RATIONAL TORUS-EQUIVARIANT STABLE HOMOTOPY I CALCULATING GROUPS OF STABLE MAPS.

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ABSTRACT. We construct an abelian category $\mathcal{A}(G)$ of sheaves over a category of closed subgroups of the r-torus G. The category $\mathcal{A}(G)$ is of injective dimension r, and can be used as a model for rational G-spectra. Indeed, we show that there is a homology theory

$$\pi_*^{\mathcal{A}}\ G\text{-}\mathbf{spectra} \longrightarrow \mathcal{A}(G)$$

on rational G-spectra with values in $\mathcal{A}(G)$ and the associated Adams spectral sequence converges for all rational G-spectra and collapses at a finite stage.

This is the first paper in a series of three. It culminates in [8] where the author and B.E.Shipley combine the Adams spectral sequence constructed here with the enriched Morita equivalence of Schwede and Shipley [9] to deduce that the category of differential graded objects of $\mathcal{A}(G)$ is Quillen equivalent to the category of rational G-spectra.

CONTENTS

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Part 1. Introduction.

1. Summary.

1.A. **The results.** The purpose of the present paper is to provide a means for calculation in the homotopy category of rational G-spectra, where G is an r-dimensional torus. In Part 2 we construct an abelian category $\mathcal{A}(G)$ and 4.4 show it is of injective dimension r. The category $\mathcal{A}(G)$ is a category of sheaves M on the space of subgroups of G, and the value $M(U)$ of a sheaf on a set U of subgroups captures the information about spaces with isotropy groups in U. In Part 3 we construct a homology theory

$$\pi_*^{\mathcal{A}} \ G\text{-spectra} \longrightarrow \mathcal{A}(G)$$

with values in $\mathcal{A}(G)$, and show that it is an effective calculational tool in that there is an Adams spectral sequence. The main theorem is as follows.

Theorem 1.1. *There is a spectral sequence*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(G)}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow [X,Y]_*^G,$$

convergent for all G-spectra X and Y.

The category $\mathcal{A}(G)$ is of injective dimension r, and so the spectral sequence concentrated between rows 0 and r it therefore collapses at the E_{r+1}-page.

The special case $r = 1$ provided the basis for the results of [5]. In addition to being a powerful tool, it is a perfectly practical one, since it is easy to make calculations in $\mathcal{A}(G)$. It essentially describes the category of rational G-spectra up to a finite filtration. For many purposes this is quite sufficient, but the other papers in the series go further. In [8] Shipley and the author combine the Adams spectral spectral sequence of the present paper with the work of Schwede and Shipley [9] to show that the category of rational G-spectra is Quillen equivalent to $dg\mathcal{A}(G)$. The paper [6] provides the information about the algebraic structure of the category $\mathcal{A}(G)$ required in [8].

Convention 1.2. Certain conventions are in force throughout the paper and the series. The most important is that *everything is rational* all spectra and homology theories are rationalized without comment. The second is the standard one that 'subgroup' means 'closed subgroup'. We attempt to let inclusion of subgroups follow the alphabet, so that when there are inclusions they are in the pattern $L \subseteq K \subseteq H \subseteq G$. The other convention beyond the usual one that H_0 denotes the identity component of H is that $\tilde{H}$ denotes a subgroup with identity component H and $\hat{H}$ denotes a subgroup in which H is cotoral (i.e., so that $H \subseteq \hat{H}$ and $\hat{H}/H$ is a torus).

1.B. **Outline of the argument.** First we must construct the the category $\mathcal{A}(G)$. This is a category of sheaves on the space of subgroups of G. In fact we consider the 'natural' open sets

$$U(K) = \{H \mid H \supseteq K\}$$

of isotropy groups, where K runs through the *connected* subgroups of G, and an object M of $\mathcal{A}(G)$ is specified by its values $M(U(K))$ and the restriction maps $M(U(K)) \longrightarrow M(U(H))$ when $U(K) \supseteq U(H)$ (i.e., when $K \subseteq H$). These are required to satisfy certain conditions

that we explain shortly, but $M(U(K))$ contains information about isotropy groups H in $U(K)$.

It is rather easy to write down the functor $\pi_*^{\mathcal{A}}(X)$.

Definition 1.3. For a G-spectrum X we define $\pi_*^{\mathcal{A}}(X)$ on U-open subsets by

$$\pi_*^{\mathcal{A}}(X)(U(K)) = \pi_*^G(DE\mathcal{F}_+ \wedge S^{\infty V(K)} \wedge X)$$

Here $E\mathcal{F}_+$ is the universal space for the family $\mathcal{F}$ of finite subgroups with a disjoint basepoint added and $DE\mathcal{F}_+ = F(E\mathcal{F}_+, S^0)$ is its functional dual. The G-space $S^{\infty V(K)}$ is defined by

$$S^{\infty V(K)} = \lim_{\to\, V^K=0} S^V,$$

so that when $K \subseteq H$ there is a map $S^{\infty V(K)} \longrightarrow S^{\infty V(H)}$ inducing the restriction map $\pi_*^{\mathcal{A}}(X)(U(K)) \longrightarrow \pi_*^{\mathcal{A}}(X)(U(H))$. □

Remark 1.4. The space $S^{\infty V(K)}$ has another role. We have written $S^{\infty V(K)}$ to emphasize its relation with Euler classes, but

$$S^{\infty V(K)} = \tilde{E}[\not\supseteq K],$$

where $[\not\supseteq K]$ is the family of subgroups of G not containing K, so that $S^{\infty V(K)}$ also has a role for the geometric fixed point functor Φ^K G-**spectra** $\longrightarrow$ G/K-**spectra**

$$\tilde{E}[\not\supseteq K] \wedge X \simeq \tilde{E}[\not\supseteq K] \wedge \Phi^K X \quad \square$$

The objects of $\mathcal{A}(G)$ have the structure of modules over the structure sheaf $\mathcal{O}$ introduced formally in Subsection 3.C. The definition of the structure sheaf is based on the ring

$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F_+),$$

where the product is over finite subgroups F of G. The sheaf $\mathcal{O}$ is defined by

$$\mathcal{O}(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$$

where $\mathcal{E}_K = \{c(V) \mid V^K = 0\} \subseteq \mathcal{O}_{\mathcal{F}}$ is the multiplicative set of Euler classes of K-essential representations, and the components $c(V)(F) = c_H(V^F) \in H^*(BG/F_+)$ of an Euler class are classical ordinary homology Euler classes.

To see that $\pi_*^{\mathcal{A}}(X)$ is a module over $\mathcal{O}$, the key is to understand S^0.

Theorem 1.5. *The image of S^0 in $\mathcal{A}(G)$ is the structure sheaf*

$$\mathcal{O} = \pi_*^{\mathcal{A}}(S^0)$$

We prove this in the course of Sections 5 to 8.

There are then two requirements on $\mathcal{O}$-modules to be objects of $\mathcal{A}(G)$. Firstly they must be *quasi-coherent*, in that

$$M(U(K)) = \mathcal{E}_K^{-1}M(U(1)),$$

where $\mathcal{E}_K$ is the set of Euler classes of K-essential representations as before. The definition of $\pi_*^{\mathcal{A}}(X)$ shows that quasi-coherence for $\pi_*^{\mathcal{A}}(X)$ is just a matter of understanding Euler classes, which we do in Section 7. The second condition involves the ring $\mathcal{O}_{\mathcal{F}}$ and its analogue

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K}_+)$$

for the quotient modulo a connected subgroup K, where $\mathcal{F}/K$ is the family of subgroups $\tilde{K}$ of G with identity component K. The second condition is that the object should be *extended*, in the sense that

$$M(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

for some $\mathcal{O}_{\mathcal{F}/K}$-module $\phi^K M$. The extendedness of $\pi_*^{\mathcal{A}}(X)$ follows from a construction of the geometric fixed point functor, and it turns out that

$$\phi^K \pi_*^{\mathcal{A}(G)}(X) = \pi_*^{G/K}(DE\mathcal{F}_+ \wedge \Phi^K(X)),$$

where Φ^K is the geometric fixed point functor. This sketches the proof of the following result, proved formally in Section 8.

