The vertex degree distribution of random intersection graphs

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Abstract

Random intersection graphs are a model of random graphs in which each vertex is assigned a subset of a set of objects independently and two vertices are adjacent if their assigned subsets are not disjoint. The number of vertices is denoted by n and the number of objects is supposed to be $\lfloor n^\alpha \rfloor$ for some $\alpha > 0$. We determine the distribution of the degree of a typical vertex and show that it changes sharply between $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$.

1 Introduction

Random intersection graphs were introduced in [4]. Given a set V of n vertices and a set W of m objects, define a bipartite graph $\mathcal{G}^*(n, m, p)$ with independent vertex sets V and W and edges between $v \in V$ and $w \in W$ existing independently and with probability p. The random intersection graph $\mathcal{G}(n, m, p)$ derived from $\mathcal{G}^*(n, m, p)$ is defined on the vertex set V with vertices $v_1, v_2 \in V$ adjacent if and only if there exists some $w \in W$ such that both v_1 and v_2 are adjacent to w in $\mathcal{G}^*(n, m, p)$. We may interpret the vertices of $W_v \subset W$ adjacent to $v \in V$ as a random subset of W, in which case two vertices $v_1, v_2 \in V$ are adjacent iff $W_{v_1} \cap W_{v_2} \neq \emptyset$.

The properties of $\mathcal{G}(n, m, p)$ were studied in [1, 4] and contrasted with the well known random graph model $\mathcal{G}(n, p)$, in which vertices are made adjacent to each other independently and with probability p. In [1, 4] the number of objects m is taken to be $m = \lfloor n^\alpha \rfloor$ for a fixed $\alpha > 0$. It is found in [4] that the thresholds for the existence of small subgraphs in $\mathcal{G}(n, m, p)$ show different behaviors from what is seen in $\mathcal{G}(n, p)$. When $\alpha > 6$ [1] showed that the total variation distance between the distributions of $\mathcal{G}(n, m, \hat{p})$ and $\mathcal{G}(n, p)$ converges to 0 when $\hat{p}$ is defined appropriately.

Intersection graphs can be viewed as relationship graphs. For example, if V represents mathematicians and W represents mathematics papers, and an edge is put between $v \in V$ and $w \in W$ iff mathematician v was an author of paper w, then the resulting intersection graph is the collaboration graph on V, where two mathematicians are connected by an edge iff they have written a paper together. The roles of V and W can be interchanged, in which case two papers are adjacent iff they have an author in common. The random graph $\mathcal{G}(n, m, p)$ is dual in this way to $\mathcal{G}(m, n, p)$. If $m = \lfloor n^\alpha \rfloor$, then $n \in [m^{1/\alpha}, (m+1)^{1/\alpha})$ and so the dual of $\mathcal{G}(n, m, p)$ with $\alpha = \beta > 1$ is basically $\mathcal{G}(n, m, p)$ with $\alpha = \beta^{-1} < 1$. Data sets for relationship graphs and models of random intersection graphs with fixed degree sequences are analyzed in [5].

Interest was expressed in [1] in further understanding the differences between $\mathcal{G}(n, m, p)$ and $\mathcal{G}(n, p)$. In [1, 4] thresholds for various quantities were looked at, but not much attention was paid to limiting distributions. A fundamental quantity that has not been studied for random intersection graphs is the distribution of the degree of a typical vertex. We give the precise distribution for $\mathcal{G}(n, m, p)$ in the form of a probability generating function in Theorem 1. The corresponding distribution for $\mathcal{G}(n, p)$ is, of course, Binomial$(n-1, p)$.

Let $X = X(n, m, p)$ be the number of vertices $V - \{v\}$ adjacent in $\mathcal{G}(n, m, p)$ to a vertex $v \in V$. The probability generating function of $X(n, m, p)$ is defined to be $\mathbb{E}x^X = \sum_{k=0}^{\infty} \mathbb{P}(X = k)x^k$.

Theorem 1 *The probability generating function $F(x) = \mathbb{E}x^X$ is given by*

$$F(x) = \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} \Big[1 - p + p(1-p)^{n-1-j}\Big]^m.$$

Theorem 1 is proved by using a generating function version of the sieve method.

