Γ-COHOMOLOGY OF RINGS OF NUMERICAL POLYNOMIALS AND E_∞ STRUCTURES ON K-THEORY

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Abstract. We investigate the Γ-cohomology of some Hopf algebroids E_*E associated with certain periodic cohomology theories. For KU and $E(1)$, the Adams summand at a prime p, we show that Γ-cohomology vanishes above degree 1 and deduce that these spectra admit unique E_∞ structures. For the Johnson-Wilson spectrum $E(n)$ with $n \geqslant 1$ we prove the analogous result for the I_n-adic completion of $E(n)$.

Introduction

In homotopy theory it is often not sufficient to have homotopy ring structures on a spectrum in order to construct for instance homotopy fixed points under a group action or quotient spectra. For this, it is necessary to have ring structures which are not just given up to homotopy but where these homotopies fulfill certain coherence conditions. We will prove the existence and uniqueness of certain E_∞ structures, *i.e.*, spectra with a coherent homotopy commutative multiplication.

Alan Robinson developed in [23] a purely algebraic obstruction theory for E_∞ structures on homotopy associative and commutative ring spectra. The device for deciding whether a spectrum possesses such a structure is a cohomology theory for commutative algebras, Γ-cohomology. When applied to the Hopf algebroid E_*E of a spectrum E, the vanishing of these cohomology groups implies the existence of an E_∞ structure on the spectrum E which extends the given homotopy ring structure.

We will apply Robinson's obstruction theory to complex K-theory KU, the Adams summand $E(1)$, and the I_n-completion of the Johnson-Wilson spectra $E(n)$.

The existence of an E_∞ structure on KU was already known: in [18] an E_∞ structure for the connected version ku was constructed and the techniques of [11, VIII] lead to an E_∞ model for KU. But as far as we know the uniqueness of this structure is not documented. The existence and uniqueness for $E(1)$ appears to be new.

By [12, 21] it is known that the Lubin-Tate spectra E_n have unique E_∞ structures. In particular, $E_1 = KU_p^\wedge$ has a unique E_∞ structure. The results for $KU_p^\wedge$ and $E(1)_p^\wedge$ follow directly from the calculation of continuous Γ-cohomology.

We also prove that for each $n \geqslant 1$ the I_n-completion of $E(n)$ possesses a unique E_∞ structure; the result for E_n then follows using ideas of [25]. However, so far we have not been able to extend this result to $E(n)$ itself since the Γ-cohomology of $E(n)_*E(n)$ appears to be very non-trivial in positive degrees for $n > 1$.

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Notation, etc. All otherwise unspecified tensor products are taken over $\mathbb{Z}$ or a localization at a prime p, $\mathbb{Z}_{(p)}$. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\widehat{\mathbb{Z}}$ denote the rings of p-adic integers, p-adic rationals and profinite integers respectively, while $\widehat{\mathbb{Q}} = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$.

For the topological part, we work in a good category of spectra with a symmetric monoidal smash product, for example that of [11]. Where necessary all ring spectra will be assumed to be fibrant.

1. Technicalities on Γ-cohomology

In this section we discuss some technical issues related to our calculation of Γ-cohomology later in the paper.

Let $\Bbbk$ be a commutative Noetherian ring and $\mathfrak{m} \triangleleft \Bbbk$ a maximal ideal. We topologize $\Bbbk$ with respect the $\mathfrak{m}$-adic topology. If

$$\bigcap_{k \geqslant 0} \mathfrak{m}^k = 0, \tag{1.1}$$

then $\Bbbk$ is Hausdorff with respect to this $\mathfrak{m}$-adic topology.

Now let A be a commutative unital $\Bbbk$-algebra. The ideal $\mathfrak{m}A = A\mathfrak{m} \triangleleft A$ also generates a topology on A which is Hausdorff if (1.1) holds. Then the unit homomorphism $\Bbbk \longrightarrow A$ is automatically continuous and if A is augmented over $\Bbbk$ then the augmentation is also continuous. Furthermore, $(A, \Bbbk)$ is a topological algebra over the topological ring $\Bbbk$.

The $\mathfrak{m}$-*adic completion* of $(A, \Bbbk)$ is $(A^{\widehat{}}_{\mathfrak{m}}, \Bbbk^{\widehat{}}_{\mathfrak{m}})$ where

$$A^{\widehat{}}_{\mathfrak{m}} = \lim_k A/\mathfrak{m}^k A, \quad \Bbbk^{\widehat{}}_{\mathfrak{m}} = \lim_k \Bbbk/\mathfrak{m}^k.$$

We say that $(A, \Bbbk)$ is $\mathfrak{m}$-*adically complete* if $A^{\widehat{}}_{\mathfrak{m}} = A$ and $\Bbbk^{\widehat{}}_{\mathfrak{m}} = \Bbbk$. When $\mathfrak{m}$ is clear from the context we will sometimes simplify notation by writing $(\widehat{A}, \widehat{\Bbbk}) = (A^{\widehat{}}_{\mathfrak{m}}, \Bbbk^{\widehat{}}_{\mathfrak{m}})$. If $(A, \Bbbk)$ is augmented over $\Bbbk$ then so is $(A^{\widehat{}}_{\mathfrak{m}}, \Bbbk^{\widehat{}}_{\mathfrak{m}})$.

For a topological left module M over such a topological algebra $(A, \Bbbk)$ we may consider the Γ-cohomology $\mathrm{H}\Gamma^*(A \mid \Bbbk; M)$; in practise, we will usually consider the $\mathfrak{m}$-adic topology on an A-module M. By [23, 24], this Γ-cohomology can be computed using a cochain complex $\mathrm{Hom}_A(C^\Gamma(A)_*, M)$, where $C^\Gamma(A)_*$ is a certain complex of free left A-modules. Topologising $C^\Gamma(A)_*$ with the $\mathfrak{m}$-adic topology, we can introduce the subcomplex

$$\mathcal{H}\mathrm{om}_A(C^\Gamma(A)_*, M) \subseteq \mathrm{Hom}_A(C^\Gamma(A)_*, M)$$

of continuous cochains whose cohomology $\mathcal{H}\Gamma^*(A \mid \Bbbk; M)$ we call the *continuous* Γ-*cohomology* of A with coefficients in M. Continuous cohomology of profinite groups is described in [26, 27]; for analogues appearing in topology see [4, 5]; our present theory is modelled closely on the presentations in those references.

Notice that the above inclusion of complexes induces a forgetful homomorphism

$$\rho \colon \mathcal{H}\Gamma^*(A \mid \Bbbk; M) \longrightarrow \mathrm{H}\Gamma^*(A \mid \Bbbk; M). \tag{1.2}$$

The continuity condition on Γ-cochains is closely connected with an inverse limit which leads to a Milnor exact sequence relating $\mathcal{H}\Gamma^*$ to ordinary Γ-cohomology. From [14] or [15, Théorème 2.2], recall that $\lim_k^s$ vanishes for $s > 1$. The proof of our next result is routine.