Corollary 1.6. *The functor $\pi_*^{\mathcal{A}}$ takes values in the abelian category $\mathcal{A}(G)$.*

This outlines the construction of the functor $\pi_*^{\mathcal{A}}$. To construct the Adams spectral sequence we need to realize an injective resolution of $\pi_*^{\mathcal{A}}(X)$ in $\mathcal{A}(G)$, and to prove the Adams spectral sequence works for maps into an injective. We must therefore first realize sufficiently many injectives. We show that there is a right adjoint f_K to the evaluation at K functor ϕ^K (described in detail in Subsection 4.A). Thus for a suitable module N over $\mathcal{O}_{\mathcal{F}/K}$ we may form an object $f_K(N)$ in $\mathcal{A}(G)$. Taking $N = H_*(BG/\tilde{K})$ for a subgroup $\tilde{K}$ with identity component K, viewed as a $\mathcal{O}_{\mathcal{F}/K}$-module via projection onto $H^*(BG/\tilde{K})$, we obtain the object $I(\tilde{K}) = f_K(H_*(BG/\tilde{K}))$. This is injective since $H_*(BG/\tilde{K})$ is injective over $H^*(BG/\tilde{K})$. It turns out (Lemma 9.2) that a suspension of $I(\tilde{K})$ is realized by the G-spectrum $E\langle\tilde{K}\rangle$ defined in 6.1 in the sense that

$$I(\tilde{K}) = \pi_*^{\mathcal{A}}(\Sigma^{-c}E\langle\tilde{K}\rangle)$$

where $\tilde{K}$ is of codimension c. Next we need to understand maps into injectives, showing that

$$\pi_*^{\mathcal{A}}\ [X, I]_*^G \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(I))$$

for these sufficiently many injectives I. This constructs a spectral sequence with the correct E_2-term. Finally we must show convergence by showing that $\pi_*^{\mathcal{A}}(X) = 0$ implies $X \simeq *$. This is an easy consequence of the geometric fixed point Whitehead theorem 9.4.

2. Formal behaviour of equivariant homology theories.

There are two ways one may hope to encode data about the homology of fixed point sets. They are close enough to be confusing, so it is worth making them explicit at the outset. We consider the case that G is a torus, and the reader may want to bear in mind the examples of stable homotopy and K theory.

Given a G-equivariant homology theory $E_*^G(\cdot)$ and a G-space X we may consider the system of values

$$H \longmapsto E_*^{G/H}(\Phi^H X),$$

where $\Phi^H X$ denotes the (geometric) H-fixed point set of X. If $K \subseteq H$ there is an inclusion $\Phi^H X \longrightarrow \Phi^K X$ of G/K-spaces and hence a map $E_*^{G/K}(\Phi^H X) \longrightarrow E_*^{G/K}(\Phi^K X)$, but in general there will not be a map $E_*^{G/H}(\Phi^H X) \longrightarrow E_*^{G/K}(\Phi^K X)$. However in favourable circumstances there are maps of this sort, and we accordingly call a *contravariant* functor F on subgroups an *inflation functor*. If we are just given the values $E_*^{G/H}(\Phi^H X)$ and no structure maps between them we refer to an *inflation system*. Because of the variance and

the motivation we sometimes write $F(G/H)$ to suggest dependence on the quotient group G/H.

On the other hand, we may always consider

$$X \wedge \tilde{E}[\not\supseteq H] \simeq \Phi^H X \wedge \tilde{E}[\not\supseteq H]$$

where $[\not\supseteq H]$ is the family of subgroups not containing H. If $K \subseteq H$ then there is a natural map $\tilde{E}[\not\supseteq K] \longrightarrow \tilde{E}[\not\supseteq H]$, and hence a map

$$E_*^G(\Phi^K X \wedge \tilde{E}[\not\supseteq K]) \longrightarrow E_*^G(\Phi^H X \wedge \tilde{E}[\not\supseteq H])$$

Since G is abelian,

$$\tilde{E}[\not\supseteq K] = \lim_{\rightarrow\, V^K=0} S^V,$$

and, under orientability hypotheses, $E_*^G(\Phi^K X \wedge \tilde{E}[\not\supseteq K])$ may be expressed as a localization $\mathcal{E}_K^{-1}E_G^*(X)$ of $E_*^G(X)$, where $\mathcal{E}_K$ is some multiplicatively closed subset of E_*^G, generated by "Euler classes" $e(V)$ with $V^K = 0$. We will call a *covariant* functor on subgroups of this form a *localization functor*.

Two major differences should be emphasized. First, inflation functors are contravariant in the subgroup whilst localization functors are covariant. Second, a localization functor takes values which are modules over E_*^G, whereas an inflation functor typically does not.

When we are fortunate enough that E gives an inflation functor and also has a localization theorem it may happen that the two structures are related in the sense that

$$E_*^G(\Phi^K X \wedge \tilde{E}[\not\supseteq K]) = \mathcal{E}_K^{-1}E_*^G \otimes_{E_*^{G/K}} E_*^{G/K}(\Phi^K X)$$

In other words, the favourable case is when we have the following structure, which will be properly defined and axiomatized in later sections.

(1) R, a ring-valued inflation functor (such as $K \longmapsto E_*^{G/K}$)
(2) M an inflation system, which is module valued functor over R, (such as $K \longmapsto E_*^{G/K}(\Phi^K X)$)
(3) EM a localization functor, which is module valued over $R(G/1)$, (such as $K \longmapsto E_*^G(\Phi^K X \wedge \tilde{E}[\not\supseteq K])$, and
(4) an isomorphism

$$EM(K) = \mathcal{E}_K^{-1}R(G/1) \otimes_{R(G/K)} M(G/K)$$

In this case we say that the localization functor EM is *extended* with associated inflation system M. However, be warned that, even if M is an inflation functor (i.e., it has contravariant structure maps), this does not supply the structure maps for $\mathcal{E}_K^{-1}R(G/1) \otimes_{R(G/K)} M(G/K)$, so that EM requires further data.

Part 2. Categories of U-sheaves.

The objects of the abelian category $\mathcal{A}(G)$ are sheaves of modules over a sheaf $\mathcal{O}$ of rings. Accordingly we begin Section 3 by describing the inflation functor on which the structure sheaf $\mathcal{O}$ is based; we can then define Euler classes and proceed with the definition. Once $\mathcal{A}(G)$ is defined we begin to control it in Section 4 first we import objects from module categories, and then show that these suffice to build all the objects and prove that $\mathcal{A}(G)$ has injective dimension equal to the rank of G.

3. The standard abelian category.

The present section leads up to the definition of the standard model as a certain category of U-sheaves of $\mathcal{O}$-modules. Before we can express the definition we need to introduce the structure sheaf $\mathcal{O}$, and before we can do this (Subsection 3.C) we need to describe its associated inflation functor (Subsection 3.A) and Euler classes (Subsection 3.B).

3.A. **The fundamental inflation functor.** The entire structure we discuss is founded on the inflation functor described in this section. We let $\mathrm{ConnSub}(G)$ denote the category of connected subgroups of G and inclusions. An inflation functor is a contravariant functor

$$M \;\; \mathrm{ConnSub}(G) \longrightarrow \mathbf{AbGp}$$

We write $M_{G/H}$ for its value on H. The purpose of this section is to introduce a ring valued inflation functor

$$\mathcal{O}_{\mathcal{F}} \;\; \mathrm{ConnSub}(G) \longrightarrow \mathbf{Rings},$$

whose value at K is written $\mathcal{O}_{\mathcal{F}/K}$. Other notations can be convenient and have been used elsewhere, for example $\mathcal{O}_{\mathcal{F}/K} = \mathcal{O}_{\mathcal{F}(K)} = \mathcal{O}(K) = R_{G/K}$, but we will stick to the above notation in this series.

For any connected subgroup K, we let

$$\mathcal{F}/K = \{\tilde{K} \mid K \text{ of finite index in } \tilde{K}\}$$

denote the set of subgroups of G with identity component K, which is in natural correspondence with the finite subgroups of G/K. Now take

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K}\in\mathcal{F}/K} H^*(BG/\tilde{K})$$

where the product is over the set of subgroups with identity component K.

To describe the inflation maps, suppose K and L are connected and $L \subseteq K$ and L is of finite index in $\tilde{L}$. The inclusion defines a quotient map $q \;\; G/L \longrightarrow G/K$ and hence

$$q_* \;\; \mathcal{F}/L \longrightarrow \mathcal{F}/K$$

The inflation map $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}/L}$ has $\tilde{L}$th component

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K}\in\mathcal{F}/K} H^*(BG/\tilde{K}) \longrightarrow H^*(G/q_*\tilde{L}) \longrightarrow H^*(BG/\tilde{L})$$

given by projection onto the term $H^*(BG/q_*\tilde{L})$ followed by the inflation map induced by the quotient $G/\tilde{L} \longrightarrow G/q_*\tilde{L}$.

Now an *inflation system* of $\mathcal{O}_{\mathcal{F}}$-modules is given by specifying an $\mathcal{O}_{\mathcal{F}/K}$-module $M_{G/K}$ for each subgroup K. No structure maps relating these modules are required.