The expectation of X is given by

$$\mathbb{E}X = (n-1)\Big[1 - (1-p^2)^m\Big] \tag{1}$$

because the expression in square brackets is the probability that two vertices v, v_1 in V are simultaneously adjacent to some vertex $w \in W$ in $\mathcal{G}^*(n, m, p)$. The derivative $F'(1)$ also gives (1). If we let

$$p = \sqrt{c} n^{-(1+\alpha)/2}, \tag{2}$$

then

$$\mathbb{E}X = (n-1)\Big[1 - (1 - mp^2 + O(mp^4))\Big] = c + o(1).$$

With respect to vertex degree, defining $p = p(n)$ by (2) for $\mathcal{G}(n, p)$ is therefore analogous to defining $p = cn^{-1}$ for $\mathcal{G}(n, p)$.

The vertex degree distribution for $\mathcal{G}(n, p)$ converges to the Poisson distribution with parameter c as $n \to \infty$ when $p = cn^{-1}$. With $p = \sqrt{c} n^{-(1+\alpha)/2}$ the vertex degree distribution of $\mathcal{G}(n, m, p)$ converges to a Poisson distribution in the limit if and only if $\alpha > 1$. We say that X_n is asymptotically almost surely a_n if $\mathbb{P}(|X_n/a_n - 1| > \epsilon) \to 0$ as $n \to \infty$ for each $\epsilon > 0$.

Theorem 2 *Let $\mathcal{G}(n, m, p)$ denote the random intersection graph with $m = \lfloor n^\alpha \rfloor$ and $p = \sqrt{c} n^{-(1+\alpha)/2}$.*

(i) If $\alpha < 1$, then the number of non-isolated vertices is asymptotically almost surely $\sqrt{c} n^{(1+\alpha)/2} = o(n)$. It follows that the degree of a fixed vertex in V has a distribution which converges to δ_0, the probability distribution with all mass at 0.

(ii) If $\alpha = 1$, then the degree of a fixed vertex in V has a distribution which converges weakly to the compound Poisson distribution of the random variable $Z_1 + Z_2 + \cdots + Z_N$, where $N, Z_1, Z_2, \ldots$ are i.i.d. Poisson($\sqrt{c}$) random variables.

(iii) If $\mathcal{G}(n,m,p)$ with $\alpha > 1$ and $p = \sqrt{c}n^{-(1+\alpha)/2}$, then the degree of a fixed vertex has distribution which converges weakly to a Poisson limiting distribution with parameter c.

Theorem 2 can roughly be explained in the following way. When $\alpha < 1$ the probability in $\mathcal{G}^*(n,m,p)$ that any one vertex $v \in V$ is connected to a vertex in W goes to 0 and so the degree distribution in $\mathcal{G}(n,m,p)$ converges to δ_0 and most of the vertices are isolated. When $\alpha = 1$ a vertex $v \in V$ will have approximately a Poisson($\sqrt{c}$) number of neighbors in W and each of those neighbors have independently about Poisson($\sqrt{c}$) number of neighbors in V, not including v. When α is large enough, [1] shows that it becomes unlikely that a vertex $w \in W$ has more than two neighbors in V and as a result the events that different edges in $\mathcal{G}(n,m,p)$ exist become independent.

Theorem 3 shows that if $\alpha > 1$ and p grows faster than $n^{-(1+\alpha)/2}$, but not as fast as $\min\left(n^{-2/3-\alpha/3}, n^{-1/3-\alpha/2}\right)$, then X converges to normal when it is rescaled.

Theorem 3 *Let $\mathcal{G}(n,m,p)$ denote the random intersection graph with $m = \lfloor n^\alpha \rfloor$, suppose that $\alpha > 1$, and suppose that p satisfies $nmp^2 \to \infty$ and $p = o\left(n^{-2/3-\alpha/3}\right)$ if $1 < \alpha \leq 2$, $p = o\left(n^{-1/3-\alpha/2}\right)$ if $\alpha > 2$. Under these assumptions,*

$$\frac{X - \mathbb{E}X}{\sigma(X)} \Rightarrow N(0,1),$$

where $\sigma(X)$ is the standard deviation of X and $N(0,1)$ is the standard normal distribution.

The formula in Theorem 1 is derived in Section 2. In Section 3 Theorem 2 is proven for $\alpha < 1$ by using Chebyshev's inequality. Section 4 proves Theorem 2 for $\alpha \geq 1$ and Theorem 3 by analyzing the probability generating function found in Section 2.