Proposition 1.1. *Let M be a complete Hausdorff topological module over $\widehat{A}$ which is finitely generated over $\widehat{\Bbbk}$. Then for each n there is a short exact sequence*

$$0 \to \lim_k{}^1 \mathrm{H}\Gamma^{n-1}(A/\mathfrak{m}^k A \mid \Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) \longrightarrow \mathcal{H}\Gamma^n(\widehat{A} \mid \widehat{\Bbbk}; M) \longrightarrow \lim_k \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A \mid \Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) \to 0.$$

This leads to some useful calculational results, versions of which have already appeared in [20, 21]. Notice that for any $\widehat{A}$-module M and $k \geqslant 1$, there is a natural reduction homomorphism

$$\mathrm{H}\Gamma^n(\widehat{A} \mid \widehat{\Bbbk}; M) \longrightarrow \mathrm{H}\Gamma^n(A \mid \Bbbk; M) \longrightarrow \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A \mid \Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M), \tag{1.3}$$

compatible with respect to different values of k. Here the right hand map arises from the isomorphism

$$\mathrm{Hom}_A(C^\Gamma(A)_*, M/\mathfrak{m}^k M) \cong \mathrm{Hom}_{A/\mathfrak{m}^k A}(C^\Gamma(A/\mathfrak{m}^k A)_*, M/\mathfrak{m}^k M)$$

which is inherent in the construction of the complex $C^\Gamma(A)_*$ and natural isomorphisms

$$A^{\otimes_\Bbbk r} \underset{\Bbbk}{\otimes} (A/\mathfrak{m}^k A) \cong (A/\mathfrak{m}^k A)^{\otimes_{\Bbbk/\mathfrak{m}^k} r} \underset{\Bbbk/\mathfrak{m}^k}{\otimes} (A/\mathfrak{m}^k A).$$

In turn there is a homomorphism

$$\mathrm{H}\Gamma^n(\widehat{A} \mid \widehat{\Bbbk}; M) \longrightarrow \mathrm{H}\Gamma^n(A \mid \Bbbk; M) \longrightarrow \lim_k \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A \mid \Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M). \tag{1.4}$$

Proposition 1.2. *Let M be an $\widehat{A}$-module which is complete and Hausdorff with respect to the $\mathfrak{m}$-adic topology and finitely generated over $\widehat{\Bbbk}$. Then the natural homomorphism induces an isomorphism*

$$\mathcal{H}\Gamma^*(\widehat{A} \mid \widehat{\Bbbk}; M) \cong \mathrm{H}\Gamma^*(\widehat{A} \mid \widehat{\Bbbk}; M).$$

Proof. Since M is $\mathfrak{m}$-adically complete, there is a short exact sequence

$$0 \to M \longrightarrow \prod_k M/\mathfrak{m}^k M \xrightarrow{\mathrm{id}-\sigma} \prod_k M/\mathfrak{m}^k M \to 0$$

where σ is the shift-reduction map. This defines the inverse limit $\lim_k M/\mathfrak{m}^k IM = M$. The cochain complex functor $\mathrm{Hom}_{\widehat{A}}(C^\Gamma(\widehat{A})_*, \ )$ commutes with limits, so applying it and taking cohomology we obtain for each n a short exact sequence

$$0 \to \lim_k{}^1 \mathrm{H}\Gamma^{n-1}(A/I \mid \Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) \longrightarrow \mathrm{H}\Gamma^n(\widehat{A} \mid \widehat{\Bbbk}; \widehat{\Bbbk}) \longrightarrow \lim_k \mathrm{H}\Gamma^n(A/I \mid \Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) \to 0.$$

Now using the naturality provided by (1.3) we obtain a diagram of short exact sequences from the exact sequence of Proposition 1.1 into the one above. As the homomorphisms at either end are identities, the natural map $\mathcal{H}\Gamma^*(\widehat{A} \mid \widehat{\Bbbk}; M) \longrightarrow \mathrm{H}\Gamma^*(\widehat{A} \mid \widehat{\Bbbk}; M)$ is an isomorphism. □

Remark 1.3. Analogous ideas apply to Hochschild cohomology for which a continuous version appears in [4].

Note that the sequence in the proof of Proposition 1.2 would be the same if one applies the cochain functor $\mathrm{Hom}_A(C^\Gamma(A)_*, \ )$, thus in the cases as above we obtain an isomorphism between $\mathrm{H}\Gamma^n(\widehat{A} \mid \widehat{\Bbbk}; \widehat{\Bbbk})$ and $\mathrm{H}\Gamma^n(A \mid \Bbbk; \widehat{\Bbbk})$. This leads to the following exact sequence which will be used in Section 4.

Proposition 1.4. *There is a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}\Gamma^{n-1}(A \mid \Bbbk; \widehat{\Bbbk}/\Bbbk) \longrightarrow \mathrm{H}\Gamma^{n}(A \mid \Bbbk; \Bbbk) \longrightarrow \mathrm{H}\Gamma^{n}(\widehat{A} \mid \widehat{\Bbbk}; \widehat{\Bbbk}) \longrightarrow \mathrm{H}\Gamma^{n}(A \mid \Bbbk; \widehat{\Bbbk}/\Bbbk) \\ \longrightarrow \mathrm{H}\Gamma^{n+1}(A \mid \Bbbk; \Bbbk) \longrightarrow \cdots$$

Finally, we record a result on the Γ-(co)homology of formally étale algebras that we will make repeated use of. We call an algebra *formally étale* if it is a colimit of étale algebras.

Lemma 1.5. *If $(A, \Bbbk)$ is a formally étale algebra then for any A-module M,*

$$\mathrm{H}\Gamma_*(A \mid \Bbbk; M) = 0 = \mathrm{H}\Gamma^*(A \mid \Bbbk; M).$$

Proof. By [24, Theorem 6.8 (3)], Γ-homology and cohomology vanishes for étale algebras. Also, Γ-homology commutes with colimits. Hence if $A = \operatorname{colim}_r A_r$ with A_r étale, for any A-module N we have

$$\mathrm{H}\Gamma_*(A \mid \Bbbk; N) = \operatorname{colim}_r \mathrm{H}\Gamma_*(A_r \mid \Bbbk; N) = 0.$$

There is a universal coefficient spectral sequence

$$\mathrm{E}_2^{*,*} = \operatorname{Ext}_A^{*,*}(\mathrm{H}\Gamma_*(A \mid \Bbbk; A), M) \Longrightarrow \mathrm{H}\Gamma^*(A \mid \Bbbk; M),$$

which has trivial E_2-term, therefore $\mathrm{H}\Gamma^*(A \mid \Bbbk; M) = 0$. □

2. Linear compactness and cohomology

We refer to [9, chapter III] for a reasonably complete (but not very compact) discussion of linear compactness.