3.B. **Euler classes.** We are now in a position to describe the Euler classes which are used in the localization process. This will allow us to discuss localization functors, and hence quasi-coherent and extended U-sheaves.

The Euler classes provide functions which are an intimate part of the structure. Since G is abelian, we need only define Euler classes $e(\alpha) \in \mathcal{O}_{\mathcal{F}}$ for one dimensional representations α the Euler class of an arbitrary representation is defined by the product formula $e(V \oplus W) = e(V)e(W)$.

We take

$$e(\alpha) \in \mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F),$$

to be defined by

$$e(\alpha)(F) = \begin{cases} 1 & \text{if } \alpha^F = 0 \\ c_1(\alpha) & \text{if } \alpha \text{ is trivial on } F. \end{cases}$$

This is not a homogeneous element. The best way to sanitize this is to introduce an invertible sheaf associated to a representation (corresponding to suspension), and make $e(\alpha)$ a section of that. Thus $e(\alpha)$ should be thought of as a section of a line bundle vanishing at a finite group F if and only if F acts trivially on α. Since H acts trivially on α if and only if all finite subgroups of H act trivially, we can think of $e(\alpha)$ as defining the 'U-closed' set of subgroups of $\ker(\alpha)$.

There are enough representations on G in the sense that if H is fixed, it is separated from all subgroups (except those containing it) by an Euler class if $H \not\subseteq K$ there is a representation α trivial over K and non-trivial over H. Accordingly, if H is connected, the open set $U(H)$ of subgroups containing H is defined by inverting the set

$$\mathcal{E}_H = \{e(\alpha) \mid \alpha^H = 0\}$$

of Euler classes of representations not arising from G/H. If $\tilde{H}$ has identity component H we let $\mathcal{E}_{\tilde{H}} = \mathcal{E}_H$.

Example 3.1. For example if G is the circle group and z is the natural representation, $e(z)$ is supposed to define $\{1\}$. We think of $e(z)$ as the function (or rather global section) given on finite subgroups by

$$e(z)(F) = \begin{cases} c & \text{if } F = 1 \\ 1 & \text{if } F \neq 1 \end{cases}$$

Remark 3.2. The correspondence with divisors can be very important (see [7]). By definition $e(\alpha)$ vanishes to the first order at finite subgroups of $\ker(\alpha)$. It is thus natural to view the line bundle of which $e(\alpha)$ is a generating section as corresponding to the 'divisor' $\overline{\ker(\alpha)}$, and call it $\mathcal{O}(\ker(\alpha))$.

3.C. **The structure sheaf and the category $\mathcal{A}(G)$.** We now turn to localization functors. We introduce terminology so that we can view them as giving sheaves of functions on the space of subgroups.

For each closed connected subgroup K of G we consider the set $U(K)$ of subgroups containing K (which can be identified with the set of subgroups of G/K). We view the collection

$$\mathcal{U} = \{U(K) \mid K \text{ a connected subgroup }\}$$

as the generating set for the U-topology on the set of subgroups of G. We carry the letter U throughout the discussion to distinguish it from a second topology introduced in [6].

A U-sheaf M is a contravariant functor $M\ \mathcal{U} \longrightarrow \mathbf{AbGp}$. (The terminology is reasonable since any cover of a set $U(K)$ by sets from $\mathcal{U}$ must involve $U(K)$ itself, so the sheaf condition is automatically satisfied). Thus if K and L are connected with $L \subseteq K$ then $U(L) \supseteq U(K)$ and there is a restriction map $M(U(L)) \longrightarrow M(U(K))$. Note that this is covariant for the inclusion of subgroups and is therefore simply another way of speaking of a localization functor.

We may construct a U-sheaf from the ring $\mathcal{O}_{\mathcal{F}}$.

Definition 3.3. (i) The structure U-sheaf $\mathcal{O}$ is defined by

$$\mathcal{O}(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}},$$

and the structure maps are the localizations. Thus $\mathcal{O}$ is a U-sheaf of rings, and its ring of global sections is $\mathcal{O}_{\mathcal{F}}$.
(ii) A sheaf of $\mathcal{O}$-modules is a U-sheaf M with the additional structure that $M(U(H))$ is a module over $\mathcal{O}(U(H)) = \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}}$. The restriction maps are required to be module maps if $L \subseteq K$, the restriction map

$$M(U(L)) \longrightarrow M(U(K)),$$

for the inclusion $U(L) \supseteq U(K)$ is required to be a map of $\mathcal{O}(U(L))$-modules.

We shall be working almost exclusively with sheaves M of $\mathcal{O}$-modules the standard model for rational G-spectra will be a category of dg sheaves of $\mathcal{O}$-modules with additional structure.

First we restrict attention to modules which are determined by their value on $U(1)$. This is analogous to forming a sheaf over $\mathrm{spec}(R)$ from an R-module its values over the open set on which x is invertible is $M[1/x]$. We also borrow the well-established and unwieldy terminology from this situation.

Definition 3.4. A *quasi-coherent* U-sheaf (qc U-sheaf) of $\mathcal{O}$-modules is one in which for each connected subgroup K, the restriction map $M(U(1)) \longrightarrow M(U(K))$ is localization so as to invert $\mathcal{E}_K$.

Remark 3.5. (i) The structure sheaf $\mathcal{O}$ is quasi-coherent.
(ii) For a quasicoherent sheaf, all values $M(U(H))$ are determined by the value $M(U(1))$.
(iii) The quasi-coherence condition has a major effect. For example, if M is a quasi-coherent module only nonzero on $U(1)$ then $M(U(1))$ is necessarily a torsion module.

The second restriction is to sheaves of $\mathcal{O}$-modules which are extended from quotient groups in the following sense.

Definition 3.6. A sheaf M of $\mathcal{O}$-modules is *extended* if we are given a tensor decomposition

$$M(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

where $\phi^K M$ is an $\mathcal{O}_{\mathcal{F}/K}$-module, so that $\{\phi^\bullet M\}$ is an inflation system of $\mathcal{O}_{\mathcal{F}}$-modules. This splitting must be compatible with restriction maps in that if $L \subseteq K$, the restriction is obtained from a map

$$\phi^L M \longrightarrow \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

by extension of scalars. A morphism of extended modules is required to arise from a map of inflation systems if $\theta \;\; M \longrightarrow N$ is a morphism of extended modules, for each K we have a diagram

$$\begin{array}{ccc} M(U(K)) & \stackrel{\theta(U(K))}{\longrightarrow} & N(U(K)) \\ =\downarrow & & \downarrow= \\ \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes \phi^K M & \stackrel{1\otimes\phi^K\theta}{\longrightarrow} & \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes \phi^K N \end{array}$$

We write e-$\mathcal{O}$-mod for the category of extended $\mathcal{O}$-modules.

Remark 3.7. (i) The condition on restriction maps makes sense since

$$\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/L}} \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/K} \otimes_{\mathcal{O}_{\mathcal{F}/K}} (\cdot) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} (\cdot)$$

The point here is that representations α of G/L with $\alpha^{K/L} = 0$ (whose Euler classes lie in $\mathcal{E}_{K/L}$) map to representations of G with $\alpha^K = 0$ (whose Euler classes lie in $\mathcal{E}_K$) under inflation.
(ii) The structure sheaf $\mathcal{O}$ is extended since

$$\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \mathcal{O}_{\mathcal{F}/K}$$

(iii) The splitting of $M(U(K))$ is specified by the *basing map*

$$\phi^K M \longrightarrow M(U(K))$$

corresponding to the inclusion of the unit.
(iv) The reason for the notation is that $\phi^K M$ is analagous to the value $E_*^{G/K}(\Phi^K X)$ of a cohomology theory on geometric fixed points (see also 3.10).

Remark 3.8. We may therefore think of a quasi-coherent extended U-sheaf M of $\mathcal{O}$-modules as an $\mathcal{O}_{\mathcal{F}}$-module $M(U(1))$ together with additional structure. The additional structure specifies particular "relative trivializations" of $\mathcal{E}_K^{-1}M(U(1))$

$$\mathcal{E}_K^{-1}M(U(1)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

The whole structure is given by $M(U(1))$ together with basing maps $\phi^K M \longrightarrow \mathcal{E}_K^{-1}M(U(1))$ giving the splittings.

Finally, we may introduce the class of sheaves directly relevant to us.

Definition 3.9. The *standard abelian category*

$$\mathcal{A} = \mathcal{A}(G) = \text{qce-}\mathcal{O}\text{-mod}$$

is the category of all quasi-coherent extended U-sheaves of $\mathcal{O}$-modules (qce $\mathcal{O}$-modules).