2 The probability generating function

We will determine $F(x)$ by using Lemma 1, which is a probability generating function version of the sieve method. Lemma 1 is used, for example, by Takács in [7], though according to [3] it may also have been known to Jordan. For completeness we give a proof of Lemma 1. It is similar to an argument in

Section 4.2 of [8]. Let P be a set of properties that a random object can take on. Let p_k be the probability that the object takes on exactly k properties in P. We are interested in the probability generating function

$$F(x) = \sum_k p_k x^k.$$

Lemma 1 *For $S \subset P$, we define N_S to be the event that the random object possesses the properties S. Define N_r to be*

$$N_r = \sum_{|S|=r} \mathbb{P}(N_S)$$

and define $N(x)$ to be

$$N(x) = \sum_{r\geq 0} N_r x^r.$$

With the definitions above, we have

$$F(x) = N(x-1).$$

Proof The proof is similar to the argument in Section 4.2 of [8], but replacing certain summations with expectations. Let I_{N_S} be the indicator function of the event N_S. Let Y be the number of properties that the random object possesses. We have

$$N_r = \sum_{|S|=r} \mathbb{P}(N_S) = \sum_{|S|=r} \mathbb{E}(I_{N_S}) = \mathbb{E}\left(\sum_{|S|=r} I_{N_S}\right) = \mathbb{E}\binom{Y}{r}.$$

Therefore,

$$\begin{aligned} N(x) &= \sum_{r\geq 0} N_r x^r = \sum_{r\geq 0} \mathbb{E}\binom{Y}{r} x^r = \mathbb{E}\left(\sum_{r\geq 0}\binom{Y}{r} x^r\right) \\ &= \mathbb{E}\left((1+x)^Y\right) = F(1+x) \end{aligned}$$

and $F(x) = N(x-1)$. ∎

In our application to random intersection graphs, there are $n-1$ properties consisting of the non-adjacency of the fixed vertex to the other $n-1$

vertices. We use non-adjacency rather than adjacency in the initial analysis for ease of calculation; the generating function for the number of adjacent vertices follows immediately.

Proof of Theorem 1
Let $G(x)$ be the generating function of p'_k, the probability that exactly k vertices in $V - \{v\}$ are not adjacent to $v \in V$. The probability that the fixed vertex v is adjacent to none of the vertices represented by $S \subset V - \{v\}$ is given by

$$\mathbb{P}(N_S) = \sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} (1-p)^{k|S|},$$

where the index k counts the number of elements of W adjacent to v in $\mathcal{G}^*(n, m, p)$. Therefore,

$$\begin{aligned} N_r &= \binom{n-1}{r} \sum_{k=0}^{m} \binom{m}{k} \left[p(1-p)^r\right]^k (1-p)^{m-k} \\ &= \binom{n-1}{r} \Big[1 - p + p(1-p)^r\Big]^m. \end{aligned}$$

Lemma 1 implies that $G(x) = N(x-1)$, where

$$N(x) = \sum_{r=0}^{n-1} N_r x^r = \sum_{r=0}^{n-1} \binom{n-1}{r} x^r \Big[1 - p + p(1-p)^r\Big]^m.$$

Hence,

$$\begin{aligned} G(x) &= \sum_{r=0}^{n-1} \binom{n-1}{r} (x-1)^r \Big[1 - p + p(1-p)^r\Big]^m \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (x-1)^{n-1-j} \Big[1 - p + p(1-p)^{n-1-j}\Big]^m. \end{aligned}$$

Now use the identity $F(x) = x^{n-1} G\left(x^{-1}\right)$. ■

3 The number of isolated vertices

In this section we prove Theorem 2 for $\alpha < 1$.