Let $\Bbbk$ be a commutative Noetherian ring and $\mathfrak{m} \lhd \Bbbk$ a maximal ideal. We will assume that $\Bbbk$ is complete and Hausdorff with respect to the $\mathfrak{m}$-adic topology. For each $k \geqslant 0$, $\mathfrak{m}^k$ is a finitely generated $\Bbbk$-module while $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a finitely generated $\Bbbk/\mathfrak{m}$-module which is therefore an Artinian $\Bbbk/\mathfrak{m}$-module.

Recall that a topological $\Bbbk$-module M is *topologically free on a countable basis* $\{b_i\}_{i\geqslant 1}$ if for each element $m \in M$ and $k \geqslant 1$, in $M/\mathfrak{m}^k M$ considered as an $\Bbbk/\mathfrak{m}^k$-module, there is a unique (finite) expansion

$$\overline{m} = \sum_{i\geqslant 1} \overline{r_i}\overline{b_i}$$

with $\overline{r_i} \in \Bbbk/\mathfrak{m}^k$, where $\overline{(\ )}$ denotes residue class modulo $\mathfrak{m}^k$. As a consequence, m has a unique expansion as a limit sum

$$m = \sum_{i\geqslant 1} t_i b_i$$

where $t_i \to 0$ as $i \to \infty$; this means that for each k, there is an n_k such that for $i > n_k$ we have $t_i \in \mathfrak{m}^k$. The linear topology on M has basic open neighbourhoods of 0 of the form $\mathfrak{m}^k M$. Now the Noetherian condition on $\Bbbk$ implies that

$$\mathfrak{m}^k M = \{\sum_{i\geqslant 1} t_i b_i : t_i \in \mathfrak{m}^k\}. \tag{2.1}$$

Proposition 2.1. *Suppose that $\Bbbk$ is complete and Hausdorff with respect to the $\mathfrak{m}$-adic topology and that M is a finitely generated $\Bbbk$-module. If L is a $\Bbbk$-module which is complete and Hausdorff with respect to the $\mathfrak{m}$-adic topology and topologically free on a countable basis then $\mathcal{H}\mathrm{om}_{\Bbbk}(L, M)$ is linearly compact $\Bbbk$-module.*

Proof. First note that

$$\mathcal{H}\mathrm{om}_{\Bbbk}(L,M)=\mathcal{H}\mathrm{om}_{\Bbbk}(L,\lim_k M/\mathfrak{m}^kM)=\lim_k \mathcal{H}\mathrm{om}_{\Bbbk}(L,M/\mathfrak{m}^kM).$$

If $\{b_j\}_{j\geqslant 1}$ is a topological basis for L, then using the Noetherian condition on $\Bbbk$ we find that the basic neighbourhoods of 0 in L are the submodules $\mathfrak{m}^kL\subseteq L$. From this we find that

$$\begin{aligned}\mathcal{H}\mathrm{om}_{\Bbbk}(L,M/\mathfrak{m}^kM)&=\mathcal{H}\mathrm{om}_{\Bbbk/\mathfrak{m}^k}(L/\mathfrak{m}^kL,M/\mathfrak{m}^kM)\\&=\mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}(L/\mathfrak{m}^kL,M/\mathfrak{m}^kM)\\&=\prod_{j\geqslant 1}\mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}((\Bbbk/\mathfrak{m}^k)b_j,M/\mathfrak{m}^kM).\end{aligned}$$

But

$$\mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}((\Bbbk/\mathfrak{m}^k)b_j,M/\mathfrak{m}^kM)=M/\mathfrak{m}^kM$$

and this is Artinian, hence linearly compact. This in turn implies that the final product above is also linearly compact. The claim now follows since $\lim_k \mathcal{H}\mathrm{om}_{\Bbbk}(L,M/\mathfrak{m}^kM)$ is a closed subspace of the product

$$\prod_{j\geqslant 1}\mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}((\Bbbk/\mathfrak{m}^k)b_j,M/\mathfrak{m}^kM).$$ □

We will apply this in the following situation.

Corollary 2.2. *Suppose further that A is a topological $\Bbbk$-algebra with respect to the $\mathfrak{m}$-adic topology inherited from $\Bbbk$ and that L and M are topological A-modules. Then $\mathcal{H}\mathrm{om}_A(L,M)\subseteq \mathcal{H}\mathrm{om}_{\Bbbk}(L,M)$ is a closed $\Bbbk$-submodule. Hence $\mathcal{H}\mathrm{om}_A(L,M)$ is linearly compact.*

Proof. The two continuous action maps

$$A\otimes_{\Bbbk}\mathcal{H}\mathrm{om}_{\Bbbk}(L,M)\longrightarrow \mathcal{H}\mathrm{om}_{\Bbbk}(L,M)$$

given by

$$a\otimes f\longmapsto af,\quad a\otimes f\longmapsto f(a(-))$$

are equalised on $\mathcal{H}\mathrm{om}_A(L,M)$, so this is a closed subset of $\mathcal{H}\mathrm{om}_{\Bbbk}(L,M)$. □

Let us consider what happens when L is not necessarily Hausdorff in Proposition 2.1. In this case, Nakayama's Lemma implies that for any $f\in\mathcal{H}\mathrm{om}_{\Bbbk}(L,M)$ we have

$$f\bigcap_{k\geqslant 1}\mathfrak{m}^kL=0.$$

Hence such an f factors through the quotient $L_0=L/\bigcap_{k\geqslant 1}\mathfrak{m}^kL$, so we might as well replace L by this Hausdorff quotient. Then we have

$$\mathcal{H}\mathrm{om}_{\Bbbk}(L,M)=\mathcal{H}\mathrm{om}_{\Bbbk}(L_0,M). \tag{2.2}$$

Similarly, in Corollary 2.2, if A is not Hausdorff then setting $A_0=A/\bigcap_{k\geqslant 1}\mathfrak{m}^kA$ we have

$$\mathcal{H}\mathrm{om}_{A}(L,M)=\mathcal{H}\mathrm{om}_{A_0}(L_0,M). \tag{2.3}$$

Proposition 2.3. *Let (C^*,δ) be a cochain complex of linearly compact and Hausdorff $\Bbbk$-modules where for each n, the coboundary $\delta^n\colon C^n\longrightarrow C^{n+1}$ is continuous. Then for each n, $\mathrm{H}^n(C^*,\delta)$ is linearly compact.*

Proof. Since each C^n is linearly compact and Hausdorff, the submodules $\mathrm{Im}\,\delta^{n-1}$ and $\mathrm{Ker}\,\delta^n$ of C^n are both closed. Therefore

$$\mathrm{H}^n(C^*,\delta)=\mathrm{Ker}\,\delta^n/\,\mathrm{Im}\,\delta^{n-1}.$$ □

3. Rings of numerical polynomials

We begin by recalling the definitions and some properties of certain rings of numerical polynomials. These appeared in a topological setting in [2, 7] and we follow these sources in our discussion. We will need various basic results on these rings.

