\langle\ \rangle$-structure on $X^\bullet$ is *commutative* if the diagram

$$\begin{array}{ccc} X^{f^{-1}(1)} \times X^{f^{-1}(2)} & \xrightarrow{\langle f\rangle} & X^T \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle =} \\ X^{f^{-1}(2)} \times X^{f^{-1}(1)} & \xrightarrow{\langle t\circ f\rangle} & X^T \end{array}$$

commutes, where τ is the switch map and t is the transposition of $\{1,2\}$.

For the associativity condition we need some notation. Let T be a totally ordered set and let $g : T \to \{1, 2, 3\}$ be a function. Define

$$\alpha : \{0, 1, 2\} \to \{1, 2\}$$

by $\alpha(1) = 1, \alpha(2) = 1, \alpha(3) = 2$ and define

$$\beta : \{0, 1, 2\} \to \{1, 2\}$$

by $\beta(1) = 1, \beta(2) = 2, \beta(3) = 2$. Let g_1 be the restriction of g to $g^{-1}\{1, 2\}$ and let g_2 be the restriction of g to $g^{-1}\{2, 3\}$.

Definition 7.5. A $\langle\ \rangle$-structure on $X^\bullet$ is *associative* if, with the notation above, the diagram

$$\begin{array}{ccc} X^{g^{-1}(1)} \times X^{g^{-1}(2)} \times X^{g^{-1}(3)} & \xrightarrow{\langle g_1 \rangle \times 1} & X^{g^{-1}\{1,2\}} \times X^{g^{-1}(3)} \\ {\scriptstyle 1 \times \langle g_2 \rangle} \downarrow & & \downarrow {\scriptstyle \langle \alpha \circ g \rangle} \\ X^{g^{-1}(1)} \times X^{g^{-1}\{2,3\}} & \xrightarrow{\langle \beta \circ g \rangle} & X^T \end{array}$$

commutes for every choice of T and of $g : T \to \{1, 2, 3\}$.

Definition 7.6. A $\langle\ \rangle$-structure on $X^\bullet$ is *unital* if there is an element $\varepsilon \in X^\emptyset$ with the property that if $f : T \to \{1, 2\}$ takes all of T to 1 then $\langle f \rangle(x, \varepsilon) = x$ for all x and if f takes all of T to 2 then $\langle f \rangle(\varepsilon, x) = x$ for all x.

Proposition 7.7. *A commutative $\boxtimes$-monoid structure on $X^\bullet$ determines, and is determined by, a $\langle\ \rangle$-structure on $X^\bullet$ which is consistent, commutative, associative and unital.*

The proof is a routine verification using the definitions in Section 6.

Remark 7.8. The $\langle f \rangle$ operations on $S^\bullet W$ defined in Section 5 give a consistent, commutative, associative and unital $\langle\ \rangle$-structure on $S^\bullet W$.

8 A filtration of Ξ by functor-operads.

In this section we describe a filtration of Ξ by functor-operads Ξ^n; the operad associated to Ξ^n will turn out to be equivalent to the little n-cubes operad $\mathcal{C}_n$.

We begin with some motivation. If T is a totally ordered set and $f : T \to \{1, 2\}$ is a function, the two totally ordered sets $f^{-1}(1)$ and $f^{-1}(2)$ are mixed together to form T. The amount of mixing can be measured by the number of times the value of f switches from 1 to 2 or from 2 to 1 as one moves through the set T. The idea in the definition of Ξ^n is to control the amount of mixing that is allowed.

Definition 8.1. Let T be a finite totally ordered set, let $k \geq 2$, and let $f : T \to \bar{k}$. We define the *complexity* of f as follows. If k is 0 or 1 the complexity is 0. If $k = 2$ let $\sim$ be the equivalence relation on T generated by

$$a \sim b \text{ if } a \text{ is adjacent to } b \text{ and } f(a) = f(b)$$

and define the complexity of f to be the number of equivalence classes minus 1. If $k > 2$ define the complexity of f to be the maximum of the complexities of the restrictions $f|_{f^{-1}(A)}$ as A ranges over the two-element subsets of $\bar{k}$.

Remark 8.2. If $k = 2$ the complexity of f is exactly the amount of mixing in f as discussed above. The definition of complexity is suggested by [21]; the reason we use it here is that it is well-adapted to the proofs of Theorems 8.5 and 9.1(a). There may be other ways of defining complexity that would also lead to Theorems 8.5 and 9.1(a), although this seems unlikely.

Now fix $n \geq 1$. Recall the category $\mathfrak{Q}_k$ from Definition 6.2.

Definition 8.3. Let $\mathfrak{Q}_k^n$ be the full subcategory of $\mathfrak{Q}_k$ whose objects are pairs (f, S) where f has complexity $\leq n$. Let

$$\iota^n : \mathfrak{Q}_k^n \to \mathfrak{Q}_k$$

be the inclusion.