It is useful to have an algebraic analogue of the fixed point functor. This is defined on extended $\mathcal{O}$-modules.

Lemma 3.10. *There is a functor*

$$\Phi^L \ \ e\text{-}\mathcal{O}_G\text{-}mod \longrightarrow e\text{-}\mathcal{O}_{G/L}\text{-}mod$$

defined by

$$(\Phi^L M)(U(K/L)) = \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M,$$

or equivalently

$$\phi^{K/L}(\Phi^L M) = \phi^K M$$

This functor takes quasi-coherent modules to quasi-coherent modules. □

Example 3.11. The fixed point functor takes the G-structure sheaf $\mathcal{O}$ to the G/L-structure sheaf there is an equivalence

$$\Phi^L\mathcal{O}_G = \mathcal{O}_{G/L}$$

of sheaves of modules on the toral chain category for G/L. □

4. A filtration of the standard abelian category $\mathcal{A}(G)$.

In this section we show that any object of the abelian category $\mathcal{A}(G)$ can be built up from objects $f_H(N)$ arising from modules N over the rings $\mathcal{O}_{\mathcal{F}/H}$ for various connected subgroups H. The object $f_H(N)$ is zero on $U(K)$ unless $K \subseteq H$, and it is constant where it is non-zero.

The underlying reason we can decompose objects in this way is the fact that all restriction maps $M(U(L)) \longrightarrow M(U(K))$ go in one direction they increase the dimension of the subgroups. The topological explanation of this phenomenon is just as in [4]. Because we work over the rationals we may express transfer maps (which would go in the other direction) entirely in terms of idempotents from Burnside rings.

This filtration is fundamental for calculation, and perhaps the first striking consequence is that the category $\mathcal{A}(G)$ has finite injective dimension (equal to the rank of G). This is the key to the power of $\mathcal{A}(G)$ in the study of G-equivariant cohomology theories.

Subsection 4.A introduces the method for constructing objects of $\mathcal{A}(G)$ from modules, Subsection 4.B shows how arbitrary objects can be constructed from these, and Subsection 4.C deduces consequences homological algebra.

4.A. **Evaluation and extension.** For a chosen connected subgroup K, evaluation gives a functor

$$\mathrm{ev}_K \quad \mathcal{O}\text{-mod} \longrightarrow \mathcal{O}(U(K))\text{-modules}$$

defined by

$$M \longmapsto M(U(K))$$

This functor has a right adjoint

$$c_K \quad \mathcal{O}(U(K))\text{-modules} \longrightarrow \mathcal{O}\text{-mod}$$

given by taking the sheaf constant below K

$$c_K(N)(U(H)) = \begin{cases} N & \text{if } H \subseteq K \\ 0 & \text{if } H \not\subseteq K \end{cases}$$

The unit of the adjunction

$$\eta \quad M \longrightarrow c_K \mathrm{ev}_K M$$

is defined to be the restriction $\eta(U(L)) \quad M(U(L)) \longrightarrow M(U(K))$ if $L \subseteq K$ and is zero otherwise. The counit

$$\epsilon \quad ev_K c_K N \longrightarrow N$$

is the identity. Thus we have an adjunction

$$\mathrm{ev}_K \quad \mathcal{O}\text{-mod} \rightleftarrows \mathcal{O}(U(K))\text{-modules} \quad f_K$$

with the left adjoint on top.

This adjunction obviously restricts to an adjunction between extended $\mathcal{O}$-modules and extended $\mathcal{O}(U(K))$-modules, and if we identify extended $\mathcal{O}(U(K))$-modules with modules for $\mathcal{O}_{\mathcal{F}/K}$, this gives the adjunction

$$\phi^K \quad \text{e-}\mathcal{O}\text{-mod} \rightleftarrows \mathcal{O}_{\mathcal{F}/K}\text{-modules} \quad f_K$$

Explicitly, $f_K(V)$ is constant below K at $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} V$. In other words,

$$f_K(V) = c_K(\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} V)$$

A little more care is necessary for quasi-coherent sheaves. Indeed, the U-sheaf $c_K(N)$ will not be quasi-coherent unless $\mathcal{E}_K$ is invertible on N and $\mathcal{E}_{K'}^{-1}N = 0$ when $K' \not\subseteq K$. We call modules of this sort $\mathcal{E}_K$-invertible K-torsion $\mathcal{O}_{\mathcal{F}}$-modules, and we call sheaves with $M(U(K')) = 0$ when $K' \not\subseteq K$, sheaves concentrated below K. Since quasi-coherent sheaves form a full subcategory, this is the only obstacle, and we have an adjunction

$$\mathrm{ev}_K \;\; \text{qc-}\mathcal{O}\text{-mod-below-}K \rightleftarrows \mathcal{E}_K\text{-inv-}K\text{-torsion-}\mathcal{O}_{\mathcal{F}}\text{-modules} \;\; c_K$$

Finally, on qce $\mathcal{O}$-modules we combine these to give the adjunction we actually need.

Lemma 4.1. *For any connected subgroup K there is an adjunction*

$$\phi^K \;\; \textit{qce-}\mathcal{O}\textit{-mod-below-}K \rightleftarrows \textit{torsion-}\mathcal{O}_{\mathcal{F}/K}\textit{-modules} \;\; f_K$$

Furthermore, for any torsion $\mathcal{O}_{\mathcal{F}/K}$-module V and an arbitrary extended module M,

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/K}}(\phi^K M, V) = \mathrm{Hom}(M, f_K(V)) \quad \square$$

4.B. **U-sheaves are constructed from constant ones.** The category of qce U-sheaves is an abelian category, and we will need to do homological algebra in it. The fact that it has finite injective dimension is fundamental, and the method for proving it in the following theorem is a practical method of calculation.

Theorem 4.2. *The qce U-sheaves constant below connected subgroups (i.e., the sheaves of the form $f_K(V)$ for some connected subgroup K and some torsion $\mathcal{O}_{\mathcal{F}/K}$-module V) generate the category of all qce U-sheaves using short exact sequences and sums.*

Proof We say that a U-sheaf is supported on a set of subgroups $\mathcal{K}$ if $M(U(K')) = 0$ when $K' \notin \mathcal{K}$. We argue by finite induction on s that qce sheaves supported on subgroups of dimension $\leq s$ are generated by U-sheaves constant below some point. The induction begins since the statement is obvious with $s = -1$, and the theorem is the case $s = r$.

Suppose then that qce U-sheaves supported on subgroups of dimension $\leq s-1$ are generated by U-sheaves constant below some point, and that M is a qce U-sheaf supported on subgroups of dimension $\leq s$. For each connected subgroup L of dimension s we note that $M(U(L))$ is torsion and lift the identity map $M(U(L))$ to a map $M \longrightarrow f_L(M(U(L)))$. Now combine these to a map

$$M \longrightarrow \prod_{\dim(L)=s} f_L(M(U(L)))$$

The product is the termwise product of vector spaces, and therefore not a qce sheaf.

The following lemma is crucial in understanding what torsion modules look like.

Lemma 4.3. *Suppose M is a qc U-sheaf. If $x \in \ker(M(U(1) \longrightarrow M(U(K)))$ for all connected subgroups K of dimension $s+1$ then x maps to zero in $M(U(L))$ for almost all connected subgroups L of dimension s.*

Proof The hypothesis states that for each $(s+1)$-dimensional connected subgroup K, there is a representation $V(K)$ with $V(K)^K = 0$, and $e(V(K))x = 0$. Accordingly x maps to zero in $M(U(L))$ whenever $V(K)^L = 0$ for some K.

However if $V(K) = \alpha_1(K) \oplus \cdots \oplus \alpha_{n(K)}(K)$, the fixed point set $V(K)^L = 0$ unless $L \subseteq \ker(\alpha_i(K))$ for some i. In particular, for each i we have $K \not\subseteq \ker(\alpha_i(K))$.

We argue by induction on t that for $t = 0, 1, \;\; , r-s$ the element x is only non-zero in $M(U(H))$ where H lies in an t-fold generic intersection of maximal subgroups from a finite

list. If we can show this when $t = r - s$ the lemma is proved, since each $(r-s)$-fold generic intersection specifies a unique connected s-dimensional subgroup H.

