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Invasion percolation and the incipient infinite cluster in 2D

Antal A. Járai[*]

The University of British Columbia
Department of Mathematics
#121-1984 Mathematics Road
Vancouver, B.C. Canada V6T 1Z2
jarai@math.ubc.ca

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Abstract: We establish two links between two-dimensional invasion percolation and Kesten's incipient infinite cluster (IIC). We first prove that the k-th moment of the number of invaded sites within the box $[-n, n] \times [-n, n]$ is of order $(n^2 \pi_n)^k$, for $k \geq 1$, where π_n is the probability that the origin in critical percolation is connected to the boundary of a box of radius n. This improves a result of Y. Zhang. We show that the size of the invaded region, when scaled by $n^2 \pi_n$, is tight.

Secondly, we prove that the invasion cluster looks asymptotically like the IIC, when viewed from an invaded site v, in the limit $|v| \to \infty$. We also establish this when an invaded site v is chosen at random from a box of radius n, and $n \to \infty$.

Key words. invasion percolation, incipient infinite cluster, self-organized criticality

1. Introduction

Invasion percolation [11,20,5,28] is a stochastic growth model that is closely related to critical Bernoulli percolation. Critical percolation clusters have a fractal geometry that has been widely studied. The invasion dynamics reproduces the critical percolation picture, without a parameter being tuned to criticality. Both heuristics and existing work on invasion [28,9,29] indicate a close relationship between the invasion cluster and the "incipient cluster" of critical percolation. In this paper we formulate and prove results that relate these two objects.

To explain our motivation in more detail, we review a few results about invasion percolation in Section 1.2 below. We only consider the simplest setting, invasion without trapping, which is defined in Section 1.1.

[*] *Present address:* CWI, PNA 3, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands, jarai@cwi.nl

1.1. The model. Consider the hypercubic lattice $\mathbb{Z}^d$ with its set of nearest neighbor bonds $\mathbb{E}^d$. For a subgraph G of $(\mathbb{Z}^d, \mathbb{E}^d)$ we write $E(G)$ for the set of bonds of G, and $v \in G$ means that v is a vertex of G. We write $e = \langle v, w \rangle$ for a bond e with endpoints v and w. We define *invasion without trapping* as follows.

Let $\{\omega(e)\}_{e \in \mathbb{E}^d}$ be i.i.d. uniform random variables on $[0,1]$, indexed by the bonds. Given the random configuration ω, we construct an increasing sequence $G_0, G_1, G_2, \dots$ of connected subgraphs of the lattice. The graph G_0 only contains the origin. If G_i has been defined, we consider its *outer boundary* ΔG_i, where for any graph G

$$\Delta G = \{e = \langle v, w \rangle \in \mathbb{E}^d : e \notin E(G), \text{ but } v \in G \text{ or } w \in G\}.$$

We select the bond e_{i+1} which minimizes ω on ΔG_i, and we define G_{i+1} by setting $E(G_{i+1}) = E(G_i) \cup \{e_{i+1}\}$. The graph G_i is called the *invasion cluster at time* i, and $\mathcal{S} = \cup_{i=0}^{\infty} G_i$ is called the *invasion cluster at time infinity* or the *invaded region.* (If in the definition of the outer boundary we only include those edges that are not separated by G_i from infinity, we obtain *invasion with trapping.*)

The dynamically defined set $\mathcal{S}$ is closely related to the static Bernoulli percolation model. Let $0 \le p \le 1$. For each bond e, if $\omega(e) < p$, we let $\omega_p(e) = 1$, and e is called *p-open.* Otherwise we let $\omega_p(e) = 0$, and e is called *p-closed.* The set of p-open bonds is then *Bernoulli percolation* at bond density p.

