Galois structure of vanishing cycles

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Dedicated to Hyman Bass with great respect

1 Introduction

The purpose of this paper is to make a few elementary remarks about the action of the Galois group of a local field on vanishing cycles in relative codimension one. There is a formula due to Deligne and Kato for the dimension of the group of vanishing cycles in this situation

$$dim_{\Lambda}(H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))) = \phi(\eta) - \phi(s) + 2\delta_k dim_{\Lambda}(\mathcal{F}).$$

Precise details of the context and of the Deligne-Kato formula are given in §3 and §4.4.

There is a canonical action of the absolute Galois group $G(\overline{k}/k)$ on the vanishing cycles $H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))$ and the problem of analysing this Galois action is posed in ([16] p.632).

Let k be a local field and Λ a finite field of characteristic l different from the residue characteristic of k. Then the cohomology group

$$H^{s+1}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})) \cong H^s(L; H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})))$$

can only possibly be non-zero for $s = 0, 1, 2$ for each finite Galois extension L/k. As a contribution to the Galois structure problem posed in ([16] p.632) we derive a relation between the annihilator ideals of the $\mathbf{Z}_l[G(L/k)]$-modules $H^j(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F}))$ in the case when the Galois group is abelian.

The structure of the paper is as follows: In §2 we explain how to derive annihilator relations (Theorem 2.7) of the sort we want from elements of the relative K-group $K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l)$. This is very simple algebra and is a slight generalisation of the phenomenon discovered in ([37] Chapter 7; see also [38]). In §3 we construct a "refined Euler characteristic", an invariant lying in $K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l)$, from the vanishing cycles in relative codimension one. When $G(L/k)$ is abelian this invariant is represented by an element

$$det(\Phi) \in \frac{\mathbf{Q}_l[G(L/k)]^*}{\mathbf{Z}_l[G(L/k)]^*}$$

which enters into the annihilator relation of Theorem 4.2. In §§4.3-4.6 we discussion some clues which point to a connection between the Kato-Swan conductor and $\det(\Phi)$.

This all adds up to a very interesting phenomenon which has the prospect of some very intriguing formulae. Unfortunately, at the moment, I have no substantial examples to present but I hope to return to this topic in a future paper.

I am very grateful to Igor Zhukov for introducing me to the topic of Swan conductors in algebraic geometry – in particular to the results and problems which appear in ([1], [2], [3], [6], [7], [15], [16], [17], [18], [19], [21], [22], [23], [24], [25], [30], [31], [32], [41], [42], [43], [44]). I am also very grateful to the Royal Society for the Joint Projects Grant which made possible my visits to St. Petersburg and Igor Zhukov's to Southampton during 2000-2002. In fact, this work was started during the first of those visits.

2 $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ and annihilator relations

2.1 Let l be a prime, G a finite group and let $f : \mathbf{Z}_l[G] \longrightarrow \mathbf{Q}_l[G]$ denote the homomorphism of group-rings induced by the inclusion of the l-adic integers into the fraction field, the l-adic rationals. Write $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ for the relative K-group of f, denoted by $K_0(\mathbf{Z}_l[G], f)$ in ([39] p.214; see also [37] Definition 2.1.5). By ([39] Lemma 15.6) elements of $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ are represented by triples $[A, g, B]$ where A, B are finitely generated, projective $\mathbf{Z}_l[G]$-modules and g is a $\mathbf{Q}_l[G]$-module isomorphism of the form $g : A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \stackrel{\cong}{\to} B \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$. subject to relations described in [39]. This group fits into a localisation sequence of the form ([29] §5 Theorem 5; [12] p.233)

$$K_1(\mathbf{Z}_l[G]) \xrightarrow{f_*} K_1(\mathbf{Q}_l[G]) \xrightarrow{\partial} K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \xrightarrow{\pi} K_0(\mathbf{Z}_l[G]) \xrightarrow{f_*} K_0(\mathbf{Q}_l[G]).$$

Assume now that G is abelian. In this case $K_1(\mathbf{Q}_l[G]) \cong \mathbf{Q}_l[G]^*$ because $\mathbf{Q}_l[G]$ is a product of fields and $K_1(\mathbf{Z}_l[G]) \cong \mathbf{Z}_l[G]^*$ ([5]I p.179 Theorem (46.24)). Under these isomorphisms f_* is identified with the canonical inclusion.

The homomorphism, $K_0(\mathbf{Z}_l[G]) \xrightarrow{f_*} K_0(\mathbf{Q}_l[G])$, is injective for all finite groups G ([34] Theorem 34 p.131; [5]II p.47 Theorem 39.10). Alternatively, when G is abelian, $\mathbf{Z}_l[G]$ is semi-local and the injectivity of f_* follows from the fact that a finitely generated projective module over a local ring is free ([8] p.124 Corollary 4.8 and p.205 Exercise 7.2). Thus the localisation sequence yields an isomorphism of the form

$$K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \cong \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

when G is abelian. From the explicit description of ∂ ([39] p.216) this isomorphism sends the coset of $\alpha \in \mathbf{Q}_l[G]^*$ to $[\mathbf{Z}_l[G], (\alpha \cdot -), \mathbf{Z}_l[G]]$. The inverse isomorphism sends $[A, g, B]$, where A and B may be assumed to be free $\mathbf{Z}_l[G]$-modules, to the coset of $det(g) \in \mathbf{Q}_l[G]^*$ with respect to any choice of $\mathbf{Z}_l[G]$-bases for A and B.

Example 2.2 We shall be particularly interested in the following source of elements of $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$.

As in §2.1, let l be a prime and let G be a finite abelian group. Suppose that

$$0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a bounded complex of finitely generated, projective $\mathbf{Z}_l[G]$-modules (i.e. a *perfect* complex of $\mathbf{Z}_l[G]$-modules), having all its homology groups finite.

