RATIONAL TORUS-EQUIVARIANT STABLE HOMOTOPY II THE ALGEBRA OF LOCALIZATION AND INFLATION.

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Abstract. In [4] we constructed an abelian category $\mathcal{A}(G)$ of sheaves over a category of closed subgroups of the r-torus G and showed it can be used as the basis of a finite Adams spectral sequence for calculating groups of stable G-maps.

In the present paper we make an algebraic study of the category $\mathcal{A}(G)$. We show how to separate information from isotropy groups with the same identity component, giving an equivalent category $\mathcal{A}^e(G)$ that is often easier to work with. We also explain how to view $\mathcal{A}$ as a category of modules over a ring R with many objects, and prove a number of results about R.

Some of these results are used in [6], where B.E.Shipley and the author use the Adams spectral sequence of [4], together with the enriched Morita theory of Schwede and Shipley [7] to establish a Quillen equivalence between the category of rational G-spectra and the category $dg\mathcal{A}(G)$ of differential graded objects of $\mathcal{A}(G)$.

Contents

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Part 1. Motivation.

1. Overview of the series.

This paper is the second in the series containing [4, 6], whose purpose is to give a small and calculable algebraic model of the category of rational G-spectra when G is a torus. The algebraic model is the derived category of an abelian category $\mathcal{A}(G)$. In [4] we constructed the abelian category, and a homology functor

$$\pi_*^{\mathcal{A}(G)} \; G\text{-spectra} \longrightarrow \mathcal{A}(G)$$

from the category of G-spectra to $\mathcal{A}(G)$ and showed that there is a finite, convergent Adams spectral sequence based on $\pi_*^{\mathcal{A}(G)}$. Thus $\mathcal{A}(G)$ gives a basis for calculating stable G-equivariant maps.

With Shipley in [6] we show that there is an equivalence

$$G\text{-}\mathbf{spectra} \simeq dg\mathcal{A}(G)$$

of homotopy theories in the sense of Quillen.

In the present paper we make an algebraic study of the category $\mathcal{A}(G)$, establishing the basis of the stabilization process and proving a number of results needed in [6].

2. Overview of the present paper

The definition of $\mathcal{A}(G)$ is designed so that the connection with topology is straightforward. Its objects M associate to each connected subgroup K of G a group $M(U(K))$ capturing the information for isotropy groups in $U(K) = \{H \mid K \subseteq H\}$. The information for all these isotropy groups is grouped together because, for a space X, the non-triviality of X^H forces X^K to be non-trivial if $K \subseteq H$. However it means that for example $M(U(1))$ contains information from all finite subgroups. In the stable context one can separate the different isotropy groups a little more by using idempotents of Burnside rings to separate information from different subgroups with the same identity component. The present paper explains how this can be done with no loss of information, and gives two ways of packaging the result.

Recall from [4, Section 2] the relationship between the cohomology of a space and of its fixed point set there are two structures here (i) the quotient $G/L \longrightarrow G/K$ when $L \subseteq K$ induces an inflation map $\inf_{G/K}^{G/L} \; H^*(BG/K) \longrightarrow H^*(BG/L)$ and (ii) the localization theorem, which shows that one cannot separate behaviour at the isotropy group H from any subgroup $\hat{H}$ in which H is cotoral (i.e., so that $H \subseteq \hat{H}$ and $\hat{H}/H$ is a torus). The first main result of the present paper is the following theorem.

Theorem 2.1. *The abelian category $\mathcal{A}(G)$ of injective dimension $r = \mathrm{rank}(G)$ is equivalent to the category $\mathcal{A}^e(G)$ with objects M specified by*

1. *An $H^*(BG/K)$-module $\phi_e^K M$ for each closed subgroup K of G.*
2. *A transitive system of isomorphisms*

$$\mathcal{E}_{K/L}^{-1}\phi_e^L M \xrightarrow{\cong} \mathcal{E}_{K/L}^{-1} H^*(BG/L) \otimes_{H^*(BG/K)} \phi_e^K M$$

whenever $L \subseteq K$ with K/L a torus, where $\mathcal{E}_{K/L}$ is the multiplicative set generated by Euler classes $c_1(\alpha) \in H^*(BG/L)$ of representations of G/L with $\alpha^{K/L} = 0$.

The organizing principle of our description is that a cohomology theory (or an object M of $\mathcal{A}(G)$) may be viewed as a sheaf on the space of subgroups, with the value $M(U)$ of an object M on a set U of subgroups encoding the behaviour of M on spaces with isotropy in U. The module $\phi_e^K M$ mentioned above is essentially the stalk of M at K. In Section 3 we give a more detailed description of the sheaf theoretic language for $\mathcal{A}(G)$, drawing parallels with the category of sheaves on an affine variety with the Zariski topology.

There are two justifications for using the sheaf-theoretic language. The principal one is that a sheaf is designed to deal with the passage from local to global information an important feature of the category $\mathcal{A}(G)$ is that we can work with small collections of isotropy groups where behaviour is simple, and then use the sheaf condition to patch this together to describe the behaviour of the entire cohomology theory. The second reason for the sheaf language is the strength of the analogy with algebraic geometry. It has already proved possible to give a functorial construction of S^1-equivariant elliptic cohomology [5] using the model of [3] the correspondence between functions on the elliptic curve with poles at points of finite order n and the values of the $\mathcal{A}(S^1)$-sheaf on the open set defined by the subgroup of order n is perfect. I expect that a similar construction with the present model will allow the construction of G-equivariant cohomology theories associated to a suitable abelian variety of dimension r.

We will in fact give four descriptions of the category $\mathcal{A}(G)$, all of which correspond to different views of sheaves.

- the first ($\mathcal{A}(G)$ itself) in terms of "U-sheaves" which has a module $M(U(K))$ for each *connected* subgroup K of G. The value $M(U(K))$ contains information from all isotropy groups in the set
$$U(K) = \{H \mid K \subseteq H\}$$
of subgroups containing K.
- the second in terms of "B-sheaves" which has a module $M(B(K))$ for *every* closed subgroup K of G. The value $M(B(K))$ contains information from all isotropy groups in the set
$$B(K) = \{H \mid K \subseteq H \text{ and } H/K \text{ a torus } \}$$
of subgroups in which K is cotoral.
- the third is the category $\mathcal{A}^e(G)$ described in Theorem 2.1 above. This is in terms of the stalks $\phi_e^K M$ for each closed subgroup K together with specialization information.
- the fourth in terms of modules M over a certain ring R with many objects (indexed by the closed subgroups K of G); the value of the module at K is again the stalk $\phi_e^K M$ at K, and the ring R encodes the holonomy.

Here the description of M as a U-sheaf has a smaller indexing set (just the *connected* subgroups K), but each value $M(U(K))$ is much more complicated, the second B-sheaf description separates non-interacting phenomena, and in fact the values $M(B(K))$ are much simpler objects. The reconstruction of the values $M(U(K))$ from the values $M(B(K))$ is the local to global principle the open set $U(K)$ can be covered by the open sets $B(\tilde{K})$ for subgroups $\tilde{K}$ with identity component K. The third description simply removes duplicated information from the second. The possibility of describing M as a module over a ring R with many objects corresponds to the fact that a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules over a discrete space is determined by its stalks over the points x we need only specify the $\mathcal{O}_x$-module

$\mathcal{F}_x$ ring for each x. The fact that M is a sheaf over a discrete *category* not a discrete *space* means that the rings over different subgroups are related. When all the structures have been properly introduced, we give provide Example 10.2 showing corresponding objects in the various categories; even at the present stage the reader may be reassured to see that the objects become progressively more familiar.

This paper is divided into three parts. Part 1 is purely motivational. Part 2 introduces B-sheaves and proves that $\mathcal{A}(G)$ is equivalent to various categories of B-sheaves and hence to the category $\mathcal{A}^e(G)$. Part 3 introduces a ring R with many objects and reformulates $\mathcal{A}^e(G)$ as the category of torsion R-modules. This point of view is the basis for the work of [6].

Convention 2.2. Certain conventions are in force throughout the paper and the series. The most important is that *everything is rational* all spectra and homology theories are rationalized without comment. The second is the standard one that ‘subgroup’ means ‘closed subgroup’. We attempt to let inclusion of subgroups follow the alphabet, so that when there are inclusions they are in the pattern $L \subseteq K \subseteq H \subseteq G$. The other convention beyond the usual one that H_0 denotes the identity component of H is that $\tilde{H}$ denotes a subgroup with identity component H and $\hat{H}$ denotes a subgroup in which H is cotoral (i.e., so that $H \subseteq \hat{H}$ and $\hat{H}/H$ is a torus).