1.2. Previous results. Numerical work by Wilkinson and Willemsen [28] in dimensions $d = 2, 3$ has indicated that the empirical distribution of the values $\{\omega(e_i)\}_{i=1}^{t}$ accepted into the invasion cluster up to time t is asymptotically uniform on $[0, p_c]$, as $t \to \infty$, where p_c is the percolation threshold. Their work has also shown the fractal nature of the invaded region. Namely, with $\mathcal{S}_n$ denoting the intersection of $\mathcal{S}$ with the box of radius n centered at the origin, their results indicated that $|\mathcal{S}_n|$ obeys a power law as $n \to \infty$. Here $|A|$ denotes the number of elements of the set A.

To the best of our knowledge, the mathematical study of invasion percolation started with two papers of Chayes, Chayes and Newman [9,10] who rigorously established, among other things, the uniformity of the empirical distribution on $[0, p_c]$. (In spatial dimensions $d \ge 3$ they proved this modulo major conjectures that later have been established [21,22,1,13].) They also obtained results regarding the fractal nature of the invaded region. They showed that it has zero volume fraction, provided there is no percolation at p_c, and that its surface to volume ratio is $(1 - p_c)/p_c$, the same as the asymptotic ratio for large critical clusters.

An object that turns out to be related to invasion arose in Kesten's analysis of the "incipient infinite cluster" [17]. Condition the cluster of the origin, in critical Bernoulli percolation, to intersect the boundary of the box of radius n. Letting $n \to \infty$, an infinite cluster is obtained, which we will call the IIC. Kesten showed that the limit exists, at least when $d = 2$. (The precise statement of his result is recalled in Section 1.5.) With π_n denoting the probability that the origin is connected to the boundary of $[-n, n] \times [-n, n]$ in critical percolation, he also proved that the k-th moment of the intersection of the IIC with $[-n, n] \times [-n, n]$ is of order $(n^2 \pi_n)^k$, for $k \ge 1$. Around the same time, in [6, p. 1102] the invaded region was proposed as another possible definition of the “incipient cluster”. It is of interest to explore the relationship between invasion and the IIC.

Zhang [29] proved results for the fractal dimension of $\mathcal{S}$ in $d = 2$. He showed that for any $\varepsilon > 0$, with probability tending to 1,

$$n^{2-\varepsilon}\pi_n \le |\mathcal{S}_n| \le n^{2+\varepsilon}\pi_n,$$

confirming observations of [28]. Recent breakthroughs[1] by Smirnov [26] and Lawler, Schramm and Werner [19] show that $\pi_n = n^{-5/48+o(1)}$ as $n \to \infty$, on the triangular site lattice. This shows that, at least on this lattice, the dimensions of the invaded region and the IIC are both 91/48. By the conjectured universality of this exponent, this presumably holds on all common 2D lattices.

In this paper, we establish further close links between invasion and the IIC in two dimensions. First we show that the k-th moment of $|\mathcal{S}_n|$ is of order $(n^2\pi_n)^k$, improving the moment bound of Zhang by a factor of n^ε. We find the improvement interesting for two reasons. We can show that the distribution of $|\mathcal{S}_n|/(n^2\pi_n)$ is tight, which establishes the correct scaling, and the scaling is the same as for the IIC. On the other hand, we hope that our refinement of the method of Zhang may be helpful in the rigorous study of invasion percolation.

Since the fractal dimension is only a crude measure of the geometry, it is of interest to compare the structures of the invaded region and the IIC in more detail. Regarding this, we prove the following. Let $0 < k < \infty$ be fixed, and let v be an invaded site far away from the origin. We look at the invaded region in a window of size k centered at v. We show that, as $|v| \to \infty$, the distribution of invaded sites inside the window approaches the distribution of sites connected to 0 in the IIC. We can also show a somewhat harder result. The same asymptotic distribution is obtained, if v is chosen uniformly at random from $\mathcal{S}_n$, as $n \to \infty$. The latter is analogous to results in [15].

We carry out our analysis for bond percolation on the square lattice. This is only a matter of convenience. The proofs do not use the lattice structure in an essential way, and work whenever the Russo-Seymour-Welsh technology and the method and results of [18, Theorem 2] are applicable.