As usual, let $Z_t = \mathrm{Ker}(d_t : F_t \longrightarrow F_{t-1})$ and $B_t = d_{t+1}(F_{t+1}) \subseteq F_t$ denote the $\mathbf{Z}_l[G]$-modules of t-dimensional cycles and boundaries, respectively. We have short exact sequences of the form

$$0 \longrightarrow B_i \xrightarrow{\phi_i} Z_i \longrightarrow H_i(F_*) \longrightarrow 0$$

and

$$0 \longrightarrow Z_{i+1} \xrightarrow{\psi_{i+1}} F_{i+1} \xrightarrow{d_{i+1}} B_i \longrightarrow 0.$$

Applying $(- \otimes \mathbf{Q}_l)$ we obtain isomorphisms

$$\phi_i : B_i \otimes \mathbf{Q}_l \xrightarrow{\cong} Z_i \otimes \mathbf{Q}_l$$

and we may choose $\mathbf{Q}_l[G]$-module splittings of the form

$$\eta_i : B_i \otimes \mathbf{Q}_l \longrightarrow F_{i+1} \otimes \mathbf{Q}_l$$

such that $(d_{i+1} \otimes 1)\eta_i = 1 : B_i \otimes \mathbf{Q}_l \longrightarrow B_i \otimes \mathbf{Q}_l$.

Then we form a $\mathbf{Q}_l[G]$-module isomorphism of the form

$$X : \oplus_j F_{2j} \otimes \mathbf{Q}_l \xrightarrow{\cong} \oplus_j F_{2j+1} \otimes \mathbf{Q}_l$$

given by the composition

$$\oplus_j F_{2j} \otimes \mathbf{Q}_l \xrightarrow{\oplus_j\ (\psi_{2j}+\eta_{2j-1})^{-1}} \oplus_j (Z_{2j} \otimes \mathbf{Q}_l) \oplus (B_{2j-1} \otimes \mathbf{Q}_l)$$

$$\xrightarrow{\oplus_j\ (\phi_{2j}^{-1} \oplus \phi_{2j-1})} \oplus_j (B_{2j} \otimes \mathbf{Q}_l) \oplus (Z_{2j-1} \otimes \mathbf{Q}_l)$$

$$\xrightarrow{(\oplus_j\ \eta_{2j}+\psi_{2j-1})} \oplus_j F_{2j-1} \otimes \mathbf{Q}_l.$$

If $w_i \in F_i \otimes \mathbf{Q}_l$ then X is given explicitly by the formula

$$X(w_0, w_2, \dots)$$

$$= (\eta_0(w_0) + d_2(w_2), \eta_2(w_2 - \eta_1(d_2(w_2))) + d_4(w_4), \dots,$$

$$\dots, \eta_{2t}(w_{2t} - \eta_{2t-1}(d_{2t}(w_{2t}))) + d_{2t+2}(w_{2t+2}), \dots).$$

This construction defines a class, $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}]$, in $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ which is well-known to be independent of the choices of the splittings used to define X ([39] Ch. 15; see also [37] Propositions 2.5.35 and 7.1.8).

We shall denote by

$$det(X) \in \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

the element which corresponds to $[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ under the isomorphism of §2.1.

2.3 Recall that, if R is a ring and M a (left) R-module, the (left) annihilator ideal $\mathrm{ann}_R(M) \lhd R$ of M is defined to be

$$\mathrm{ann}_R(M) = \{r \in R \mid r \cdot m = 0 \text{ for all } m \in M\}.$$

Let us recall from ([26] Appendix; see also [40]) the properties of the Fitting ideal (referred to as the initial Fitting invariant in [28]).

Let R be a commutative ring with identity and let M be a finitely presented R-module, in our applications M will actually be finite. Suppose that M has a presentation of the form

$$R^a \xrightarrow{f} R^b \longrightarrow M \longrightarrow 0$$

with $a \geq b$ then the Fitting ideal of the R-module M, denoted by $F_R(M)$, is the ideal of R generated by all $b \times b$ minors of any matrix representing f. The Fitting ideal $F_R(M)$ is independent of the presentation chosen for M.

The following result yields relations between the annihilator ideals and Fitting ideals of the homology modules in Example 2.2 in the special case when each $H_i(F_*)$ is finite and zero except for $i = 0, 1$.

Theorem 2.4 *([38] Theorem 2.4; see also [37] Chapter VII)*

Let G be a finite abelian group and l a prime. Suppose that

$$0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a bounded, perfect complex of $\mathbf{Z}_l[G]$-modules, as in Example 2.2, having $H_i(F_*)$ finite for $i = 0, 1$ and zero otherwise. Let

$$[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \cong \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

be as in Example 2.2. Then:

(i) if $t_i \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_i(F_*))$,

$$det(X)^{(-1)^i} t_i^{m_i} \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_{1-i}(F_*)) \lhd \mathbf{Z}_l[G]$$

for $i = 0, 1$. Here m_0, m_1 is the minimal number of generators required for the $\mathbf{Z}_l[G]$-module $H_0(F_*)$, $Hom(H_1(F_*), \mathbf{Q}_l/\mathbf{Z}_l)$, respectively,

(ii) if the Sylow l-subgroup of G is cyclic then in (i) $\mathrm{ann}_{\mathbf{Z}_l[G]}(H_{1-i}(F_*))$ may be replaced by $F_{\mathbf{Z}_l[G]}(H_{1-i}(F_*))$.

