Hit-and-Run is Fast and Fun *

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Abstract

The hit-and-run algorithm is one of the fastest known methods to generate a random point in a high dimensional convex set. In this paper we study a natural extension of the hit-and-run algorithm to sampling from a logconcave distribution in n dimensions. After appropriate preprocessing, hit-and-run produces a point from approximately the right distribution in amortized time $O^*(n^3)$.

1 Introduction

In recent years, the problem of sampling a convex body has received much attention, and many efficient solutions have been proposed [4, 7, 12, 10, 9], all based on random walks. Of these, the *hit-and-run* random walk, first proposed by Smith [19], has the same worst-case time complexity as the more thoroughly analyzed ball walk, but seems to be fastest in practice.

The random walk approach for sampling can be extended to the class of *logconcave distributions*. For our purposes, it suffices to define these as probability distributions on the Borel sets of $\mathbb{R}^n$ which have a density function f and the logarithm of f is concave. Such density functions play an important role in stochastic optimization [17] and other applications [8]. We assume that the function is given by an *oracle*, i.e., by a subroutine that returns the value of the function at any point x. We measure the complexity of the algorithm by the number of oracle calls.

For the lattice walk and the ball walk, the extension to logconcave functions was introduced and analyzed in [1, 6] under additional smoothness assumptions about the density function. In [15], it was shown that the ball walk can be used without such assumptions, with running time bounds that are essentially the same as for sampling from convex bodies. In this paper we analyze the behavior of the hit-and-run walk for logconcave distributions. Our main result is that after appropriate preprocessing (bringing the distribution to isotropic position), we can generate a sample using $O^*(n^4)$ steps (oracle calls) and in

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$O^*(n^3)$ steps from a warm start. (This means that we start the walk from a random point whose density function is at most a constant factor larger than the target density f; cf. [15].) As in [15], we make no assumptions on the smoothness of the density function. The analysis uses the smoothing technique introduced in [15], and in main lines of follows the analysis of the hit-and-run walk in [13], with substantial additional difficulties.

2 Results

2.1 Preliminaries.

A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is *logconcave* if it satisfies

$$f(\alpha x + (1-\alpha)y) \geq f(x)^\alpha f(y)^{1-\alpha}$$

for every $x, y \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$. This is equivalent to saying that the support K of f is convex and $\log f$ is concave on K.

An integrable function $f : \mathbb{R}^n \to \mathbb{R}_+$ is a *density function*, if $\int_{\mathbb{R}^n} f(x)\,dx = 1$. Every non-negative integrable function f gives rise to a probability measure on the measurable subsets of $\mathbb{R}^n$ defined by

$$\pi_f(S) = \int_S f(x)\,dx \Big/ \int_{\mathbb{R}^n} f(x)\,dx\ .$$

The *centroid* of a density function $f : \mathbb{R}^n \to \mathbb{R}_+$ is the point

$$z_f = \int_{\mathbb{R}^n} f(x)x\,dx;$$

the *covariance matrix* of the function f is the matrix

$$V_f = \int_{\mathbb{R}^n} f(x)xx^T\,dx$$

(we assume that these integrals exist).

For any logconcave function $f :\ \mathbb{R} \to \mathbb{R}^n$, we denote by M_f its maximum value. We denote by

$$L_f(t) = \{x \in \mathbb{R}^n :\ f(x) \geq t\}$$

its level sets, and by

$$f_t(x) = \begin{cases} f(x), & \text{if } f(x) \geq t, \\ 0, & \text{if } f(x) < t, \end{cases}$$

its restriction to the level set. It is easy to see that f_t is logconcave. In M_f and L_f we omit the subscript if f is understood.

A density function $f : \mathbb{R}^n \to \mathbb{R}_+$ is *isotropic*, if its centroid is 0, and its covariance matrix is the identity matrix. This latter condition can be expressed in terms of the coordinate functions as

$$\int_{\mathbb{R}^n} x_i x_j f(x)\,dx = \delta_{ij}$$

for all $1 \le i, j \le n$. This condition is equivalent to saying that for every vector $v \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} (v \cdot x)^2 f(x)\, dx = |v|^2.$$

In terms of the associated random variable X, this means that

$$\mathsf{E}(X) = 0 \quad \text{and} \quad \mathsf{E}(XX^T) = I.$$

We say that f is *near-isotropic up to a factor of* C, if $(1/C) \le \int (u^T x)^2 \, d\pi_f(x) \le C$ for every unit vector u. As in [15], the notions of "isotropic" and "non-isotropic" extend to non-negative integrable functions f, in which case we mean that the density function $f / \int_{\mathbb{R}^n} f$ is isotropic or near-isotropic.

Given any density function f with finite second moment $\int_{\mathbb{R}^n} \|x\|^2 f(x)\, dx$, there is an affine transformation of the space bringing it to isotropic position, and this transformation is unique up to an orthogonal transformation of the space.

For two points $u, v \in \mathbb{R}^n$, we denote by $d(u, v)$ their euclidean distance. For two probability distributions σ, τ on the same underlying σ-algebra, let

$$d_{\mathrm{tv}}(\sigma, \tau) = \sup_A (\sigma(A) - \tau(A))$$

be their total variation distance.

2.2 The random walk.

Let f be a logconcave distribution in $\mathbb{R}^n$. For any line ℓ in $\mathbb{R}^n$, let $\mu_{\ell,f}$ be the measure induced by f on ℓ, i.e.

$$\mu_{\ell,f}(S) = \int_{p+tu \in S} f(p + tu) dt,$$

where p is any point on ℓ and u is a unit vector parallel to ℓ. We abbreviate $\mu_{\ell,f}$ by μ_ℓ if f is understood, and also $\mu_\ell(\ell)$ by μ_ℓ. The probability measure $\pi_\ell(S) = \mu_\ell(S)/\mu_\ell$ is the *restriction* of f to ℓ.

We study the following generalization of the *hit-and-run* random walk.

- Pick a uniformly distributed random line ℓ through the current point.
- Move to a random point y along the line ℓ chosen from the distribution π_ℓ.

Let us remark that the first step is easy to implement. For example, we can generate n independent random numbers $U_1, \ldots, U_n$ from the standard normal distribution, and use the vector $(U_1, \ldots, U_n)$ to determine the direction of the line.

In connection with the second step, we have to discuss how the function is given: we assume that it is given by an *oracle*. This means that for any $x \in \mathbb{R}^n$, the oracle returns the value $f(x)$. (We ignore here the issue that if the value of the function is irrational, the oracle only returns an approximation of f.) It would be enough to have an oracle which returns the value $C \cdot f(x)$ for some unknown constant $C > 0$ (this situation occurs in many sampling problems e.g. in statistical mechanics and simulated annealing).

For technical reasons, we also need a "guarantee" from the oracle that the centroid z_f of f satisfies $\|z_f\| \le Z$ and that all the eigenvalues of the covariance matrix are between r and R, where Z, r and R are given positive numbers.

One way to carry out the second step is to use a binary search to find the point p on ℓ where the function is maximal, and the points a and b on both sides of p on ℓ where the value of the function is $\varepsilon f(p)$. We allow a relative error of ε, so the number of oracle calls is only $O(\log(1/\varepsilon))$.

Then select a uniformly distributed random point y on the segment $[a, b]$, and independently a uniformly distributed random real number in the interval $[0, 1]$. Accept y if $f(y) > rf(p)$; else, reject y and repeat.

The distribution of the point generated this way is closer to the desired distribution than ε, and the expected number of oracle calls needed is $O(\log(1/\varepsilon))$.

Our main theorem concerns functions that are near-isotropic (up to some fixed constant factor c).