Definition 8.4. For each $n \geq 1$ and each $k \geq 0$ define a functor Ξ_k^n as follows. Given augmented cosimplicial spaces $X_1^\bullet, \ldots, X_k^\bullet$, let $X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet$ be the functor defined in Definition 6.3 and let $\Xi_k(X_1^\bullet, \ldots, X_k^\bullet)$ be the Kan extension

$$\mathrm{Lan}_{\Phi \circ \iota^n}((X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet) \circ \Psi \circ \iota^n)$$

Theorem 8.5. *For each $n \geq 1$, Ξ^n is a (non-strict) functor-operad in the category of augmented cosimplicial spaces.*

Proof. We define σ_* as in the proof of Theorem 6.8, using the fact that the complexity of $\sigma^{-1} \circ f$ is the same as that of f. We define $\Gamma_{j_1,\ldots,j_k}$ as in the proof of Theorem 6.8, but we must verify that if the complexities of $e|_{e^{-1}(\bar{\jmath}_i)}$ and f in diagram (6.2) are $\leq n$ then the complexity of $\chi \circ e$ will be also be $\leq n$. For this we need to show that for each two-element subset A of $\bar{\jmath}_1 \coprod \ldots \coprod \bar{\jmath}_k$ the complexity of $\chi \circ e|_{e^{-1}(A)}$ will be $\leq n$; but this is true when A is contained in some $\bar{\jmath}_i$ (because the complexity of $e|_{e^{-1}(\bar{\jmath}_i)}$ is $\leq n$) and it is also true if A is not contained in any $\bar{\jmath}_i$ (because the complexity of f is $\leq n$). □

Remark 8.6. Theorem 8.5 and its proof are valid for the category of augmented cosimplicial objects in any symmetric monoidal category $\mathcal{A}$, with Cartesian products in Top replaced by the symmetric monoidal product in $\mathcal{A}$.

Remark 8.7. In the special case $n = 1$, the functor-operad Ξ^1 is closely related to the monoidal product $\square$ defined in Section 2. First observe that order-preserving maps $f : T \to \bar{k}$ have filtration 1 and that every map of filtration 1 can be written uniquely as the composite of an order-preserving map and a permuation of $\bar{k}$. If we use order-preserving maps in Definitions 8.3 and 8.4 instead of maps of filtration 1, we get a nonsymmetric strict functor-operad Υ which is related to both $\square$ and Ξ^1:

(a) The restriction of Υ_k to the category of (unaugmented) cosimplicial spaces is naturally isomorphic to the iterated $\square$-product $\square^k$.

(b) Ξ_k^1 is naturally isomorphic to

$$\coprod_{\sigma \in \Sigma_k} \Upsilon_k \circ \sigma_\#$$

That is, Ξ^1 is obtained by extending the nonsymmetric functor-operad Υ in the obvious way to a (symmetric) functor-operad.

9 An operad which acts on Tot of a Ξ^n-algebra.

Applying Proposition 4.4 with $\mathcal{F} = \Xi^n$ and $A = \Delta^\bullet$ we get an operad $\mathcal{D}_n$ with k-th space $\mathcal{D}_n(k) = \mathrm{Tot}(\Xi^n_k(\Delta^\bullet, \ldots, \Delta^\bullet))$.

Theorem 9.1. (a) *$\mathcal{D}_n$ is weakly equivalent in the category of operads to $\mathcal{C}_n$.*
(b) *If $X^\bullet$ is an algebra over Ξ^n then $\mathcal{D}_n$ acts on $Tot(X^\bullet)$.*

The statement of part (a) means that there is a chain of operads and weak equivalences of operads

$$\mathcal{D}_n \leftarrow \cdots \rightarrow \mathcal{C}_n$$

Part (b) of the Theorem is immediate from Propositions 4.6 and 4.8. Part (a) will be proved in Section 12.

Remark 9.2. The analog of Theorem 9.1(b) is valid, with the same proof, when $X^\bullet$ is an augmented cosimplicial spectrum.

In the remainder of this section we give an explicit characterization of Ξ^n-algebras, analogous to that given in Section 7 for commutative $\boxtimes$-monoids.

Definition 9.3. Let $X^\bullet$ be an augmented cosimplicial space. An n-structure on $X^\bullet$ consists of a map

$$\langle f \rangle : X^{f^{-1}(1)} \times \cdots \times X^{f^{-1}(k)} \to X^T$$

for each totally ordered set T, each $k \geq 0$, and each $f : T \to \bar{k}$ with complexity $\leq n$.

Definition 9.4. An n-structure on $X^\bullet$ is *consistent* if, for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & T' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \bar{k} & \end{array}$$

in which f and g have complexity $\leq n$, the diagram

$$\begin{array}{ccc} \prod_{i=1}^k X^{f^{-1}(i)} & \xrightarrow{\langle f \rangle} & X^T \\ {\scriptstyle \prod (h_i)_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \prod_{i=1}^k X^{g^{-1}(i)} & \xrightarrow{\langle g \rangle} & X^{T'} \end{array}$$

commutes, where h_i is the restriction of h to $f^{-1}(i)$.

It's easy to check (using the fact that Lan_Φ is left adjoint to Φ^*) that a consistent n-structure on $X^\bullet$ is the same thing as a collection of maps

$$\Xi^n_k(X^\bullet, \ldots, X^\bullet) \to X^\bullet,$$

one for each $k \geq 0$. It remains to translate the rest of the definition of Ξ^n-algebra into this language.

Definition 9.5. An n-structure on $X^\bullet$ is *commutative* if, for each f with complexity $\le n$ and each $\sigma \in \Sigma_k$, the diagram

$$\begin{array}{ccc}
\prod_{i=1}^{k} X^{f^{-1}(i)} & \xrightarrow{\langle f \rangle} & X^T \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle =} \\
\prod_{i=1}^{k} X^{f^{-1}(\sigma(i))} & \xrightarrow{\langle \sigma^{-1} \circ f \rangle} & X^T
\end{array}$$

commutes (where s is the evident permutation of the factors).