2.5 Shortening perfect complexes

Now consider once more the situation of Example 2.2. Let G be a finite abelian group and l a prime. Suppose that

$$0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} F_{k-2} \xrightarrow{d_{k-2}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a bounded perfect complex of $\mathbf{Z}_l[G]$-modules with $H_i(F_*)$ finite for each i. Let $t_0 \in \mathbf{Q}_l[G]^* \bigcap \mathrm{ann}_{\mathbf{Z}_l[G]}(H_0(F_*))$. As before, choose $\mathbf{Q}_l[G]$-modules splittings

$$\eta_i : B_i \otimes \mathbf{Q}_l \longrightarrow F_{i+1} \otimes \mathbf{Q}_l$$

so that $(d_{i+1} \otimes 1)\eta_i = 1$. Thus we obtain a $\mathbf{Q}_l[G]$-module isomorphism of the form

$$X : \oplus_i F_{2i} \otimes \mathbf{Q}_l \xrightarrow{\cong} \oplus_i F_{2i+1} \otimes \mathbf{Q}_l$$

given, if $w_i \in F_i \otimes \mathbf{Q}_l$, by

$$X(w_0, w_2, \ldots)$$

$$= (\eta_0(w_0) + d_2(w_2), \eta_2(w_2 - \eta_1(d_2(w_2))) + d_4(w_4), \ldots,$$

$$\ldots, \eta_{2t}(w_{2t} - \eta_{2t-1}(d_{2t}(w_{2t}))) + d_{2t+2}(w_{2t+2}), \ldots).$$

Assuming that F_0 is free if necessary, choose η_0 so that

$$\eta_0(t_0 \cdot -) = t_0\eta_0 : F_0 \subset F_0 \otimes \mathbf{Q}_l = B_0 \otimes \mathbf{Q}_l \longrightarrow F_1 \otimes \mathbf{Q}_l$$

lands in F_1. Now form the chain complex

$$P_* : 0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} F_{k-2} \xrightarrow{d_{k-2}} \ldots \xrightarrow{(d_3,0)} F_2 \oplus F_0 \xrightarrow{d_2 + t_0\eta_0} F_1 \longrightarrow 0.$$

Write $P_{i-1} = F_i$ for $i = 1, 3, 4, \ldots, k$ and $P_1 = F_2 \oplus F_0$ with differentials $\tilde{d}_{i-1} = d_i$ for $i = 4, 5, \ldots$, $\tilde{d}_2 = (d_3, 0)$ and $\tilde{d}_1 = d_2 + t_0\eta_0$. For $i = 3, 4, 5, \ldots$ $Ker(\tilde{d}_{i-1}) = Z_i$. Also $\tilde{d}_2(w_2, w_0) = 0$ implies that $0 = d_2(w_2) + t_0\eta_0(w_0)$ so that $0 = d_1(d_2(w_2) + t_0\eta_0(w_0)) = t_0 w_0$, which implies that $w_0 = 0$ since $t_0 \in \mathbf{Q}_l[G]^*$. Hence $Ker(\tilde{d}_1) = Z_2 \subseteq F_2 \subseteq F_2 \oplus F_0$. The boundaries of P_* satisfy $\tilde{d}_j(P_j) = d_{j+1}(F_{j+1}) = B_j$ for $j = 3, 4, \ldots$ and $\tilde{d}_2(P_2) = d_3(F_3) = B_2 \subseteq F_2 \oplus F_0$ while $\tilde{d}_1(P_1) = B_1 + t_0\eta_0(F_0)$.

Therefore we may choose $\mathbf{Q}_l[G]$-modules splittings of the form

$$\tilde{\eta}_{j-1} : \tilde{d}_j(P_j) \otimes \mathbf{Q}_l = B_j \otimes \mathbf{Q}_l \xrightarrow{\eta_j} F_{j+1} \otimes \mathbf{Q}_l = P_j \otimes \mathbf{Q}_l$$

for $j = 2, 3, \ldots, k-1$. Finally define

$$\tilde{\eta}_0 : P_0 \otimes \mathbf{Q}_l = F_1 \otimes \mathbf{Q}_l \longrightarrow F_2 \otimes \mathbf{Q}_l \oplus F_2 \otimes \mathbf{Q}_l$$

on $z \in P_0 \otimes \mathbf{Q}_l$ by

$$\tilde{\eta}_0(z) = (\eta_1(z - \eta_0(d_1(z))), t_0^{-1} d_1(z))$$

which satisfies

$$\begin{aligned}\tilde{d}_1(\tilde{\eta}_0(z)) &= d_2(\eta_1(z - \eta_0(d_1(z)))) + t_0\eta_0(t_0^{-1}d_1(z)) \\ &= z - \eta_0(d_1(z)) + \eta_0(d_1(z)) \\ &= z,\end{aligned}$$

as required for a $\mathbf{Q}_l[G]$-module splitting.

Denote by Y the $\mathbf{Q}_l[G]$-module isomorphism

$$Y : \oplus_i \ P_{2i} \otimes \mathbf{Q}_l \xrightarrow{\cong} \oplus_i \ P_{2i+1} \otimes \mathbf{Q}_l$$

defined using the $\tilde{\eta}_i$'s and $\tilde{d}_i$'s. Now $P_{2i} = F_{2i+1}$ so for $w_{2i} = z_{2i+1} \in F_{2i+1}$ we have, in $\oplus_i \ F_{2i} \otimes \mathbf{Q}_l$,

$$\begin{aligned}&Y(z_1, z_3, \ldots) \\ &= Y(w_0, w_2, \ldots) \\ &= (\tilde{\eta}_0(w_0) + \tilde{d}_2(w_2), \tilde{\eta}_2(w_2 - \tilde{\eta}_1(\tilde{d}_2(w_2))) + \tilde{d}_4(w_4), \ldots, \\ &\qquad \ldots, \tilde{\eta}_{2t}(w_{2t} - \tilde{\eta}_{2t-1}(\tilde{d}_{2t}(w_{2t}))) + \tilde{d}_{2t+2}(w_{2t+2}), \ldots) \\ &= (t_0^{-1}d_1(z_1), \eta_1(z_1 - \eta_0(d_1(z_1))) + d_3(z_3), \ldots, \\ &\qquad \ldots, \eta_{2t+1}(z_{2t+1} - \eta_{2t}(d_{2t+1}(z_{2t+1}))) + d_{2t+3}(z_{2t+3}), \ldots).\end{aligned}$$