Theorem 2.1 *Let f be a logconcave density function in $\mathbb{R}^n$ that is near-isotropic up to a factor of c. Let σ be a starting distribution and let σ^m be the distribution of the current point after m steps of the hit-and-run walk. Assume that there is a $D > 0$ such that $\sigma(S) \le D\pi_f(S)$ for every set S. Then for*

$$m > 10^{10} c^2 D^2 \frac{n^3}{\varepsilon^2} \log \frac{1}{\varepsilon},$$

the total variation distance of σ^m and π_f is less than ε.

2.3 Distance and Isoperimetry.

To analyze the hit-and-run walk, we need a notion of distance according to a density function. Let f be a logconcave density function. For two points $u, v \in \mathbb{R}^n$, let $\ell(u, v)$ denote the line through them. Let $[u, v]$ denote the segment connecting u and v, and let $\ell^+(u, v)$ denote the semiline in ℓ starting at u and not containing v. Furthermore, let

$$\begin{aligned} f^+(u,v) &= \mu_{\ell,f}(\ell^+(u,v)), \\ f^-(u,v) &= \mu_{\ell,f}(\ell^+(v,u)), \\ f(u,v) &= \mu_{\ell,f}([u,v]). \end{aligned}$$

We introduce the following "distance":

$$d_f(u, v) = \frac{f(u, v) f(\ell(u, v))}{f^-(u, v) f^+(u, v)}.$$

The function $d_f(u, v)$ does not satisfy the triangle inequality in general, but we could take $\ln(1 - d_f(u, v))$ instead, and this quantity would be a metric; however, it will be more convenient to work with d_f.

Suppose f is the uniform distribution over a convex set K. Let u, v be two points in K and p, q be the endpoints of $\ell(u, v) \cap K$, so that the points appear in the order p, u, v, q along $\ell(u, v)$. Then,

$$d_f(u, v) = d_K(u, v) = \frac{|u - v||p - q|}{|p - u||v - q|}.$$

2.4 An isoperimetric inequality

The next theorem is an extension of Theorem 6 from [13] to logconcave functions.

Theorem 2.2 *Let f be a logconcave density function on $\mathbb{R}^n$ with support K. For any partition of K into three measurable sets S_1, S_2, S_3,*

$$\pi_f(S_3) \geq d_K(S_1, S_2)\pi_f(S_1)\pi_f(S_2).$$

3 Preliminaries

3.1 Spheres and balls

Lemma 3.1 *Let H be a halfspace in $\mathbb{R}^n$ and B, a ball whose center is at a distance $t > 0$ from H. Then*

(a) *if $t \leq 1/\sqrt{n}$, then*

$$\mathrm{vol}(H \cap B) > \left(\frac{1}{2} - \frac{t\sqrt{n}}{2}\right) \mathrm{vol}(B);$$

(b) *if $t > 1/\sqrt{n}$ then*

$$\frac{1}{10t\sqrt{n}}(1 - t^2)^{(n+1)/2}\mathrm{vol}(B) \;<\; \mathrm{vol}(H \cap B) \;<\; \frac{1}{t\sqrt{n}}(1 - t^2)^{(n+1)/2}\mathrm{vol}(B).$$

Let C be a cap on the unit sphere S in $\mathbb{R}^n$, with radius r and $\mathrm{vol}_{n-1}(C) = c\mathrm{vol}_{n-1}(S)$, $c < 1/2$. We can write its radius as $r = \pi/2 - t(c)$. The function $t(c)$ is difficult to express exactly, but for our purposes, the following known bounds will be enough:

Lemma 3.2 *If $0 < c < 2^{-n}$, then*

$$\frac{1}{2}c^{1/n} < t(c) < 2c^{1/n};$$

if $2^{-n} < c < 1/4$, then

$$\frac{1}{2}\sqrt{\frac{\ln(1/c)}{n}} < t(c) < 2\sqrt{\frac{\ln(1/c)}{n}};$$

if $1/4 < c < 1/2$, *then*

$$\frac{1}{2}\left(\frac{1}{2}-c\right)\frac{1}{\sqrt{n}} < t(c) < 2\left(\frac{1}{2}-c\right)\frac{1}{\sqrt{n}}.$$

Using this function $t(c)$, we can formulate a fact that can be called "strong expansion" on the sphere:

Lemma 3.3 *Let T_1 and T_2 be two sets on the unit sphere S in $\mathbb{R}^n$, so that* $\mathrm{vol}_{n-1}(T_i) = c_i \mathrm{vol}_{n-1}(S)$. *Then the angular distance between T_1 and T_2 is at most $t(c_1) + t(c_2)$.*

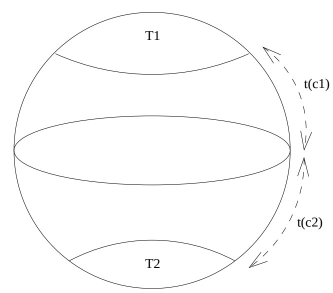

Figure 1: Caps at maximal angular distance.

Proof. Let d denote the angular distance between T_1 and T_2. The measure of T_i corresponds to the measure of a spherical cap with radius $\pi/2 - t(c_1)$. By spherical isoperimetry, the measure of the d-neighborhood of T_1 is at least as large as the measure of the d-neighborhood of the corresponding cap, which is a cap with radius $\pi/2 - t(c_1) + d$. The complementary cap has radius $\pi/2 + t(c_1) - d$ and volume at least c_2, and so it has radius at least $\pi/2 - t(c_2)$. Thus $\pi/2 + t(c_1) - d \geq \pi/2 - t(c_2)$, which proves the lemma. $\square$

Lemma 3.4 *Let K be a convex body in $\mathbb{R}^n$ containing the unit ball B, and let $r > 1$. If $\phi(r)$ denotes the fraction of the sphere rS that is contained in K, then*

$$t(1-\phi(r)) + t(\phi(2r)) \geq \frac{3}{8r}.$$

Proof. Let $T_1 = (rS) \cap K$ and $T_2 = (1/2)((2rS) \setminus K)$. We claim that the angular distance of T_1 and T_2 is at least $3/(8r)$. Consider any $y_1 \in T_1$ and $y_2 \in T_2$, we want to prove that the angle α between them is at least $3/(8r)$. We may assume that this angle is less than $\pi/4$ (else, we have nothing to prove). Let y_0 be the nearest point to 0 on the line through $2y_2$ and y_1. Then $y_0 \notin K$ by convexity, and so $s = |y_0| > 1$. Let α_i denote the angle between y_i and y_0. Then

$$\sin\alpha = \sin(\alpha_2 - \alpha_1) = \sin\alpha_2 \cos\alpha_1 - \sin\alpha_1 \cos\alpha_2.$$

Here $\cos\alpha_1 = s/r$ and $\cos\alpha_1 = s/(2r)$; expressing the sines and substituting, we get

$$\sin\alpha = \frac{s}{r}\sqrt{1 - \frac{s^2}{4r^2}} - \frac{s}{2r}\sqrt{1 - \frac{s^2}{r^2}}.$$

To estimate this by standard tricks from below:

$$\begin{aligned}
\sin\alpha &= \frac{\frac{s^2}{r^2}\left(1 - \frac{s^2}{4r^2}\right) - \frac{s^2}{4r^2}\left(1 - \frac{s^2}{r^2}\right)}{\frac{s}{r}\sqrt{1 - \frac{s^2}{4r^2}} + \frac{s}{2r}\sqrt{1 - \frac{s^2}{r^2}}} > \frac{\frac{s^2}{r^2}\left(1 - \frac{s^2}{4r^2}\right) - \frac{s^2}{4r^2}\left(1 - \frac{s^2}{r^2}\right)}{\frac{s}{r} + \frac{s}{2r}} \\
&= \frac{s}{2r} > \frac{1}{2r}
\end{aligned}$$

Since $\alpha > \sin\alpha$, this proves the lemma. $\square$

The way this lemma is used is exemplified by the following:

Corollary 3.5 *Let K be a convex body in $\mathbb{R}^n$ containing the unit ball B, and let $1 < r < \sqrt{n}/10$. If K misses 1% of the sphere rS, then it misses at least 99% of the sphere $2rS$.*

3.2 Logconcave functions: a recap

In this section we recall some folklore geometric properties of logconcave functions as well as some that were proved in [15].