For the next definition we need some notation. Suppose we are given a partially ordered set T, numbers $k, j_1, \ldots, j_k \ge 0$, and maps

$$f : T \to \bar{k}$$

and

$$g_i : f^{-1}(i) \to \overline{j_i}$$

for $1 \le i \le k$. Let $j = \sum j_i$. The maps g_i determine a map

$$g : T \to \bar{j}$$

in an evident way; the formula for g is

$$g(a) = g_i(a) + \sum_{i' < i} j_{i'} \quad \text{if } a \in f^{-1}(i)$$

Definition 9.6. An n-structure on $X^\bullet$ is *associative* if the following diagram commutes for every choice of f and $g_1, \ldots, g_k$ with complexity $\le n$:

$$\begin{array}{ccc}
\prod_{i=1}^{k} \prod_{b=1}^{j_i} X^{g_i^{-1}(b)} & \xrightarrow{\prod \langle g_i \rangle} & \prod_{i=1}^{k} X^{f^{-1}(i)} \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \langle f \rangle} \\
\prod_{c=1}^{j} X^{g^{-1}(c)} & \xrightarrow{\langle g \rangle} & X^T
\end{array}$$

In order to state the unitality condition we need some more notation. If $i \in \bar{k}$ let $\lambda_i : \overline{k-1} \to \bar{k}$ be the order-preserving monomorphism whose image does not contain i.

Definition 9.7. An n-structure on $X^\bullet$ is *unital* if there is an element $\varepsilon \in X^\emptyset$ with the following property:

$$\langle \lambda_i \circ f \rangle (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{k-1}) = \langle f \rangle (x_1, \ldots, x_{i-1}, x_i, \ldots, x_{k-1})$$

for all $f : T \to \overline{k-1}$ with complexity $\le n$, all $i \in \bar{k}$, and all choices of $x_1, \ldots, x_{k-1}$.

Proposition 9.8. *A Ξ^n-algebra structure on $X^\bullet$ determines, and is determined by, an n-structure on $X^\bullet$ which is consistent, commutative, associative and unital.*

The proof is a routine verification using the definitions in Section 6.

10 The structure of $\Xi_k(\Delta^\bullet, \ldots, \Delta^\bullet)$.

Throughout this section we write $Y_k^\bullet$ for the augmented cosimplicial space $\Xi_k(\Delta^\bullet, \ldots, \Delta^\bullet)$ and Y_k^S for the value of $Y_k^\bullet$ at the finite totally ordered set S. We want to investigate the structure of Y_k^S and $Y_k^\bullet$.

Recall (equation (6.1)) that Y_k^S is $\operatorname{colim}_{\mathcal{A}_1} G_1$, where $\mathcal{A}_1$ is the category defined just before equation (6.1) and G_1 is the functor which takes the diagram

$$\bar{k} \xleftarrow{f} T \xrightarrow{h} S$$

to $\prod_{1\le i\le k} \Delta^{f^{-1}(i)}$.

Notation 10.1. A diagram of the form

$$\bar{k} \xleftarrow{f} T \xrightarrow{h} S,$$

where h is ordered, will be denoted from now on by (f, T, h).

The elements of Y_k^S are equivalence classes of pairs $((f, T, h), u)$ with $u \in \prod_i \Delta^{f^{-1}(i)}$; we think of u as a tuple indexed by T, subject to the condition that $\sum_{a\in f^{-1}(i)} u_a = 1$ for each $i \in \bar{k}$. Note that f must be surjective because $\Delta^\emptyset = \emptyset$.

Definition 10.2. (a) A diagram (f, T, h) is *nondegenerate* if $T = [m]$ for some m, f is surjective, and for each $j < m$ either $f(j) \neq f(j+1)$ or $h(j) \neq h(j+1)$.

(b) A pair $((f, T, h), u)$ with $u \in \prod_i \Delta^{f^{-1}(i)}$ is nondegenerate if (f, T, h) is nondegenerate and $u_a \neq 0$ for all $a \in T$.

Proposition 10.3. *Each point in Y_k^S is represented by a unique nondegenerate pair.*

Corollary 10.4. *Y_k^S is a CW complex with one cell of dimension $m + 1 - k$ for each nondegenerate $(f, [m], h)$; the characteristic map of the cell corresponding to $(f, [m], h)$ is a homeomorphism from $\prod \Delta^{f^{-1}(i)}$ to the closure of the cell (and thus Y_k^S is a regular CW complex).* □

Proof of 10.3. We define a function Υ from pairs to pairs as follows. Given a pair $((f, T, h), u)$, let $\sim$ be the equivalence relation on T generated by

$$a \sim b \quad \text{if } a \text{ is adjacent to } b,\ f(a) = f(b),\ \text{and } h(a) = h(b)$$

and let T_1 be the subset

$$\{a \in T \mid u_a \neq 0\}.$$

Let $\hat{T}$ be $T_1/\sim$. Then f and h induce maps $\hat{f} : \hat{T} \to \bar{k}$ and $\hat{h} : \hat{T} \to S$. Also, let $\pi : T_1 \to \hat{T}$ be the projection and define $\hat{u} \in \prod_i \Delta^{\hat{f}^{-1}(i)}$ by $\hat{u}_c = \sum_{a\in\pi^{-1}(c)} u_a$ for $c \in \hat{T}$. Finally, let $m = |T| - 1$ and let $g : [m] \to T$ be the unique ordered bijection. We define $\Upsilon((f, T, h), u) = ((\hat{f} \circ g, [m], \hat{h} \circ g), \hat{u} \circ g)$. The proposition is immediate from the following properties of Υ:

(i) $\Upsilon((f, T, h), u)$ is nondegenerate.