Let

$$\begin{aligned} \mathbb{A} &= \{f(w) \in \mathbb{Q}[w] : \forall n \in \mathbb{Z},\ f(n) \in \mathbb{Z}\}, \\ \mathbb{A}^{\mathrm{s}} &= \{f(w) \in \mathbb{Q}[w, w^{-1}] : \forall n \in \mathbb{Z} - \{0\},\ f(n) \in \mathbb{Z}[1/n]\}. \end{aligned}$$

We refer to these as the rings of *numerical* and *stably numerical* polynomials (over $\mathbb{Z}$). If x, y are indeterminates, we can work in any of the rings $\mathbb{A}[x,y]$, $\mathbb{A}^{\mathrm{s}}[x,y]$ or $\mathbb{Q}[w,w^{-1}][x,y]$.

We will make use of the binomial coefficient functions

$$c_n(w) = \binom{w}{n} = \frac{w(w-1)\cdots(w-n+1)}{n!} \in A \subseteq \mathbb{Q}[w]$$

which can be encoded in the generating function

$$(1+x)^w = \sum_{n\geqslant 0} c_n(w) x^n \in \mathbb{A}[x] \subseteq \mathbb{Q}[w][x].$$

Notice that this satisfies the formal identity

$$(1+x)^w(1+y)^w = (1+(x+y+xy))^w. \tag{3.1a}$$

Thus we have

$$c_m(w)c_n(w) = \binom{m+n}{m} c_{m+n}(w) + (\text{terms of lower degree}) \quad (m,n \geqslant 0). \tag{3.1b}$$

Theorem 3.1. ([1, 7])

(a) *$\mathbb{A}$ is a free $\mathbb{Z}$-module with a basis consisting of the $c_n(w)$ for $n \geqslant 0$.*
(b) *$\mathbb{A}^{\mathrm{s}}$ is the localization $\mathbb{A}^{\mathrm{s}} = \mathbb{A}[w^{-1}]$ and it is a free $\mathbb{Z}$-module on a countable basis.*

Describing explicit $\mathbb{Z}$-bases for $\mathbb{A}^{\mathrm{s}}$ seems to be a non-trivial exercise; see [10, 16]. On the other hand, the multiplicative structure of the $\mathbb{Z}$-algebra $\mathbb{A}^{\mathrm{s}}$ is in some ways more understandable. Our next result describes some generators for $\mathbb{A}^{\mathrm{s}}$.

Theorem 3.2. [2, 7]

(a) *The $\mathbb{Z}$-algebra $\mathbb{A}$ is generated by the elements $c_m(w)$ with $m \geqslant 1$ subject to the relations of* (3.1).
(b) *The $\mathbb{Z}$-algebra, $\mathbb{A}^{\mathrm{s}}$ is generated by the elements w^{-1} and $c_m(w)$ with $m \geqslant 1$.*
(c) *We have*

$$\mathbb{A} \otimes \mathbb{Q} = \mathbb{Q}[w], \quad \mathbb{A}^{\mathrm{s}} \otimes \mathbb{Q} = \mathbb{Q}[w, w^{-1}].$$

It is much easier to work with the localizations of these rings at a prime $p \geqslant 2$, $\mathbb{A}_{(p)}$ and $\mathbb{A}^{\mathrm{s}}_{(p)}$.

$$\mathbb{A}_{(p)} = \{f(w) \in \mathbb{Q}[w] : \forall u \in \mathbb{Z}_{(p)},\ f(u) \in \mathbb{Z}_{(p)}\}, \tag{3.2a}$$

$$\mathbb{A}^{\mathrm{s}}_{(p)} = \{f(w) \in \mathbb{Q}[w] : \forall u \in \mathbb{Z}_{(p)}^{\times},\ f(u) \in \mathbb{Z}_{(p)}\}. \tag{3.2b}$$

Theorem 3.3. [1, 7]

(a) *$\mathbb{A}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module with a basis consisting of the monomials in the binomial coefficient functions*

$$w^{r_0} c_p(w)^{r_1} c_{p^2}(w)^{r_2} \cdots c_{p^\ell}(w)^{r_\ell},$$

where $r_k = 0, 1, \ldots, p-1$.

(b) *The* $\mathbb{Z}_{(p)}$*-algebra* $\mathbb{A}_{(p)}$ *is generated by the elements* $c_{p^m}(w)$ *with* $m \geqslant 0$ *subject to relations of the form*

$$c_{p^m}(w)^p - c_{p^m}(w) = p\, d_{m+1}(w),$$

where $d_{m+1}(w) \in \mathbb{A}_{(p)}$ *has* $\deg d_{m+1}(w) = p^{m+1}$*. In fact the monomials*

$$w^{r_0} d_1(w)^{r_1} d_2(w)^{r_2} \cdots d_\ell(w)^{r_\ell},$$

where $r_k = 0, 1, \ldots, p-1$*, form a basis of* $\mathbb{A}_{(p)}$ *over* $\mathbb{Z}_{(p)}$ *and are subject to multiplicative relations of the form*

$$d_m(w)^p - d_m(w) = p\, d'_{m+1}(w),$$

where $\deg d'_{m+1}(w) = p^{m+1}$.

(c) $\mathbb{A}^{\mathrm{s}}_{(p)}$ *is the localization* $\mathbb{A}^{\mathrm{s}}_{(p)} = \mathbb{A}_{(p)}[w^{-1}]$ *and it is a free* $\mathbb{Z}_{(p)}$*-module on a countable basis.*

(d) *The* $\mathbb{Z}_{(p)}$*-algebra,* $\mathbb{A}^{\mathrm{s}}_{(p)}$ *is generated by the elements* w *and* $e_m(w) \in \mathbb{A}^{\mathrm{s}}_{(p)}$ *for* $m \geqslant 1$ *defined recursively by*

$$w^{p-1} - 1 = pe_1(w), \qquad e_m(w)^p - e_m(w) = pe_{m+1}(w) \quad (m \geqslant 1).$$

Corollary 3.4. *Let* p *be a prime.*

(a) *As* $\mathbb{F}_p$*-algebras,*

$$\mathbb{A}/p\mathbb{A} = \mathbb{F}_p[c_{p^m}(w) : m \geqslant 0]/(c_{p^m}(w)^p - c_{p^m}(w) : m \geqslant 0),$$
$$\mathbb{A}^{\mathrm{s}}/p\mathbb{A}^{\mathrm{s}} = \mathbb{F}_p[w, e_m(w) : m \geqslant 0]/(w^{p-1} - 1, e_m(w)^p - e_m(w) : m \geqslant 1).$$

Hence these algebras are formally étale over $\mathbb{F}_p$*.*

(b) *For* $n \geqslant 1$*,* $\mathbb{A}/p^n\mathbb{A}$ *and* $\mathbb{A}^{\mathrm{s}}/p^n\mathbb{A}^{\mathrm{s}}$ *are formally étale over* $\mathbb{Z}/p^n$*.*