We need some definitions. The *marginals* of a function $f : \mathbb{R}^n \to \mathbb{R}_+$ are defined by

$$G(x_1, \ldots, x_k) = \int_{\mathbb{R}^{n-k}} f(x_1, \ldots, x_n)\, dx_{k+1} \ldots dx_n.$$

The *first marginal*

$$g(t) = \int_{x_2,\ldots,x_n} f(t, x_2, \ldots, x_n)\, dx_2 \ldots dx_n$$

will be used most often. It is easy to check that if f is in isotropic position, then so are its marginals. The *distribution function* of f is defined by

$$F(t_1, \ldots, t_n) = \int_{x_1 \le t_1, \ldots, x_n \le t_n} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

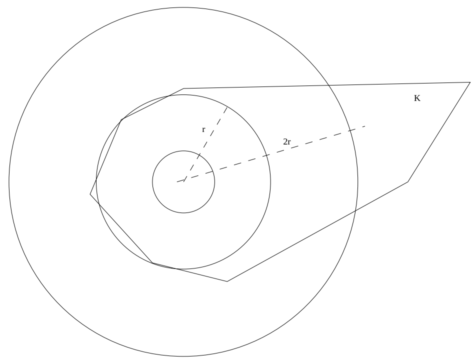

Figure 2: An illustration of Corollary 3.5.

Clearly, the product and the minimum of logconcave functions is logconcave. The sum of logconcave functions is not logconcave in general; but the following fundamental properties of logconcave functions, proved by Dinghas [3] and Prékopa [16], can make up for this in many cases.

Theorem 3.6 *All marginals as well as the distribution function of a logconcave function are logconcave. The convolution of two logconcave functions is logconcave.*

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable function such that $g(x)$ tends to 0 faster than any polynomial as $x \to \infty$. Define its moments, as usual, by

$$M_g(n) = \int_0^\infty t^n g(t)\, dt.$$

Lemma 3.7 (a) *The sequence $(M_g(n) :\ n = 0, 1, \ldots)$ is logconvex.*

(b) *If g is monotone decreasing, then the sequence defined by*

$$M'_g(n) = \begin{cases} nM_g(n-1)), & \text{if } n > 0, \\ g(0), & \text{if } n = 0 \end{cases}$$

is also logconvex.

(c) *If g is logconcave, then the sequence $M_g(n)/n!$ is logconcave.*
(d) *If g is logconcave, then*

$$g(0)M_g(1) \le M_g(0)^2.$$

(i.e., we could append $g(0)$ at the beginning of the sequence in (c) and maintain logconcavity).

Lemma 3.8 *Let X be a random point drawn from a one-dimensional logconcave distribution. Then*

$$\mathsf{P}(X \ge \mathsf{E}X) \ge \frac{1}{e}.$$

Lemma 3.9 *Let $g : \mathbb{R} \to \mathbb{R}_+$ be an isotropic logconcave density function.*
(a) *For all x, $g(x) \le 1$.*
(b) *$g(0) \ge \frac{1}{8}$.*

Lemma 3.10 *Let X be a random point drawn from a logconcave density function $g : \mathbb{R} \to \mathbb{R}_+$.*
(a) *For every $c \ge 0$,*

$$\mathsf{P}(g(X) \le c) \le \frac{c}{M_g}.$$

(b) *For every $0 \le c \le g(0)$,*

$$\mathsf{P}(\min g(2X), g(-2X) \le c) \ge \frac{c}{4g(0)}.$$

The next set of Lemmas were proven in [15] for higher dimensional logconcave functions.

Theorem 3.11 *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an isotropic logconcave density function.*
(a) *For every $v \in \mathbb{R}^n$ with $0 \le |v| \le 1/9$, we have $2^{-9n|v|}f(0) \le f(v) \le 2^{9n|v|}f(0)$.*
(b) *$f(x) \le 2^{4n}f(0)$ for every x.*
(c) *There is an $x \in \mathbb{R}^n$ such that $f(x) > 1/\sqrt{2e\pi}^n$.*
(d) *$2^{-7n} \le f(0) \le 10^n n^{n/2}$.*
(e) *$f(x) \le 2^{8n}n^{n/2}$ for every x.*

Lemma 3.12 *Let X be a random point drawn from a distribution with a logconcave density function $f : \mathbb{R}^n \to \mathbb{R}_+$. If $\beta \ge 2$, then*

$$\mathsf{P}(f(X) \le e^{-\beta(n-1)}) \le (e^{1-\beta}\beta)^{n-1}.$$

Lemma 3.13 *Let $X \in \mathbb{R}^n$ be a random point from an isotropic logconcave distribution. Then for any $R > 1$, $\mathsf{P}(|X| > R) < e^{-R}$.*

4 Comparing distances

For the next lemmas, it will be convenient to introduce the following notation: for a function $g: \mathbb{R} \to \mathbb{R}_+$ and $a < b$, let

$$g(a,b) = \int_a^b g(t)\,dt.$$

Furthermore, for $a < b < c < d$, we consider the cross-ratio

$$(a:c:b:d) = \frac{(d-a)(c-b)}{(b-a)(d-c)},$$

and its generalized version

$$(a:c:b:d)_g = \frac{g(a,d)g(b,c)}{g(a,b)g(c,d)}.$$

(The strange order of the parameters was chosen to conform with classical notation.) Clearly, $(a:c:b:d)_g = (a:c:b:d)$ if g is a constant function.

We start with a simple bound:

Lemma 4.1 *Let $g: \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c < d$. Then*

$$(a:c:b:d)_g \geq \frac{g(b)}{g(c)} - 1.$$

Proof. We may assume that $g(b) > g(c)$ (else, there is nothing to prove). Let $h(t)$ be an exponential function such that $h(b) = g(b)$ and $h(c) = g(c)$. By logconcavity, $g(x) \leq h(x)$ for $x \leq a$ and $g(x) \geq h(x)$ for $b \leq x \leq c$. Hence

$$\begin{aligned} (a:c:b:d)_g &= \frac{g(a,d)g(b,c)}{g(a,b)g(c,d)} \geq \frac{g(b,c)}{g(a,b)} \\ &\geq \frac{h(b,c)}{h(a,b)} = \frac{h(c)-h(b)}{h(b)-h(a)} \geq \frac{h(c)-h(b)}{h(b)} \\ &= \frac{g(b)}{g(c)} - 1. \end{aligned}$$

□

Lemma 4.2 *Let $g: \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c < d$. Then*

$$(a:c:b:d)_g \geq (a:c:b:d).$$

Proof. By Lemma 2.6 from [10], it suffices to prove this in the case when $g(t) = e^t$. Furthermore, we may assume that $a = 0$. Then the assertion is just Lemma 7 in [13]. □

Lemma 4.3 *Let $g:\ \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c$. Then*

$$\frac{g(a,b)}{b-a} \le \left(1 + \left|\ln \frac{g(b)}{g(c)}\right|_+\right) \frac{g(a,c)}{c-a}.$$

Proof. Let $h(t) = \beta e^{\gamma t}$ be an exponential function such that

$$\int_a^b h(t)\,dt = g(a,b) \qquad \text{and} \qquad \int_b^c h(t)\,dt = g(b,c).$$

It is easy to see that such β and γ exist, and that $\beta > 0$. The graph of h intersects the graph of g somewhere in the interval $[a,b]$, and similarly, somewhere in the interval $[b,c]$. By logconcavity, this implies that $h(b) \le g(b)$ and $h(c) \ge g(c)$.