The assertion is vacuous if $t = 0$, so suppose $0 < t \leq r - s$ and that any H lies in some $(t-1)$-fold generic intersection of the maximal subgroups $M_1, \ , M_{N(t-1)}$. Now since $r - (t-1) \geq s + 1$ for each $(t-1)$-fold generic intersection M^*_λ, we may choose an $(s+1)$-dimensional connected subgroup $K_\lambda \subseteq M^*_\lambda$. Thus if x is non-zero in $M(U(H))$ then H lies in an intersection $M^*_\lambda \cap \bigcap_\mu \ker(\alpha_{k(j)}(K_\mu))$ and hence in the t-fold generic intersection $M^*_\lambda \cap \ker(\alpha_{k(j)}(K_\lambda))$. This gives the required assertion and hence completes the inductive step. □

Since the particular module M that concerns us is supported in dimension $\leq s$, it follows that the map into the product actually maps into the sum, and we obtain

$$g \ M \longrightarrow \bigoplus_{\dim(L)=s} f_L(M(U(L)))$$

The first point is that the sum is a qce U-sheaf since localization and tensor products commute with direct sum.

Next the map g is an isomorphism at $U(H)$ whenever $\dim(H) \geq s$. The kernel and cokernel are then supported on subgroups of dimension $\leq s - 1$ and hence constructed from constant sheaves by induction. □

4.C. **Homological algebra of categories of sheaves.** We use the modules constant below some point to import convenient objects into the category of $\mathcal{O}$-modules from categories of modules over suitable rings. Since we want to use them to construct injective resolutions, it is very convenient that these constructions are right adjoints to evaluation on suitable subcategories.

Theorem 4.4. *The category of qce $\mathcal{O}$-modules has injective dimension equal to the rank r of G.*

Proof Using torsion modules for $H^*(BG) = \mathbb{Q}[x_1, \ , x_r]$ it is easy to see the injective dimension is at least r, so we concentrate on showing this is an upper bound.

We prove by induction on the dimension that any sheaf supported on subgroups of dimension $\leq s$ is of injective dimension at most r.

The result is true if $s = 0$ since the category of sheaves supported at 1 is equivalent to the category of torsion $\mathcal{O}_\mathcal{F}$-modules. Indeed, by Lemma 4.3, any torsion $\mathcal{O}_\mathcal{F}$-module T is a sum $\bigoplus_F T_F$ where T_F is a torsion $H^*(BG/F)$-module.

Now if M is supported in dimension s, we may consider the subgroups L of dimension s and the map

$$M \longrightarrow E = \bigoplus_{\dim L=s} f_L(\phi^L M)$$

of 4.2. By definition this map is an isomorphism at each subgroup L of dimension s, so that its kernel and cokernel are supported in dimension $\leq s - 1$. Thus we have two short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow I \longrightarrow 0$$

and

$$0 \longrightarrow I \longrightarrow E \longrightarrow M'' \longrightarrow 0$$

Note that E is of injective dimension $\leq r$ since this is true of $H^*(BG/L)$-modules (again using Lemma 4.3). Now M' and M'' are supported in dimension $\leq s-1$ by construction, and hence of injective dimension $\leq r$ by induction. Thus I is of injective dimension $\leq s$ from the second exact sequence, and so M is of injective dimension $\leq s$ from the first. □

When we come to connections with G-spectra we need to know we can realize enough injectives, and accordingly it is good to have a small list of injectives.

Lemma 4.5. *There are enough injective quasi-coherent extended $\mathcal{O}$-modules which are sums of those of the form*

$$I(\tilde{L}) = f_L(H_*(BG/\tilde{L}))$$

where $\tilde{L}$ is a subgroup with identity component L.

Proof First note that $H_*(BG/\tilde{L})$ is the $\mathbb{Q}$-dual of $H^*(BG/\tilde{L})$ and therefore injective over it. Any torsion $H^*(BG/\tilde{L})$-module may be embedded in a product of suspensions of $H_*(BG/\tilde{L})$ and takes values in the sum. Accordingly enough injective $H^*(BG/\tilde{L})$-modules are sums of suspensions of $H_*(BG/\tilde{L})$.

Now, even though $\mathcal{O}_{\mathcal{F}/L} = \prod_{\tilde{L}} H^*(BG/\tilde{L})$, Lemma 4.3 shows a torsion module M is a sum of its idempotent pieces $M = \bigoplus_{\tilde{L}} e_{\tilde{L}}M$. Therefore $H_*(BG/\tilde{L})$ is also an injective $\mathcal{O}_{\mathcal{F}/L}$-module, and any torsion module can be embedded in a sum of suspensions of modules $H_*(BG/\tilde{L})$ for subgroups $\tilde{L}$ with identity component L. □

Part 3. The Adams spectral sequence.

In Part 2 we introduced the algebraic category $\mathcal{A}(G)$, and in Part 3 we provide the connection with G-equivariant cohomology theories by defining the functor $\pi^{\mathcal{A}}$ and constructing an Adams spectral sequence based on it.

The basis of the connection is the calculation

$$\mathcal{O}_{\mathcal{F}} = [E\mathcal{F}_+, E\mathcal{F}_+]^G_*$$

of the endomorphism ring of $E\mathcal{F}_+$. This is completed in Section 6. In preparation we begin by understanding the basic building blocks and how they are related to each other.

5. Basic cells.

The familiar generators in topology are the natural cells G/K_+. It is more convenient to consider the basic cells.

Definition 5.1. The *basic cell* for the closed subgroup K is defined by

$$\sigma^0_K = e_{K/K_1} G/K_+,$$

where $e_{K/K_1} \in A(K/K_1)$ is the primitive idempotent in the Burnside ring corresponding to the group K/K_1 of components of K.

The usefulness of the basic cells is that they provide decompositions of all the natural cells.

Lemma 5.2. *Suppose $\tilde{K}$ is a subgroup with identity component K_1. There is a decomposition*

$$G/\tilde{K}_+ \simeq \bigvee_{K_1 \subseteq K \subseteq \tilde{K}} \sigma^0_K$$

where the splitting is indexed by subgroups K/K_1 of the group $\tilde{K}/K_1$ of components of $\tilde{K}$.

Proof We follow the pattern of [5, 2.1.5]. It suffices to show that if $K_1 \subseteq K \subseteq \widetilde{\tilde{K}}$ then $G_+ \wedge_K e_K S^0 = \sigma^0_K \simeq G_+ \wedge_{\tilde{K}} e_K S^0$. Indeed, we need only show that $G_+ \wedge_{\tilde{K}} e_K \widetilde{\tilde{K}/K}$ is contractible where $\widetilde{\tilde{K}/K} = \text{cofibre}(\tilde{K}/K_+ \longrightarrow S^0)$.

We suppose $G = G' \times G''$ with $\tilde{K} \subseteq G'$ a product of inclusions of a cyclic group in a circle. It suffices to prove the analogous result with G replaced by G'. The analogue of [5, 2.1.4] replaces a single cofibre sequence by $r' = \text{rank}(G')$ of them. For each of the cyclic factors we apply the method of [5, 2.1.5] to the permutation representation of $\tilde{K}/K$. □

Lemma 5.3. *Maps between basic cells in degree 0 are as follows*

$$[\sigma^0_K, \sigma^0_L]^G_0 = \begin{cases} \mathbb{Q} & \text{if } K \text{ is cotoral in } L \\ 0 & \text{otherwise} \end{cases}$$

Proof We need only apply idempotents to the corresponding statements with natural cells. Indeed $[G/K_+, \sigma^0_L]^G = [S^0, \sigma^0_L]^K$. This is zero unless L is of finite index in K. If L is of finite index the idempotents e_K and e_L are orthogonal unless $K = L$. Finally, if $K = L$ we note that G/L_+ is L-equivariantly obtained from S^0 by attaching cells of dimension ≥ 1, and hence the desired group is a quotient of $e_K[S^0, S^0]^K = \mathbb{Q}$. It is non-trivial since σ^0_K is not contractible. □

Lemma 5.4. *If F is finite, the endomorphism ring of σ^0_F is exterior on r generators,*

$$[\sigma^0_F, \sigma^0_F]^G_* = \Lambda(H_1(G/F))$$

Proof Additively the calculation is correct since $[\sigma^0_F, \sigma^0_F]^G_* = [G/F_+, \sigma^0_F]^G_* = [S^0, G/F_+]^F_*$ and G/F_+ is F-fixed and a torus with added basepoint. The ring structure may be seen by passing to non-equivariant homology

$$[\sigma^0_F, \sigma^0_F]^G_* \longrightarrow \text{Hom}(H_*(\sigma^0_F), H_*(\sigma^0_F))$$

This is a ring map and, since $H_*(\sigma^0_F) \cong H_*(G/F_+)$, the codomain is exterior. It remains to note that this is surjective in degree 1. This in turn follows from the rank 1 case by the Künneth theorem. The rank 1 case is clear since the degree 1 map is tautologously detected in F-equivariant homotopy and hence in homology. □

Finally we record the Whithehead theorem for spectra with stable isotropy only at F.