Therefore, if $y_{2i} \in F_{2i} \otimes \mathbf{Q}_l$,

$$\begin{aligned}&Y(X(y_0, y_2, \ldots)) \\ &= Y(\eta_0(y_0) + d_2(y_2), \eta_2(y_2 - \eta_1(d_2(y_2))) + d_4(w_4), \ldots, \\ &\qquad \ldots, \eta_{2t}(y_{2t} - \eta_{2t-1}(d_{2t}(y_{2t}))) + d_{2t+2}(y_{2t+2}), \ldots) \\ &= (t_0^{-1}d_1(\eta_0(y_0)) + t_0^{-1}d_1(d_2(y_2)), \eta_1(\eta_0(y_0) + d_2(y_2) - \eta_0(d_1(\eta_0(y_0) + d_2(y_2)))) \\ &\qquad + d_3(\eta_2(y_2 - \eta_1(d_2(y_2)))), \ldots) \\ &= (t_0^{-1}y_0, \eta_1(\eta_0(y_0) + d_2(y_2) - \eta_0(y_0)) + y_2 - \eta_1(d_2(y_2)), \ldots) \\ &= (t_0^{-1}y_0, \eta_1(d_2(y_2)) + y_2 - \eta_1(d_2(y_2)), \ldots) \\ &= (t_0^{-1}y_0, y_2, y_4, \ldots)\end{aligned}$$

so that

$$det(Y)det(X) = t_0^{-rank(F_0)} = t_0^{-m_0}$$

where, by modifying F_0 if necessary (see [38] Theorem 2.4(proof)), we may assume that $m_0 = rank_{\mathbf{Z}_l[G]}(F_0)$ is equal to the minimal number of generators of $H_0(F_*)$ as a $\mathbf{Z}_l[G]$-module.

2.6 The case of three homology groups

Now consider a perfect complex with only three non-zero, finite homology groups $H_j(F_*)$ for $j = 0, 1, 2$, in this case we may assume without changing the class in $K_0(\mathbf{Z}_l[G], \mathbf{Q}_l)$ that the complex has the form

$$F_* : \ 0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

with F_i free. The construction of §2.5 makes a new perfect complex of the form

$$P_* : \ 0 \longrightarrow F_3 \stackrel{(d_3,0)}{\longrightarrow} F_2 \oplus F_0 \stackrel{d_2+t_0\eta_0}{\longrightarrow} F_1 \longrightarrow 0$$

with $det(Y) = det(X)^{-1}t_0^{-m_0}$ where m_0 is the minimal number of generators of $H_0(F_*)$. Now, from the discussion of §2.5,

$$H_1(P_*) = \frac{Ker(d_2 + t_0\eta_0)}{d_3(F_3)} = \frac{Ker(d_2)}{d_3(F_3)} = H_2(F_*).$$

However

$$H_0(P_*) = \frac{F_1}{d_2(F_2) + t_0\eta_0(F_0)} = \frac{F_1}{B_1 + t_0\eta_0(F_0)}.$$

We have $Z_1 \bigcap (B_1 + t_0\eta_0(F_0)) = B_1$ so that

$$H_1(F_*) = \frac{Z_1}{B_1} = \frac{Z_1}{Z_1 \bigcap (B_1 + t_0\eta_0(F_0))} = \frac{Z_1 + t_0\eta_0(F_0))}{B_1 + t_0\eta_0(F_0)}.$$

Therefore we have a short exact sequence of the form

$$0 \longrightarrow H_1(F_*) \longrightarrow \frac{F_1}{B_1 + t_0\eta_0(F_0)} \longrightarrow \frac{F_1}{Z_1 + t_0\eta_0(F_0)} \longrightarrow 0.$$

Also d_1 induces an isomorphism of the form

$$d_1 : \frac{F_1}{Z_1 + t_0\eta_0(F_0)} \stackrel{\cong}{\longrightarrow} \frac{B_0}{t_0F_0}$$

so the the short exact sequence has the form

$$0 \longrightarrow H_1(F_*) \longrightarrow H_0(P_*) \longrightarrow \frac{B_0}{t_0F_0} \longrightarrow 0.$$

We may apply Theorem 2.4 to the shortened complex of §2.5 and §2.6 to obtain the following result.

Theorem 2.7

Let G be a finite abelian group and l a prime. Suppose that

$$0 \longrightarrow F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a bounded, perfect complex of $\mathbf{Z}_l[G]$-modules, as in Example 2.2, having $H_i(F_*)$ finite for $i = 0, 1, 2$ and zero otherwise. Let

$$[\oplus_j F_{2j}, X, \oplus_j F_{2j+1}] \in K_0(\mathbf{Z}_l[G], \mathbf{Q}_l) \cong \frac{\mathbf{Q}_l[G]^*}{\mathbf{Z}_l[G]^*}$$

be as in Example 2.2. Then:

(i) if $t_i \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_i(F_*))$ for $i = 0, 2$

$$det(X) t_0^{m_0} t_2^{m_2} \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_1(F_*)) \lhd \mathbf{Z}_l[G].$$

Here m_0, m_2 is the minimal number of generators required for the $\mathbf{Z}_l[G]$-module $H_0(F_*), Hom(H_2(F_*), \mathbf{Q}_l/\mathbf{Z}_l)$, respectively,

(ii) if the Sylow l-subgroup of G is cyclic then in (i) $\mathrm{ann}_{\mathbf{Z}_l[G]}(H_1(F_*))$ may be replaced by $F_{\mathbf{Z}_l[G]}(H_1(F_*))$.

Proof

Firstly, we may assume that $t_0 \in \mathbf{Q}_l[G]^* \bigcap \mathrm{ann}_{\mathbf{Z}_l[G]}(H_0(F_*))$ (see [38] Theorem 2.4(proof)) by replacing t_0 by $t_0 + N|H_0(F_*)||H_1(F_*)|$ for some suitably large integer N.