(ii) $((f, T, h), u)$ and $\Upsilon((f, T, h), u)$ represent the same point in Y_k^S.

(iii) If $((f, T, h), u)$ and $((f', T', h'), u')$ represent the same point in Y_k^S then $\Upsilon((f, T, f), u) = \Upsilon((f', T', h'), u')$

□

Our next goal is to show for each k that $Y_k^\bullet$ is isomorphic as an augmented cosimplicial space to $\Delta^\bullet \times Y_k^0$ (this is the analog for ⊠ of Lemma 3.6).

Definition 10.5. For each S, η_S is the unique map $S \to [0]$. This will also be denoted by η when S is clear from the context.

Definition 10.6. Define

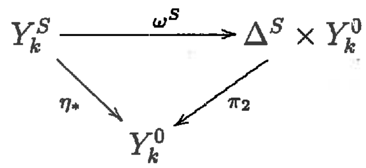
$$\omega^S : Y_k^S \to \Delta^S \times Y_k^0$$

by letting the projection on the second factor be the map η_* induced by η and letting the projection on the first factor take the equivalence class of $((f, T, h), u)$ to the element $v \in \Delta^S$ with $v_a = \frac{1}{k} \sum_{b \in h^{-1}(a)} u_b$. The ω^S fit together to give a cosimplicial map

$$\omega : Y_k^\bullet \to \Delta^\bullet \times Y_k^0$$

Proposition 10.7. *ω is an isomorphism of augmented cosimplicial spaces.*

Proof. The diagram



commutes, where π_2 is the projection. We begin by showing

(1) for each point $y \in Y_k^0$ the map $\eta_*^{-1}(y) \to \pi_2^{-1}(y)$ induced by ω^S is a bijection.

For this, it suffices to show

(2) the composite $\eta_*^{-1}(y) \xrightarrow{\omega^S} \Delta^S \times Y_k^0 \xrightarrow{\pi_1} \Delta^S$ is a bijection.

So let y be a point of Y_k^0 and let

$$((f, [m], \eta_{[m]}), u)$$

be the nondegenerate pair which represents it.

We will define an inverse

$$\lambda : \Delta^S \to \eta_*^{-1}(y)$$

of $\pi_1 \circ \omega^S$ as follows. Let $v \in \Delta^S$. For each $j \in [m]$, let $a_j \in S$ be the smallest element for which

$$\sum_{a \leq a_j} v_a \geq \frac{1}{k} \sum_{i=0}^{j} u_i$$

Define a totally ordered set T by adjoining to S an immediate successor of a_j, denoted $\tilde{a}_j$, for each j. Define $g : T \to [m]$ by $g(b) = j$ if $\tilde{a}_{j-1} \leq b \leq a_j$. Define $f' : T \to \bar{k}$ to be $f \circ g$. Define $h : T \to S$ by $h(b) = b$ if $b \in S$ and $h(\tilde{a}_j) = a_j$. Define

$$u'_b = \begin{cases} \sum_{i=0}^{j} u_i - k \sum_{a<a_j} v_a \text{ if } b = a_j \\ k \sum_{a \leq a_j} v_a - \sum_{i=0}^{j} u_i \text{ if } b = \tilde{a}_j \\ k v_b \text{ otherwise} \end{cases}$$

We define $\lambda(v)$ to be the point represented by the pair $((f', T, h), u')$. It is easy to check that this point is in $\eta_*^{-1}(y)$ (this amounts to showing that $\Upsilon((f', T, \eta_T), u') = ((f, [m], \eta_{[m]}), u))$ and that λ is an inverse of $\pi_1 \circ \omega^S$; this completes the proof of (2).

Next let e be a cell of Y_k^0 and let $\bar{e}$ be its closure. Then $\eta_*^{-1}(\bar{e})$ is a finite union of closed cells of Y_k^S, and in particular it is compact. This together with (1) implies that ω^S induces a homeomorphism

$$\eta_*^{-1}(\bar{e}) \to \pi_2^{-1}(\bar{e})$$

Since the closure of each cell of $\Delta^S \times Y_k^0$ is contained in a set of the form $\pi_2^{-1}(\bar{e})$, it follows that $(\omega^S)^{-1}$ is continuous on the closure of each cell of $\Delta^S \times Y_k^0$, and from this it follows that ω^S is a homeomorphism. □

We can now complete the proof of Theorem 7.1(a) by showing:

Corollary 10.8. *Y_k^0 is contractible for each $k \geq 0$.*

Proof. First observe that if A is a space then

$$\Xi_k(X_1^\bullet, \ldots, X_i^\bullet, \ldots, X_k^\bullet) \times A \cong \Xi_k(X_1^\bullet, \ldots, X_i^\bullet \times A, \ldots, X_k^\bullet)$$