(c) *The* p*-adic completions* $\mathbb{A}_p = \lim_n \mathbb{A}/p^n\mathbb{A}$ *and* $\mathbb{A}^{\mathrm{s}}_p = \lim_n \mathbb{A}^{\mathrm{s}}/p^n\mathbb{A}^{\mathrm{s}}$ *are formally étale over* $\mathbb{Z}_p$*.*

Proof. Parts (b) and hence (c) can be proved by induction on $n \geqslant 1$ using the *infinite dimensional Hensel lemma* of [4, 3.9]. The case $n = 1$ is immediate from (a). Suppose that we have found a sequence of elements $s_0, s_1, \ldots, s_k, \ldots \in \mathbb{A}_{(p)}$ satisfying

$$s_m^p - s_m \equiv 0 \pmod{p^n} \quad (m \geqslant 0).$$

Taking $s'_m = s_m + (s_m^p - s_m)$ we find that

$$\begin{aligned} {s'_m}^p - s'_m &= (s_m + (s_m^p - s_m))^p - (s_m + (s_m^p - s_m)) \\ &\equiv s_m^p - (s_m + (s_m^p - s_m)) \pmod{p^{n+1}} \\ &= 0. \end{aligned}$$

Hence for every n we can inductively produce such elements $s_{n,m} \in \mathbb{A}_{(p)}$ for which

$$\begin{aligned} \mathbb{A}/p^n\mathbb{A} &= \mathbb{Z}/p^n[s_{n,m} : m \geqslant 0]/(s_{n,m}^p - s_{n,m} : m \geqslant 0) \\ &= \bigotimes_{m \geqslant 0} \mathbb{Z}/p^n[s_{n,m}]/(s_{n,m}^p - s_{n,m}). \end{aligned}$$

Now passing to p-adic limits we obtain elements $s_m = \lim_{n\to\infty} s_{n,m} \in \mathbb{A}_p$ for which

$$s_m^p - s_m = 0.$$

In these cases we obtain for the module of 1-forms

$$\Omega^1_{(\mathbb{A}/p^n\mathbb{A})/\mathbb{Z}/p^n} = 0 = \Omega^1_{\mathbb{A}_p/\mathbb{Z}_p}.$$

□

There are two natural choices of augmentation for $\mathbb{A}$, namely evaluation at 0 or 1,

$$\varepsilon_+\colon \mathbb{A} \longrightarrow \mathbb{Z};\quad \varepsilon_+ f(w) = f(0),$$
$$\varepsilon_\times\colon \mathbb{A} \longrightarrow \mathbb{Z};\quad \varepsilon_\times f(w) = f(1).$$

For our purposes, the latter augmentation will be used. Notice that there is a ring automorphism

$$\varphi\colon \mathbb{A} \longrightarrow \mathbb{A};\quad \varphi f(w) = f(w+1)$$

for which $\varepsilon_+\varphi = \varepsilon_\times$, so these augmentations are not too dissimilar. In fact, they correspond to different bialgebra structures on $\mathbb{A}$, one of which extends to a Hopf algebra structure on $\mathbb{A}^{\mathrm{s}}$. There are also coproducts

$$\begin{aligned}
\psi_+\colon \mathbb{A} &\longrightarrow \mathbb{A}\otimes\mathbb{A}; & \psi_+ f(w) &= f(w\otimes 1 + 1\otimes w),\\
\psi_\times\colon \mathbb{A} &\longrightarrow \mathbb{A}\otimes\mathbb{A}; & \psi_\times f(w) &= f(w\otimes w),\\
\psi_\times\colon \mathbb{A}^{\mathrm{s}} &\longrightarrow \mathbb{A}^{\mathrm{s}}\otimes\mathbb{A}^{\mathrm{s}}; & \psi_\times f(w) &= f(w\otimes w),
\end{aligned}$$

and antipodes

$$\begin{aligned}
\chi_+\colon \mathbb{A} &\longrightarrow \mathbb{A}; & \chi_+ f(w) &= f(-w),\\
\chi_\times\colon \mathbb{A}^{\mathrm{s}} &\longrightarrow \mathbb{A}^{\mathrm{s}}; & \chi_\times f(w) &= f(w^{-1}).
\end{aligned}$$

Theorem 3.5. *$(\mathbb{A}, \psi_+, \chi_+, \varepsilon_+)$ and $(\mathbb{A}^{\mathrm{s}}, \psi_\times, \chi_\times, \varepsilon_\times)$ are cocommmutative Hopf algebras over $\mathbb{Z}$, while $(\mathbb{A}, \psi_+, \varepsilon_+)$ is a cocommutative bialgebra.*

4. The Γ-cohomology of numerical polynomials

Recall that $\widehat{\mathbb{Z}}/\mathbb{Z}$ and $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ for any prime p are torsion-free divisible groups, so they are both $\mathbb{Q}$-vector spaces which have the same cardinality and (uncountable) dimensions; thus they are isomorphic. Similarly, we have $\widehat{\mathbb{Z}}/\mathbb{Z} \cong \widehat{\mathbb{Q}}/\mathbb{Q}$ and $\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}$.

As remarked earlier, we will use the augmentations $\varepsilon_\times\colon \mathbb{A} \longrightarrow \mathbb{Z}$ and $\varepsilon_\times\colon \mathbb{A}^{\mathrm{s}} \longrightarrow \mathbb{Z}$ and their analogues for the p-localized versions.

Theorem 4.1. *We have*

$$\mathrm{H}\Gamma^n(\mathbb{A}^{\mathrm{s}} \mid \mathbb{Z};\mathbb{Z}) = \mathrm{H}\Gamma^n(\mathbb{A} \mid \mathbb{Z};\mathbb{Z}) = \begin{cases} \widehat{\mathbb{Z}}/\mathbb{Z} & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

For a prime p,

$$\mathrm{H}\Gamma^n(\mathbb{A}^{\mathrm{s}}_{(p)} \mid \mathbb{Z}_{(p)};\mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^n(\mathbb{A}_{(p)} \mid \mathbb{Z}_{(p)};\mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_p/\mathbb{Z}_{(p)} & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Since

$$\mathrm{H}\Gamma^*(\mathbb{A}[w^{-1}] \mid \mathbb{A};\mathbb{Z}) = 0 = \mathrm{H}\Gamma^*(\mathbb{A}_{(p)}[w^{-1}] \mid \mathbb{A}_{(p)};\mathbb{Z}),$$

the Transitivity Theorem [24, 3.4] implies that there are isomorphisms

$$\mathrm{H}\Gamma^*(\mathbb{A} \mid \mathbb{Z};\mathbb{Z}) \cong \mathrm{H}\Gamma^*(\mathbb{A}[w^{-1}] \mid \mathbb{Z};\mathbb{Z}) = \mathrm{H}\Gamma^*(\mathbb{A}^{\mathrm{s}} \mid \mathbb{Z};\mathbb{Z}),$$
$$\mathrm{H}\Gamma^*(\mathbb{A}_{(p)} \mid \mathbb{Z}_{(p)};\mathbb{Z}_{(p)}) \cong \mathrm{H}\Gamma^*(\mathbb{A}_{(p)}[w^{-1}] \mid \mathbb{Z}_{(p)};\mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^*(\mathbb{A}^{\mathrm{s}}_{(p)} \mid \mathbb{Z}_{(p)};\mathbb{Z}_{(p)}),$$

hence it suffices to prove the result for $\mathbb{A}$ and $\mathbb{A}_{(p)}$.