In the following section we introduce some more notation. Two simple observations, that are well known, but crucial for our analysis are given in Section 1.4. The precise formulation of our main theorems are given in Section 1.5.

1.3. Notation. Restricting from now on to the case $d = 2$, we denote the underlying probability measure (resp. expectation) of our model by P (resp. E). The space of configurations is denoted by $([0,1]^{\mathbb{E}^2}, \mathcal{G})$, where $\mathcal{G}$ is the natural σ-field on $[0,1]^{\mathbb{E}^2}$.

We fix some notation regarding Bernoulli percolation; for more background see [12]. The event that some site in the set A is connected by p-open bonds to some site in the set B is denoted by $A \overset{p}{\longleftrightarrow} B$. The event that there is an infinite p-open path starting at the vertex v is denoted by $v \overset{p}{\longleftrightarrow} \infty$. The *percolation probability* is defined by

$$\theta(p) = P\big(0 \overset{p}{\longleftrightarrow} \infty\big),$$

[1] The first version of our paper was submitted before these results were announced.

and the *critical probability* is $p_c = \inf\{p : \theta(p) > 0\}$. The special feature that $p_c = 1/2$ for the square lattice will not be used. For $v = (v_1, v_2) \in \mathbb{Z}^2$ we let

$$|v| = \max\{|v_1|, |v_2|\}.$$

We define the box

$$B(n, v) = \{w \in \mathbb{Z}^2 : |w - v| \leq n\},$$

and write $B(n)$ for $B(n, 0)$. The boundary of the box is defined by

$$\partial B(n) = \{w \in \mathbb{Z}^2 : |w| = n\}.$$

An important quantity for us is the *point-to-box connectivity*

$$\pi(p, n) = P\big(0 \overset{p}{\longleftrightarrow} \partial B(n)\big).$$

We abbreviate $\pi(p_c, n)$ to π_n, and write

$$s(n) = n^2 \pi_n.$$

1.4. Two observations. The following two facts form the basis of the understanding of why invasion percolation is critical [9, 29].

(A) Let $p > p_c$. Then there exists, with probability 1, an infinite p-open cluster. Suppose that for some i the graph G_i in the definition of the invasion process contains a vertex of this cluster. Then all edges invaded after time i have ω-value less than p. In other words, once the invasion process reaches the infinite p-open cluster, it cannot leave it.

(B) Fix some time i, and consider the set of bonds:

$$\mathcal{H}_i = \big\{e = \langle v, w\rangle \in \mathbb{E}^2 : \omega(e) < p_c, \text{ and } v \overset{p_c}{\longleftrightarrow} G_i\big\}.$$

In words, $\mathcal{H}_i$ is the set of edges that have a p_c-open connection to G_i (and are themselves p_c-open). For common two-dimensional lattices, including the square lattice, it has been established that $\theta(p_c) = 0$ [12, Section 11.3], which implies that $|\mathcal{H}_i| < \infty$ almost surely. This means that (almost surely) all edges in $\mathcal{H}_i$ will be invaded before an ω-value $\geq p_c$ is selected. In other words, the entire p_c-open cluster of any invaded site is also invaded.

1.5. Main results. All constants that appear below are strictly positive and finite. Constants denoted by C_i in different theorems may be different.

Recall that $\mathcal{S}_n = \mathcal{S} \cap B(n)$ denotes the set of invaded sites in the box $B(n)$. In [29, Theorem 1] the following bounds on the moments of $|\mathcal{S}_n|$ are shown. For any $t \geq 1$ there is $C_1(t)$, such that

$$E|\mathcal{S}_n|^t \geq C_1(t)\left(n^2 \pi_n\right)^t, \tag{1.1}$$

and for any $t \geq 1$ and $\varepsilon > 0$ there is $C_2(t, \varepsilon)$, such that

$$E|\mathcal{S}_n|^t \leq C_2(t, \varepsilon)\left(n^{2+\varepsilon} \pi_n\right)^t. \tag{1.2}$$

Our first theorem improves the upper bound.