3. The toral Zariski topology.

In equivariant topology it is convenient to describe behaviour according to isotropy groups. Accordingly we consider the collection of $\mathrm{Sub}(G)$ of all closed subgroups of G as an indexing set. For some purposes the Hausdorff metric on this space is important, but since G is abelian in our case, we can ignore this.

We can make $\mathrm{Sub}(G)$ into a category in two ways. The first, $\mathrm{Sub}_U(G)$ has a morphism $K \longrightarrow H$ whenever $K \subseteq H$. This is most important in unstable homotopy theory. The second, $\mathrm{Sub}_B(G)$ has a morphism $K \longrightarrow H$ when K is cotoral in H (i.e., when K is a normal subgroup of H and H/K is a torus). This is most important in stable homotopy theory, and gives the toral chain category $\mathrm{Sub}_B(G) = \mathcal{TC}G$ considered in [2].

In each case we consider a Grothendieck topology arising from the morphisms. The archetype is the Zariski topology on the prime spectrum $X = \mathrm{spec}(A)$ of a commutative ring A, so we recall its relevant features. The points of $\mathrm{spec}(A)$ are prime ideals, and this set can be made into a category by *reverse* inclusion. The maximal ideals are the closed points, and in general the closure of a prime ideal is the set $V(\wp)$ of all primes containing $\wp$. We say $\wp$ is the generic point of $V(\wp)$. When we work in the geometric context (finitely generated algebras over an algebraically closed field), a prime is determined by the maximal ideals in its closure, and we identify $\wp$ with the set of closed points in $V(\wp)$. The topology of open sets is generated by the sets $D(a)$ of primes not containing an element a, which are the primes of $A[1/a]$. Furthermore the structure sheaf $\mathcal{O}$ of functions on X has global sections A, local rings $\mathcal{O}_\wp = A_\wp$, and the sheaf on a closed set $V(I)$ is associated to A/I. Since the ring A is the ring of global sections, it can be recovered from the local rings together with the specialization information used to form an étale space from the stalks. This information states how a function on the generic point of a set specializes to its values on the closed points, or equivalently, when functions on closed points patch to give a function on the whole set. In the geometric case, a function is determined by its values on closed points. Finally, a sheaf

$\mathcal{F}$ on X is specified by its sections $\mathcal{F}(D(a))$ on the basic open sets $D(a)$. The sheaf is called quasicoherent if it is obtained from an A-module M in the sense that $\mathcal{F}(D(a)) = M[1/a]$.

Now return to the set $\mathrm{Sub}(G)$ of closed subgroups, and consider the two categories with this object set. The U-closure operator V_U is given by

$$V_U(Y) = \{K \subseteq G \mid K \subseteq H \in Y\}$$

for a collection Y of closed subgroups. The idea here is that for a G-space X, if the fixed point space $\Phi^H X$ is essential then $\Phi^K X$ must be non-trivial too since $\Phi^G X \subseteq \Phi^K X$. On the other hand the B-closure operator V_B is given by

$$V_B(Y) = \{K \subseteq G \mid K \text{ is cotoral in } H \in Y\}$$

In this case, if X is a G-spectrum with $\Phi^H X$ non-trivial, the localization theorem forces $\Phi^K X$ to be non-trivial if K is cotoral in H.

Thus the only U-closed point is the trivial subgroup 1, but the B-closed points are the finite subgroups. Since every subgroup H is determined by the finite subgroups it contains, H is determined by $V_B(\{H\})$, and the stable situation is analogous to the geometric case. We can therefore hope that functions should be determined by their values at the closed points.

The open sets associated to these two topologies are the eponymous ones. The U open sets are generated by the *natural* open sets

$$U(K) = \{H \subseteq G \mid K \subseteq H\}$$

of all groups containing K. The B-open sets are generated by the *basic* open sets

$$B(K) = \{H \subseteq G \mid K \text{ is cotoral in } H\}$$

Both classes are closed under intersections, and each generates a topology by taking unions. Since each natural open set $U(K)$ is a union of basic open sets, the B-topology is finer than the U-topology. We note one significant difference between these topologies and the Zariski topology the Zariski topology is compact (for example it is only complements of a finite number of closed points that are open), whereas neither the U nor the B topology is compact.

Example 3.1. (i) If G is the circle, $U(1)$ consists of all subgroups whereas $B(1) = \{1, G\}$. The open set $U(1)$ is covered by the sets $B(F)$ with F finite, and these all intersect in $B(T)$.
(ii) Generally the elements of $U(H)$ correspond to all subgroups of G/H, whilst $B(H)$ corresponds to the connected subgroups of G/H. Since any subgroup of G has a finite subgroup with the same components, $U(H)$ admits a cover by the sets $B(\tilde{H})$ where H is a subgroup of finite index in $\tilde{H}$. □

We now wish to consider sheaves in these topologies. Indeed, we considered U-sheaves in [4], and the present discussion explains the terminology. As usual, a presheaf is a contravariant functor on the open sets, and a sheaf is a presheaf satisfying an appropriate unique patching condition. Once this sheaf condition is made explicit a sheaf is determined by its values on a generating collection of open sets, although in general not every contravariant functor on the generating open sets will be compatible with the sheaf condition.

In our case, any cover of a set $U(K)$ by the generating U-open sets must contain $U(K)$ itself so that any contravariant functor on these sets determines a U-sheaf, and similarly for the B-topology. This explains why we refer to a contravariant functor

$$M\ \mathcal{U} \longrightarrow \mathbf{AbGp}$$

as a U-sheaf and a contravariant functor

$$N\ \mathcal{B} \longrightarrow \mathbf{AbGp}$$

as a B-sheaf. In general we shall only consider the values of sheaves on sets of the form $U(H)$ and $B(H)$.

On the other hand, the main point of introducing the B-topology is to formalize the globalization process in going from basic to natural open sets a B-sheaf N precisely codifies how to recover the value $N(U(1))$ from the values $N(B(H))$. The advantage is that since the open sets $B(H)$ are smaller, distinct phenomena can be separated. On the other hand, any B-sheaf N gives a U-sheaf $u(N)$ by restriction.

We have seen in [4] that a cohomology theory naturally provides a U-sheaf M of modules over a sheaf of rings. The idempotents in the sheaf of rings allow one to construct a B-sheaf $b(M)$. By restriction this determines a new U-sheaf $ub(M)$, and our first main result states 7.1 that for qce U-sheaves M, this recovers the original U-sheaf

$$M = ub(M)$$

It is obvious that $bu(N) = N$, so we obtain an equivalence of categories.

Part 2. Categories of B-sheaves.

Having given a formal definition of the abelian category $\mathcal{A}(G)$ in [4, Section 3], we begin the process of finding simpler descriptions of it. In the category of U-sheaves of $\mathcal{O}$-modules we have to deal with modules $\phi^K M$ over rings $\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K}\in\mathcal{F}/K} H^*(BG/\tilde{K})$. Such infinite products are unpleasant to work with. However, for each subgroup $\tilde{K}$ with identity component K, the module $\phi^K M$ has a summand $e_{\tilde{K}}\phi^K M$ which is a module over the polynomial ring $e_{\tilde{K}}\mathcal{O}_{\mathcal{F}/K} = H^*(BG/\tilde{K})$. For quasi-coherent modules, it turns out that $\phi^K M$ can be reconstructed from these summands, and more generally the entire U-sheaf can be reconstructed in an analogous way from a leaner and more accessible structure. We introduce the structure in Section 4, and construct a B-sheaf of $\mathcal{O}$-modules from a U-sheaf of $\mathcal{O}$-modules. In Section 5 we introduce a minor variation which applies only to extended modules; this is obviously equivalent to the first construction but much more economical. Finally in Sections 6 and 7 we show the new category of qce B-sheaves of $\mathcal{O}$-modules is equivalent to $\mathcal{A}(G)$.

4. B-sheaves.

In this section we define B-sheaves and explain how a U-sheaf of $\mathcal{O}$-modules gives rise to a B-sheaf by use of idempotents.

Let $\mathrm{Sub}(G)$ be the collection of closed subgroups of G. For each subgroup K of G we consider the set

$$B(K) = \{H \mid K \text{ cotoral in } H\}$$

of subgroups containing K. This can be identified with the set of connected subgroups of G/K. We view the collection

$$\mathcal{B} = \{B(K) \mid K \text{ a subgroup }\}$$

as the collection of generating open subsets for the B-topology on the set of subgroups of G. As discussed in Section 3, B-sheaf M is a contravariant functor $M\ \mathcal{B} \longrightarrow \mathbf{AbGp}$. Thus if K and L are subgroups with L cotoral in K then $B(L) \supseteq B(K)$ and there is a restriction map $M(B(L)) \longrightarrow M(B(K))$. Note that this is covariant in the inclusion of subgroups.