For each natural number n, on writing $n = \prod_p p^{\mathrm{ord}_p n}$ where the product is taken over all primes p, the Chinese Remainder Theorem gives splittings

$$\mathbb{Z}/n = \prod_p \mathbb{Z}/p^{\mathrm{ord}_p n},\quad \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

Since $\mathrm{H}\Gamma(\mathbb{A} \mid \mathbb{Z};\ )$ commutes with limits, by Proposition 1.2 we have

$$\mathrm{H}\Gamma^*(\mathbb{A} \mid \mathbb{Z}; \widehat{\mathbb{Z}}) = \prod_p \mathrm{H}\Gamma^*(\mathbb{A} \mid \mathbb{Z}; \mathbb{Z}_p) = \prod_p \mathcal{H}\Gamma^*(\mathbb{A}_p \mid \mathbb{Z}_p; \mathbb{Z}_p).$$

Now by Corollary 3.4, for each $k \geqslant 1$ we have

$$\mathrm{H}\Gamma^*(\mathbb{A}/p^k\mathbb{A} \mid \mathbb{Z}/p^k; \mathbb{Z}/p^k) = 0.$$

Therefore we obtain

$$\mathcal{H}\Gamma^*(\mathbb{A}_p \mid \mathbb{Z}_p; \mathbb{Z}_p) = \lim_{k\geqslant 1} \mathrm{H}\Gamma^*(\mathbb{A}/p^k\mathbb{A} \mid \mathbb{Z}/p^k; \mathbb{Z}/p^k) = 0 = \lim_{k\geqslant 1}{}^1 \mathrm{H}\Gamma^*(\mathbb{A}/p^k\mathbb{A} \mid \mathbb{Z}/p^k; \mathbb{Z}/p^k).$$

Now for each n, Proposition 1.4 implies that

$$\mathrm{H}\Gamma^n(\mathbb{A}_{(p)} \mid \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^{n-1}(\mathbb{A}_{(p)} \mid \mathbb{Z}_{(p)}; \mathbb{Z}_p/\mathbb{Z}_{(p)}),$$
$$\mathrm{H}\Gamma^n(\mathbb{A} \mid \mathbb{Z}; \mathbb{Z}) = \mathrm{H}\Gamma^{n-1}(\mathbb{A} \mid \mathbb{Z}; \widehat{\mathbb{Z}}/\mathbb{Z}).$$

By [21, 4.1] and the fact that $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ and $\widehat{\mathbb{Z}}/\mathbb{Z}$ are $\mathbb{Q}$-vector spaces, we have

$$\mathrm{H}\Gamma^*(\mathbb{A}_{(p)} \mid \mathbb{Z}_{(p)}; \mathbb{Z}_p/\mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^*(\mathbb{A}\otimes\mathbb{Q} \mid \mathbb{Q}; \mathbb{Z}_p/\mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^*(\mathbb{Q}[w] \mid \mathbb{Q}; \mathbb{Z}_p/\mathbb{Z}_{(p)}) = \mathbb{Z}_p/\mathbb{Z}_{(p)}$$

and

$$\mathrm{H}\Gamma^*(\mathbb{A} \mid \mathbb{Z}; \widehat{\mathbb{Q}}/\mathbb{Q}) = \mathrm{H}\Gamma^*(\mathbb{A}\otimes\mathbb{Q} \mid \mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z}) = \mathrm{H}\Gamma^*(\mathbb{Q}[w] \mid \mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z}) = \widehat{\mathbb{Z}}/\mathbb{Z}.$$

Thus we obtain

$$\mathrm{H}\Gamma^n(\mathbb{A}_{(p)} \mid \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_p/\mathbb{Z}_{(p)} & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{H}\Gamma^n(\mathbb{A} \mid \mathbb{Z}; \mathbb{Z}) = \begin{cases} \widehat{\mathbb{Z}}/\mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$ □

5. The ring of $\mathbb{Z}/(p-1)$-invariants in $\mathbb{A}^{\mathrm{s}}_{(p)}$

In this section, p always denotes an *odd* prime.

Since polynomial functions $\mathbb{Z}_{(p)}^{\times} \longrightarrow \mathbb{Q}$ are continuous with respect to the p-adic topology they extend to continuous functions $\mathbb{Z}_p^{\times} \longrightarrow \mathbb{Q}_p$; such functions which also map $\mathbb{Z}_{(p)}^{\times}$ into $\mathbb{Z}_{(p)}$ give continuous functions $\mathbb{Z}_p^{\times} \longrightarrow \mathbb{Z}_p$. Hence we can regard $\mathbb{A}^{\mathrm{s}}_{(p)}$ as a subring of $\mathbb{Q}_p[w, w^{-1}]$ which in turn can be viewed as a space of continuous functions on the p-adic units $\mathbb{Z}_p^{\times}$.

There is a splitting of topological groups

$$\mathbb{Z}_p^{\times} \cong \mathbb{Z}/(p-1) \times (1 + p\mathbb{Z}_p) \quad (p \geqslant 3),$$

where $\mathbb{Z}/(p-1)$ identifies with a subgroup generated by a primitive $(p-1)$-st root of unity ω. There is also a bicontinuous isomorphism $1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$.

For an odd prime p, the group $\langle\omega\rangle \cong \mathbb{Z}/(p-1)$ acts continuously on $\mathbb{Q}_p[w, w^{-1}]$ by

$$\omega \cdot f(w) = f(\omega w)$$

and it is immediate that this action sends elements of $\mathbb{A}^{\mathrm{s}}_{(p)}$ to continuous functions $\mathbb{Z}_p^{\times} \longrightarrow \mathbb{Z}_p$. It then makes sense to ask for the subring of $\mathbb{A}^{\mathrm{s}}_{(p)}$ fixed by this action, ${}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}$.

Recall the elements $e_m(w)$ of Theorem 3.3(e). We will write $\overline{e}_m(w)$ for $w^{-1}e_m(w)$.