If $\gamma > 0$ then $h(t)$ is monotone increasing, and so

$$\frac{g(a,b)}{b-a} = \frac{h(a,b)}{b-a} \le \frac{h(a,c)}{c-a} = \frac{g(a,c)}{c-a},$$

and so the assertion is trivial. So suppose that $\gamma < 0$. For notational convenience, we can rescale the function and the variable so that $\beta = 1$ and $\gamma = -1$. Also write $u = b - a$ and $v = c - b$. Then we have

$$g(a,b) = 1 - e^{-u} \qquad \text{and} \qquad g(a,c) = 1 - e^{-u-v}.$$

Hence

$$\frac{g(a,b)}{g(a,c)} = \frac{1-e^{-u}}{1-e^{-u-v}} \le \frac{u(v+1)}{u+v} = (v+1)\frac{b-a}{c-a}.$$

(The last step can be justified like this: $(1-e^{-u})/(1-e^{-u-v})$ is monotone increasing in u if we fix v, so replacing e^{-u} by $1-u < e^{-u}$ both in the numerator and denominator increases its value; similarly replacing e^{-v} by $1/(v+1)$ in the denominator decreases its value). To conclude, it suffices to note that

$$\ln \frac{g(b)}{g(c)} \ge \ln \frac{h(b)}{h(c)} = \ln \frac{e^{-u}}{e^{-u+v}} = v.$$

$\square$

The following lemma is a certain converse to Lemma 4.2:

Lemma 4.4 *Let $g:\ \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c < d$. Let $C = 1 + \max\{\ln(g(b)/g(a)), \ln(g(c)/g(d))\}$. If*

$$(a:c:b:d) \le \frac{1}{2C},$$

then

$$(a:c:b:d)_g \le 6C(a:c:b:d).$$

Proof. By the definition of $(a : c : b : d)$ and Lemma 4.3,

$$(a : c : b : d) = \frac{(d-a)(c-b)}{(b-a)(d-c)} > \frac{c-b}{b-a} > \frac{c-b}{c-a} \geq \frac{1}{C}\frac{g(b,c)}{g(a,c)}.$$

Hence by the assumption on $(a : c : b : d)$,

$$\frac{g(b,c)}{g(a,c)} = \frac{g(b,c)}{g(a,b)+g(b,c)} \leq \frac{1}{2},$$

which implies that $g(a,b) \geq g(b,c)$. Similarly, $g(c,d) \geq g(b,c)$. We may assume by symmetry that $g(a,b) \leq g(c,d)$. Then $g(a,d) = g(a,b) + g(b,c) + g(c,d) \leq 3g(c,d)$, and so we have

$$(a : c : b : d)_g = \frac{g(a,d)g(b,c)}{g(a,b)g(c,d)} \leq \frac{3g(b,c)}{g(a,b)} \leq \frac{6g(b,c)}{g(a,c)}.$$

Using Lemma 4.3 again, we get

$$(a : c : b : d)_g \leq 6C\frac{c-b}{c-a} \leq 6C\frac{c-b}{b-a} \leq 6C\frac{(c-b)(d-a)}{(b-a)(d-c)} = 6C(a : b : c : d).$$

□

5 Taming the function

5.1 A smoother version

In [15], we defined "smoothed-out" version of the given density function f as

$$\hat{f}(x) = \min_C \frac{1}{\mathrm{vol}(C)} \int_C f(x+u)\, du,$$

where C ranges over all convex subsets of the ball $x + rB$ with $\mathrm{vol}(C) = V_0/16$. The quotient

$$\delta(x) = \frac{\hat{f}(x)}{f(x)}$$

is a certain measure of the smoothness of the function f at x. The value

$$\rho(x) = \frac{r}{16t(\delta(x))} \approx \frac{r}{16\sqrt{\ln(1/\delta(x))}}$$

will also play an important role; the function is well behaved in a ball with radius $\rho(x)$ about x.

As shown in [15], the somewhat complicated definition of the function $\hat{f}$ serves to assure its logconcavity (Lemma 5.2). Further, it is not much smaller than f on the average (Lemma 5.3). We recall that we could (at the cost of a factor of 2) replace equality in the condition on C by inequality. The next 3 lemmas are from [15].

Lemma 5.1 *For every convex subset $D \subseteq rB$ with* $\mathrm{vol}(D) \geq \mathrm{vol}(B)/16$, *we have*

$$\frac{1}{\mathrm{vol}(D)} \int_D f(x+u)\, du \geq \frac{1}{2}\hat{f}(x).$$

Lemma 5.2 *The function $\hat{f}$ is logconcave.*

Lemma 5.3 *We have*

$$\int_{\mathbb{R}^n} \hat{f}(x)\, dx \geq 1 - 64r^{1/2}n^{1/4}.$$

The value of $\delta(x)$ governs the local smoothness of the function f in more natural ways than its definition, as we'll show below.

Lemma 5.4 *For every $x, y \in \mathbb{R}^n$ with $|x-y| \leq \frac{r}{2\sqrt{n}}$, we have*

$$\frac{\hat{f}(x)}{2} \leq f(y) \leq 2\frac{f(x)^2}{\hat{f}(x)}.$$

Proof. Let a be the closest point to x with $f(a) \leq \hat{f}(x)/2$. Consider the supporting hyperplane of the convex set $\{y \in \mathbb{R}^n : \ f(y) \geq \hat{f}(x)/2\}$, and the open halfspace H bounded by this hyperplane that does not contain x. Clearly $f(y) < \hat{f}(x)$ for $y \in H$. By the definition of $\hat{f}$, it follows that the volume of the convex set $H \cap rB$ must be less than $V_0/16$. On the other hand, by Proposition 3.1 the volume of this set is at least

$$\left(\frac{1}{2} - \frac{|a|\sqrt{n}}{2r}\right) V_0.$$

Comparing these two bounds, it follows that

$$|a| > \frac{7}{8}\frac{r}{\sqrt{n}} > \frac{r}{2\sqrt{n}}.$$

This proves the first inequality. The second follows easily, since for the point $y' = 2x - y$ we have $|y' - x| = |y - x| < r/(2\sqrt{n})$, and so by the first inequality,

$$f(y') \geq \frac{\hat{f}(x)}{2}.$$

Then logconcavity implies that

$$f(y) \leq \frac{f(x)^2}{f(y')} \leq 2\frac{f(x)^2}{\hat{f}(x)},$$

as claimed. ☐

Lemma 5.5 (a) *Let $0 < q < \delta(x)^{1/n} r$. Then the fraction of points y on the sphere $x + qS$ with $f(y) < \hat{f}(x)/2$ is less than $1 - \delta(x)/4$.*

(b) *Let $0 < q \le \rho(x)$. Then the fraction of points y on the sphere $x + qS$ with $f(y) < \hat{f}(x)/2$ is less than $\delta(x)/4$.*

Proof. (a) We may assume, for notational convenience, that $x = 0$. Let $\delta = \delta(x)$ and $L = L(\hat{f}(0)/2)$. To prove (a), suppose that this fraction is larger than $1-\delta/4$. Let $C = rB \cap H$, where H is a halfspace with $\mathrm{vol}(H \cap (rB) = V_0/16$ which avoids the points y with $f(y) > f(0)$. We can write

$$\int_C f(y)\, dy = \int_{C \setminus L} + \int_{C \cap L}.$$

Since $f(y) \le f(0)$ for all $y \in C$ and $f(y) \le \hat{f}(0)/2$ on the first set,

$$\int_C f(y)\, dy \le \frac{\hat{f}(0)}{2} \mathrm{vol}(C \setminus L)) + f(0)\mathrm{vol}(C \cap L).$$