Theorem 1. *For $t \geq 1$ there is a constant $C(t)$ such that*

$$E|\mathcal{S}_n|^t \leq C(t)\left(n^2\pi_n\right)^t. \tag{1.3}$$

Once Theorem 1 is proved, it is not hard to obtain the following tightness result.

Theorem 2. *We have*

$$\lim_{\varepsilon\downarrow 0}\inf_n P\left(\varepsilon \leq \frac{|\mathcal{S}_n|}{n^2\pi_n} \leq \frac{1}{\varepsilon}\right) = 1. \tag{1.4}$$

The analogues of (1.1), (1.3) and (1.4) are known to hold for the intersection of the IIC with $B(n)$ by the results of [17, Theorem 8].

In order to compare the local geometry of the invaded region with the IIC, we recall Kesten's construction. For this, let $\mathcal{F} = \sigma(\omega_{p_c}(e); e \in \mathbb{E}^2)$. The σ-field $\mathcal{F}$ is generated by the collection $\mathcal{F}_0$ of events that only depend on finitely many of the values $\omega_{p_c}(\cdot)$. It is shown in [17] that for any $E \in \mathcal{F}_0$ the limit

$$\nu(E) = \lim_{n\to\infty} P\big(E \mid 0 \xleftrightarrow{p_c} \partial B(n)\big) \tag{1.5}$$

exists. It follows that ν has a unique extension to a probability measure on $\mathcal{F}$, and under the measure ν, the cluster

$$\mathcal{C}(0) = \left\{v \in \mathbb{Z}^2 : v \xleftrightarrow{p_c} 0\right\}$$

is almost surely infinite. The distribution of the cluster $\mathcal{C}(0)$ under ν is called the IIC.

By (A) and (B) in Section 1.4, it is plausible that if a site v with $|v|$ large is in the invaded region, then the invasion neighborhood of v typically coincides with a large critical percolation cluster, and therefore with the IIC. To formulate this statement, we need some more notation. For $v \in \mathbb{Z}^2$ let τ_v denote the translation of the lattice by v. For a configuration ω and an edge $\langle x, y\rangle$ we let $\tau_v\omega(\langle x, y\rangle) = \omega(\langle x-v, y-v\rangle)$, and for an event A we let $\tau_v A = \{\tau_v\omega : \omega \in A\}$. Let $\mathcal{K}$ be a finite set of edges. Define the event

$$E_{\mathcal{K}} = \{\mathcal{K} \subset \mathcal{S}\},$$

and for $v \in \mathbb{Z}^2$ let

$$T_v E_{\mathcal{K}} = \{\tau_v\mathcal{K} \subset \mathcal{S}\}.$$

The latter is the event that the edges in the translated set $\tau_v\mathcal{K}$ are invaded. (We use the notation T_v, since this event is not the same as $\tau_v E_{\mathcal{K}}$.) To make the connection with the IIC define

$$E'_{\mathcal{K}} = \{\mathcal{K} \subset \mathcal{C}(0)\} \in \mathcal{F}.$$

As $\mathcal{K}$ varies over all finite sets of edges the events $E'_{\mathcal{K}}$ provide all information about the set $\mathcal{C}(0)$.

Theorem 3. *For any $E \in \mathcal{F}_0$ we have*

$$\lim_{|v|\to\infty} P(\tau_v E \,|\, v \in \mathcal{S}) = \nu(E).$$

Also, for any finite $\mathcal{K} \subset \mathbb{E}^2$ we have

$$\lim_{|v|\to\infty} P(T_v E_{\mathcal{K}} \,|\, v \in \mathcal{S}) = \nu(E'_{\mathcal{K}}).$$

The content of the first statement is that asymptotically, the only information the condition $v \in \mathcal{S}$ gives about the neighborhood of v is that v lies in a large p_c-open cluster. The second statement says that the distribution of invaded bonds near v is given by the IIC measure. Instead of a deterministic site v, we can prove such a result for a site chosen uniformly at random from $\mathcal{S}_n$.