Proposition 5.1. *As a $\mathbb{Z}_{(p)}$-algebra, ${}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}$ is generated by the elements w^{p-1} and $\overline{e}_m(w)$ for $m \geqslant 1$.*

Proof. It is clear that

$$^{\omega}\mathbb{Q}[w, w^{-1}] = \mathbb{Q}[w^{p-1}, w^{-(p-1)}].$$

Also, by construction of the $e_m(w)$,

$$\overline{e}_m(w) \in {}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)} \subseteq \mathbb{Q}[w^{p-1}, w^{-(p-1)}].$$

Consider the multiplicative idempotent

$$E_\omega\colon \mathbb{Q}[w, w^{-1}] \longrightarrow \mathbb{Q}[w, w^{-1}]; \quad E_\omega f(w) = \frac{1}{p-1}\sum_{r=1}^{p-1} f(\omega^r w).$$

Then we have

$$^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)} = E_\omega \mathbb{A}^{\mathrm{s}}_{(p)}.$$

Each element $f(w) \in \mathbb{Q}[w, w^{-1}]$ has the form

$$f(w) = f_0(w^{p-1}) + w f_1(w^{p-1}) + \cdots + w^{p-2} f_{p-2}(w^{p-1})$$

where $f_k(x) \in \mathbb{Q}[x, x]$, hence

$$E_\omega f(w) = f_0(w^{p-1}).$$

From this it easily follows that $^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}$ is generated as an $\mathbb{Z}_{(p)}$-algebra by the stated elements. □

Corollary 5.2. *The following hold.*

(a) *As* $\mathbb{F}_p$*-algebras,*

$$^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}/p({}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}) = \mathbb{F}_p[w, \overline{e}_m(w) : m \geqslant 1]/(w^{p-1} - 1, \overline{e}_m(w)^p - \overline{e}_m(w) : m \geqslant 1).$$

Hence this algebra is formally étale over $\mathbb{F}_p$.

(b) *For* $n \geqslant 1$, $^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}/p^n({}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)})$ *is formally étale over* $\mathbb{Z}/p^n$.

(c) *The p-adic completion* $^{\omega}\mathbb{A}^{\mathrm{s}}_{p} = \lim_n {}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}/p^n({}^{\omega}\mathbb{A}^{\mathrm{s}})_{(p)}$ *is formally étale over* $\mathbb{Z}_p$.

These results allow us to calculate the Γ-cohomology of $^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}$ over $\mathbb{Z}_{(p)}$ directly as was done above for $\mathbb{A}^{\mathrm{s}}$. Alternatively, we may use the fact that the extension $\mathbb{A}^{\mathrm{s}}_{(p)}/{}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}$ is étale since it has the form B/A, where $B = A[t]/(t^{p-1} - v)$ for a unit $v \in A$ where A is a $\mathbb{Z}_{(p)}$-algebra. We can now determine the Γ-cohomology of $^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}$ since the Transitivity Theorem of [24, 3.4] gives

Proposition 5.3. *For an odd prime p,*

$$\mathrm{H}\Gamma^*({}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)} \mid \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^*(\mathbb{A}^{\mathrm{s}}_{(p)} \mid \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}).$$

6. Applications to E_∞ structures on K-theory

Robinson [23] has developed an obstruction theory for E_∞ structures on a homotopy commutative ring spectrum E. Provided E satisfies a Künneth theorem and a universal coefficient theorem for E_*E (both are true if E_*E is E_* projective), then the obstructions lie in groups $\mathrm{H}\Gamma^{n,2-n}(E_*E \mid E_*; E_*)$, while the extensions are determined by classes in $\mathrm{H}\Gamma^{n-1,2-n}(E_*E \mid E_*; E_*)$; here the bigrading (s,t) involves cohomological degree s and internal degree t. Moreover, relevant values of n are for $n \geqslant 4$.

We want to apply this to the cases of complex KU-theory and the Adams summand $E(1)$ of $KU_{(p)}$ at a prime p. Recall that

$$KU_* = \mathbb{Z}[t, t^{-1}], \quad KU_{(p)*} = \mathbb{Z}_{(p)}[t, t^{-1}], \quad E(1)_* = \mathbb{Z}_{(p)}[u, u^{-1}],$$

where $t \in KU_2$ and $u \in E(1)_{2(p-1)}$ The next result implies that the relevant conditions mentioned above are both satisfied for KU and $E(1)$.

Proposition 6.1. *We have*

$$KU_0KU \cong \mathbb{A}^{\mathrm{s}}, \quad {KU_{(p)}}_0KU_{(p)} \cong \mathbb{A}^{\mathrm{s}}_{(p)}, \quad E(1)_0E(1) \cong {}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}.$$

Hence,

$$KU_*KU \cong KU_* \otimes \mathbb{A}^{\mathrm{s}}, \quad {KU_{(p)}}_*KU_{(p)} \cong {KU_{(p)}}_* \otimes \mathbb{A}^{\mathrm{s}}_{(p)}, \quad E(1)_*E(1) \cong E(1)_* \otimes {}^{\omega}\mathbb{A}^{\mathrm{s}}_{(p)}.$$

Thus we may use Theorem 4.1 to show that

Theorem 6.2. *For a prime p and $n \geqslant 4$,*

$$\mathrm{H}\Gamma^{n,2-n}(KU_*KU \mid KU_*; KU_*) = 0 = \mathrm{H}\Gamma^{n-1,2-n}(KU_*KU \mid KU_*; KU_*),$$
$$\mathrm{H}\Gamma^{n,2-n}({KU_{(p)}}_*KU_{(p)} \mid {KU_{(p)}}_*; {KU_{(p)}}_*) = 0 = \mathrm{H}\Gamma^{n-1,2-n}({KU_{(p)}}_*KU_{(p)} \mid {KU_{(p)}}_*; {KU_{(p)}}_*),$$
$$\mathrm{H}\Gamma^{n,2-n}(E(1)_*E(1) \mid E(1)_*; E(1)_*) = 0 = \mathrm{H}\Gamma^{n-1,2-n}(E(1)_*E(1) \mid E(1)_*; E(1)_*).$$

Hence KU, $KU_{(p)}$, and $E(1)$ each have a unique E_∞ structure.

7. E_∞ STRUCTURES ON THE I_n-ADIC COMPLETION OF $E(n)$

In this section we describe what we can prove about E_∞ structures on the I_n-adic completion of Johnson-Wilson spectrum $E(n)$ for a prime p and $n \geqslant 1$.

The coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$$

is Noetherian and contains the maximal ideal

$$I_n = (p, v_1, \ldots, v_{n-1}) \lhd E(n)_*.$$

Here the v_i denote the images of the Araki generators of BP_* and we sometimes write $v_0 = p$. There is a commutative ring spectrum $\widehat{E(n)}$ for which the coefficient ring $\widehat{E(n)}_*$ is the I_n-adic completion of $E(n)_*$, *i.e.*, its completion at I_n. It is known from [8, 13] that $\widehat{E(n)}$ is the $K(n)$-localization of $E(n)$. We also know from [4] that for each prime p, $\widehat{E(n)}$ possesses a unique A_∞ structure and the canonical map $\widehat{E(n)} \longrightarrow \widehat{E(n)}/I_n \simeq K(n)$ to the n-th Morava K-theory is a map of A_∞ ring spectra for any of the A_∞ structures on $K(n)$ shown to exist in [22]. Actually these results were only claimed for odd primes but the arguments also work for the prime 2.