Lemma 2 *Consider a random intersection graph $\mathcal{G}(n, m, p)$ with $\alpha < 1$ and $p = o(n^{-\alpha})$. Let Y be the number of non-isolated vertices. If $nmp \to \infty$, then asymptotically almost surely $Y \sim nmp$. In particular, if $\alpha < 1$ and $p = \sqrt{c}n^{-(1+\alpha)/2}$, then $Y \sim \sqrt{c}n^{(1+\alpha)/2} = o(n)$.*

Proof Write $W = \sum_{v \in V} I_v$, where I_v is the indicator that vertex $v \in V$ is isolated, so that the number of non-isolated vertices is $Y = n - W$. The probability that v is isolated is

$$\begin{aligned} \mathbb{E}I_v &= \sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} (1-p)^{(n-1)k} \\ &= \Big[1 - p + p(1-p)^{n-1}\Big]^m, \end{aligned}$$

where the index k represents the number of vertices in W which are adjacent to v in $\mathcal{G}^*(n, m, p)$. Hence

$$\mathbb{E}W = n\Big[1 - p + p(1-p)^{n-1}\Big]^m,$$

a formula computed in [6] by different means. When $\alpha < 1$ and $p = o(n^{-\alpha})$ the expectation of Y is

$$\begin{aligned} \mathbb{E}Y &= n - \mathbb{E}W \\ &= n - n\Big[1 - p + p(1-p)^{n-1}\Big]^m \\ &= n - n(1-p)^m \left(1 + O(mpe^{-(n-2)p})\right) \\ &= n - n(1-p)^m \left(1 + O(mp\exp\left(-n^{1-\alpha}\right))\right) \\ &= nmp + O(nm^2p^2). \end{aligned}$$

Next we calculate the variance of Y. We have

$$\begin{aligned} \mathbb{E}\, W(W-1) &= \sum_{v_2 \neq v_2} \mathbb{E}I_{v_1} I_{v_2} \\ &= n(n-1) \sum_{s=0}^{m} \binom{m}{s} (1-p)^{2s} [2p(1-p)]^{m-s} (1-p)^{(m-s)(n-2)} \\ &= n(n-1) \Big[(1-p)^2 + 2p(1-p)^{n-1}\Big]^m, \end{aligned}$$

where s counts the number of vertices in W adjacent to neither v_1 or v_2 in $\mathcal{G}^*(n,m,p)$, leaving $m-s$ vertices in W adjacent to exactly one of v_1, v_2. The factor $(1-p)^{(m-s)(n-2)}$ is the probability that none fo the $m-s$ vertices are adjacent to other vertices in V. Now,

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(W) \\
&= n(n-1)\Big[(1-p)^2 + 2p(1-p)^{n-1}\Big]^m + n\Big[1-p+p(1-p)^{n-1}\Big]^m \\
&\quad - \Big(n\Big[1-p+p(1-p)^{n-1}\Big]^m\Big)^2 \\
&= n(n-1)(1-p)^{2m}\left(1+O(\exp\left(-n^{1-\alpha}\right)\right) \\
&\quad +n(1-p)^m\left(1+O(\exp\left(-n^{1-\alpha}\right)\right) \\
&\quad -n^2(1-p)^{2m}\left(1+O(\exp\left(-n^{1-\alpha}\right)\right) \\
&= n(1-p)^m - n(1-2p+p^2)^m + O\left(n^2\exp\left(-n^{1-\alpha}\right)\right) \\
&= O(nmp).
\end{aligned}$$

An application of Chebyshev's inequality completes the proof. ■

4 Limit laws for the vertex degree

In this section we prove Theorem 2 for $\alpha \geq 1$ and Theorem 3.

Lemma 3 *Consider the random intersection graph $\mathcal{G}(n,m,p)$ with $m = \lfloor n^\alpha \rfloor$ and $p = \sqrt{c}n^{-(1+\alpha)/2}$ with $\alpha = 1$. When $x \leq 1$, the probability generating function $\sum_k \mathbb{P}(X(n,m,p)=k)x^k$ satisfies*

$$F(x) = \exp\left(-\sqrt{c} + \sqrt{c}e^{-\sqrt{c}(1-x)}\right) + O\left(n^{-1/4}\right)$$

It follows that the probability that a fixed vertex of V in $\mathcal{G}(n,m,p)$ equals $k \geq 0$ asymptotically approaches the compound Poisson distribution given by $\mathbb{P}(Z_1 + Z_2 + \cdots + Z_N = k)$, where $N, Z_1, Z_2, \ldots$ are i.i.d. Poisson($\sqrt{c}$) distributed random variables.