The first term can be estimated simply by $(\hat{f}(0)/2)\mathrm{vol}(C)$. The second term we split further:

$$\mathrm{vol}(C \cap L) \le \mathrm{vol}((C \cap L) \setminus (qB)) + \mathrm{vol}(C \cap (qB)).$$

Since the fraction of every sphere tS, $t \ge q$, inside L is at most δ, it follows that the first term is at most $\delta\mathrm{vol}(C)$. We claim that also the second term is less than $\delta\mathrm{vol}(C)$. Indeed,

$$\mathrm{vol}(C \cap L \cap (qB)) \le \frac{1}{16}\mathrm{vol}(qB) = \frac{1}{16}\left(\frac{q}{r}\right)^n \mathrm{vol}(rB) \le \delta\mathrm{vol}(C).$$

Thus

$$\int_C f(y)\, dy < \frac{\hat{f}(0)}{2}\mathrm{vol}(C) + 2f(x)\delta\mathrm{vol}(C) \le \hat{f}(0)\mathrm{vol}(C),$$

which contradicts the definition of $\hat{f}$. This proves (a).

To prove (b), suppose that a fraction of more than δ of the sphere qS is not in L. On the other hand, a fraction of at least δ of the sphere $2qB$ is in L. This follows from part (a) if $q < \delta^{1/n} r$. If this is not the case, then we have

$$q \le \rho(x) < \frac{r}{16\sqrt{\ln(1/\delta(x))}} < \frac{r}{16n\sqrt{\ln(r/q))}},$$

from where it is easy to conclude that $q < r/(2\sqrt{n})$. From Lemma 5.4 we get that all of the sphere qS is in L.

Now Lemma 3.4 implies that

$$2t(\delta) \ge \frac{3r}{16q},$$

which contradicts the assumption on q. $\square$

5.2 Cutting off small parts

We shall assume that the function f is isotropic; the arguments are similar if we only assume that the function is near-isotropic. Let $\varepsilon_0 = e^{-3(n-1)}$ and $K = \{x \in \mathbb{R}^n : \ |X| < R, \ f(x) > \varepsilon_0\}$.

Lemma 5.6

$$\pi_f(K) > 1 - 2e^{-R}.$$

Proof. Let $U = \{x \in \mathbb{R}^n : \ f(x) \le \varepsilon_0\}$ and $V = \mathbb{R}^n \setminus RB$. Then by Lemma 3.12,

$$\pi_f(U) \le (3e^{-2})^{n-1} < e^{-R},$$

and by Lemma 3.13,

$$\pi_f(V) \le e^{-R},$$

and so

$$\pi_f(K) \ge 1 - \pi_f(U) - \pi_f(V) \ge 1 - 2e^{-R}.$$

□

This lemma shows that we can replace the distribution f by its restriction to the convex set K: the restricted distribution is logconcave, very close to isotropic, and the probability that we ever step outside K is negligible. So from now on, we assume that $f(x) = 0$ for $x \notin K$. This assumption implies some important relations between three distance functions we have to consider: the euclidean distance $d(u,v) = |u - v|$, the f-distance

$$d_f(u,v) = \frac{\mu_f^*(u,v)\mu_f(u,v)}{\mu_f^-(u,v)\mu_f^+(u,v)}.$$

and the K-distance as defined in [12]:

$$d_K(u,v) = \frac{|u-v| \cdot |u'-v'|}{|u-u'| \cdot |v-v'|},$$

where u' and v' are the intersection points of the line through u and v with the boundary of K, labeled so that u lies between v and u'. Equivalently, this is the f-distance if f is the density function of the uniform distribution on K.

Lemma 5.7 *For any two points $u, v \in K$,*
(a) $d_K(u,v) \le d_f(u,v)$;
(b) $d_K(u,v) \ge \frac{1}{2R} d(u,v)$;
(c) $d_K(u,v) \ge \frac{1}{8n\log n} \min(1, d_f(u,v))$.

Proof. (a) follows from Lemma 4.2; (b) is immediate from the definition of K. For (c), we may suppose that $d_K(u,v) \le 1/(8n\log n)$ (else, the assertion is obvious). By Theorem 3.11(e) and the definition of K, we have for any two points $x, y \in K$

$$\frac{f(x)}{f(y)} \leq \frac{2^{8n} n^{n/2}}{e^{-3(n-1)}} < e^{2n \ln n}.$$

So we can apply Lemma 4.4, and get that

$$d_K(u,v) \geq \frac{1}{3+6n\ln n} d_f(u,v) > \frac{1}{8n\log n} d_f(u,v),$$

proving (c). $\square$

6 One step of hit-and-run

6.1 Steps are long

For any point u, let P_u be the distribution obtained on taking one hit-and-run step from u. It is not hard to see that

$$P_u(A) = \frac{2}{n\pi_n} \int_A \frac{f(x)\,dx}{\mu_f(u,x)|x-u|^{n-1}}. \tag{1}$$

Let ℓ be any line, x a point on ℓ. We say that (x, ℓ) is *ordinary*, if both points $u \in \ell$ with $|u - x| = \rho(x)$ satisfy $f(u) \geq \hat{f}(x)/2$.

Lemma 6.1 *Suppose that (x,ℓ) is ordinary. Let p, q be intersection points of ℓ with the boundary of $L(F/8)$ where F is the maximum value of f along ℓ, and let $s = \max\{\rho(x)/4, |x-p|/32, |x-q|/32\}$. Choose a random point y on ℓ from the distribution π_ℓ. Then*

$$\mathsf{P}(|x-y| > s) > \frac{\sqrt{\delta(x)}}{8}.$$

Proof. We may assume that $x = 0$. Suppose first that the maximum in the definition of s is attained by $\rho(0)/4$. Let y be a random step along ℓ, and apply lemma 3.10(b) with $c = \sqrt{f(0)\hat{f}(0)/2}$. We get that the probability that $f(2y) \leq c$ or $f(-2y) \leq c$ is at least

$$\frac{c}{4f(0)} = \frac{\sqrt{\delta(0)}}{8}.$$

Suppose $f(2y) \leq c$. Then logconcavity implies that $f(4y) \leq c^2/f(0) = \hat{f}(0)/4$. Since ℓ is ordinary, this means that in such a case $|4y| > \rho(x)$, and so $|x - y| = |y| > \rho(x)/4$.

So suppose that the maximum in the definition of s is attained by (say) $|p|/32$. We have the trivial estimates

$$\int_{|y|<s} f(y)\,dy \leq 2sF,$$

but
$$\int_\ell f(y)\,dy \ge |p-q|\frac{F}{8},$$
and so
$$\mathsf{P}(|y|\le s) \le \frac{16s}{|p-q|}.$$
Hence if $|p-q|>24s$, then the conclusion of the lemma is valid.

So we may assume that $|p-q|<24s$. Then q is between 0 and p, and so for every point y in the interval $[p,q]$, we have $|y|\ge 8s$. Since the probability of $y\in[p,q]$ is at least $7/8$ the lemma follows again. □

Lemma 6.2 *Let $x\in\mathbb{R}^n$ and let ℓ be a random line through x. Then with probability at least $1-\delta(x)/2$, (x,ℓ) is ordinary.*

Proof. If (x,ℓ) is not ordinary, then one of the points u on ℓ at distance $\rho(x)$ has $f(u)<f(x)/2$. By Lemma 5.5, the fraction of such points on the sphere $x+\rho(x)S$ is at most $\delta(x)/4$. So the probability that ℓ is not ordinary is at most $\delta(x)/2$. □

For a point $x\in K$, define $\alpha(x)$ as the smallest $s\ge 3$ for which
$$\mathsf{P}(f(y)\ge sf(x)) \le \frac{1}{16},$$
where y is a hit-and-run step from x.