Let $t_2 \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_2(F_*)) = \mathrm{ann}_{\mathbf{Z}_l[G]}(H_1(P_*))$, in the notation of §2.6. Then we may apply Theorem 2.4 to conclude that

$$det(Y)^{-1} t_2^{m_2} \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_0(P_*)) \lhd \mathbf{Z}_l[G]$$

and, by §2.5, $det(Y)^{-1} t_2^{m_2} = det(X) t_0^{m_0} t_2^{m_2}$. From §2.6 $H_1(F_*) \subseteq H_0(P_*)$ and both are finite groups, since $t_0 \in \mathbf{Q}_l[G]^*$. This inclusion always implies that

$$\mathrm{ann}_{\mathbf{Z}_l[G]}(H_0(P_*)) \subseteq \mathrm{ann}_{\mathbf{Z}_l[G]}(H_1(F_*))$$

and, providing the l-Sylow subgroup of G is cyclic it also implies

$$F_{\mathbf{Z}_l[G]}(H_0(P_*)) \subseteq F_{\mathbf{Z}_l[G]}(H_1(F_*)),$$

as explained in ([38] Proposition 2.8). □

Remark 2.8 In the proof of Theorem 2.7 we have only used one of the annihilator relations on Theorem 2.4. However, we may modify the proof of ([38] Theorem 2.4) to obtain another relation. Applying the proof to P_* when $i = 0$ we replace F_1 by a free module which is the direct sum of the first modules in a minimal resolutions for $H_1(F_*)$ and $B_0/(t_0 F_0)$ and then replace multiplication by t_0 by the sum of multiplication by t_1 and t_0 on these summands. Then, under the assumption that $t_0 \in \mathbf{Q}_l[G]^* \bigcap \mathrm{ann}_{\mathbf{Z}_l[G]}(H_0(F_*))$, one obtains

$$det(X)^{-1} t_1^{m_1} t_0^{m(B_0/(t_0F_0))-m_0} \in \mathrm{ann}_{\mathbf{Z}_l[G]}(H_2(F_*))$$

where $m(B_0/(t_0F_0))$ is the minimal number of generators for the finite module $B_0/(t_0F_0)$. When the l-Sylow subgroup of G is cyclic we may once again replace the annihilator ideal by the Fitting ideal.

In the absence of a good estimate for $m(B_0/(t_0F_0))$ this relation does not appear to be very useful.

3 The invariant

3.1 In [16] the following situation is considered: $\mathcal{O}_k$ is an excellent, henselian discrete valuation ring with fraction field, k, and residue field, F. Then $\mathcal{O}_k[T]$ is a local ring with maximal ideal $\mathcal{M} =< T, \mathcal{P} = ker(\mathcal{O}_k \longrightarrow F) >$. Let $A = \mathcal{O}_k\{T\}$ denote the henselianisation of $\mathcal{O}_k[T]$ at $\mathcal{M}$ ([27] p.36). We also write $\mathcal{P} \lhd A$ for the kernel of $\mathcal{O}_k\{T\} \longrightarrow F\{T\}$ so that $\mathcal{P} \bigcap \mathcal{O}_k = \mathcal{P}$. If $X = Spec(A) - \{\mathcal{M}, \mathcal{P}\}$ and $P_1, P_2, \ldots, P_t$ is a non-empty finite set of primes of A different from $\mathcal{M}$ and $\mathcal{P}$ then we have an open immersion

$$U = SpecA - \{\mathcal{M}, \mathcal{P}, P_1, P_2, \ldots, P_t\} \stackrel{u}{\hookrightarrow} X \stackrel{i}{\hookleftarrow} Y = \{P_1, P_2, \ldots, P_t\}$$

where u is an open immersion and i is a closed immersion.

Let Λ be a field of positive characteristic different from that of F and let $\mathcal{F}$ be a locally constant sheaf of Λ-modules of finite rank on $U_{ét}$. Fix $\overline{k}$, an algebraic closure of k. Then [16] gives a formula for the Λ-dimension of the space of vanishing cycles

$$H^q(X \times_{Spec(k)} Spec(\overline{k}); u_!(\mathcal{F}))$$

in terms of the Kato-Swan conductor. In general this group is zero except when $q = 0, 1$ ([13] p.14) and is also zero when $q = 0$ in the above situation because $U \neq X$ ([16] p.631).

Now suppose that k is a complete discrete valuation field of finite cohomological dimension. For example, k might be an n-dimensional local field (see [10], [11]).

Consider $\mathrm{Spec}(A \otimes_{\mathcal{O}_k} k)$ and observe that $A \otimes_{\mathcal{O}_k} k = A(\mathcal{O}_k - \{0\})^{-1}$ so that the primes in this localisation come from localisation of primes in A. Hence the primes of $A \otimes_{\mathcal{O}_k} k$ correspond to the primes, $P \lhd A$, which satisfy $P \bigcap \mathcal{O}_k = \{0\}$. The primes $\mathcal{M}$ and $\mathcal{P}$ both localise to give the whole of $A \otimes_{\mathcal{O}_k} k$ since they contain $\mathcal{P}$. Any other prime ideal of $P \lhd A$ must have $P \bigcap \mathcal{O}_k$ equal to $\mathcal{P}$, $\mathcal{O}_k$ or $\{0\}$. The middle case means $P = A$, which is not allowed, the first case means that $P = \mathcal{P}$ or $\mathcal{M}$, since A is two-dimensional. Hence the topological space of $\mathrm{Spec}(A \otimes_{\mathcal{O}_k} k)$ is equal to $X = \mathrm{Spec}(A) - \{\mathcal{M}, \mathcal{P}\}$ and it is clear from the definitions that, for all open $W \subseteq X$, $\mathcal{O}_X(W) = \mathcal{O}(W)$ where $\mathcal{O}$ is the sheaf of $\mathrm{Spec}(A)$.