(because $\times A$ preserves colimits). Thus we have

$$\begin{aligned} Y_k^0 \times Y_j^0 &= \Xi_k(\Delta^\bullet, \ldots, \Delta^\bullet)^0 \times Y_j^0 \\ &\approx \Xi_k(\Delta^\bullet, \ldots, \Delta^\bullet \times Y_j^0, \ldots, \Delta^\bullet)^0 \\ &\approx \Xi_k(\Delta^\bullet, \ldots, \Xi_j(\Delta^\bullet, \ldots), \ldots)^0 \quad \text{by Proposition 5.3} \\ &\approx \Xi_{k+j-1}(\Delta^\bullet, \ldots, \Delta^\bullet)^0 \quad \text{by Theorem 6.8} \\ &= Y_{k+j-1}^0 \end{aligned}$$

It therefore suffices to prove the corollary when $k = 2$. Y_2^0 has the special property that the $(n-1)$-skeleton is contained in the closure of either of the two n-cells. Corollary 10.4 implies that the closure of a cell is contractible, so the inclusion of the $(n-1)$-skeleton is nullhomotopic for each n. Thus any map from a sphere into Y_2^0 is nullhomotopic, so Y_2^0 is contractible. □

11 The structure of $\Xi^n_k(\Delta^\bullet,\ldots,\Delta^\bullet)$.

For use in Section 12, we prove the analogs for Ξ^n_k of the results of Section 10.

Fix n and denote the augmented cosimplicial space $\Xi^n_k(\Delta^\bullet,\ldots,\Delta^\bullet)$ by $Z^\bullet_k$; thus $\mathcal{D}_n(k) = \mathrm{Tot}(Z^\bullet_k)$. Recall Notation 10.1. The elements of Z^S_k are equivalence classes of pairs $((f,T,h),u)$ with $u \in \prod_i \Delta^{f^{-1}(i)}$, where f has complexity $\le n$.

We define nondegenerate pairs exactly as in Definition 10.2, and the proof of Proposition 10.3 goes through to show

Proposition 11.1. *Each point in Z^S_k is represented by a unique nondegenerate pair.* □

Corollary 11.2. *Z^S_k is a CW complex with one cell of dimension $m+1-k$ for each nondegenerate $(f,[m],h)$ for which f has complexity $\le n$; the characteristic map of the cell corresponding to $(f,[m],h)$ is a homeomorphism from $\prod \Delta^{f^{-1}(i)}$ to the closure of the cell.* □

Proposition 11.1 also gives a useful relationship between $Z^\bullet_k$ and the augmented cosimplicial space $Y^\bullet_k$ defined in Section 10:

Corollary 11.3. *The map $Z^S_k \to Y^S_k$ is a monomorphism for all S, and the diagram*

$$\begin{array}{ccc} Z^S_k & \longrightarrow & Y^S_k \\ \downarrow & & \downarrow \\ Z^0_k & \longrightarrow & Y^0_k \end{array}$$

is a pullback. □

Next we define

$$\omega : Z^\bullet_k \to \Delta^\bullet \times Z^0_k$$

as in Section 10: the projection of ω^S on Δ^S takes the equivalence class of $((f,T,h),u)$ to v, where $v_a = \frac{1}{k}\sum_{b\in h^{-1}(a)} u_b$, and the projection of ω^S on Z^0_k is η_* (see Notation 10.5). The diagram

$$\begin{array}{ccc} Z^\bullet_k & \xrightarrow{\omega} & \Delta^\bullet \times Z^0_k \\ \downarrow & & \downarrow \\ Y^\bullet_k & \xrightarrow{\omega} & \Delta^\bullet \times Y^0_k \end{array}$$

commutes, and this together with Corollary 11.3 implies

Proposition 11.4. *$\omega : Z^\bullet_k \to \Delta^\bullet \times Z^0_k$ is an isomorphism of augmented cosimplicial spaces.*

12 Proof of Theorem 9.1(a).

In this section we prove Theorem 9.1(a). As motivation for the method, recall that one way to show that two spaces are weakly equivalent is to show that they have contractible open covers with the same nerve, or more generally to show that they can be decomposed into homotopy colimits of contractible pieces over the same indexing category. We will show that the cosimplicial space $\Xi^n_k(\Delta^\bullet, \ldots, \Delta^\bullet)$ can be decomposed as a homotopy colimit of contractible cosimplicial spaces indexed over a certain category $\mathcal{K}^n_k$ considered by Berger [5]; Berger has shown that $\mathcal{C}_n(k)$ is a homotopy colimit of contractible pieces indexed by $\mathcal{K}^n_k$, and from this we will deduce Theorem 9.1(a).

We begin by recalling some definitions from [5] (but our notation differs somewhat from that in [5]).

Definition 12.1. For each $k \geq 0$, let $P_2\bar{k}$ be the set of subsets of $\bar{k}$ that have two elements.

Definition 12.2. (a) Let $\mathcal{K}_k$ be the set whose elements are pairs (b, T), where b is a function from $P_2\bar{k}$ to the nonnegative integers and T is a total ordering of $\bar{k}$. We give $\mathcal{K}_k$ the partial order for which $(a, S) \leq (b, T)$ if $a(\{i,j\}) \leq b(\{i,j\})$ for each $\{i,j\} \in P_2\bar{k}$ and $a(\{i,j\}) < b(\{i,j\})$ for each $\{i,j\}$ with $i < j$ in the order S but $i > j$ in the order T. Let $\mathcal{K}^n_k$ be the subset of pairs (b, T) such that $b\{i,j\} < n$ for each $\{i,j\} \in P_2\bar{k}$. The set $\mathcal{K}^n_k$ inherits an order from $\mathcal{K}_k$.