Lemma 6.3 *Let u be a random point from the stationary distribution π_f. For every $t>0$,*
$$\mathsf{P}(\alpha(u)\ge t) \le \frac{16}{t}.$$

Proof. If $t\le 3$, then the assertion is trivial, so let $t\ge 3$. Then for every x with $a(x)\ge t$, we have
$$\mathsf{P}(f(y)\ge \alpha(x)f(x)) = \frac{1}{16},$$
and hence $\alpha(x)\ge t$ if and only if
$$\mathsf{P}(f(y)\ge tf(x)) \ge \frac{1}{16}.$$
Let $\mu(x)$ denote the probability on the left hand side. By Lemma 3.10(a), for any line ℓ, a random step along ℓ will go to a point x such that $f(x)\le (1/t)\max_{y\in\ell} f(y)$ with probability at most $1/t$. Hence for every point u, the probability that a random step from u goes to a point x with $f(x)\le(1/t)f(u)$ is again at most $1/t$. By time-reversibility, for the random point u we have
$$\mathsf{E}(\mu(u)) \le \frac{1}{t}.$$

On the other hand,

$$\mathsf{E}(\mu(u)) \geq \frac{1}{16}\mathsf{P}\left(\mu(u) \geq \frac{1}{16}\right) = \frac{1}{16}\mathsf{P}(\alpha(u) \leq t),$$

which proves the lemma. $\square$

6.2 Geometric distance and probabilistic distance.

Here we show that if two points are close in a geometric sense, then the distributions obtained after making one step of the random walk from them are also close in total variation distance.

Lemma 6.4 (Main Lemma) *Let u, v be two points in $\mathbb{R}^n$ such that*

$$d_f(u,v) < \frac{1}{128\ln(3+\alpha(u))} \quad \textit{and} \quad d(u,v) < \frac{r}{64\sqrt{n}}.$$

Then

$$d_{tv}(P_u, P_v) < 1 - \frac{1}{500}\delta(u).$$

Proof. Let $\delta = \delta(u)$ and $\alpha = \alpha(u)$. We will show that there exists a set $A \subseteq K$ such that $P_u(A) \geq \sqrt{\delta}/32$ and for every subset $A' \subset A$,

$$P_v(A') \geq \frac{\sqrt{\delta}}{16} P_u(A').$$

To this end, we define certain "bad" lines through u. Let σ be the uniform probability measure on lines through u.

Let B_0 be the set of non-ordinary lines through u. By Lemma 6.2, $\sigma(B_0) \leq 2\delta$.

Let B_1 be the set of lines that are not almost orthogonal to $u - v$, in the sense that for any point $x \neq u$ on the line,

$$|(x-u)^T(u-v)| > \frac{2}{\sqrt{n}}|x-u||u-v|.$$

The measure of this subset can be bounded as $\sigma(B_1) \leq 1/8$.

Next, let B_2 be the set of all lines through u which contain a point y with $f(y) > 2\alpha f(u)$. By Lemma 3.10, if we select a line from B_2, then with probability at least $1/2$, a random step along this line takes us to a point x with $f(x) \geq \alpha f(u)$. From the definition of α, this can happen with probability at most $1/16$, which implies that $\sigma(B_2) \leq 1/8$.

Let A be the set of points in K which are not on any of the lines in $B_0 \cup B_1 \cap B_2$, and which are far from u in the sense of Lemma 6.1:

$$|x-u| \geq \frac{1}{4}\max\left\{\rho(u), \frac{1}{32}|u-p|, \frac{1}{32}|u-q|\right\}.$$

Applying Lemma 6.1 to each such line, we get

$$P_u(A) \geq (1 - \frac{1}{8} - \frac{1}{8} - \frac{\delta}{2})\frac{\sqrt{\delta}}{8} \geq \frac{\sqrt{\delta}}{32}.$$

We are going to prove that if we do a hit-and-run step from v, the density of stepping into x is not too small whenever $x \in A$. By the formula (1), we have to treat $|x - v|$ and $\mu_f(v, x)$.

We start with noticing that $f(u)$ and $f(v)$ are almost equal. Indeed, Lemma 4.1 implies that

$$\frac{64}{65} \leq \frac{f(v)}{f(u)} \leq \frac{65}{64}.$$

Claim 1. For every $x \in A$,

$$|x - v|^n \leq \sqrt{\frac{1}{\delta}}|x - u|^n.$$

Indeed, since $x \in A$, we have

$$|x - u| \geq \frac{1}{4}\rho(u) \geq \frac{r}{4\sqrt{\ln(1/\delta)}} \geq 8\frac{\sqrt{n}}{\sqrt{\ln(1/\delta)}}|u - v|.$$

On the other hand,

$$\begin{aligned}
|x - v|^2 &= |x - u|^2 + |u - v|^2 + 2(x - u)^T(u - v) \\
&\leq |x - u|^2 + |u - v|^2 + \frac{4}{\sqrt{n}}|x - u||u - v| \\
&\leq |x - u|^2 + \frac{\ln(1/\delta)}{64n}|x - u|^2 + \frac{\sqrt{\ln(1/\delta)}}{2n}|x - u|^2 \\
&\leq (1 + \frac{\ln(1/\delta)}{n})|x - u|^2
\end{aligned}$$

Hence the claim follows:

$$|x - v|^n \leq \left(1 + \frac{\ln(1/\delta)}{n}\right)^{\frac{n}{2}} |x - u|^n < \sqrt{\frac{1}{\delta}}|x - u|^n.$$

The main part of the proof of Lemma 6.4 is the following Claim:

Claim 2. For every $x \in A$,

$$\mu_f(v, x) < 32\frac{|x - v|}{|x - u|}\mu_f(u, x).$$

To prove this, let y, z be the points where $\ell(u, v)$ intersects the boundary of $L(f(u)/2)$, so that these points are in the order y, u, v, z. Let y', z' be the points where $\ell(u,v)$ intersects the boundary of K. By $f(y) = f(u)/2$, we have $f(y',u) \le 2f(y,u)$, and so

$$d_f(u,v) = \frac{f(u,v)f(y',z')}{f(y',u)f(v,z')} \ge \frac{f(u,v)}{f(y',u)} \ge \frac{f(u,v)}{2f(y,u)} \ge \frac{|u-v|}{4|y-u|}.$$

It follows that

$$|y-u| \ge \frac{|u-v|}{4d_f(u,v)} \ge 32\ln(3+\alpha)\cdot|u-v| > 32|u-v|. \tag{2}$$

A similar argument shows that

$$|z-v| \ge 32\ln(3+\alpha)\cdot|u-v| > 32|u-v|. \tag{3}$$

Next, we compare the function values along the lines $\ell(u,x)$ and $\ell(v,x)$. Let F denote the maximum value of f along $\ell(u,x)$, and let p,q be the intersection points of $\ell(u,x)$ with the boundary of $L(F/8)$, so that q is in the same direction from p as x is from u. Since $x \in A$, we know that

$$|u-p|, |u-q| \le 32|x-u|. \tag{4}$$

For each point $a \in \ell(u,x)$ we define two points $a', a'' \in \ell(v,x)$ as follows. If a is on the semiline of $\ell(u,x)$ starting from x containing u, then we obtain a' by projecting a from y to $\ell(v,x)$, and we obtain a'' by projecting a from z. If a is on the complementary semiline, then the other way around, we obtain a' by projecting from z and a'' by projecting from y.

Simple geometry shows that if

$$|a-u| < \frac{|y-u|}{|u-v|}|x-u|, \frac{|z-u|}{|u-v|}|x-u|,$$

then a', a'' exist and a'' is between v and a'. Furthermore, $a \mapsto a'$ and $a \mapsto a''$ are monotone mappings in this range.