Proof Write the formula for $F(x)$ given by Theorem 1 for fixed $x \leq 1$ as

$$F(x) = \sum_{|j-nx| \leq n^{3/4}} \binom{n-1}{j} x^j (1-x)^{n-1-j}\Big[1-\sqrt{c}n^{-1} + \sqrt{c}n^{-1}(1-\sqrt{c}n^{-1})^{n-1-j}\Big]^n$$

$$+\sum_{|j-nx|>n^{3/4}} \binom{n-1}{j} x^j(1-x)^{n-1-j}\Big[1-\sqrt{c}n^{-1}+\sqrt{c}n^{-1}(1-\sqrt{c}n^{-1})^{n-1-j}\Big]^n.$$

The second sum is bounded by $\sum_{|j-nx|>n^{3/4}} \binom{n-1}{j} x^j(1-x)^{n-1-j}$, which is $o(1)$ by large deviation bounds for the binomial; see Theorem 2.1 of [2], for example.

As for the first sum, we have

$$(1-\sqrt{c}n^{-1})^{n-1-j} = e^{-\sqrt{c}(1-x)} + O\left(n^{-1/4}\right).$$

uniformly for all j such that $|j-nx| \leq n^{3/4}$. Hence,

$$\Big[1-\sqrt{c}n^{-1}+\sqrt{c}n^{-1}(1-\sqrt{c}n^{-1})^{n-1-j}\Big]^n = \exp\left(-\sqrt{c}+\sqrt{c}e^{-\sqrt{c}(1-x)}\right)+O\left(n^{-1/4}\right)$$

uniformly for all j such that $|j-nx| \leq n^{3/4}$ and

$$\begin{aligned}&\sum_{|j-nx|\leq n^{3/4}} \binom{n-1}{j} x^j(1-x)^{n-1-j}\Big[1-\sqrt{c}n^{-1}+\sqrt{c}n^{-1}(1-\sqrt{c}n^{-1})^{n-1-j}\Big]^n \\ = \; & \exp\left(-\sqrt{c}+\sqrt{c}e^{-\sqrt{c}(1-x)}\right)+O\left(n^{-1/4}\right).\end{aligned}$$

The Laplace transform $F(e^{-t})$ converges pointwise to $\exp\left(-\sqrt{c}+\sqrt{c}e^{-\sqrt{c}(1-e^{-t})}\right)$ which, as is easily checked, is the Laplace transform of $Z_1+Z_2+\cdots+Z_N$. ■

Lemma 4 *Consider the random intersection graph $\mathcal{G}(n,m,p)$ with $m = \lfloor n^\alpha \rfloor$ and $p = \sqrt{c}n^{-(1+\alpha)/2}$ with $\alpha > 1$. When $x \leq 1$, the probability generating function $\sum_k \mathbb{P}(X(n,m,p)=k)x^k$ satisfies*

$$F(x) = e^{-c+cx} + O\left(n^{-1}\right) + O\left(n^{(1-\alpha)/2}\right).$$

It follows that the probability p_k that a fixed vertex of V in $\mathcal{G}(n,m,p)$ equals $k \geq 0$ is asymptotically Poisson: $p_k \sim e^{-c}c^k/k!$.

Proof Expand $(1-p)^{n-1-j}$ for $j \in [1,n]$ as

$$(1-p)^{n-1-j} = 1-\sqrt{c}(n-1-j)n^{-(1+\alpha)/2} + O\left(n^{1-\alpha}\right).$$

It follows that

$$\begin{aligned}\left[1-p+p(1-p)^{n-1-j}\right]^m &= \left[1-c(n-1-j)n^{-1-\alpha}+O\left(n^{(1-3\alpha)/2}\right)\right]^m \\ &= \exp\left(-c(n-1-j)n^{-1}+O\left(n^{(1-\alpha)/2}\right)\right)\end{aligned}$$

uniformly for $j \in [1,n]$. Now, for $x \in [0,1]$,

$$\begin{aligned}F(x) &= \sum_{j=0}^{n-1}\binom{n-1}{j}x^j(1-x)^{n-1-j}e^{-c(n-1-j)/n}+O\left(n^{(1-\alpha)/2}\right) \\ &= e^{-c(n-1)/n}\left(1-x+xe^{c/n}\right)^{n-1}+O\left(n^{(1-\alpha)/2}\right) \\ &= e^{-c(n-1)/n}\exp\left(xc(n-1)n^{-1}+O\left(n^{-1}\right)\right)+O\left(n^{(1-\alpha)/2}\right) \\ &= e^{-c+cx}+O\left(n^{-1}\right)+O\left(n^{(1-\alpha)/2}\right).\end{aligned}$$