(b) Let $\mathcal{K}$ denote the collection of partially ordered sets $\mathcal{K}_k$, $k \geq 0$, and let $\mathcal{K}^n$ denote the collection of partially ordered sets $\mathcal{K}^n_k$, $k \geq 0$.

It is shown in [5] that $\mathcal{K}$ is an operad in the category of partially ordered sets with the following structure maps. The right action of Σ_k on $\mathcal{K}_k$ is given by

$$(b, T)\rho = (b \circ \rho_2, T\rho)$$

where $\rho_2 \colon P_2\bar{k} \to P_2\bar{k}$ is the function $\rho_2(\{i,j\}) = \{\rho(i), \rho(j)\}$ and where $i < j$ in the total order $T\rho$ if $\rho(i) < \rho(j)$ in the total order T. The operad composition

$$\mathcal{K}_k \times \mathcal{K}_{a_1} \times \cdots \times \mathcal{K}_{a_k} \to \mathcal{K}_{\Sigma a_i}$$

takes $((b, T); (b_1, T_1), \ldots, (b_k, T_k))$ to the pair $(b(b_1, \ldots, b_k), T(T_1, \ldots, T_k))$, where $b(b_1, \ldots, b_k)$ is the function which takes $\{r, s\}$ to

$$\begin{cases} b_i(\{r,s\}) & \text{if } \{r,s\} \subset \bar{a}_i \\ b(\{i,j\}) & \text{if } r \in \bar{a}_i,\ s \in \bar{a}_j \text{ and } i \neq j \end{cases}$$

and $T(T_1, \ldots, T_k)$ is the total order of $\coprod \bar{a}_i$ for which $r < s$ if either $r < s$ in the order T_i or $r \in \bar{a}_i$, $s \in \bar{a}_j$ and $i < j$.

Note that, for each n, $\mathcal{K}^n$ is a suboperad of $\mathcal{K}$.

Let us write N for the functor that takes a partially ordered set to the geometric realization of its nerve. Then $\mathrm{N}\,\mathcal{K}^n$ is an operad of spaces and Berger shows ([5, Theorem 1.16]) that it is weakly equivalent to the little n-cubes operad $\mathcal{C}_n$. To complete the proof of Theorem 9.1(a) it therefore suffices to show that $\mathrm{N}\,\mathcal{K}^n$ is weakly equivalent to $\mathcal{D}_n$. We will do this by finding a homotopy colimit decomposition of the functor-operad Ξ^n; first we need some definitions.

Definition 12.3. Let $f \colon [m] \to \bar{k}$. Then $(b_f, T_f) \in \mathcal{K}_k$ is the pair where $b_f(\{i,j\})$ is one less than the complexity of the restriction of f to a map $f^{-1}(\{i,j\}) \to \{i,j\}$ and where $i < j$ in the total order T_f if the smallest element of $f^{-1}(i)$ is less than the smallest element of $f^{-1}(j)$.

Recall the definition of the category $\mathcal{Q}_k$ (Definition 6.2).

Definition 12.4. For each pair $(b,T) \in \mathcal{K}_k$ let $\mathcal{Q}_{(b,T)}$ be the full subcategory of $\mathcal{Q}_k$ whose objects are the maps f with $(b_f, T_f) \le (b,T)$. Let

$$\iota_{(b,T)} : \mathcal{Q}_{(b,T)} \to \mathcal{Q}_k$$

be the inclusion functor.

The next definition uses the notation of Definition 6.3.

Definition 12.5. Let $X_1^\bullet, \ldots, X_k^\bullet$ be augmented cosimplicial spaces.

(a) For each $(b,T) \in \mathcal{K}_k$, define $\Xi_{(b,T)}(X_1^\bullet, \ldots, X_k^\bullet)$ to be the Kan extension

$$\mathrm{Lan}_\Phi((X_1^\bullet \bar{\times} \cdots \bar{\times} X_k^\bullet) \circ \Psi \circ \iota_{(b,T)})$$

(b) Define $\Lambda^n(X_1^\bullet, \ldots, X_k^\bullet)$ to be

$$\mathrm{hocolim}_{\mathcal{K}_k^n} \Xi_{(b,T)}(X_1^\bullet, \ldots, X_k^\bullet)$$

Lemma 12.6. *Λ^n is a functor-operad.*

Proof. First note that the natural transformations defining the functor-operad Ξ restrict to natural transformations

$$\sigma_* : \Xi_{(b,T)} \to \Xi_{(b,T)\sigma} \circ \sigma_\# \tag{12.1}$$

for $\sigma \in \Sigma_k$ and

$$\Gamma : \Xi_{(b,T)}(\Xi_{(b_1,T_1)}, \ldots, \Xi_{(b_k,T_k)}) \to \Xi_{(b(b_1,\ldots,b_k),T(T_1,\ldots,T_k))} \tag{12.2}$$

Next recall the definition of hocolim given in [12, Section 20.1]: if $\mathcal{A}$ is a category and $F : \mathcal{A} \to \mathrm{Top}$ is a functor then

$$\mathrm{hocolim}_{\mathcal{A}} F = F \otimes_{\mathcal{A}} U$$

where $\otimes_{\mathcal{A}}$ denotes the coend and U is the contravariant functor $\mathcal{A} \to \mathrm{Top}$ which takes an object $a \in \mathcal{A}$ to $\mathrm{N}(a \downarrow \mathcal{A})$.