A key observation is that if $|a-u| \le 32|x-u|$, then

$$f(a') < 2f(a). \tag{5}$$

To prove this, let $b = \phi(a)$. We have to distinguish three cases.

(a) $a \in [u,x]$. Then, using (2),

$$\frac{|a-b|}{|y-b|} \le \frac{|u-v|}{|y-v|} \le \frac{|u-v|}{|y-u|} \le \frac{1}{128\ln(3+\alpha)}.$$

Further, by the logconcavity of f,

$$f(a) \ge f(b)^{\frac{|y-a|}{|y-b|}} f(y)^{\frac{|a-b|}{|y-b|}}.$$

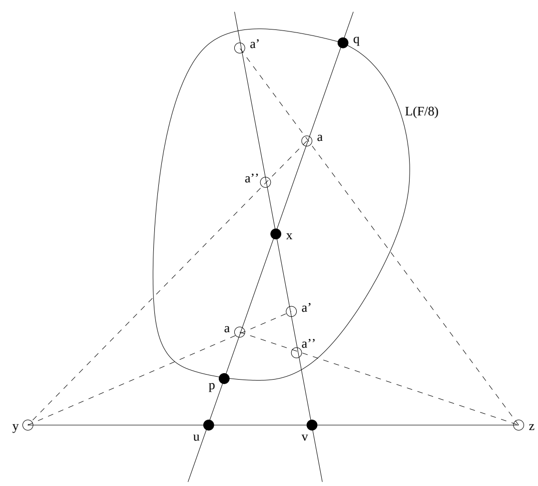

Figure 3: Comparing steps from nearby points.

Thus,

$$f(b) \leq \frac{f(a)^{\frac{|y-b|}{|y-a|}}}{f(y)^{\frac{|a-b|}{|y-a|}}} \quad = \quad f(a)\left(\frac{f(a)}{f(y)}\right)^{\frac{|a-b|}{|y-a|}}.$$

Here

$$f(a) \leq 2\alpha f(u) \leq 4\alpha f(y),$$

since $x \in A_2$. Thus

$$f(b) \leq f(a)(4\alpha)^{\frac{1}{128\ln(3+\alpha)}} < 2f(a).$$

(b) $a \in \ell^+(x,u)$. By Menelaus' theorem,

$$\frac{|a-b|}{|b-z|} = \frac{|x-a|}{|x-u|} \cdot \frac{|u-v|}{|v-z|}.$$

By (4), $|x-a|/|x-u| \leq 16$, and so by (3),

$$\frac{|a-b|}{|b-z|} \leq 16 d_f(u,v) \leq \frac{1}{4\ln(3+\alpha)}.$$

By logconcavity,

$$f(a) \geq f(b)^{|a-z|/|b-z|} f(z)^{|a-b|/|b-z|}$$

Rewriting, we get

$$\begin{aligned} f(b) \leq \frac{f(a)^{|b-z|/|a-z|}}{f(z)^{|a-b|/|a-z|}} &= f(a)\left(\frac{f(a)}{f(z)}\right)^{|a-b|/|a-z|} \\ &\leq f(a)(4\alpha)^{\frac{1}{4\ln(3+\alpha)-1}} \leq 2f(a). \end{aligned}$$

(c) $a \in \ell^+(u, x)$. By Menelaus' theorem again,

$$\frac{|a-b|}{|b-y|} = \frac{|x-a|}{|x-u|} \cdot \frac{|u-v|}{|v-y|}.$$

Again by (4), $|x - a|/|x - u| \leq 16$. Hence, using (2) again,

$$\frac{|a-b|}{|b-z|} \leq 16 d_f(u, v) \leq \frac{1}{4\ln(3+\alpha)}.$$

By logconcavity,

$$f(a) \geq f(b)^{|a-y|/|b-y|} f(z)^{|a-b|/|b-y|}$$

Rewriting, we get

$$\begin{aligned} f(b) \leq \frac{f(a)^{|b-y|/|a-y|}}{f(y)^{|a-b|/|a-y|}} &= f(a)\left(\frac{f(a)}{f(y)}\right)^{|a-b|/|a-y|} \\ &\leq f(a)(4\alpha)^{\frac{1}{4\ln(3+\alpha)-1}} \leq 2f(a). \end{aligned}$$

This proves inequality (5).

Similar argument shows that if $|a - u| \leq 32|x - u|$, then

$$f(a'') > \frac{1}{2} f(a). \tag{6}$$

Let $a \in \ell(u, x)$ be a point with $f(a) = F$. Then $a \in [p, q]$, and hence $|a - u| < \max\{|p - u|, |q - u|\} \leq 32|x - u|$ (since $x \in A$).

These considerations describe the behavior of f along $\ell(v, x)$ quite well. Let $r = p'$ and $s = q'$. (5) implies that $f(r), f(r) \leq F/4$. On the other hand, $f(a'') > F/2$ by (6).

Next we argue that $a'' \in [r, s]$. To this end, consider also the point $b \in \ell(u, x)$ defined by $b' = a''$. It is easy to see that such a b exists and that b is between u and a. This implies that $|b - u| < 32|x - u|$, and so by (5), $f(b) > f(b')/2 = f(a'')/2$. Thus $f(b) > F/4$, which implies that $b \in [p, q]$, and so $b' \in [p', q'] = [r, s]$.

Thus f assumes a value at least $F/2$ in the interval $[r, s]$ and drops to at most $F/4$ at the ends. Let c be the point where f attains its maximum along the line $\ell(v, x)$. It follows that $c \in [r, s]$ and so $c = d'$ for some $d \in [p, q]$. Hence by (5), $f(c) \leq 2f(b) \leq 2F$. Thus we know that the maximum value F' of f along $\ell(v,x)$ satisfies

$$\frac{1}{2}F \leq F' \leq 2F. \tag{7}$$

Having dealt with the function values, we also need an estimate of the length of $[r,s]$:

$$|r-s| \leq 2\frac{|x-v|}{|x-u|}|p-q|. \tag{8}$$

To prove this, assume e.g. that the order of the points along $\ell(u,x)$ is p,u,x,q (the other cases are similar). By Menelaus' theorem,

$$\frac{|x-r|}{|v-r|} = \frac{|u-y|}{|v-y|} \cdot \frac{|x-p|}{|u-p|} = \left(1 - \frac{|v-u|}{|v-y|}\right)\frac{|x-p|}{|u-p|}.$$

Using (2), it follows that

$$\frac{|x-r|}{|v-r|} \geq \frac{31}{32}\frac{|x-p|}{|u-p|}.$$

Thus,

$$\begin{aligned}
\frac{|x-v|}{|v-r|} &= \frac{|x-r|}{|v-r|} - 1 \geq \frac{31}{32}\frac{|x-p|}{|u-p|} - 1 \\
&= \frac{|x-u|}{|u-p|} - \frac{1}{32}\frac{|x-p|}{|u-p|} \\
&= \frac{|x-u|}{|u-p|}\left(1 - \frac{1}{32}\frac{|x-p|}{|x-u|}\right) \\
&> \frac{|x-u|}{|u-p|}\left(1 - \frac{1}{32}\cdot 16\right) = \frac{1}{2}\frac{|x-u|}{|u-p|}.
\end{aligned}$$

In the last line above, we have used (4). Hence,

$$|v-r| < 2\frac{|x-v|}{|x-u|}|u-p|. \tag{9}$$

Similarly,

$$|v-s| < 2\frac{|x-v|}{|x-u|}|u-q|.$$

Adding these two inequalities proves (8).