Theorem 4. *Let I_n denote a vertex of $\mathcal{S}_n = \mathcal{S} \cap B(n)$ chosen uniformly at random, given $\mathcal{S}_n$. For any $E \in \mathcal{F}_0$ we have*

$$\lim_{n\to\infty} P(\tau_{I_n} E) = \nu(E),$$

and for any finite $\mathcal{K} \subset \mathbb{E}^2$ we have

$$\lim_{n\to\infty} P(T_{I_n} E_{\mathcal{K}}) = \nu(E'_{\mathcal{K}}).$$

The proof of Theorem 4 is similar to that of Theorem 3. We do not think that Theorem 4 could be deduced directly from Theorem 3. The choice of I_n contains information about $|\mathcal{S}_n|$, and a priori one does not know the influence of this on the configuration near I_n. We shall return to this issue when we discuss the proof of Theorem 4.

The next section contains the proofs of Theorems 1 and 2. Some preliminary results are summarized in Section 2.2. Theorems 3 and 4 are proved in Section 3.

2. Upper bound on the number of invaded sites

2.1. Heuristic argument. We describe the main idea of the proof of Theorem 1 in the case $t = 1$, given some scaling assumptions. The actual proof is the translation of the argument below into rigorous statements valid on a number of 2D lattices. Our two main assumptions are that $\theta(p)$ scales like $\pi(p_c, \xi(p))$ for $p > p_c$, where ξ is the correlation length, and that $\pi(p_c, m)$ obeys a power law. For simplicity, we even assume that the latter scales like $m^{-5/48}$. This is in fact known for the triangular site lattice by the work of Smirnov [26] and Lawler, Schramm and Werner [19] which has led to enormous progress [27]. However, in the rest of the paper we will use an argument independent of the lattice.

Assume that $n = 2^k$, and let X_k be the number of invaded sites in the annulus $A_k = B(2^k) \setminus B(2^{k-1})$. We show that $EX_k \le C\, n^2 \pi_n = C\, s(n)$, which is essentially what we need. In using (A) of Section 1.4 for an upper bound, we have to find $p_k > p_c$ so that with high probability the invasion is already in the infinite p_k-open cluster by the time it reaches A_k. An event which ensures that this cluster is reached is

$$H_k = \big\{\text{there is a } p_k\text{-open circuit } \mathcal{D} \text{ in } A_{k-1}, \text{ and } \mathcal{D} \overset{p_k}{\longleftrightarrow} \infty\big\}. \tag{2.1}$$

We want to choose p_k as close to p_c as possible to get a good upper bound on X_k in terms of the infinte cluster, but we also need $P(H_k^c)$ to be small. The proof of (1.2) was essentially based on the optimal choice of p_k. Choose p_k to satisfy

$$\xi(p_k) = \frac{n}{C_1 \log n} = \frac{2^k}{C_1' k},$$

where C_1 is a large constant. As it will be clear from computations in the next paragraph, this leads to a bound of the form $EX_k \le Cs(n)(\log n)^c$ with $c > 0$. Such a bound is implicit in [29].

One can improve this using several p_k's. For example, take $p_k(0) > p_k(1) > p_c$ satisfying

$$\xi(p_k(0)) = \frac{n}{C_1 \log n} \qquad \xi(p_k(1)) = \frac{n}{C_1 \log \log n}.$$

Define the events $H_k(0)$ and $H_k(1)$ by replacing p_k in (2.1) by $p_k(0)$ and $p_k(1)$. To bound the probabilities of $H_k(j)^c$ first note that for $p > p_c$ the crossing probability of a squares of linear scale $m > \xi(p)$ is $1 - O(\exp(-am/\xi(p)))$, for some constant $a > 0$. Since the shortest scale on which connections are required for the event $H_k(0)$ (resp. $H_k(1)$) is of order $(\log n)\xi(p_k(0))$ (resp. $(\log \log n)\xi(p_k(1))$), this leads to the bounds

$$P(H_k(0)^c) \le C \exp(-c \log n) \qquad P(H_k(1)^c) \le C \exp(-c \log \log n). \tag{2.2}$$