We will extend the definition of a B-sheaf to the B-open sets $U(K)$ by a sheaf condition. Indeed, the set $U(K)$ for a connected subgroup K can be covered by B-open sets

$$U(K) = \bigcup_{\tilde{K}} B(\tilde{K})$$

where $\tilde{K}$ runs through subgroups with identity component K. In Section 6 we describe a patching condition determining the value $M(U(K))$ from the values $M(B(\tilde{K}))$, and hence a B-sheaf is a U-sheaf by restriction.

There are two related constructions of B-sheaves from U-sheaves. They contain the same information, but the first works for any U-sheaf of $\mathcal{O}$-modules whilst the second construction applies only to extended modules. The advantage of the second is that it removes a large amount of duplication, leaving the essential core. We describe the first construction in this section, and the second in Section 5.

The key to both constructions is the use of idempotents. We use the notation $e_Z \in \mathcal{O}_{\mathcal{F}}$ for the idempotent supported on a set $Z \subseteq \mathcal{F}$ of finite subgroups. In particular, for a subgroup H we write $e_H \in \mathcal{O}_{\mathcal{F}}$ for the idempotent

$$e_H(F) = \begin{cases} 1 & \text{if } F \subseteq H \\ 0 & \text{if } F \not\subseteq H \end{cases}$$

Construction 4.1. If M is a U-sheaf of $\mathcal{O}$-modules and $\tilde{K}$ is a subgroup with identity component K, then we define

$$M(B(\tilde{K})) = e_{\tilde{K}} M(U(K))$$

Note that if $\tilde{K}$ is cotoral in $\tilde{H}$ then $e_{\tilde{K}}$ is a refinement of $e_{\tilde{H}}$ so there is a restriction map

$$M(B(\tilde{K})) = e_{\tilde{K}} M(U(K)) \longrightarrow e_{\tilde{H}} M(U(H)) = M(B(\tilde{H}))$$

and we have a B-sheaf.

Considering the basic example, we see that the construction still involves a lot of duplication.

Example 4.2. If we apply this construction to the structure sheaf $\mathcal{O}$ we find

$$\mathcal{O}(B(\tilde{H})) = \mathcal{E}_H^{-1} e_{\tilde{H}} \mathcal{O}_{\mathcal{F}} = \mathcal{E}_H^{-1} \prod_{F \subseteq \tilde{H}} H^*(BG/F)$$

It is also awkward to define a quasi-coherent B-sheaf of $\mathcal{O}$-modules, and we shall do so only indirectly. However extendedness is straightforward. First we record a consistency statement.

Lemma 4.3. *The primitive idempotent $e_{\tilde{K}} \in \mathcal{O}_{\mathcal{F}/K}$ inflates to the idempotent with the same name in $\mathcal{O}_{\mathcal{F}}$. Accordingly, for any subgroup $\tilde{K}$ with identity component K, and any $\mathcal{O}_{\mathcal{F}/K}$-module N,*

$$e_{\tilde{K}} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} N = \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} e_{\tilde{K}} N \quad \square$$

Definition 4.4. A B-sheaf M of $\mathcal{O}$-modules is *extended* if

$$M(B(\tilde{K})) = \mathcal{E}_K^{-1} e_{\tilde{K}} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi_e^{\tilde{K}} M = e_{\tilde{K}} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi_e^{\tilde{K}} M$$

for some $H^*(BG/\tilde{K})$-module $\phi_e^{\tilde{K}} M$. The restriction maps are required to be compatible with this tensor decomposition in the sense that if $\tilde{L}$ is cotoral in $\tilde{K}$ so that $B(\tilde{L}) \supseteq B(\tilde{K})$ then the restriction map arises from

$$\phi_e^{\tilde{L}} M \longrightarrow \mathcal{E}_{\tilde{K}/\tilde{L}}^{-1} \mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi_e^{\tilde{K}} M$$

The definition is arranged so that extendedness for a B-sheaf corresponds to that for the associated U-sheaf.

Lemma 4.5. *If M is extended as a U-sheaf, it is extended as a B-sheaf.* □

5. The essential B-sheaf.

In this section we finally reach the most economical and practical form of the category $\mathcal{A}(G)$, as the category of qce B-sheaves of $\mathcal{O}^e$-modules. It is obtained by using more idempotents to remove duplication in the construction of a B-sheaf from an *extended* U-sheaf of $\mathcal{O}$-modules.

Before we can proceed to define the functor on extended modules we need to describe the category in which it takes values, so we first deal with inflation systems as described in [4, Subsection 3.A].

Construction 5.1. If M is an $\mathcal{O}$-module valued inflation system, we define

$$M^e_{\tilde{K}} = e_{\tilde{K}} M_K$$

If $\tilde{L}$ is cotoral in $\tilde{K}$ then the restriction of $e_{\tilde{K}} \in \mathcal{O}_{\mathcal{F}/K}$ to $\mathcal{O}_{\mathcal{F}/L}$ has $e_{\tilde{L}}$ as a summand. Accordingly, if M is actually an inflation functor, there is an inflation map

$$M^e_{\tilde{K}} = e_{\tilde{K}} M_K \longrightarrow e_{\tilde{L}} M_L = M^e_{\tilde{L}},$$

and M^e is also an inflation functor.

Example 5.2. The archetype of an inflation functor is $\mathcal{O}_{\mathcal{F}}$, and the above construction gives

$$\mathcal{O}^e_{\mathcal{F}/\tilde{K}} = H^*(BG/\tilde{K})$$

and if $\tilde{L}$ is cotoral in $\tilde{K}$ the structure map

$$\mathcal{O}^e_{\mathcal{F}/\tilde{K}} = H^*(BG/\tilde{K}) \longrightarrow H^*(BG/\tilde{L}) = \mathcal{O}^e_{\mathcal{F}/\tilde{L}}$$

induced by the quotient map $G/\tilde{L} \longrightarrow G/\tilde{K}$.

If M is an inflation system module over $\mathcal{O}_{\mathcal{F}}$ then M^e is an inflation system module over $\mathcal{O}^e_{\mathcal{F}}$, and similarly for inflation functors. □

Based on this, we can define the basic structure sheaf and Euler classes, which will then allow us to define the required category of qce B-sheaves.

Definition 5.3. (i) If K is an arbitrary closed subgroup of G, we write

$$\mathcal{E}_K = \{c_1(V) \mid V^{K_0} = 0\} \subseteq H^*(BG)$$

for the Euler classes of representations with no points fixed by the *identity component* K_0 of K.
(ii) The essential structure B-sheaf is defined by

$$\mathcal{O}^e(B(K)) = \mathcal{E}_K^{-1} H^*(BG)$$

If L is cotoral in K the associated restriction map

$$\mathcal{O}(B(K)) = \mathcal{E}_K^{-1} H^*(BG) \longrightarrow \mathcal{E}_L^{-1} H^*(BG/L) = \mathcal{O}(B(L))$$

is inversion of $\mathcal{E}_L$.

We may now introduce notions of quasi-coherence and extendedness for suitable B-sheaves.

Definition 5.4. (i) A B-sheaf M^e of $\mathcal{O}^e$-modules is *quasi-coherent* if whenever L is cotoral in K

$$M^e(B(K)) = \mathcal{E}_K^{-1} M^e(B(L))$$

(ii) A B-sheaf M^e of $\mathcal{O}^e$-modules is *extended* if

$$M^e(B(K)) = \mathcal{E}_K^{-1} H^*(BG) \otimes_{H^*(BG/K)} \phi_e^K M$$

for some $H^*(BG/K)$-module $\phi_e^K M$. The restriction maps are required to be compatible with this tensor decomposition in the sense that if L is cotoral in K so that $B(L) \supseteq B(K)$ then the restriction map arises from

$$\phi_e^L M \longrightarrow \mathcal{E}_{K/L}^{-1} \mathcal{O}^e_{\mathcal{F}/L} \otimes_{\mathcal{O}^e_{\mathcal{F}/K}} \phi_e^K M = \mathcal{E}_{K/L}^{-1} H^*(BG/L) \otimes_{H^*(BG/K)} \phi_e^K M$$

(iii) The category $\mathcal{A}^e(G)$ is the category of qce B-sheaves of $\mathcal{O}^e$-modules.

Remark 5.5. It should finally be clear why we want to consider B-sheaves. All the structure is based on modules $\phi_e^K M$ for the polynomial rings $\mathcal{O}^e_{\mathcal{F}/K} = H^*(BG/K)$ for subgroups K of G. Additional structure is described in terms of inflation maps between these rings.