Lemma 5.5. *If F is finite, and X is a spectrum so that*

(1) *X has stable isotropy only at F and*
(2) *$[\sigma^0_F, X]^G_* = 0$*

then X is contractible.

Proof By (i) we have $\Phi^K X$ trivial unless K is a subgroup of F. It therefore suffices to show that $[G/K_+, X]^G_* = 0$ if $K \subseteq F$. However, G/K_+ splits as a wedge of basic cells for finite subgroups. Since X only has isotropy at F, the only possible contribution is from the summand $[\sigma^0_F, X]^G_*$, and this is zero by hypothesis. □

6. Endomorphisms of injective spectra.

The basis for the correspondence between algebra and topology is the universal space $E\mathcal{F}_+$ for the collection $\mathcal{F}$ of finite subgroups of G. This plays a central role because its endomorphism ring is so well behaved the simplicity we see here will have even more power in [8].

We follow the strategy of [5], adapted to account for the fact that the exterior algebra $H_*(G_+)$ and the polynomial algebra $H^*(BG_+)$ now have r generators, rather than the single generator in the case of the circle.

First it is convenient to introduce injective counterparts of the basic cells.

Definition 6.1. For any subgroup K we define the G-space $E\langle K\rangle$ by

$$E\langle K\rangle = \text{cofibre}(E[\subset K]_+ \longrightarrow E[\subseteq K]_+)$$

Example 6.2. (i) If $K = 1$ we have $E\langle 1\rangle = EG_+$.
(ii) If $K = G$ we have $E\langle G\rangle = \tilde{E}\mathcal{P}$ where $\mathcal{P}$ is the family of proper subgroups of G.

Between them these give the general picture.

Lemma 6.3. *If K is a subgroup with identity component K_0, then there is an equivalence*

$$\Phi^{K_0} E\langle K\rangle \simeq E\langle K/K_0\rangle$$

of G/K_0-spaces and an equivalence

$$E\langle K\rangle \simeq S^{\infty V(K_0)} \wedge E\langle K/K_0\rangle$$

of G-spaces.

Proof For any family $\mathcal{H}$ of subgroups and any subgroup L

$$\Phi^L E\mathcal{H}_+ \simeq E\mathcal{H}/L_+$$

where $\mathcal{H}/L$ is the family of subgroups of G/L which are images of those of $\mathcal{H}$. This gives the first statement.

For the second, note that nothing is changed by smashing with $S^{\infty V(K_0)}$. □

We are now ready to identify homotopy endomorphism rings.

Theorem 6.4. *The homotopy endomorphism ring of $E\mathcal{F}_+$ is given by*

$$[E\mathcal{F}_+, E\mathcal{F}_+]^G_* = \mathcal{O}_{\mathcal{F}} = \prod_{F\in\mathcal{F}} H^*(BG/F_+)$$

Proof First, as in [3] we may use idempotents to split $E\mathcal{F}_+$

$$E\mathcal{F}_+ \simeq \bigvee_{F\in\mathcal{F}} E\langle F\rangle$$

It therefore suffices to prove the corresponding result, Theorem 6.5, about the summands. □

Theorem 6.5. *The homotopy endomorphism ring of $E\langle F\rangle$ is given by*

$$[E\langle F\rangle, E\langle F\rangle]_*^G = H^*(BG/F_+)$$

The first tool is a characterization of $E\langle F\rangle$.

Proposition 6.6. *If F is finite, the spaces $E\langle F\rangle$ are characterized by*

(1) *$E\langle F\rangle$ has isotropy only at F and*
(2) *$[\sigma^0_F, E\langle F\rangle]_*^G = \mathbb{Q}$.*

Proof First note that $[\sigma^0_F, E\langle F\rangle]_*^G = \mathbb{Q}$. Now we proceed by cellular approximation to construct a map $X \longrightarrow E\langle F\rangle$, where X is constructed from cells σ^0_F which is an isomorphism of $[\sigma^0_F, \cdot]_*^G$. This is an equivalence by the Whitehead theorem 5.5. □

We may now identify the endomorphism ring of $E\langle F\rangle$.

Proof of 6.5 Note that $[E\langle F\rangle, E\langle F\rangle]_*^G = [E\langle F\rangle, S^0]_*^G$ so the result will follow additively if we can construct $E\langle F\rangle$ with basic cells in even degrees corresponding to the monomials in $H^*(BG/F_+)$. The proof is by killing homotopy groups.

In the proof of 6.6 we noted that $E\langle F\rangle$ can be constructed using the basic cell σ^0_F. We repeat the proof, but this time keep track of the cells. By 5.4, the endomorphism ring of σ^0_F is exterior on r generators. Indeed, let

$$\mathbb{Q} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

be the standard Koszul resolution of $\mathbb{Q}$ by free $\Lambda H_1(G/F)$-modules. Thus

$$P = \Lambda(H_1G/F)[c_1, c_2, \ , c_r]$$

Note that the kernel of each map $P_n \longrightarrow P_{n-1}$ is generated by its bottom degree elements and these are in bijective correspondence with monomials of degree n.

We argue inductively that we may construct (1) a $2n$-dimensional complex $X^{(2n)}$ with basic cells in bijective correspondence with monomials of degree $\leq n$ in the $c_1, c_2, \ , c_r$ so that its cellular chain complex is the first n stages of the Koszul resolution and (2) a map $X^{(2n)} \longrightarrow E\langle F\rangle$ which is $2n$-connected. This is certainly true for $r = 0$, so we need only describe the inductive step. However, by construction the bottom degree homotopy generates nth syzygy in the Koszul resolution, so there is no obstruction.

It remains to comment on the ring structure. Consider the cellular filtration, and the resulting spectral sequence for $[S^0, \cdot]_*^G$. We obtain a ring map

$$[E\langle F\rangle, E\langle F\rangle]_*^G \longrightarrow \mathrm{Hom}(H^*(BG/F_+), H^*(BG/F_+))$$

Each generator $c_i \in H^2(BG/F_+)$ corresponds to a map of resolutions, and we may realize this by a map $E\langle F\rangle \longrightarrow \Sigma^2 E\langle F\rangle$. It follows that the ring map is surjective. By the additive result, it is an isomorphism. □

We also need to know this identification is natural for quotient maps.

Now there is a natural map $E\mathcal{F}_+ \longrightarrow E\mathcal{F}/K_+$ of G-spaces, since every finite subgroup of G has finite image in G/K. Viewing this is a map of spectra and dualizing, we obtain a map $DE\mathcal{F}/K_+ \longrightarrow DE\mathcal{F}_+$. Combined with the inflation map $[S^0, DE\mathcal{F}/K_+]_*^{G/K} \longrightarrow$

$[S^0, DE\mathcal{F}/K_+]^G_*$, and using 6.4 for G and G/K, we obtain a ring homomorphism $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$.

Lemma 6.7. *The geometrically induced ring homomorphism coincides with the map* q^* $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$ *described in Subsection 3.A, which is the product of the ring homomorphisms*

$$q^*_{\tilde{K}} \ \mathcal{O}_{\tilde{K}} \longrightarrow \mathcal{O}_{q_*^{-1}(\tilde{K})} = \prod_{q_*(F)=\tilde{K}/K} H^*(BG/F_+)$$

where $q_* \ \mathcal{F} \longrightarrow \mathcal{F}/K$ *is reduction mod* K*, and the components of* $q^*_{\tilde{K}}$ *are induced by the quotient maps* $G/F \longrightarrow G/\tilde{K}$*.*

Proof We have the splitting $E\mathcal{F}_+ \simeq \bigvee_{F\in\mathcal{F}} E\langle F\rangle$ of rational G-spectra [3]. Similarly the stable rational G/K-splitting $E\mathcal{F}/K_+ \simeq \bigvee_{\tilde{K}/K\in\mathcal{F}/F} E\langle \tilde{K}/K\rangle$ may be inflated to a G-splitting. ¿From fixed points one sees that the map $E\mathcal{F}_+ \longrightarrow E\mathcal{F}/K_+$ respects the splitting in the sense that $E\langle F\rangle$ maps trivially to $E\langle \tilde{K}/K\rangle$ unless $q(F) = \tilde{K}/K$. Since duality takes sums to products, 6.5 completes the proof. □

7. Topology of Euler classes.

The next ingredient is to show that the inclusions $S^0 \longrightarrow S^V$ induce suitable Euler classes.

The relevant input from topology comes from the Thom isomorphism for an individual stalk. We once again use the basic injectives 6.1.

Lemma 7.1. *For any finite group* F *there is an equivalence*

$$S^V \wedge E\langle F\rangle \simeq S^{|V^F|} \wedge E\langle F\rangle$$

Proof The cofibre of the map $S^{V^F} \longrightarrow S^V$ is built from cells with isotropy not containing F. It is therefore contractible when smashed with $E\langle F\rangle$. We may thus suppose V is F-fixed.