In [13, §1], Hovey and Strickland asked whether one this result can be improved by showing that $\widehat{E(n)}$ has a unique E_∞ structure. We will give an elementary proof which relies on Robinson's obstruction theory [23].

For the completed Johnson-Wilson spectrum $\widehat{E(n)}$ we have a *continuous* universal coefficient theorem, *i.e.*, our obstruction groups live in the *continuous* $\widehat{E(n)}$-cohomology of $(X_m)_+ \wedge_{\Sigma_m} E^{\wedge m}$ where X_m is some topological space arising as the filtration quotient of an E_∞ operad. These cohomology groups can be identified with the continuous $\widehat{E(n)}_*$-homomorphisms from the corresponding $\widehat{E(n)}$-homology groups (compare [23, 5.4] and [4, §1]). Thus the obstruction groups for an E_∞ structure on $\widehat{E(n)}$ live in *continuous* Γ*-cohomology* $\mathcal{H}\Gamma^*$. Here the long exact sequence of Proposition 1.2 becomes

$$0 \to \lim_k{}^1 \mathrm{H}\Gamma^{\ell-1,*}(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; E(n)_*/I_n^k)$$
$$\longrightarrow \mathcal{H}\Gamma^{\ell,*}(\widehat{E(n)}_*\widehat{E(n)} \mid E(n)_*/I_n^k; \widehat{E(n)}_*)$$
$$\longrightarrow \lim_k \mathrm{H}\Gamma^{\ell,*}(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; E(n)_*/I_n^k) \to 0.$$

Theorem 7.1. *The* Γ*-cohomology of* $E(n)_*E(n)/I_n^k$ *over* $E(n)_*/I_n^k$ *is trivial,*

$$\mathrm{H}\Gamma^{\ell,*}(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; \widehat{E(n)}_*/I_n^k) = 0.$$

Proof. We will use that the algebra $E(n)_*E(n)/I_n^k$ is formally étale.

We can now apply the infinite dimensional Hensel lemma [4, 3.9] (see the proof of Corollary 3.4) to split $E(n)_*E(n)/I_n^k$ into an infinite tensor product of $E(n)_*/I_n^k$-algebras,

$$E(n)_*E(n)/I_n^k = \bigotimes_{j\geqslant 1} E(n)_*/I_n^k[s_j]/(v_n s_j^{p^n} - v_n^{p^j} s_j).$$

We can write $E(n)_*E(n)/I_n^k$ as a colimit of finite tensor products,

$$E(n)_*E(n)/I_n^k = \operatorname*{colim}_m \bigotimes_{j=1}^{m} E(n)_*/I_n^k[s_j]/(v_n s_j^{p^n} - v_n^{p^j} s_j).$$

We claim that each algebra $E(n)_*/I_n^k[s_j]/(v_n s_j^{p^n} - v_n^{p^j} s_j)$ is étale over $E(n)_*/I_n^k$. Notice that it is flat over $E(n)_*/I_n^k$ and is finitely generated by s_j. As the ground ring $E(n)_*/I_n^k$ is Noetherian, the only thing that remains to be shown is that the module of Kähler differentials is trivial.

The Kähler differentials are generated by the symbol ds_j, but in $E(n)_*E(n)/I_n^k$ we have the relation $v_n s_j^{p^n} = v_n^{p^j} s_j$. The element v_n is a unit in the ring $E(n)_*/I_n$ and thus we can deduce

$$ds_j = v_n^{1-p^j} d(s_j^{p^n}) = p^n v_n^{1-p^j} s_j^{p^n-1} ds_j.$$

In the quotient $E(n)_*/I_n^k$, p^k is zero. So either n already exceeds k or we can iterate this reformulation as long as the power of p is big enough to make ds_j vanish. By Lemma 1.5, we now have

$$\mathrm{H}\Gamma_{\ell,*}(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; \widehat{E(n)}_*/I_n^k) = 0 = \mathrm{H}\Gamma^{\ell,*}(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; \widehat{E(n)}_*/I_n^k),$$

proving Theorem 7.1. □

Combining this result with the Milnor exact sequence for Γ-cohomology groups we obtain the following result.

Theorem 7.2. *For* p *a prime and* $n \geqslant 1$*, the spectrum* $\widehat{E(n)}$ *possesses a unique* E_∞ *structure.*

To extend this result to cover $E(n)$ seems not to be straightforward since the Γ-cohomology of $E(n)_*E(n)$ appears to be non-trivial in positive degrees. We expect to return to this issue in future work.

8. Some remarks on André-Quillen cohomology

In this section we remark that our methods lead to computations of André-Quillen cohomology for rings of numerical polynomials and other examples related to periodic cohomology theories.

Recall the definitions of André-Quillen homology and cohomology from [17, 27] where it is denoted $\mathrm{D}_*(\ )$ and $\mathrm{D}_*(\ )$. In particular, for a commutative algebra A over a commutative ring $\Bbbk$ and and A-module M, when A is smooth over $\Bbbk$ we have

$$\mathrm{AQ}_*(A \mid \Bbbk; M) = \Omega^1_{A|\Bbbk} \underset{A}{\otimes} M, \quad \mathrm{AQ}^*(A \mid \Bbbk; M) = \mathrm{Der}_{\Bbbk}(A, M), \tag{8.1}$$

concentrated in degree 0.

First we record the André-Quillen homology and cohomology of $\mathbb{A}$ and $\mathbb{A}^{\mathrm{s}}$.

Theorem 8.1. *We have*

$$\mathrm{AQ}_n(\mathbb{A}^{\mathrm{s}} \mid \mathbb{Z};\mathbb{Z}) = \mathrm{AQ}_n(\mathbb{A} \mid \mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Q} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases}$$

$$\mathrm{AQ}^n(\mathbb{A}^{\mathrm{s}} \mid \mathbb{Z};\mathbb{Z}) = \mathrm{AQ}^n(\mathbb{A} \mid \mathbb{Z};\mathbb{Z}) = \begin{cases} \widehat{\mathbb{Q}}/\mathbb{Q} & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

Using the ideas of Section 1, we can also deduce

Theorem 8.2. *For $n \geqslant 1$ and $k \geqslant 1$, we have*

$$\mathrm{AQ}_*(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; E(n)_*/I_n^k) = 0 = \mathrm{AQ}^*(E(n)_*E(n)/I_n^k \mid E(n)_*/I_n^k; E(n)_*/I_n^k),$$

$$\mathrm{AQ}_*(E(n)_*E(n)\widehat{_{I_n}} \mid \widehat{E(n)}_*; \widehat{E(n)}_*) = 0 = \mathrm{AQ}^*(E(n)_*E(n)\widehat{_{I_n}} \mid \widehat{E(n)}_*; \widehat{E(n)}_*).$$

Further calculation of André-Quillen homology and cohomology for $E(n)_*E(n)$ over $E(n)_*$ is complicated and we will return to it in future work.

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