Comparing $\mathcal{O}$ and $\mathcal{O}^e$, notice that if K is the identity component of $\tilde{K}$ the inflation map $H^*(BG/K) \xrightarrow{\cong} H^*(BG/\tilde{K})$ is always an isomorphism. Thus there is a map

$$I \quad \mathcal{O}^e(B(K)) = \mathcal{E}_K^{-1} H^*(BG) \longrightarrow \mathcal{E}_K^{-1} \prod_{F \subseteq K} H^*(BG/F) = \mathcal{O}(B(K))$$

whose components are given by the inverse of inflation. This can be viewed as a localization of the diagonal map. Because inflation is transitive, we obtain a map

$$I \quad \mathcal{O}^e \longrightarrow \mathcal{O}$$

of B-sheaves of rings. This gives a restriction functor

$$I^* \quad \mathcal{O}\text{-mod} \longrightarrow \mathcal{O}^e\text{-mod}$$

and an extension of scalars functor

$$I_* \quad \mathcal{O}^e\text{-mod} \longrightarrow \mathcal{O}\text{-mod}$$

This allows us to show that our two constructions of a B-sheaf from an extended U-sheaf contain the same information.

Lemma 5.6. *The adjunction*

$$I_* \quad \mathcal{O}\text{-}mod \rightleftarrows \mathcal{O}^e\text{-}mod \quad I^*$$

gives an equivalence

$$e\text{-}\mathcal{O}\text{-}mod \simeq e\text{-}\mathcal{O}^e\text{-}mod$$

of categories of extended modules. □

We use this equivalence to give meaning to qce B-sheaves of $\mathcal{O}$-modules.

Definition 5.7. (i) If M is an extended U-sheaf of $\mathcal{O}$-modules we define an extended B-sheaf M^e of $\mathcal{O}^e$-modules on the open set associated to a subgroup $\tilde{K}$ with identity component K by

$$M^e(B(\tilde{K})) = \mathcal{E}_K^{-1}H^*(BG) \otimes_{H^*(BG/\tilde{K})} e_{\tilde{K}}\phi^K M$$

In other words

$$\phi_e^{\tilde{K}} M^e = e_{\tilde{K}}\phi^K M$$

(ii) If $\tilde{L}$ is cotoral in $\tilde{K}$ then the inflation of $e_{\tilde{K}} \in \mathcal{O}_{\mathcal{F}/K}$ to $\mathcal{O}_{\mathcal{F}/L}$ has $e_{\tilde{L}}$ as a summand. Accordingly, the map

$$\phi_e^{\tilde{L}} M^e = e_{\tilde{L}}\phi^L M \longrightarrow e_{\tilde{K}}\phi^L M \longrightarrow \mathcal{E}_{\tilde{K}/\tilde{L}}^{-1} H^*(BG/\tilde{L}) \otimes_{H^*(BG/\tilde{K})} \phi_e^{\tilde{K}} M^e$$

gives a restriction map

$$M^e(B(\tilde{L})) = \mathcal{E}_L^{-1}H^*(BG) \otimes_{H^*(BG/\tilde{L})} e_{\tilde{L}}\phi^L M \longrightarrow \mathcal{E}_L^{-1}H^*(BG) \otimes_{H^*(BG/\tilde{L})} e_{\tilde{K}}\phi^K M = M^e(B(\tilde{K}))$$

In circumstances where $M(U(H))$ is abbreviated to $M(H)$, $M^e(B(H))$ is abbreviated to $M^e(H)$, following the convention used for Mackey functors in [2].

These definitions are designed so that the following consistency statement is clear.

Lemma 5.8. *(i) If M is an extended $\mathcal{O}$-module as a U-sheaf then M^e is a $\mathcal{O}^e$-module as a B-sheaf.*
(ii) If the extended $\mathcal{O}$-module M is quasi-coherent as a U-sheaf M^e is quasi-coherent as a B-sheaf.
(iii) If M is an extended U-sheaf, M^e is extended as a B-sheaf. □

Finally, we need to check that the two constructions of a B-sheaf from an extended U-sheaf contain the same information.

Lemma 5.9. *If M is an extended U-sheaf of $\mathcal{O}$-modules then we have natural isomorphisms*

$$I^*M \cong M^e \text{ and } M \cong I_*M^e$$

of B-sheaves, where the first is an isomorphism of $\mathcal{O}^e$-modules and the second an isomorphism of $\mathcal{O}$-modules. □

6. B-Čech cohomology.

The purpose of this section is to formulate the patching condition which allows us to find the value of a B-sheaf M on the open set $U(H)$ from its values $M(B(\tilde{H}))$ for subgroups $\tilde{H}$ with identity component H. This corresponds to the cover of $U(H)$ by sets $B(\tilde{H})$. In Section 7 we show that if the B-sheaf M arose from a qce U-sheaf as in Section 4 then the reconstructed value is the original value $M(U(H))$, so that the notation is consistent. Despite our very limited aims it is convenient to introduce Čech cohomology.

First, we consider the set

$$\mathcal{F}(H) = \{\tilde{H} \mid H \text{ is a subgroup of finite index in } \tilde{H}\},$$

indexing the cover of $U(H)$, and then let

$$\mathcal{F}^{(n)}(H) = \{(\tilde{H}_1, \tilde{H}_2, \ \ , \ \ \tilde{H}_n) \mid \tilde{H}_i \in \mathcal{F}(H) \text{ and } \tilde{H}_i \neq \tilde{H}_j \text{ if } i \neq j\}$$

be the set indexing the n-fold intersections.

The first thought is that the sheaf condition should be expressed by requiring the diagram

$$M(U(H)) \longrightarrow \prod_{\tilde{H}} M(B(\tilde{H})) \rightrightarrows \prod_{\tilde{H}_1 \neq \tilde{H}_2} M(B(\tilde{H}_1) \cap B(\tilde{H}_2))$$

to be an equalizer. However, this is wrong an example will point this out and suggest the correct version.

Example 6.1. If G is the circle, $H = 1$, and N is a torsion module over $\mathcal{O}_{\mathcal{F}} = \prod_F \mathbb{Q}[c]$ then $N = \bigoplus_F e_F N$, where each $e_F N$ if a torsion module over $\mathbb{Q}[c]$. The associated quasicoherent sheaf M has $M(U(1)) = N$ and $M(U(T)) = 0$. Thus the above diagram becomes

$$N \longrightarrow \prod_F e_F N \rightrightarrows \prod_{F_1 \neq F_2} 0$$

This is not an equalizer if infinitely many of the $e_F N$ are non-zero.

The solution in this case is to replace this diagram by

$$M(U(1)) \longrightarrow M(B(T)) \times \prod_F M(B(F)) \rightrightarrows \mathcal{E}_T^{-1} \prod_F M(B(F))$$

We return to justify it in Example 7.6, but it suggests the general case. □

We start by considering the full subcategory $\mathcal{C}(H)$ of $\mathcal{TCG}$ on subgroups containing H. We then consider the nerve of $\mathcal{C}(H)$ whose i-simplexes are composable sequences of maps of $\mathcal{C}(H)$, and the subset

$$\mathcal{C}'_i(H) = \{(f_i, f_{i-1}, \ \ , f \ \ _0) \mid f_j \text{ is a non-identity map in } \mathcal{C}(H) \text{ with } \mathrm{cod}(f_j) = \mathrm{dom}(f_{j+1})\}$$

of non-degenerate simplexes. The advantage here is that $\mathcal{C}'_i(H) = \emptyset$ if $i > dim(G/H)$. Now for a simplex $\mathbf{f} = (f_i, \ \ , f \ \ _0)$ we consider its domain $\mathrm{dom}(\mathbf{f}) = \mathrm{dom}(f\ _0)$ and its codomain $\mathrm{cod}(\mathbf{f}) = \mathrm{cod}(f\ _i)$. This lets us consider the usual cohomology of a contravariant functor on $\mathcal{C}(H)$,

$$H^*(\mathcal{C}(H); M) = H^*(\prod_{\mathbf{f} \in \mathcal{C}_0(H)} M(\mathrm{cod}(\mathbf{f})) \longrightarrow \prod_{\mathbf{f} \in \mathcal{C}_1(H)} M(\mathrm{cod}(\mathbf{f})) \longrightarrow \prod_{\mathbf{f} \in \mathcal{C}_2(H)} M(\mathrm{cod}(\mathbf{f})) \longrightarrow \cdots)$$

The homology is unaltered if we replace $\mathcal{C}$ by $\mathcal{C}'$. However we need a slight variation for $\mathcal{O}$-modules M. The notation is chosen to suggest the cohomology is the Čech cohomology associated to the B-cover of $U(H)$.