Now $E\langle F\rangle$ may be built from basic cells σ^0_F. Since

$$G/F_+ \wedge S^{V^F} \simeq G_+ \wedge_F S^{|V^F|} \simeq G/F_+ \wedge S^{|V^F|},$$

we find that

$$\sigma^0_F \wedge S^{V^F} \simeq \sigma^0_F \wedge S^{|V^F|}$$

Accordingly, $E\langle F\rangle \wedge S^{V^F}$ is also built from cells σ^0_F and

$$[\sigma^0_F, E\langle F\rangle \wedge S^{V^F}]^G_* = [\sigma^0_F, E\langle F\rangle \wedge S^{|V^F|}]^G_* = \Sigma^{|V^F|}\mathbb{Q}$$

Thus the result follows from 6.6. □

Remark 7.2. Note that the proof displays a specific equivalence on the bottom cell, and hence determines the homotopy class of the equivalence.

As usual, the Thom isomorphism gives rise to an Euler class.

Definition 7.3. The F Euler class $c(V)(F)$ of a representation V is the map

$$S^0 \wedge E\langle F\rangle \longrightarrow S^V \wedge E\langle F\rangle \simeq S^{|V^F|} \wedge E\langle F\rangle$$

We may identify these Euler classes in familiar terms.

Lemma 7.4. *Under the identification* $[E\langle F\rangle, E\langle F\rangle]^G_* = H^*(BG/F_+)$*, the Euler class* $c(V)(F)$ *is the ordinary cohomology Euler class* $c_H(V^F)$*.*

Proof Since both Euler classes take sums of representations to products, it suffices to consider a 1-dimensional representation V. If $V^F = 0$, both Euler classes are 1. If V is fixed by F, then V is a faithful representation of G/K for some $(r-1)$-dimensional subgroup K containing F. Both maps are given by multiplication by a degree 2 class.

It therefore suffices to consider the case of the circle and the representation z^n. The standard generator is the first Euler class $c_H(z)$ and the additive formal group shows $c_H(z^n) = nc_H(z)$. On the other hand, the identification of $G_+ \wedge S^{z^n} \simeq G_+ \wedge S^2$ lets $t(g \wedge x) = tg \wedge x$ in $G_+ \wedge S^2$ correspond to $t(g \wedge x) = tg \wedge t^n x$ in $G_+ \wedge S^{z^n}$, which is a map of degree n. □

In view of the splitting theorem $E\mathcal{F}_+ \simeq \bigvee_{F\in\mathcal{F}} E\langle F\rangle$ we obtain a general Thom isomorphism.

Corollary 7.5. *For any virtual complex representation* V *and associated dimension function* $v\ \mathcal{F} \longrightarrow \mathbb{Z}$ *defined by* $v(F) = \dim_{\mathbb{R}}(V^F)$*, there are equivalences*

$$S^V \wedge E\mathcal{F}_+ \simeq \bigvee_F S^{v(F)} \wedge E\langle F\rangle,$$

and

$$S^V \wedge DE\mathcal{F}_+ \simeq \prod_F S^{v(F)} \wedge DE\langle F\rangle$$

We may now define the global Euler class.

Definition 7.6. The Euler class of a complex representation V is

$$S^0 \wedge E\mathcal{F}_+ \longrightarrow S^V \wedge E\mathcal{F}_+ \simeq \bigvee_F S^{v(F)} \wedge E\langle F\rangle,$$

as a non-homogeneous element of $\mathcal{O}_{\mathcal{F}}$.

Corollary 7.7. *The Euler class, viewed as an element of* $\mathcal{O}_{\mathcal{F}}$ *has* F*th component.*

$$c(V)(F) = c_H(V^F) \in H^*(BG/F_+) \quad \square$$

8. Sheaves from spectra.

Now that we understand the homotopy endomorphism ring of $E\mathcal{F}_+$ we may forge the link with algebra since $[E\mathcal{F}_+, E\mathcal{F}_+]^G_* = \mathcal{O}_{\mathcal{F}}$ by 6.4, any spectrum $X \wedge DE\mathcal{F}_+$ has homotopy groups which are $\mathcal{O}_{\mathcal{F}}$-modules. In this section we give the proof that $\pi^{\mathcal{A}}_*$ takes values in $\mathcal{A}(G)$ (stated as 1.6).

¿From the definition of Euler classes we see that $\pi^{\mathcal{A}}_*(X)$ is quasi-coherent.

Proposition 8.1. *For any* G*-spectrum* X *the object* $\pi^{\mathcal{A}}_*(X)$ *is quasi-coherent in the sense that for any connected subgroup* K*,*

$$\pi^{\mathcal{A}}_*(X)(U(K)) = \mathcal{E}_K^{-1}\pi^{\mathcal{A}}_*(X)(U(1))$$

Proof We combine the definition of $\pi^{\mathcal{A}}_*$ with that of Euler classes to obtain

$$\pi^{\mathcal{A}}_*(X)(U(K)) = \pi^G_*(X \wedge DE\mathcal{F}_+ \wedge S^{\infty V(K)}) = \mathcal{E}_K^{-1}\pi^G_*(X \wedge DE\mathcal{F}_+) = \mathcal{E}_K^{-1}\pi^{\mathcal{A}}_*(X)(U(1))$$

□

We may now complete the proof that S^0 corresponds to the structure sheaf $\mathcal{O}$.

Proof of Theorem 1.5 By 6.4, $\pi_*^{\mathcal{A}}(S^0)(U(1)) = \mathcal{O}_{\mathcal{F}}$. By 8.1

$$\pi_*^{\mathcal{A}}(S^0)(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} = \mathcal{O}(U(K))$$

□

Lemma 8.2. *The quasi-coherent U-sheaf $\pi_*^{\mathcal{A}}(X)$ of $\mathcal{O}$-modules is extended. In fact, the value at $U(K)$ splits with*

$$\phi^K\pi_*^{\mathcal{A}}(X) = \pi_*^{G/K}(\Phi^K X \wedge D E\mathcal{F}/K_+)$$

since

$$\pi_*^G(X \wedge D E\mathcal{F}_+ \wedge S^{\infty V(K)}) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \pi_*^{G/K}(\Phi^K X \wedge D E\mathcal{F}/K_+)$$

Proof There is a natural transformation arising from

$$\inf_{G/K}^G(\Phi^K X \wedge D E\mathcal{F}/K_+) \longrightarrow X \wedge D E\mathcal{F}_+ \wedge S^{\infty V(K)}$$

This gives a natural transformation of homology theories of X, so we need only check it is an isomorphism for various cells $X = G/H_+$. If $X = S^0 = G/G_+$ the map is an isomorphism by definition. The general case follows by the $Rep(G)$-isomorphism argument (Theorem 10.2) since we have Thom isomorphisms on both sides. □

9. Adams spectral sequences.

It is clear that $\pi_*^{\mathcal{A}}$ is functorial and exact, and therefore by 1.6 it defines a homology functor

$$\pi_*^{\mathcal{A}}\ G\text{-}\mathbf{spectra} \longrightarrow \mathcal{A}(G)$$

with values in the abelian category $\mathcal{A}(G)$ with injective dimension r.

Theorem 9.1. *The homology theory $\pi_*^{\mathcal{A}}$ gives a convergent Adams spectral sequence which collapses at E_{r+1}.*

In the usual way, we attempt to construct an Adams spectral sequence based on a homology theory H with values in an abelian category $\mathcal{A}$ by geometrically realizing an algebraic resolution of the homology.

We need to prove

(1) enough injective objects I of $\mathcal{A}$ are realized (i.e., in the sense that there are spectra X with $H_*(X) = I$)
(2) the injective case of the spectral sequence is correct in that homology gives an isomorphism
$$[X, Y] \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{A}}(H_*(X), H_*(Y))$$
if $H_*(Y)$ is injective, and
(3) the homology theory detects isomorphisms in the sense that $H_*(X) = 0$ implies that X is contractible. This will give convergence of the spectral sequence, at least when resolutions are of finite length.

To proceed, we need the basic injective G-spectrum $E\langle K\rangle$ of 6.1 which has stable isotropy only at the subgroup K. The essential property is that this realizes the basic injectives $I(K)$ in $\mathcal{A}(G)$.

Lemma 9.2. *If K is of codimension c we have*

$$I(K) = f_{K_1}(\Sigma^c H_*(BG/K_+)) = \pi_*^{\mathcal{A}}(E\langle F\rangle)$$

and hence there are enough realizable injectives.