Now let $\sigma \in \Sigma_k$ and observe that σ induces a functor

$$\sigma_\downarrow : ((b,T) \downarrow \mathcal{K}_k^n) \to ((b,T)\sigma \downarrow \mathcal{K}_k^n)$$

We define

$$\sigma_* : \Lambda_k^n(X_1^\bullet, \ldots, X_k^\bullet) \to \Lambda_k^n(X_{\sigma(1)}^\bullet, \ldots, X_{\sigma(k)}^\bullet)$$

to be the map induced by the collection of maps

$$\Xi_{(b,T)}(X_1^\bullet, \ldots, X_k^\bullet) \times \mathrm{N}((b,T) \downarrow \mathcal{K}_k^n) \xrightarrow{\sigma_* \times \mathrm{N}(\sigma_\downarrow)} \Xi_{(b,T)\sigma}(X_{\sigma(1)}^\bullet, \ldots, X_{\sigma(k)}^\bullet) \times \mathrm{N}((b,T)\sigma \downarrow \mathcal{K}_k^n)$$

Finally, we define the structural map

$$\Gamma : \Lambda^n_k(\Lambda^n_{j_1}, \ldots, \Lambda^n_{j_k}) \to \Lambda^n_{j_1+\cdots+j_k}$$

to be the map induced by the collection of maps

$$\Xi_{(b,T)}(\Xi_{(b_1,T_1)}, \ldots, \Xi_{(b_k,T_k)}) \times \mathrm{N}((b,T) \downarrow \mathcal{K}^n_k) \times \prod_{i=1}^{k} \mathrm{N}((b_i,T_i) \downarrow \mathcal{K}^n_{j_i}) \xrightarrow{\cong}$$

$$\Xi_{(b,T)}(\Xi_{(b_1,T_1)}, \ldots, \Xi_{(b_k,T_k)}) \times \mathrm{N}\Big(((b,T) \downarrow \mathcal{K}^n_k) \times \prod_{i=1}^{k}((b_i,T_i) \downarrow \mathcal{K}^n_{j_i})\Big) \xrightarrow{\Gamma\times\mathrm{N}(\gamma_\downarrow)}$$

$$\Xi_{(b(b_1,\ldots,b_k),T(T_1,\ldots,T_k))} \times \mathrm{N}((b(b_1,\ldots,b_k), T(T_1,\ldots,T_k)) \downarrow \mathcal{K}^n_{j_1+\cdots+j_k})$$

where $\gamma_\downarrow$ is induced by the composition map

$$\gamma : \mathcal{K}^n_k \times \prod_{i=1}^{k} \mathcal{K}^n_{j_i} \to \mathcal{K}^n_{j_1+\cdots+j_k}$$

of the Cat-operad $\mathcal{K}^n$. □

Now let $\mathcal{B}_n$ be the operad obtained by applying Proposition 4.4 with $\mathcal{F} = \Lambda^n$ and $A = \Delta^\bullet$. To complete the proof of Theorem 9.1(a) it remains to show:

Lemma 12.7. *(a) There is a weak equivalence of operads*

$$\mathcal{B}_n \to \mathcal{D}_n$$

(b) There is a weak equivalence of operads

$$\mathcal{B}_n \to \mathrm{N}\,\mathcal{K}^n$$

For the proof of part (a), we first observe that $\mathcal{Q}^n_k$ is the union of $\mathcal{Q}_{(b,T)}$ for $(b,T) \in \mathcal{K}^n_k$; it follows that

$$\Xi^n_k(X^\bullet_1, \ldots, X^\bullet_k) = \mathrm{colim}_{\mathcal{K}^n_k}\, \Xi_{(b,T)}(X^\bullet_1, \ldots, X^\bullet_k)$$

for all $X^\bullet_1, \ldots, X^\bullet_k$. The projection from hocolim to colim gives a map of functor-operads

$$\Lambda^n \to \Xi^n$$

and an induced map of the associated operads:

$$\phi : \mathcal{B}_n \to \mathcal{D}_n$$

We need to show that for each $k \geq 0$ the map

$$\phi(k) : \mathcal{B}_n(k) \to \mathcal{D}_n(k)$$

is a weak equivalence of spaces. Recall from Proposition 11.4 that there is an isomorphism of cosimplicial spaces

$$\Xi^n_k(\Delta^\bullet,\ldots,\Delta^\bullet) \cong \Delta^\bullet \times \Xi^n_k(\Delta^\bullet,\ldots,\Delta^\bullet)^0$$

The proof of Proposition 11.4 shows that for each (b,T) we have an isomorphism of cosimplicial spaces

$$\Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet) \cong \Delta^\bullet \times \Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0$$

It follows that we have homeomorphisms

$$\mathcal{D}_n(k) \approx \mathrm{Tot}(\Delta^\bullet) \times \mathrm{colim}_{\mathcal{K}^n_k} \Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0$$

and

$$\mathcal{B}_n(k) \approx \mathrm{Tot}(\Delta^\bullet) \times \mathrm{hocolim}_{\mathcal{K}^n_k} \Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0 \tag{12.3}$$

Thus it suffices to show that the map

$$\mathrm{hocolim}_{\mathcal{K}^n_k} \Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0 \to \mathrm{colim}_{\mathcal{K}^n_k} \Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0$$

is a weak equivalence, and this follows from a standard fact about homotopy colimits [12, Theorem 20.9.1]; the "Reedy cofibrancy" condition needed for [12, Theorem 20.9.1] is satisfied in our case because the map

$$\bigcup_{(b',T')<(b,T)} \Xi_{(b',T')}(X^\bullet_1,\ldots,X^\bullet_k)^0 \to \Xi_{(b,T)}(X^\bullet_1,\ldots,X^\bullet_k)^0$$

is the inclusion of a sub-CW-complex (cf. Corollary 11.2).