Here c can be made large by choosing C_1 large in the definition of $p_k(0)$ and $p_k(1)$. We write

$$EX_k = E(X_k; H_k(0)^c) + E(X_k; H_k(0) \cap H_k(1)^c) + E(X_k; H_k(1)). \tag{2.3}$$

Using (2.2), the first term is bounded above by

$$|A_k|\, P(H_k(0)^c) = O(n^2 n^{-c}).$$

Recalling that $s(n) = n^2 \pi_n \approx n^2 n^{-5/48}$, we see that the right hand side is $o(s(n))$, if c is large enough. For the second term of (2.3), on the event $H_k(0)$ we can bound X_k from above by the intersection of A_k with the $p_k(0)$-open infinite cluster. Let $Z_k(0)$ denote the size of this intersection. Then the second term is bounded above by

$$E(Z_k(0); H_k(1)^c) \le E(Z_k(0))\, P(H_k(1)^c) \le |A_k|\, \theta(p_k(0))\, C\, (\log n)^{-c}, \tag{2.4}$$

where we used the FKG inequality in the first step, and (2.2) in the second. By our scaling assumptions and the definition of $p_k(0)$, we have

$$\theta(p_k(0)) \approx \pi(p_c, \xi(p_k(0))) \approx \left(\frac{n}{\log n}\right)^{-5/48} \approx n^{-5/48} (\log n)^{5/48}.$$

It follows that the right hand side of (2.4) and therefore the second term of (2.3) are bounded above by $C\, n^2 n^{-5/48} (\log n)^{5/48} (\log n)^{-c}$. This quantity is again $o(s(n))$, if c is large enough. Finally, the third term of (2.3) is bounded above by

$$E(Z_k(1)) = |A_k|\, \theta(p_k(1)) \approx |A_k|\, \pi(p_c, \xi(p_k(1))) = O(n^2 n^{-5/48} (\log \log n)^{5/48}).$$

We have shown that $EX_k \le Cs(n)(\log \log n)^{5/48}$. By similar arguments we can prove an upper bound with any number of logarithms. Furthermore, a careful look at the argument will show that the bound $Cs(n)$ in fact holds.

2.2. Preliminaries. Our main tool for the rigorous argument will be the *finite-size scaling correlation length* introduced in [8, Section 3] and further studied in [18]. (See also [4] for a recent account.) Let

$$\sigma(n, m, p) = P(\text{there is a } p\text{-open horizontal crossing of } [0, n] \times [0, m]),$$

where it is assumed that the open crossing does not use bonds lying on the top and bottom sides of the rectangle. Given $\varepsilon > 0$, we define

$$L(p, \varepsilon) = \min\{n : \sigma(n, n, p) \geq 1 - \varepsilon\}, \quad \text{for } p > p_c. \tag{2.5}$$

It is known [18, (1.24)] that there exists an $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ the scaling of $L(p, \varepsilon)$ is independent of ε, in the sense that

$$L(p, \varepsilon_1) \asymp L(p, \varepsilon_2), \quad \text{for fixed } 0 < \varepsilon_1, \varepsilon_2 \leq \varepsilon_0. \tag{2.6}$$

The symbol $\asymp$ means that the ratio of the two sides is bounded away from 0 and ∞ as $p \downarrow p_c$. It is also known that $L(p, \varepsilon_0)$ scales like the usual correlation length [18, Corollary 2], but we will not use this fact explicitly. We are going to take $\varepsilon = \varepsilon_0$, and let $L(p) = L(p, \varepsilon_0)$ for the entire proof. We summarize the properties of $L(p)$ that we need.