$$\check{H}^*_B(U(H);M) = H^*\Big(\prod_{\mathbf{f}\in\mathcal{C}'_0(H)} M(B(\mathrm{dom}(\mathbf{f}))) \longrightarrow \mathcal{E}_{\mathcal{C}}^{-1}\prod_{\mathbf{f}\in\mathcal{C}'_1(H)} M(B(\mathrm{dom}(\mathbf{f}))) \longrightarrow \mathcal{E}_{\mathcal{C}}^{-1}\prod_{\mathbf{f}\in\mathcal{C}'_2(H)} M(B(\mathrm{dom}(\mathbf{f}))) \longrightarrow \cdots\Big)$$

Here

$$\mathcal{E}_{\mathcal{C}}^{-1}\prod_{\mathbf{f}\in\mathcal{C}'_i(H)} M(B(\mathrm{dom}(\mathbf{f}))) = \prod_{K\in\mathcal{C}'(H)} \mathcal{E}_K^{-1}\prod_{\mathrm{cod}(\mathbf{f})=K} M(B(\mathrm{dom}(\mathbf{f})))$$

If there are only finitely many simplexes in $\mathcal{C}_i(H)$ the two constructions agree for quasi-coherent modules M, since then the localization commutes with the product. However for a non-zero torus there are infinitely many simplices, and the second definition is what we need.

By definition there is a natural map

$$\nu \;\; M(U(H)) \longrightarrow \check{H}^0_B(U(H);M)$$

which we want to show is an isomorphism for qce U-sheaves M.

Definition 6.2. We say that a U-sheaf M of $\mathcal{O}$-modules *satisfies the B-sheaf condition at H* if

$$\nu \;\; M(U(H)) \xrightarrow{\cong} \check{H}^0_B(U(H);M)$$

is an isomorphism.

Finally we record the usual exactness property.

Lemma 6.3. *If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is a short exact sequence of U-sheaves then there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \check{H}^0_B(U(H);M') \longrightarrow \check{H}^0_B(U(H);M) \longrightarrow \check{H}^0_B(U(H);M'') \\ \longrightarrow \check{H}^1_B(U(H);M') \longrightarrow \check{H}^1_B(U(H);M) \longrightarrow \check{H}^1_B(U(H);M'') \\ \longrightarrow \check{H}^2_B(U(H);M') \longrightarrow \check{H}^2_B(U(H);M) \longrightarrow \check{H}^2_B(U(H);M'') \longrightarrow \cdots \end{aligned}$$

7. U-sheaves and B-sheaves.

In this section we prove that any qce U-sheaf M satisfies the B-sheaf condition at all subgroups H. This justifies saying that M *is* a B-sheaf. The practical consequence is that the values $M(U(H))$ for any object of the standard category can be reconstructed from the values $M(B(\tilde{H}))$ (or $M^e(B(\tilde{H}))$, so that we can work entirely with modules over finitely generated polynomial rings.

For the purposes of the following statement, if M is a U-sheaf of $\mathcal{O}$-modules, we write $b(M)$ for the B-sheaf of $\mathcal{O}$-modules defined in 4.1, and if N is a B-sheaf of $\mathcal{O}$-modules we write $u(N)$ for the U-sheaf defined by the patching condition

$$b(M)(B(\tilde{K})) = e_{\tilde{K}}M(U(K))$$

and

$$u(N)(U(K)) = \check{H}^0_B(U(K);N)$$

Theorem 7.1. *The forgetful and idempotent constructions*

$$u \;\; qce\text{-}B\text{-}\mathcal{O}\text{-}mod \rightleftarrows qce\text{-}U\text{-}\mathcal{O}\text{-}mod \;\; b$$

induce an adjoint equivalence of categories

$$qce\text{-}U\text{-}\mathcal{O}\text{-}mod \simeq qce\text{-}B\text{-}\mathcal{O}\text{-}mod$$

Combining this with 5.9 we obtain the desired simplification of the standard abelian category.

Corollary 7.2. *There is an equivalence*

$$\mathcal{A}(G) = qce\text{-}U\text{-}\mathcal{O}\text{-}mod \simeq qce\text{-}B\text{-}\mathcal{O}^e\text{-}mod = \mathcal{A}^e(G)$$

of abelian categories. □

Proof First note that $bu(N) = N$. Indeed, the forgetful functor u just omits the values of a sheaf except on U-open subsets and b fills them in again, More formally, multiplying by an idempotent is exact, so $bu(N)(B(\tilde{K})) = e_{\tilde{K}}\check{H}^0_B(U(K); M)$ is calculated from the complex in which only terms $M(\mathrm{cod}(\mathbf{f}))$ with $\tilde{K}$ cotoral in $\mathrm{dom}(f_0)$ occur this is the complex formed from the category with initial object $\tilde{K}$, and hence $bu(N)(B(\tilde{K})) = N(B(\tilde{K}))$ as required.

The main thing we need show is that if M is a qce U-sheaf then M can be reconstructed from the B-sheaf $b(M)$ in the sense that $ub(M) = M$. This will suffice, since the definitions of quasi-coherence and extendedness are compatible. Indeed, we observed in 5.8 that if M is quasi-coherent or extended as a U-sheaf then $b(M)$ has the corresponding property. The fact that if N is quasi-coherent or extended as a B-sheaf then $u(N)$ has the corresponding property follows from the fact that $u(b(\mathcal{O})) = \mathcal{O}$.

It remains to prove that $ub(M) = M$, and the proof may be quickly reduced to a very special case. Indeed, we know any U-sheaf is built from sheaves of the form $f_K(T)$ for torsion $\mathcal{O}_{\mathcal{F}/K}$-modules T (see [4, Subsection 4.A]) using short exact sequences and very special sums. Because short exact sequences of sheaves induce long exact sequences in cohomology it suffices to show that for any fixed s, the U-sheaves of the form $\bigoplus_{\dim L=s} e_L(T_L)$ are B-sheaves. Indeed, we may argue by induction on s that a U-sheaf supported in dimensions $\leq s$ is a B-sheaf. For $s = 0$ any such sheaf is $f_1(T)$, and hence a B-sheaf from the special case. The case $s = r$ is the general case.

Now if M is supported in dimensions $\leq s$ the construction of [4, Theorem 4.2] gives a map $M \longrightarrow E$, where $E = \bigoplus_{\dim L=s} f_L(\phi^L M)$ which is an isomorphism at each subgroup of dimension s. Hence we have short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow I \longrightarrow 0 \text{ and } 0 \longrightarrow I \longrightarrow E \longrightarrow M'' \longrightarrow 0$$

These can be compared to the corresponding exact sequences

$$0 \longrightarrow \check{H}^0_B(M') \longrightarrow \check{H}^0_B(M) \longrightarrow \check{H}^0_B(I) \text{ and } 0 \longrightarrow \check{H}^0_B(I) \longrightarrow \check{H}^0_B(E) \longrightarrow \check{H}^0_B(M'')$$

By the inductive hypothesis the comparison is an isomorphism for M' and M'' and by the special case it is an isomorphism for E. It follows that the comparison is an isomorphism, first for I and then for M as required.

We return to prove the result in the basic special case.

Lemma 7.3. *For any $s \leq r$, the U-sheaf $\bigoplus_{\dim L=s} f_K(T_L)$ is a B-sheaf.*

Remark 7.4. It is easy to see that $M = f_G(V)$ has no higher B-cohomology, Since $\mathcal{E}_H$ is invertible on $M(U(H))$, and M is constant, the inversion of Euler classes in the complex defining cohomology may be omitted. However, the U-sheaves $f_K(T)$ often have B-cohomology in higher degrees when K is a proper subgroup. For example if G is the circle, $f_1(\bigoplus_H \mathbb{I}(H))$ has non-trivial first B-cohomology, where $\mathbb{I}(H) = H_*(BG/H)$ with $\mathcal{O}_{\mathcal{F}}$-module structure via projection onto $H^*(BG/H)$.

Proof Suppose $M = \bigoplus_K f_K(T_K)$ where K runs through connected subgroups of dimension s. The Čech complex begins

$$\prod_{i=0}^{s} \prod_{\dim(L)=i} M(B(L)) \longrightarrow \prod_{i=1}^{r} \prod_{\dim(H)=i} \mathcal{E}_H^{-1} \prod_{L \lhd_t H} M(B(L))$$

We pick off small quotient complexes

$$d_{i,s} \quad \prod_{\dim(L)=i} M(B(L)) \longrightarrow \prod_{\dim K=s} \mathcal{E}_K^{-1} \prod_{L \lhd_t K, \dim(L)=i} M(B(L))$$

for $i = 0, 1, \ , s-1$ in turn. This does not change $\check{H}^0_B$ by the following lemma.