Next we prove part (b). Consider the map

$$\psi(k) : \mathcal{B}_n(k) = \mathrm{Hom}(\Delta^\bullet, \Lambda^n_k(\Delta^\bullet,\ldots,\Delta^\bullet)) \to \mathrm{Hom}(\Delta^\bullet, \Lambda^n_k(*,\ldots,*)) = \mathrm{N}\,\mathcal{K}^n$$

where the arrow is induced by the projection $\Delta^\bullet \to *$ and the second equality follows from the fact that $\Lambda^n_k(*,\ldots,*)$ is the constant cosimplicial space with value $\mathrm{N}\,\mathcal{K}^n$. It is easy to check that the collection $\{\psi(k)\}$ is an operad map; it remains to show that each $\psi(k)$ is a weak equivalence. Using equation (12.3) and the fact that $\mathrm{Tot}(\Delta^\bullet)$ is contractible it suffices to show that the map

$$\mathrm{hocolim}_{\mathcal{K}^n_k} \Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0 \to \mathrm{hocolim}_{\mathcal{K}^n_k} \Xi_{(b,T)}(*,\ldots,*)^0 = \mathrm{hocolim}_{\mathcal{K}^n_k} * = \mathrm{N}\,\mathcal{K}^n_k$$

(where the arrow is induced by $\Delta^\bullet \to *$) is a weak equivalence; and this is a consequence of the following lemma.

Lemma 12.8. *For each choice of b and T the space $\Xi_{(b,T)}(\Delta^\bullet,\ldots,\Delta^\bullet)^0$ is weakly equivalent to a point.*

Proof. The proof is by induction on k.

Since the map (12.1) is an isomorphism we may assume that T is the standard total order on $\bar{k}$.

For each $f : [m] \to \bar{k}$ we define $c(f) : [m+1] \to \bar{k}$ to be the function which takes 1 to 1 and p to $f(p-1)$ if $p > 1$. This construction gives a functor, also called c, from $\Omega_{(b,T)}$ to itself.

Next let $C : \Delta \to \Delta$ be the functor which takes $[m]$ to $[m+1]$ and takes a morphism $h : [m] \to [n]$ to the morphism $C(h) : [m+1] \to [n+1]$ defined by

$$C(h)(p) = \begin{cases} 0 & \text{if } p = 0 \\ h(p-1)+1 & \text{if } p > 0 \end{cases}$$

We can define a map

$$\alpha : \Xi_{(b,T)}(\Delta^\bullet, \dots, \Delta^\bullet)^0 \to \Xi_{(b,T)}(\Delta^\bullet \circ C, \Delta^\bullet, \dots, \Delta^\bullet)^0$$

as follows: if $f : [p] \to \bar{k}$ and $u_i \in \Delta^{f^{-1}(i)}$ for $1 \le i \le k$, let α take the equivalence class of $(f, u_1, \dots, u_n)$ to that of $(f, d^0 u_1, u_2, \dots, u_k)$. We can also define a map

$$\beta : \Xi_{(b,T)}(\Delta^\bullet \circ C, \Delta^\bullet, \dots, \Delta^\bullet)^0 \to \Xi_{(b,T)}(\Delta^\bullet, \dots, \Delta^\bullet)^0$$

by letting β take the equivalence class of $(f, u_1, \dots, u_n)$ to that of $(cf, u_1, \dots, u_k)$. It is easy to check that α and β are well-defined and that $\beta \circ \alpha$ is the identity; that is, $\Xi_{(b,T)}(\Delta^\bullet, \dots, \Delta^\bullet)^0$ is a retract of $\Xi_{(b,T)}(\Delta^\bullet \circ C, \Delta^\bullet, \dots, \Delta^\bullet)^0$. It therefore suffices to show that the latter is weakly equivalent to a point.

But $\Delta^\bullet \circ C$ is isomorphic to the degreewise cone on $\Delta^\bullet$, and in particular it is homotopic as a cosimplicial space to a point, so we have

$$\Xi_{(b,T)}(\Delta^\bullet \circ C, \Delta^\bullet, \dots, \Delta^\bullet)^0 \simeq \Xi_{(b,T)}(*, \Delta^\bullet, \dots, \Delta^\bullet)^0$$

and an inspection of Definition 12.5 shows that

$$\Xi_{(b,T)}(*, \Delta^\bullet, \dots, \Delta^\bullet) \cong \Xi_{(b',T')}(\Delta^\bullet, \dots, \Delta^\bullet)$$

where b' is the restriction of b to $P_2(\bar{k} - \{1\})$ and T' is the restriction of T to $\bar{k} - \{1\}$. The inductive hypothesis shows that $\Xi_{(b',T')}(\Delta^\bullet, \dots, \Delta^\bullet)^0$ is weakly equivalent to a point, and this concludes the proof. □

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