The Laplace transform $F(e^{-t})$ converges pointwise to $\exp\left(-c+ce^{-t}\right)$, the Laplace transform of the Poisson distribution with parameter c. ■

Proof of Theorem 3

Suppose that x_n is a sequence of complex numbers such that $|x_n| = O(1)$. By the assumptions on p we have $n^2mp^3 = o(1)$, $nm^2p^4 = o(1)$, $n^2m^2p^5 = o(1)$, and $n^2m^3p^6 = o(1)$. Therefore,

$$(1-p)^{n-1-j} = 1-p(n-1-j)+O(n^2p^2)$$

and

$$\begin{aligned}\left[1-p+p(1-p)^{n-1-j}\right]^m &= \left[1-p^2(n-1-j)+O(n^2p^3)\right]^m \\ &= \exp\left(-mp^2(n-1-j)+O(n^2mp^3)\right) \\ &= (1+o(1))\exp\left(-mp^2(n-1-j)\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}F(x_n) &= (1+o(1))\sum_{j=0}^{n-1}\binom{n-1}{j}x_n^j(1-x_n)^{n-1-j}e^{-mp^2(n-1-j)} \\ &= (1+o(1))e^{-mp^2(n-1)}\sum_{j=0}^{n-1}\binom{n-1}{j}(x_ne^{mp^2})^j(1-x_n)^{n-1-j}\end{aligned}$$

$$
\begin{aligned}
&= (1+o(1))e^{-mp^2(n-1)}(1-x_n+x_ne^{mp^2})^{n-1}\\
&= (1+o(1))e^{-mp^2(n-1)}(1+x_nmp^2+O(m^2p^4))^{n-1}\\
&= (1+o(1))e^{-\mu+\mu x_n},
\end{aligned}
$$

with $\mu = mp^2(n-1)$. The equality (1) shows that $\mathbb{E}X = \mu + o(1)$, Suppose that $\mu/\sigma(X)^2 \to 1$ as $n \to \infty$. Writing σ for $\sigma(X)$, the characteristic function of $(X - \mathbb{E}X)/\sigma$ is

$$
\begin{aligned}
e^{-it\mathbb{E}X/\sigma}F(e^{it/\sigma}) &= (1+o(1))e^{-it\mu/\sigma}F(e^{it/\sigma})\\
&= (1+o(1))e^{-it\mu/\sigma}\exp\big(-\mu+\mu e^{it/\sigma}\big)\\
&= (1+o(1))e^{-it\mu/\sigma}\exp\big(\mu it/\sigma - \mu t^2/(2\sigma^2) + O(\mu/\sigma^3)\big)\\
&= (1+o(1))e^{-t^2/2},
\end{aligned}
$$

which converges to the characteristic function of the standard normal distribution.

It remains to be shown that $\mu/\sigma^2 \to 1$. By Theorem 1, the second derivative of $F(x)$ at 1 equals

$$
F''(1) = \mathbb{E}X(X-1) = (n-1)(n-2)\Big[1-2(1-p^2)^m + (1-2p^2+p^3)^m\Big],
$$

from which

$$
\begin{aligned}
\sigma^2 &= (n-1)(n-2)\Big[1-2(1-p^2)^m + (1-2p^2+p^3)^m\Big]\\
&\quad + (n-1)\Big[1-(1-p^2)^m\Big] - (n-1)^2\Big[1-(1-p^2)^m\Big]^2\\
&= (n-1)(1-p^2)^m + (n-1)(n-2)(1-2p^2+p^3)^m\\
&\quad - (n-1)^2(1-2p^2+p^4)^m\\
&= (n-1)\Big[1-mp^2+O(m^2p^4)\Big]\\
&\quad + (n-1)(n-2)\Big[1-2mp^2+2m^2p^4+O(mp^3)+O(m^2p^5)+O(m^3p^6)\Big]\\
&\quad - (n-1)^2\Big[1-2mp^2+2m^2p^4+O(mp^4)+O(m^3p^6)\Big]\\
&= \mu + o(1).
\end{aligned}
$$

■

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