Proof First, by 6.3, we have

$$E\langle K\rangle = S^{\infty V(K_1)} \wedge E\langle K/K_1\rangle,$$

so that

$$\pi_*^G(DE\mathcal{F}_+ \wedge E\langle K\rangle) = \mathcal{E}_{K_1}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \pi_*^{G/K_1}(DE(\mathcal{F}/K_1)_+ \wedge E\langle K/K_1\rangle)$$

The result therefore follows from the special case in which $K = F$ is finite and $c = r$.

Now $DE\mathcal{F}_+ \wedge E\langle F\rangle \simeq E\langle F\rangle$ and therefore

$$\pi_*^G(DE\mathcal{F}_+ \wedge E\langle F\rangle) = H_*(\Sigma^r BG/F_+)$$

Since this is a torsion module

$$\pi_*^{\mathcal{A}}(E\mathcal{F}_+) = f_1(\Sigma^r BG/F_+)$$

□

We may now prove the injective case of the Adams spectral sequence.

Lemma 9.3. *For any G-spectrum X, application of $\pi_*^{\mathcal{A}}$ induces an isomorphism*

$$[X, E\langle K\rangle]^G \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(E\langle K\rangle))$$

Proof Let $N = \pi_*^{\mathcal{A}}(X)$, and argue by induction on the dimension of G.

For $E\langle K\rangle$ we combine the following diagram

$$\begin{array}{ccc} [X, E\langle K\rangle]^G & \longrightarrow & \mathrm{Hom}(N, f_K(\Sigma^c H_*(BG/K_+)) \\ \downarrow{\scriptstyle =} & & {\scriptstyle =}\downarrow \\ [\Phi^K X, EG/K_+]^{G/K} & \longrightarrow & \mathrm{Hom}(\phi^K N, \Sigma^c H_*(BG/K_+)) \end{array}$$

with a result for G/K to show the bottom horizontal is an isomorphism.

For notational simplicity we treat the case $K = 1$, where we are left to show

$$\pi_*^G\ [X, EG_+]^G \xrightarrow{\cong} \mathrm{Hom}(\pi_*^G(X \wedge DEG_+), \Sigma^r H_*(BG_+))$$

It is easy to see the groups are isomorphic for $X = S^0$. Passage to homology is injective because $\pi_*^G(S^0) \longrightarrow \pi_*^G(S^0 \wedge DEG_+)$ is a monomorphism in degree 0. Since π_*^G compares rational vector spaces of equal finite dimension when $X = S^0$, it is an isomorphism. There are Thom isomorphisms in algebra and topology, so it follows that passage to homology is an isomorphism for $X = S^V$ for any complex representation V. By the $Rep(G)$-isomorphism argument (Theorem 10.2) it is an isomorphism for $X = G/K_+$ for any subgroup K and hence in general. □

Finally, we may prove the universal Whitehead theorem.

Lemma 9.4. *The functor $\pi_*^{\mathcal{A}}$ detects isomorphisms in the sense that if $f\ \ Y \longrightarrow Z$ is a map of G spectra inducing an isomorphism $f_*\ \ \pi_*^{\mathcal{A}}(Y) \longrightarrow \pi_*^{\mathcal{A}}(Z)$ then f is an equivalence.*

Proof Since $\pi_*^{\mathcal{A}}$ is exact, it suffices to prove that if $\pi_*^{\mathcal{A}}(X) = 0$ then $X \simeq *$. We argue by induction on the dimension of G.

Suppose $\pi_*^{\mathcal{A}}(X) = 0$. From the geometric fixed point Whitehead theorem it suffices to show that $\Phi^K X$ is non-equivariantly contractible for all K. Since $\phi^K \pi_*^{\mathcal{A}}(X) = \pi_*^{G/K}(\Phi^K X)$, and $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$ is faithfully flat over $\mathcal{O}_{\mathcal{F}/K}$, the result follows by induction from the G/K-equivariant result provided $K \neq 1$.

It remains to deduce $X \wedge EG_+$ is contractible. Since we have Thom isomorphisms we note that by the $Rep(G)$-isomorphism argument (Theorem 10.2), it suffices to show that $\pi_*^G(X \wedge EG_+) = 0$. Now $E\mathcal{F}_+ \longrightarrow S^0$ is a non-equivariant equivalence so that $DE\mathcal{F}_+ \wedge EG_+ \simeq DS^0 \wedge EG_+ = EG_+$, so that it is enough to show $\pi_*^G(X \wedge DE\mathcal{F}_+ \wedge EG_+) = 0$.

However if Y has Thom isomorphisms and $\mathbb{C}$ is a one dimensional representation then the cofibre sequence

$$Y \wedge S(\infty\mathbb{C})_+ \longrightarrow Y \longrightarrow Y \wedge S^{\infty\mathbb{C}}$$

shows that if $\pi_*^G(Y) = 0$ then also $\pi_*^G(Y \wedge S(\infty\mathbb{C})_+) = 0$. Writing $EG_+ = S(\infty\mathbb{C}_1)_+ \wedge \ \ \wedge S(\infty\mathbb{C}_r)_+$ we reach the desired conclusion in r steps. □

10. The $Rep(G)$-isomorphism argument.

The present section records a method that is useful rather generally in equivariant topology. It has nothing to do with the fact that we are working rationally.

When trying to establish an object is contractible or a map is an equivalence we want to use the most convenient test objects. The Whitehead theorem says it suffices to use the set $\{G/H_+ \mid H \subseteq G\}$ if $[G/H_+, X]_*^G = \pi_*^H(X) = 0$ for all H then X is contractible.

Definition 10.1. A G-spectrum is $Rep(G)$-contractible if $[S^V, X]_*^G = 0$ for all complex representations V.

It is often easy to see $\pi_*^G(X) = 0$. If we happen to have Thom isomorphisms $S^V \wedge X \simeq S^{|V|} \wedge X$ for complex representations V this shows that X is $Rep(G)$-contractible. It does not necessarily follow that X is contractible even if X is rational and G is abelian (for example if G is cyclic of order 3 and X is a Moore spectrum for a two dimensional simple representation [2]) but it is useful to have a sufficient condition.

Theorem 10.2. *Suppose G is an abelian compact Lie group. If X is a $Rep(G)$ contractible G-spectrum and G/H acts trivially on $\pi_*^H(S^{-V} \wedge X)$ for all H and all desuspensions of X then X is contractible. If G is a torus the condition of trivial action may be omitted.*

Proof We argue by induction on the size of G since compact Lie groups satisfy the descending chain condition on subgroups we can assume the result is true for all proper subgroups. We know $\pi_*^G(X) = 0$ by hypothesis, so by the Whitehead theorem it suffices to show that $\pi_*^K(X) = 0$ for all proper subgroups. Now any proper subgroup K lies in a subgroup H with G/H a subgroup of the circle. It therefore suffices by induction to establish that X is $Rep(H)$-contractible. Since $H \subseteq G$, any trivial action condition will certainly be inherited by subgroups.

If G/H is a circle, we use the cofibre sequence $G/H_+ \longrightarrow S^0 \longrightarrow S^{V(H)}$ where $V(H)$ has kernel H. We conclude that $[G/H_+, X]_*^G = 0$, and more generally, by smashing with S^V, that $[S^V, X]_*^H = 0$ for any representation V of G. Since G is abelian, every representation

of H extends to one of G, and so X is $Rep(H)$-contractible, and hence we conclude X is H-contractible by induction.

If G/H is a finite cyclic group we choose a faithful representation $W(H)$ of G/H and use the cofibre sequence $S(W(H))_+ \longrightarrow S^0 \longrightarrow S^{W(H)}$ and the stable cofibre sequence

$$G/H_+ \xrightarrow{1-g} G/H_+ \longrightarrow S(W(H))_+,$$

where g is a generator of G/H_+. The first shows that $[S(W(H))_+, X]^G_* = 0$, and the second shows that $1-g$ gives an isomorphism of $[G/H_+, X]^G_*$. By the trivial action condition we conclude $[G/H_+, X]^G_* = 0$, and more generally $[S^V, X]^H_* = 0$. □

Remark 10.3. (i) It suffices to assume that G acts unipotently on $\pi^H_*(X)$ for all H. This is useful for p-groups in characteristic p.
(ii) Variants on this theorem are useful in other contexts. For instance any nilpotent or supersoluble finite group has maximal subgroups which are normal with cyclic quotient. However, not every representation of a maximal subgroup extends to one for G, so additional hypotheses are necessary.
(iii) If we admit real representations, then no trivial action condition is necessary for subgroups of index 2 since the mapping cone of $G/H_+ \longrightarrow S^0$ is S^V for a real representation V.

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