Now Claim 2 follows easily. We have

$$\mu(\ell(u,x)) \geq \frac{F}{8}|p-q|,$$

while we know by Lemma 3.10(a) that

$$\mu(\ell(v,x)) \leq 2f[r,s].$$

By (7) and (8),

$$f(r,s) \leq 2F|r-s| \leq 4F|p-q|\frac{|x-v|}{|x-u|},$$

and hence

$$\mu(\ell(v,x)) < 32\frac{|x-v|}{|x-u|}\mu(\ell(u,x)),$$

proving Claim 2.

Using Claims 1 and 2, we get for any $A' \subset A$,

$$\begin{aligned}
P_v(A') &= \frac{2}{n\pi_n}\int_{A'}\frac{f(x)\,dx}{\mu_f(v,x)|x-v|^{n-1}} \\
&\geq \frac{2}{32n\pi_n}\int_{A'}\frac{|x-u|f(x)\,dx}{\mu_f(u,x)|x-v|^{n}} \\
&\geq \frac{\sqrt{\delta}}{32n\pi_n}\int_{A'}\frac{f(x)\,dx}{\mu_f(u,x)|x-u|^{n-1}} \\
&\geq \frac{\sqrt{\delta}}{32}P_u(A').
\end{aligned}$$

This concludes the proof of Lemma 6.4. □

7 Proof of the isoperimetric inequality.

Let h_i be the characteristic function of S_i for $i = 1, 2, 3$, and let h_4 be the constant function 1 on K. We want to prove that

$$d_K(S_1, S_2)\left(\int fh_2\right)\left(\int fh_2\right) \leq \left(\int fh_3\right)\left(\int fh_4\right).$$

Let $a, b \in K$ and g be a nonnegative linear function on $[0,1]$. Set $v(t) = (1-t)a + tb$, and

$$J_i = \int_0^1 h_i(v(t))f(v(t))g^{n-1}(v(t))\,dt.$$

By Theorem 2.7 of [11], it is enough to prove that

$$d_K(S_1, S_2)J_1 \cdot J_2 \leq J_3 \cdot J_4. \tag{10}$$

A standard argument [10, 13] shows that it suffices to prove the inequality for the case when J_1, J_2, J_3 are integrals over the intervals $[0, u_1]$, $[u_2, 1]$ and (u_1, u_2) respectively $(0 < u_1 < u_2 < 1)$.

Consider the points $c_i = (1-u_i)a + u_i b$. Since $c_i \in S_i$, we have $d_K(c_1, c_2) \geq \varepsilon$. It is easy to see that

$$d_K(c_1, c_2) \leq (a : u_2 : u_1 : b)$$

while

$$\frac{J_3 \cdot J_4}{J_1 \cdot J_2} = (a : u_2 : u_1 : b)_f.$$

Thus (10) follows from Lemma 4.2.

8 Proof of the mixing bound.

Let $K = S_1 \cup S_2$ be a partition into measurable sets with $\pi_f(S_1), \pi_f(S_2) > \varepsilon$. We will prove that

$$\int_{S_1} P_u(S_2)\, d\pi_f \quad \geq \quad \frac{r}{2^{18}\sqrt{n}R}(\pi_f(S_1) - \varepsilon)(\pi_f(S_2) - \varepsilon) \tag{11}$$

We can read the left hand side as follows: we select a random point X from distribution π and make one step to get Y. What is the probability that $X \in S_1$ and $Y \in S_2$? It is well known that this quantity remains the same if S_1 and S_2 are interchanged.

For $i \in \{1, 2\}$, let

$$\begin{aligned} S_i' &= \{x \in S_i : \; P_x(S_{3-i}) < \frac{1}{1000}\delta(x), \\ S_3' &= K \setminus S_1' \setminus S_2'. \end{aligned}$$

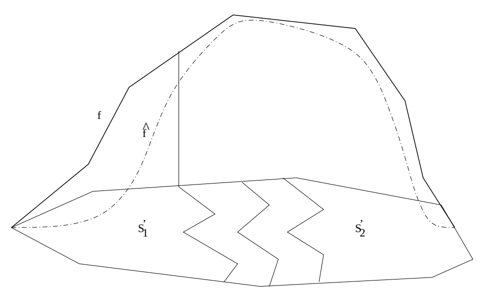

Figure 4: The mixing proof.

First, suppose that $\pi_{\hat{f}}(S_1') \leq \pi_{\hat{f}}(S_1)/2$. Then the left hand side of (11) is at least

$$\frac{1}{1000}\int_{u \in S_1 \setminus S_1'} \frac{\hat{f}(u)}{f(u)} f(u)\, du = \frac{1}{1000}\pi_{\hat{f}}(S_1 \setminus S_1') \geq \frac{1}{2000}\pi_{\hat{f}}(S_1).$$

Lemma 5.3 implies that

$$\pi_{\hat{f}}(S_1) \geq \pi_f(S_1) - \varepsilon/4.$$

Hence,

$$\int_{S_1} P_u(S_2)\, d\pi_f \geq \frac{1}{2000}(\pi_f(S_1) - \frac{\varepsilon}{4})$$

which implies (11).

So we can assume that $\pi_f(S_1') \geq \pi_f(S_1)/2$, and similarly $\pi_f(S_2') \geq \pi_f(S_2)/2$. Let W be the subset of $\mathbb{R}^n$ with $\alpha(u) > 2^{18}nR/r\varepsilon$. Then by lemma 6.3,

$$\pi_f(W) \leq \frac{\varepsilon r}{2^{20}nR}.$$

By Lemma 6.4, for any two points $u_1 \in S_1' \setminus W$, $u_2 \in S_2' \setminus W$, one of the following holds:

$$d_f(u,v) \geq \frac{1}{64\ln(3+\alpha(u))} \geq \frac{1}{2^8 \log n} \tag{12}$$

$$d(u,v) \geq \frac{r}{2^6\sqrt{n}} \tag{13}$$

In either case, we get a lower bound on $d_K(u,v)$:

$$d_K(u,v) \geq \frac{r}{2^7\sqrt{n}R}. \tag{14}$$

Indeed, in the first case, Lemma 5.7(c) implies that

$$d_K(u,v) \geq \frac{1}{8n\log n} \cdot \frac{1}{2^8 \log n} > \frac{r}{2^7\sqrt{n}R};$$

in the second, Lemma 5.7(b) implies that

$$d_K(u,v) \geq \frac{1}{2R} \cdot \frac{r}{2^6\sqrt{n}} = \frac{r}{2^7\sqrt{n}R}.$$

Applying Theorem 2.2 to $\hat{f}$, we get

$$\pi_{\hat{f}}(S_3') \geq \frac{r}{2^7\sqrt{n}R}\pi_{\hat{f}}(S_1' \setminus W)\pi_{\hat{f}}(S_2' \setminus W) \geq \frac{r}{2^7\sqrt{n}R}(\pi_f(S_1) - \frac{\varepsilon}{2})(\pi_f(S_2) - \frac{\varepsilon}{2}).$$

Therefore,

$$\begin{aligned}
\int_{S_1} P_u(S_2)\, d\pi_f &\geq \frac{1}{2}\int_{S_3'} \frac{\hat{f}(u)}{1000 f(u)} f(u) du - \pi_f(W) \\
&\geq \frac{1}{2000}\pi_{\hat{f}}(S_3') - \pi_f(W) \\
&\geq \frac{r}{2^{18}\sqrt{n}R}(\pi_f(S_1) - \varepsilon)(\pi_f(S_2) - \varepsilon)
\end{aligned}$$

and (11) is proved.

By Corollary 1.5 in [14] it follows that for all $m \geq 0$, and every measurable set S,

$$|\sigma^m(S) - \pi_f(S)| \leq 2\varepsilon + \exp\left(-\frac{mr^2}{2^{42}nR^2}\right),$$

which proves Theorem 2.1.

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