Lemma 7.5. *The map $d_{i,s}$ is injective.*

Proof We suppose $d_{i,s}((m_L)_L) = 0$ for $m_L \in M(B(L))$, and show that $m_L = 0$. In our case $M(B(L)) = \bigoplus_{L \trianglelefteq_t K} e_L \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} T_K$, and the contribution of the Kth summand to $M(B(L))$ already has $\mathcal{E}_K$ invertible on it. Accordingly, this summand is detected on the $(L, \tilde{K})$th component where $\tilde{K}$ is a subgroup with identity component K containing L with torus quotient. □

Accordingly,

$$\check{H}^0_B(U(1); M) = \ker \left[\prod_{\dim(L)=s} M(B(L)) \longrightarrow \prod_{\dim H=s+1} \mathcal{E}_H^{-1} \prod_{L \subseteq H \, \dim(L)=i} M(B(L)) \right]$$

We must check this kernel is $\bigoplus_{\dim L=s} M(B(L))$ as required. First, it is clear that the direct sum is in the kernel, because any element of $M(U(L)) = T_L$ is torsion.

Finally, we suppose $x = (x_L)_L$ lies in the kernel, and we must show almost all of the $x_L \in M(B(L))$ are zero. This argument with intersections of maximal subgroups in general position is similar to the one in [4, Lemma 4.3]. First note that for any K of dimension s and any connected subgroup H of dimension $s+1$ there is a subgroup $\tilde{H}$ with identity component H, containing K with torus quotient. The hypothesis therefore states that for each connected $(s+1)$-dimensional H there is a representation $V(H)$ with $V(H)^H = 0$ and $\chi(V(H))x = 0$. However, if $V(H) = \alpha_1(H) \oplus \cdots \oplus \alpha_{n(H)}(H)$ the Euler class $\chi(V(H))$ is the identity on L unless L is contained in one of the kernels $\ker(\alpha_i(H))$; on the other hand we know H does not lie in any of these kernels.

We argue by induction on t that for $t = 0, 1, \ , r-s$ the element x is only non-zero in $M(B(L))$ where L lies in an t-fold generic intersection of maximal subgroups from a finite list. If we can show this when $t = r-s$ the lemma is proved, since each $(r-s)$-fold generic intersection specifies an s-dimensional subgroup H which has only finitely many s-dimensional subgroups.

The assertion is vacuous if $t = 0$, so suppose $0 < t \leq r-s$ and that any L lies in a $(t-1)$-fold generic intersection of the maximal subgroups $M_1, \ , M_{N(t-1)}$. Now since $r-(t-1) \geq s+1$ for each $(t-1)$-fold generic intersection M^*_λ we may choose an $(s+1)$-dimensional connected subgroup $L_\lambda \subseteq M^*_\lambda$. Thus if x is non-zero in $M(B(L))$ then L lies in an intersection $M^*_\lambda \cap \bigcap_\mu \ker(\alpha_{k(j)}(L_\mu)$ and hence in the t-fold generic intersection $M^*_\lambda \cap \ker(\alpha_{k(j)}(L_\lambda)$. This gives the required assertion and hence completes the inductive step. □

This completes the proof of Theorem 7.1. □

The main point of considering B-sheaves is this reconstruction process, obtaining sections over natural open sets $U(H)$ from sections over basic open sets. We can illustrate the power of this for the circle group group this process appeared rather mysterious in [3].

Example 7.6. Let G be the circle group and consider an object $\beta \ \ N \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ of the G-equivariant standard model of [3] (i.e., β is the map of $\mathcal{O}_{\mathcal{F}}$-modules which inverts $\mathcal{E}_G$). We may define a sheaf M on $\mathcal{TC}$ by taking $M(U(1)) = N$, and $M(U(G)) = \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ on natural open sets, and $M(B(F)) = e_F N$ on basic open sets. It was shown in [3, 5.10.2] that the nub N was Hausdorff in the sense that it embeds in $N^\wedge_{\mathcal{F}} = \prod_F e_F N$. This is the first part of the sheaf condition. No attempt was made in [3] to give a complete reconstruction of the object from its basic pieces $e_F N$ and V.

Theorem 7.1 states that M is an acyclic B-sheaf. Explicitly, for global sections this means

$$M(U(1)) \longrightarrow M(B(T)) \times \prod_F M(B(F)) \rightrightarrows \mathcal{E}_T^{-1} \prod_F M(B(F))$$

This could be rewritten as a pullback square

$$\begin{array}{ccc} M(U(1)) & \longrightarrow & M(\coprod_F B(F)) \\ \downarrow & & \downarrow \\ M(B(T)) & \longrightarrow & \mathcal{E}_T^{-1} M(\coprod_F B(F)) \end{array}$$

corresponding to the pushout

$$\begin{array}{ccc} U(1) & \longleftarrow & \coprod_F B(F) \\ \uparrow & & \uparrow \\ B(T) & \longleftarrow & (\coprod_F B(F)) \times_{\mathcal{TC}} B(T) \end{array}$$

We repeat the proof in this special case so that its relation to [3] is highlighted.

Proof We will construct the pullback diagram. Consider the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & N^\wedge_{\mathcal{F}} & \longrightarrow & N^\wedge_{\mathcal{F}}/N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathcal{E}_T^{-1}N & \longrightarrow & \mathcal{E}_T^{-1}N^\wedge_{\mathcal{F}} & \longrightarrow & \mathcal{E}_T^{-1}N^\wedge_{\mathcal{F}}/\mathcal{E}_T^{-1}N & \longrightarrow & 0 \end{array}$$

Here the map $N \longrightarrow N^\wedge_{\mathcal{F}}$ is a monomorphism since N is Hausdorff [3, 5.10.2], and the second row is a localization of the first. The new information is that the left hand vertical is an isomorphism. This follows from the Snake Lemma because inverting $\mathcal{E}_T$ is effected by a direct limit of maps which are isomorphisms away from a finite set of subgroups, whilst $N \longrightarrow N^\wedge_{\mathcal{F}}$ is also an isomorphism on all finite sets of subgroups. A geometric version of this result is the fact that $X \wedge \Pi_F E\langle F\rangle \longrightarrow \Pi_F X \wedge E\langle F\rangle$ is an $\mathcal{F}$-equivalence since for each finite subgroup F', only finitely many of the $E\langle F\rangle$ are F'-equivariantly essential. We therefore have a short exact sequence

$$N \longrightarrow N^{\wedge}_{\mathcal{F}} \longrightarrow \mathcal{E}_T^{-1}N^{\wedge}_{\mathcal{F}}/(\mathcal{E}_T^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

or equivalently

$$M(U(1)) \longrightarrow \Pi_F M(B(F)) \longrightarrow (\mathcal{E}_T^{-1}\Pi_F M(B(F)))/M(B(T)),$$

allowing us to reconstruct $M(U(1))$ from the values $M(B(F))$ and $M(B(T))$. □

Part 3. Categories of R-modules.

We already have two ways to look at the standard abelian category $\mathcal{A}(G)$ as a category of U-sheaves and as a category of B-sheaves. Finally, in this part we show that it can also be viewed as a category of modules over a ring R with many objects. This is only a minor change in terminology, but it is necessary to make contact with the language of Morita equivalences used in [1] and [7].

8. The injectives $I(H)$.

The ring with many objects is an endomorphism ring of a certain subcategory of $\mathcal{A}(G)$. It is the purpose of this section to prepare for its introduction in Section 9 by describing these elements.

8.A. **The injectives $I(H)$ as U-sheaves.** For each subgroup $\tilde{H}$, we may consider the injective $H^*(BG/\tilde{H})$-module

$$\mathbb{I}(G/\tilde{H}) = H_*(BG/\tilde{H}),$$

and view it as a module over $\mathcal{O}_{\mathcal{F}/H}$, where H is the identity component of $\tilde{H}$. The module is injective amongst the modules occurring as $\phi^H M$ for objects M of the standard abelian category. We then take $I(\tilde{H})$ to be the U-sheaf constant below H

$$I(\tilde{H}) = e_H(\mathbb{I}(G/\tilde{H})),$$

or explicitly, for connected subgroups K,

$$I(\tilde{H})(U(K)) = \begin{cases} \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/H}} \mathbb{I}(G/\tilde{H}) & \text{if } K \subseteq H \\ 0 & \text{if } K \not\subseteq H \end{cases}$$

8.B. **The injectives $I(H)$ as B-sheaves.** We may immediately write down the corresponding B-sheaf of $\mathcal{O}^e$-modules.