Therefore we have

$$U = SpecA - \{\mathcal{M}, \mathcal{P}, P_1, P_2, \ldots, P_t\} \overset{u}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y = \{P_1, P_2, \ldots, P_t\}$$

where u is an open immersion and i is a closed immersion. Here P_i is a prime ideal of A which is not maximal and such that $\mathcal{O}_k \bigcap P_i = \{0\}$. The structure sheaf on U, for all open $W \subseteq U$, $\mathcal{O}_U(W) = \mathcal{O}(W)$ where $\mathcal{O}$ is the sheaf of Spec(A). Then i corresponds to the closed immersions induced by the local ring homomorphisms $A(\mathcal{O}_k - \{0\})^{-1} \longrightarrow A/P_i$ and $Y = \text{Spec}(\prod_{i=1}^t A/P_i)$. Since i corresponds to the sum of local ring homomorphisms of the form $A(\mathcal{O}_k - \{0\})^{-1} \longrightarrow A/P_i$ we see that we must have multiplication by any $\lambda \in \mathcal{O}_k - \{0\}$ inducing an isomorphism on A/P_i or, equivalently, $A = A\lambda + P_i \lhd A$ for each $1 \leq i \leq t$.

We have a diagram of homomorphisms of k-algebras

$$\begin{array}{ccc} \text{k} & \longrightarrow & \overline{k} \\ \downarrow & & \downarrow \\ A \otimes_{\mathcal{O}_k} k & \xrightarrow{f} & A \otimes_{\mathcal{O}_k} \overline{k} \end{array}$$

We are studying the vanishing cycle cohomology groups

$$H^q(X \times_{\text{Spec}(k)} \text{Spec}(\overline{k}); u_!(\mathcal{F})).$$

There is an action on this cohomology of the absolute Galois group, $G(\overline{k}/k)$ (cf. [20]).

For each finite Galois extension, $k \subseteq L \subset \overline{k}$, write $A_L = A \otimes_{\mathcal{O}_k} L$ and $B_L = A \otimes_{\mathcal{O}_k} \mathcal{O}_L$. Since $\mathcal{O}_L$ is a finite $\mathcal{O}_k$-module there is a decomposition ([27] Theorem 4.2 p.32)

$$B_L \cong \prod_{\mathcal{M} \lhd B_L,\ \mathcal{M} \text{ maximal}} B_L(B_L - \mathcal{M})^{-1}.$$

Since A was a Hensel local ring so is each of the factors and furthermore

$$A_L = B_L \otimes_{\mathcal{O}_L} L \cong \prod_{\mathcal{M} \lhd B_L,\ \mathcal{M} \text{ maximal}} B_L(B_L - \mathcal{M})^{-1} \otimes_{\mathcal{O}_L} L$$

in which each of the factors is a Dedekind domain. Therefore

$$A \otimes_{\mathcal{O}_k} \overline{k} = \lim_{\substack{\rightarrow \\ k \subseteq L \subset \overline{k}}} B_L \otimes_{\mathcal{O}_L} L$$

so that

$$H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})) \cong \lim_{\substack{\rightarrow \\ k \subseteq L \subset \overline{k}}} H^1(\mathrm{Spec}(B_L \otimes_{\mathcal{O}_L} L)); u_!(\mathcal{F}))$$

which is a continuous $\Lambda[G(\overline{k}/k)]$-module in the sense of [33].

Choose a bounded resolution of $H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))$

$$0 \to H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})) \to I^0 \to I^1 \to I^2 \to \ldots \to I^m \to 0$$

by cohomologically trivial $\Lambda[G(\overline{k}/k)]$-modules, which may be taken to be finitely generated if necessary.

Form the cochain complex of $G(\overline{k}/L)$-invariants

$$(I^\bullet)^{G(\overline{k}/L)} : \quad 0 \to (I^0)^{G(\overline{k}/L)} \to (I^1)^{G(\overline{k}/L)} \to \ldots \to (I^m)^{G(\overline{k}/L)} \to 0,$$

which is a complex of cohomologically trivial $\Lambda[G(L/k)]$-modules (which may be taken to be finitely generated if necessary). From the spectral sequence

$$E_2^{s,t} = H^s(L; H^t(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))) \Longrightarrow H^{s+t}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F}))$$

and the fact that $H^t(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})) = 0$ for $t \neq 1$ we find that the s-th cohomology group of this complex is equal to

$$H^{s+1}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})).$$

This is a finite-dimensional Λ-vector space.

Also the complex $(I^\bullet)^{G(\overline{k}/L)} \otimes \mathbf{Q}$ is exact.

Suppose that $\Lambda = \mathbf{F}_{l^e}$, the field with l^e elements for some prime l. In this case $(I^\bullet)^{G(\overline{k}/L)}$ is homotopy equivalent to a complex of finitely generated, cohomologically trivial $\mathbf{Z}_l[G(L/k)]$-modules

$$P^\bullet : \quad 0 \to P^u \to P^{u+1} \to \ldots \to P^v \to 0.$$

This complex defines an Euler characteristic in the relative K-group

$$\chi(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})) = [P^{od}, \Phi, P^{ev}] \in K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l)$$

by the construction described in Example 2.2 (see also [37] §2.1.8).

The following result is standard (see [37]).

Theorem 3.2

The element

$$\chi(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})) = [P^{od}, \Phi, P^{ev}] \in K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l)$$

is independent of all choices made in its construction. Furthermore its image under the canonical homomorphism

$$K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l) \xrightarrow{\pi} K_0(\mathbf{Z}_l[G(L/k)]) \xrightarrow{c} G_0(\mathbf{Z}_l[G(L/k)])$$

is equal to ([37] §2.1.8)

$$\sum_s (-1)^s \, H^s(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})).$$

Here π is the homomorphism in the localisation sequence and c is the Cartan map.