Lemma 8.1.

$$I(H)^e(B(K)) = \begin{cases} \mathcal{E}_H^{-1}H^*(BG) \otimes_{H^*(BG/H)} \mathbb{I}(G/H) & \textit{if } K \textit{ is cotoral in } H \\ 0 & \textit{if } K \textit{ is not cotoral in } H \end{cases}$$

Proof We first show that $I(H)(B(K)) = 0$ unless K is cotoral in H. Since $I(H)(B(K)) = f_K I(H)(U(K_1))$, we see the answer is only non-zero if $K_1 \subseteq H_1$. Now, if the answer is to be non-zero, the pullback of e_H along $\mathrm{map}(\mathcal{F}/H_1, \mathbb{Q}) \longrightarrow \mathrm{map}(\mathcal{F}/K_1, \mathbb{Q})$ must contain e_K, and so $K \cdot H_1/H_1 = H/H_1$, which is to say that $K \cdot H_1 = H$, or that K is cotoral in H.

The answer when K is cotoral in H follows from the constancy and the value when $K = H$. □

9. The ring R.

We are now ready to introduce the ring R. The set of objects of R is the set of closed subgroups of G, and in fact R is an endomorphism ring. We let $\mathcal{BI}$ denote the set of basic injectives

$$\mathcal{BI} = \{I(H) \mid H \text{ is a subgroup of } G\}$$

Definition 9.1. We define R to be the opposite of the endomorphism ring of $\mathcal{BI}$

$$R(L,K) = \mathrm{Hom}_{\mathcal{A}}(I(K), I(L))$$

We may now make this quite explicit.

Lemma 9.2.

$$R(L,K) = \begin{cases} \mathrm{Hom}_{H^*(BG/K)}(\mathcal{E}^{-1}_{K/L}H^*(BG/L), H^*(BG/K)) & \textit{if } L \textit{ is cotoral in } K \\ 0 & \textit{if } L \textit{ is not cotoral in } K \end{cases}$$

Remark 9.3. Note that the vector space is usually uncountable in each degree. The vector space $R(L,K)$ should be viewed as an $(R(K,K), R(L,L))$-bimodule. From the nature of bimodule structure (and the fact that $R(L,L)$ is much bigger than $R(K,K)$) it is enough to consider it as an $R(L,L)$-module. As such it is the inverse limit of copies of $R(L,L) \otimes_{R(K,K)} R(K,K)^\vee$, under a sequence of surjective maps.

Proof By definition, $I(L)$ is the U-sheaf constant below L_0, or the B-sheaf constant below L

$$I(L) = f^U_{L_0}(\mathbb{I}(G/L)) = f^B_L(\mathbb{I}(G/L))$$

Thus

$$R(L,K) = \begin{cases} \mathrm{Hom}_{H^*(BG/L)}(\mathcal{E}^{-1}_{K/L}H^*(BG/L) \otimes_{H^*(BG/K)} \mathbb{I}(G/K), \mathbb{I}(G/L)) & \text{if } L \text{ is cotoral in } K \\ 0 & \text{otherwise} \end{cases}$$

Next, note that

$$\mathrm{Hom}_{H^*(BG/L)}(\mathcal{E}^{-1}_{K/L}H^*(BG/L) \otimes_{H^*(BG/K)} \mathbb{I}(G/K), \mathbb{I}(G/L)) = (\mathcal{E}^{-1}_{K/L}H^*(BG/L) \otimes_{H^*(BG/K)} \mathbb{I}(G/K))^\vee$$

Finally, to obtain the required expression we use the adjunction

$$\mathrm{Hom}_k(A \otimes_R B, k) = \mathrm{Hom}_R(A, \mathrm{Hom}_k(B,k))$$

with $R = H^*(BG/K), k = \mathbb{Q}, A = \mathcal{E}^{-1}_{K/L}H^*(BG/L)$ and $B = H_*(BG/K)$, together with the fact that $H^*(BG/K)$ is locally finite and hence equal to its double dual. □

Remark 9.4. Some readers may be reassured by a comparison between the calculation performed with U-sheaves and the same calculation performed with B-sheaves

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/L_0}}(\mathcal{E}^{-1}_{K/L_0}\mathcal{O}_{\mathcal{F}/L_0}\otimes_{\mathcal{O}_{\mathcal{F}/K_0}}\mathbb{I}(G/K),\mathbb{I}(G/L))=\\&\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/L_0}}(\mathcal{E}^{-1}_{K/L_0}H^*(BG/L)\otimes_{\mathcal{O}_{\mathcal{F}/K_0}}\mathbb{I}(G/K),\mathbb{I}(G/L))=\\&\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/L_0}}(\mathcal{E}^{-1}_{K/L_0}H^*(BG/L)\otimes_{H^*(BG/L)}\mathbb{I}(G/K),\mathbb{I}(G/L))=\\&\mathrm{Hom}_{H^*(BG/L)}(\mathcal{E}^{-1}_{K/L_0}H^*(BG/L)\otimes_{H^*(BG/L)}\mathbb{I}(G/K),\mathbb{I}(G/L))\quad\Box\end{aligned}$$

If L is cotoral in K there is a simpler but less natural expression for $R(L,K)$. Indeed we have an extension

$$1\longrightarrow K/L\stackrel{i}{\longrightarrow}G/L\stackrel{p}{\longrightarrow}G/K\longrightarrow 1$$

giving maps

$$H^*(BK/L)\longleftarrow H^*(BG/L)\longleftarrow H^*(BG/K)$$

Since K/L is a torus, i is split by some map s, and

$$H^*(G/L)=p^*(H^*(BG/K))\otimes s^*(H^*(BK/L))$$

Thus

$$H^*(G/L)\otimes_{H^*(BG/K)}(\cdot)=s^*H^*(BK/L)\otimes_{\mathbb{Q}}(\cdot)$$

Furthermore, if V is a representation of G/L with $V^K=0$, we find $i^*c(V)=c(i^*V)$, and so

$$\mathcal{E}^{-1}_{K/L}H^*(G/L)\otimes_{H^*(BG/K)}(\cdot)=s^*\mathcal{E}^{-1}_{K/L}H^*(BK/L)\otimes_{\mathbb{Q}}(\cdot)$$

Lemma 9.5. *A splitting s of the inclusion $K/L\longrightarrow G/L$ induces an isomorphism*

$$R(L,K)=\mathrm{Hom}_{H^*(BG/K)}(\mathcal{E}^{-1}_{K/L}H^*(BG/L),H^*(BG/K))=(\mathcal{E}^{-1}_{K/L}H^*(BK/L))^{\vee}$$

Proof We saw in the proof of 9.2 that

$$\mathrm{Hom}_{H^*(BG/K)}(H^*(BG/L),H^*(BG/K))=\mathrm{Hom}_{H^*(BG/K)}(s^*H^*(BK/L),\mathbb{Q}),$$

and multiplication by Euler classes in the cohomology groups corresponds. □

10. The category $\mathcal{A}(G)$ as a category of R-modules.

We are now equipped to describe $\mathcal{A}(G)$ as the category of torsion R-modules. We work with the description of $\mathcal{A}(G)$ as the category of qce B-sheaves of $\mathcal{O}^e$-modules.

First, to become comfortable with variance, note that $R(L,1)=0$ unless $L=1$, whilst (for example) $R(1,G)\neq 0$. Thus a *left* R-module M will include various maps $M(1)\longrightarrow M(G)$. This is how $\mathcal{A}(G)$ works, and it turns out that $\mathcal{A}(G)$ is equivalent to a category of left R-modules.

We will describe a category of torsion R-modules and prove the following theorem.

Theorem 10.1. *There is an equivalence of abelian categories*

$$\mathcal{A}(G)\simeq \textit{tors-}R\textit{-mod}$$

Before proceeding with the proof we illustrate the structure sheaf in its several manifestations in the case of the circle group.

Example 10.2. For the circle group G, we make explicit the structure sheaf $\mathcal{O}$ in the four different formulations of the category $\mathcal{A}(G)$.

qce U-sheaves of $\mathcal{O}$-modules (i.e., $\mathcal{A}(G)$ itself) The structure sheaf is the single diagram

$$\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}$$

corresponding to the inclusion

$$U(1) \supseteq U(G)$$

qce B-sheaves of $\mathcal{O}$-modules The structure sheaf is given by specifying, for each finite subgroup H, the diagram

$$\mathbb{Q}[c] \longrightarrow \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}$$

corresponding to the inclusion

$$B(H) \supseteq B(G)$$

qce B-sheaves of $\mathcal{O}^e$-modules (i.e., $\mathcal{A}^e(G)$) The structure sheaf is given by specifying, for each finite subgroup H, the diagram

$$\mathbb{Q}[c] \longrightarrow \mathbb{Q}[c, c^{-1}] \otimes \mathbb{Q}$$

corresponding to the inclusion

$$B(H) \supseteq B(G)$$

torsion R-modules The structure sheaf is given by specifying, for each finite subgroup H, the diagram

$$\mathbb{Q}[c] \stackrel{\mathbb{Q}[c,c^{-1}]}{\longrightarrow} \mathbb{Q}$$

corresponding to the inclusion

$$H \subseteq G$$

We draw attention to the way in which the picture becomes simpler and simpler as we proceed down the list. Roughly speaking the first formulation is closest to topology and the last is closest to algebra.