4 The case when k is a local field

4.1 In this section we shall makes a few prefatory remarks about the situation which is the intersection of Theorem 2.7 and Theorem 3.2. Consider the situation in which k is a (one-dimensional) local field of residue characteristic p and $\Lambda = \mathbf{F}_{l^e}$ where p and l are distinct primes. In this case, for each finite Galois extension L/k in §3.1 the group

$$H^{s+1}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})) \cong H^s(L; H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})))$$

can only possibly be non-zero for $s = 0, 1, 2$. Furthermore, if N is the finite Galois extension of L such that $G(\overline{k}/N) = G_1(\overline{k}/L)$, the first wild ramification subgroup of $G(\overline{k}/L)$, we have an isomorphism

$$H^s(L; H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})))$$

$$\cong H^s(G(N/L); H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))^{G(\overline{k}/N)})$$

because $G(\overline{k}/N)$ is a pro-p-group while $H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))$ is a finite group of order prime to p. Also, replacing L by N, we have

$$H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))^{G(\overline{k}/N)} \cong H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(N); u_!(\mathcal{F})).$$

Note that this group may be re-expressed as a vanishing cyclic group in the manner of ([16] foot of p.655).

Now let us make the further assumption that $G(L/k)$ is abelian. In this case we may apply Theorem 2.7 to the invariant of Theorem 3.2 to obtain the following result.

Theorem 4.2

Suppose that $G(L/k)$ is abelian in Theorem 3.2 and that

$$\chi(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})) = [P^{od}, \Phi, P^{ev}] \in K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l) \cong \frac{\mathbf{Q}_l[G(L/k)]^*}{\mathbf{Z}_l[G(L/k)]^*}$$

is represented by $det(\Phi) \in \mathbf{Q}_l[G(L/k)]^*$. Then, for any

$$t_j \in \mathrm{ann}_{\mathbf{Z}_l[G(L/k)]}(H^j(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})))$$

($j = 1, 3$), we have

(i) $det(\Phi) t_1^{m_1} t_3^{m_3} \in \mathrm{ann}_{\mathbf{Z}_l[G(L/k)]}(H^2(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F}))) \triangleleft \mathbf{Z}_l[G(L/k)]$.

Here m_1, m_3 is the minimal number of generators required for the $\mathbf{Z}_l[G(L/k)]$-module $H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F}))$, $Hom(H^3(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})), \mathbf{Q}_l/\mathbf{Z}_l)$, respectively,

(ii) if the Sylow l-subgroup of $G(L/k)$ is cyclic then in (i)

$$\mathrm{ann}_{\mathbf{Z}_l[G(L/k)]}(H^2(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})))$$

may be replaced by $F_{\mathbf{Z}_l[G(L/k)]}(H^2(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F})))$.

4.3 The annihilator relation of Theorem 4.2 becomes interesting only when we can determine the identity of $det(\Phi)$. I hope to return to this topic with some more convincing examples worked out in detail. However, for the moment, I shall content myself with recounting a few observations concerning the nature of $det(\Phi)$.

Firstly we may ask what sort of data determines the invariant $\chi(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F}))$? This question is too hard! On the other hand we may ask the same question about various images of the invariant in other class-groups. We may use the isomorphism

$$K_0(\mathbf{Z}[G(L/k)], \mathbf{Q}) \cong \oplus_{s \text{ prime}} K_0(\mathbf{Z}_s[G(L/k)], \mathbf{Q}_s)$$

to consider $\chi(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L); u_!(\mathcal{F}))$ as an element of $K_0(\mathbf{Z}[G(L/k)], \mathbf{Q})$ whose s-component is trivial when $s \neq l$. Now let $\mathcal{M}_{max}$ denote the maximal $\mathbf{Z}$-order of $\mathbf{Q}[G(L/k)]$ and consider the image of the invariant under the canonical homomorphism

$$K_0(\mathbf{Z}[G(L/k)], \mathbf{Q}) \longrightarrow K_0(\mathbf{Z}[G(L/k)]) \longrightarrow K_0(\mathcal{M}_{max}).$$

One way in which to distinguish classes in $K_0(\mathcal{M}_{max})$ which are images from $K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l)$ is the canonical factorisability method of [14] (see also [35] and [36]). Every class in $K_0(\mathbf{Z}_l[G(L/k)], \mathbf{Q}_l)$ may be represented by a class $[P', \lambda, P'']$ where P', P'' are two finitely generated, projective $\mathbf{Z}_l[G(L/k)]$-modules and $\lambda : P' \longrightarrow P''$ is an inclusion with finite cokernel. If $Coker(\lambda) = M$, say, then M is a finite l-group which is cohomologically trivial as a $\mathbf{Z}[G(L/k)]$-module. The image of $[P', \lambda, P'']$ in $K_0(\mathcal{M}_{max})$ is distinguished by the following fixed-pixed data. Let $C \subseteq G(L/K)$ be any cyclic subgroup and write $C = C_l \times C'$ where C_l is the l-Sylow subgroup of C. Let $e \in \mathbf{Z}_l[C']$ denote an irreducible idempotent. Then M^{C_l} is a finite $\mathbf{Z}_l[C']$-module and so is eM^{C_l}. By ([14]; [35] Theorem 7.3.12; [36] Theorem 5.2.2) the function on pairs (C, e) given by the formula

$$\rho_M : (C, e) \mapsto |eM^{C_l}|$$

has the propoerty that $[P', \lambda, P'']$ and $[P_0', \lambda_0, P_0'']$ have the same image in $K_0(\mathcal{M}_{max})$ if and only if $\rho_{Coker(\lambda)} = \rho_{Coker(\lambda_0)}$.

In our situation M will be a vector space over the finite field Λ so that ρ_M is equivalent to the function $(C, e) \mapsto \mathrm{dim}_\Lambda(\mathrm{eM}^{\mathrm{C_l}})$. This suggests that the formula of Deligne and Kato ([16], [22]) for

$$\mathrm{dim}_\Lambda(\mathrm{H}^1(\mathrm{X} \times_{\mathrm{Spec(k)}} \mathrm{Spec}(\overline{\mathrm{k}}); \mathrm{u}_!(\mathcal{F})))$$

gives an important clue to the nature of $det(\Phi)$. The results of [16] and [22] were recently proved by a different method in [24].

4.4 The Deligne-Kato dimension formula

In the situation of §3.1 a triple $(A, U, \mathcal{F})$ will be called *stable* if it satisfies the following three conditions (see [16] (5.1.1) and (6.1.1)-(6.1.2)):

(i) There exists an étale connected Galois covering $\tilde{U}$ of U such that the inverse image of $\mathcal{F}$ on $\tilde{U}$ is a constant sheaf.