Proof We construct a functor

$$r \;\; \text{e-}\mathcal{O}\text{-mod} \simeq R\text{-mod}$$

on extended modules, and observe it is full and faithful. For the present we define torsion R-modules as those in the image of $\mathcal{A}(G)$ (up to isomorphism). We will then turn to the task of describing this image in R-module terms.

The functor r is obvious on objects we define

$$r(M)(L) = \phi^L M$$

To define

$$R(L, K) \otimes r(M)(L) \longrightarrow r(M)(K)$$

we note that there is nothing to be done unless L is cotoral in K, and in that case

$$R(L, K) = \mathrm{Hom}_{\mathcal{A}}(I(K), I(L)) = \mathrm{Hom}_{H^*(BG/K)}(\mathcal{E}_{K/L}^{-1} H^*(BG/L), H^*(BG/K))$$

Supposing that $x \in R(L,K) = \mathrm{Hom}_{H^*(BG/K)}(\mathcal{E}_{K/L}^{-1} H^*(BG/L), H^*(BG/K))$ we define

$$x \quad \phi^{\ L} M \longrightarrow \phi^K M$$

by

$$\phi^L M \longrightarrow \mathcal{E}_{K/L}^{-1} H^*(BG/L) \otimes_{H^*(BG/K)} \phi^K M \stackrel{x \otimes 1}{\longrightarrow} H^*(BG/K) \otimes_{H^*(BG/K)} \phi^K M = \phi^K M$$

One may now check this is compatible with multiplication in R.

Lemma 10.3. *The above construction defines a functor r e-$\mathcal{O}$-mod $\longrightarrow R$-mod.* □

It is obvious that the functor is full and faithful. □

We pause to observe that the functor r is related to another construction.

Lemma 10.4. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(M, I(L)) = (\phi^L M)^{\vee} \quad \square$$

It is evident that $\mathrm{Hom}_{\mathcal{A}}(M, I(\cdot))$ is a right R-module.

Lemma 10.5. *The comparison map*

$$\phi^L M \longrightarrow ((\phi^L M)^{\vee})^{\vee}$$

induces a map

$$r(M) \longrightarrow ((\phi^{\bullet} M)^{\vee})^{\vee}$$

of left R-modules. □

We now begin to identify the category of torsion R-modules. First, it turns out that not all R-modules are of the form $r(M)$ for extended $\mathcal{O}$-modules M. Suppose N is an R-module, and we attempt to define an extended $\mathcal{O}$-module $s(N)$.

Definition 10.6. An R-module N is *quasi-finite* if whenever L is cotoral in K the adjoint structure map

$$N(L) \stackrel{\hat{\beta}}{\longrightarrow} \mathrm{Hom}_{\mathbb{Q}}(R(L,K), N(K)),$$

has image in the image of the comparison map

$$\mathcal{E}_{K/L}^{-1} H^*(BG/L) \otimes_{H^*(BG/K)} N(K) = \mathrm{Hom}_{\mathbb{Q}}(R(L,K), \mathbb{Q}) \otimes N(K) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(R(L,K), N(K))$$

The definition is designed so that the following lemma is obvious.

Lemma 10.7. *The image of r e-$\mathcal{O}$-mod $\longrightarrow R$-mod consists of quasi-finite R-modules, and induces an equivalence between extended $\mathcal{O}$-modules and quasi-finite R-modules.* □

Remark 10.8. We comment that the condition is non-trivial. The required maps

$$N(L) \longrightarrow \mathcal{E}_{K/L}^{-1} H^*(BG/L) \otimes_{H^*(BG/K)} N(K) = \mathcal{E}_{K/L}^{-1} H^*(BK/L) \otimes_{\mathbb{Q}} N(K)$$

can be inferred from the special case in which the codimension of L in K is 1. In this case $\mathcal{E}_{K/L}^{-1} H^*(BG/L) = \mathbb{Q}[c, c^{-1}]$, which is self-dual. Now

$$R(L,K) \otimes N(L) \longrightarrow N(K)$$

corresponds to

$$N(L) \xrightarrow{\hat{\beta}} \mathrm{Hom}_{\mathbb{Q}}(R(L,K), N(K)),$$

and we have a comparison map

$$\mathcal{E}_{K/L}^{-1}H^*(BG/L)\otimes_{H^*(BG/K)}N(K) = \mathrm{Hom}_{\mathbb{Q}}(R(L,K),\mathbb{Q})\otimes N(K) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(R(L,K),N(K))$$

We need to require that $\hat{\beta}$ maps into the image, which is to say that $\hat{\beta}(n)$ is only non-zero on c^i for finitely many i. □

It is now easy to define torsion R-modules to make the theorem true.

Definition 10.9. A torsion R-module N is a quasi-finite R-module with the property that the adjoint structure map

$$N(L) \longrightarrow \mathcal{E}_{K/L}^{-1}H^*(BG/L) \otimes_{H^*(BG/K)} N(K)$$

is localization away from $\mathcal{E}_{K/L}^{-1}$ whenever L is cotoral in K.

Remark 10.10. We may also describe torsion R-modules as those in the thick category generated by sums of R-modules $I(H)$.

Lemma 10.11. *There is a torsion functor*

$$\Gamma \;\; R\text{-}mod \longrightarrow tors\text{-}R\text{-}mod$$

right adjoint to the inclusion

$$i \;\; tors\text{-}R\text{-}mod \longrightarrow R\text{-}mod,$$

Proof The existence of the functor is a formality, since the union of torsion modules is a torsion module. □

Remark 10.12. The explicit construction of a torsion functor gives more information, but is correspondingly more intricate. The particular case of the circle is treated in detail in [3, Chapter 20].

11. The injectives $I(L)$ as R-modules.

Now that we have an alternative view of $\mathcal{A}(G)$, we should identify the basic injectives in it. The main point is that they are like the $H^*(BG)$-module $H_*(BG)$ on a crude level this is the $\mathbb{Q}$-dual of the ring itself, but its significance is as the injective hull of the simple module.

Now that we are considering a ring with many objects we need to proceed with care. For any fixed subgroup L, we may form the *left* R-module $R(L,\cdot)$ and the *right* R-module $R(\cdot,L)$.

First we record a formula for the objects $I(K)$ as R-modules.

Lemma 11.1. *The R-module $I(K)$ is defined by*

$$I(K)(L) = \begin{cases} \mathcal{E}_{K/L}^{-1}H^*(BG/L) \otimes_{H^*(G/K)} \mathbb{I}(G/K) & \text{if } L \text{ is cotoral in } K \\ 0 & \text{otherwise} \end{cases}$$ □

Lemma 11.2. *There is an isomorphism of right R-modules*

$$I(K)^\vee = R(\cdot, K)$$

Proof Both R-modules are zero at L unless, as we now suppose, L is cotoral in K. We then have

$$R(L, K) = \mathrm{Hom}_{\mathcal{A}}(I(K), I(L)) = (\phi^L I(K))^\vee$$

as required. □

Because $R(\cdot, K)$ is usually infinite dimensional in each degree, it does not follow that $R(\cdot, K)^\vee$ is $I(K)$. However $I(K)$ can be identified as a submodule of $R(\cdot, K)^\vee$.

Proposition 11.3.

$$I(K) = \Gamma R(\cdot, K)^\vee$$

Proof Following 11.2, it suffices to show that the natural map $\nu \quad I(K) \longrightarrow (I(K) \quad ^\vee)^\vee$ is the inclusion of the maximal torsion submodule. Certainly the map is a monomorphism and $I(K)$ is a torsion module so it suffices to show that a map $f \quad M \longrightarrow R(\cdot, K) \ ^\vee$ from a torsion module factors through $f' \quad M \longrightarrow I(K)$. This follows immediately, since

$$\mathrm{Hom}_R(M, I(K)) = \mathrm{Hom}_{\mathbb{Q}}(\phi^K M, \mathbb{Q}) = \mathrm{Hom}_R(M, R(\cdot, K)^\vee),$$

where the left hand adjunction only holds for torsion modules M.

□

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