(ii) The integral closure B of A in $\tilde{U}$ is a two-dimensional hensel normal local ring over $\mathcal{O}_k$. If $\mathcal{M}_B$ and $\mathcal{M}_k$ are the maximal ideals of B and $\mathcal{O}_k$, respectively, then B is isomorphic over $\mathcal{O}_k$ to the strict henselianisation of $\mathcal{O}_{X,x}$ for some scheme X of finite type over $\mathcal{O}_k$ and for some closed point x lying over $\mathcal{M}_k$ such that $X - \{x\}$ is smooth over $\mathcal{O}_k$.

(iii) The residue fields $\kappa(\mathcal{P})$ for all $\mathcal{P} \in \mathrm{Spec}(\mathrm{B} \otimes_{\mathcal{O}_k} \mathrm{k}) - \tilde{\mathrm{U}}$ are separable over k.

For any $(A, U, \mathcal{F})$, *not necessarily* satisfying conditions (i)-(iii) the dimension formula ([16] Theorem 6.7) takes the form

$$dim_\Lambda(H^0(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))) - dim_\Lambda(H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F})))$$

$$= \phi(s) - \phi(\eta) - 2\delta_k dim_\Lambda(\mathcal{F}).$$

In the case of interest to us $U \neq X$ and ([16] p.631) the formulae simplifies to the form

$$dim_\Lambda(H^1(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}); u_!(\mathcal{F}))) = \phi(\eta) - \phi(s) + 2\delta_k dim_\Lambda(\mathcal{F}).$$

The ingredients in this formula are defined in the following manner, where the subtlety of [16] lies in the definition of $\phi(s)$. Let $P(A)$ denote the set of height one primes $\mathcal{P}$ of A such that $\mathcal{P} \notin U$. Define $P_s(A)$ to be the finite set

$$P_s(A) = \{\mathcal{P} \in P(A) \mid \mathcal{P} \bigcap \mathcal{O}_k = \mathcal{M}_k\}$$

and set $P_\eta(A) = P(A) - P_s(A)$, which may be identified with the set of maximal ideals of $A \otimes_{\mathcal{O}_k} k = A_k$. For $\mathcal{P} \in P_\eta(A)$ define

$$dimtot_{\mathcal{P},k}(\mathcal{F}) = [\kappa(\mathcal{P}) : k](sw_{\mathcal{P}}(\mathcal{F}) + dim_\Lambda(\mathcal{F}))$$

where $sw_{\mathcal{P}}(\mathcal{F})$ is the Swan conductor of $\mathcal{F}$ for the discrete valuation ring $A_{\mathcal{P}}$.

For a triple $(A, U, \mathcal{F})$ take any finite extension k'/k such that the associated extension $(A', U', \mathcal{F}')$ is stable (i.e. satisfies (i)-(iii)), which is possible by a result of [9] ([16] Proposition 6.3). With these choices let

$$\phi(\eta) = \sum_{\mathcal{P} \in P_\eta(A'),\ \mathcal{P} \notin U'} dimtot_{\mathcal{P},k'}(\mathcal{F}').$$

Similarly $\phi(s)$ is defined by a formula

$$\phi(s) = \sum_{\mathcal{P} \in P_s(A'),\ \mathcal{P} \notin U'} dimtot_{\mathcal{P},k'}(\mathcal{F}').$$

Here the terms $dimtot_{\mathcal{P},k'}(\mathcal{F}')$ in the definition of $\phi(s)$ are definition in terms of the Swan conductor ([16] §3) of the Galois representation corresponding to $\mathcal{F}$ of the two-dimensional discrete valuation ring $V^h(\mathcal{P})$ given by the henselianisation of the subring of $A_{\mathcal{P}}$ consisting of the elements whose images in $\kappa(\mathcal{P})$ lie in the normalisation of $A/\mathcal{P}$.

The integer δ_k is also defined in terms of the valuation theory of the two-dimensional rings $V^h(\mathcal{P})$ ([16] §5).

Question 4.5 This discussion of §4.3 and §4.4 suggests, to me at least, the following question: In the situation of Theorem 4.2 is there a formula for

$$det(\Phi) \in \frac{\mathbf{Q}_l[G(L/k)]^*}{\mathbf{Z}_l[G(L/k)]^*}$$

in terms of the functions $\phi(\eta), \phi(s), \delta_{k'}$ as k' varies through suitable extensions of k or L?

Remark 4.6 Á propos of Question 4.5, the Swan conductor which underlies the functions $\phi(s)$ and $\phi(\eta)$ may be thought of as a function on Galois representations ([18], [16], [4], see also [35] Chapter 6). Therefore, in order to make sense of the formula for $det(\Phi)$ in Question 4.5, it may be useful to recall that $\mathbf{Q}_l[G(L/k)]^*$ is also describeable in terms of functions on representations. Let $\Omega_{\mathbf{Q}_l}$ denote the absolute Galois group of $\mathbf{Q}_l$ and let $R(G(L/K))$ denote the Grothendieck group of finite-dimensional representations of $G(L/K)$ defined over $\overline{\mathbf{Q}}_l$, the algebraic closure of $\mathbf{Q}_l$. Then there is an isomorphism of the form [38]

$$\lambda : \mathrm{Hom}_{\Omega_{\mathbf{Q}_l}}(R(G(L/k)), \overline{\mathbf{Q}}_l^*) \xrightarrow{\cong} \mathbf{Q}_l[G]^*$$

given by

$$\lambda(h) = \sum_{\chi:G(L/k)\to\overline{\mathbf{Q}}_l^*} h(\chi)e_\chi$$

where

$$e_\chi = [L:k]^{-1} \sum_{g\in G(L/k)} \chi(g)g^{-1} \in \overline{\mathbf{Q}}_l[G(L/k)].$$

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