1. From the definition it is clear that

$$L(p) \text{ is decreasing, right continuous and } L(p) \to \infty \text{ as } p \downarrow p_c. \tag{2.7}$$

2. If ε_0 is small enough, there are constants C_1 and C_2 such that

$$\sigma(2mL(p), mL(p), p) \geq 1 - C_1 \exp(-C_2 m), \quad \text{for } m > 1. \tag{2.8}$$

This can be shown using ideas of [2, 8, 6] by the rescaling argument of [6, Lemma 2.7]. Indeed, in [6] the rescaling bound

$$\sigma(2n, n, p) \geq 1 - \frac{1}{16}\lambda \quad \text{implies} \quad \sigma(4n, 2n, p) \geq 1 - \frac{1}{16}\lambda^2 \tag{2.9}$$

is shown. One can iterate this starting with $n = L(p)$, and use the the Russo-Seymour-Welsh Lemma (RSW Lemma) [12, Section 11.7] to get an initial bound when $n = L(p)$.

3. It will be important for us that the jumps of $L(p)$ are bounded on a logarithmic scale; there is a constant D such that

$$\lim_{\delta \downarrow 0} \frac{L(p - \delta)}{L(p)} \leq D \quad \text{for } p > p_c. \tag{2.10}$$

For the subcritical version of the finite-size scaling length this was observed in [4]. In two dimensions their proof is easily adapted to the supercritical case. Indeed, the rescaling bound (2.9) implies that (2.10) holds for the quantity

$$\widetilde{L}(p, \varepsilon) = \min\{n : \sigma(2n, n, p) \geq 1 - \varepsilon\}, \quad p > p_c,$$

with $D = 2$, when $\varepsilon < 1/16$. It is simple to deduce from this that (2.10) also holds for $L(p)$. The simple inequality $\sigma(n, n, p) \geq \sigma(2n, n, p)$ shows that $L(p, \varepsilon) \leq \widetilde{L}(p, \varepsilon)$. On the other hand, using the RSW Lemma one can show that for some function $f(\varepsilon)$ we have $\widetilde{L}(p, f(\varepsilon)) \leq L(p, \varepsilon)$, where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Together with (2.6) this establishes (2.10).

4. Finally, the following theorem makes it precise that $\theta(p)$, $\pi(p, L(p))$ and $\pi(p_c, L(p))$ obey the same scaling when $p > p_c$.

Theorem. [18, Theorem 2] *There are constants C_1 and C_2 such that for $p > p_c$*

$$\begin{aligned}\pi(p_c, L(p)) &\leq \pi(p, L(p)) \leq C_1\theta(p) \leq C_1\pi(p, L(p)) \\ &\leq C_2\pi(p_c, L(p)).\end{aligned} \tag{2.11}$$

As for the behavior of $\pi(p_c, n)$, it will be enough for us to have a power law lower bound. Using the idea of [3, Corollary 3.15] one can show that there exists a constant D_1, such that

$$\frac{\pi(p_c, m)}{\pi(p_c, n)} \geq D_1\sqrt{\frac{n}{m}}, \quad m \geq n \geq 1. \tag{2.12}$$

2.3. Proof of Theorem 1. We first prove the case $t = 1$; the extension to higher moments will not pose extra difficulties.

We still write $s(n) = n^2\pi_n$ for short. By monotonicity of π_n we may assume that n is a power of 2. Indeed, if $2^K \leq n < 2^{K+1}$, then $s(2^{K+1}) \leq 4s(n)$. Assuming $n = 2^K$, we divide $B(n)$ into disjoint annuli; $B(n) = \cup_{k=1}^K A_k$, where

$$A_k = B(2^k) \setminus B(2^{k-1}) = \{v \in \mathbb{Z}^2 : 2^{k-1} < |v| \leq 2^k\}, \quad \text{for } k \geq 2,$$

and $A_1 = B(2)$. Letting $X_k = |\mathcal{S} \cap A_k|$ we can write

$$|\mathcal{S}_n| = X_1 + \cdots + X_K. \tag{2.13}$$

We are going to bound EX_k.

Following the idea in Section 2.1 we start by defining a suitable sequence $p_k(0) > p_k(1) > \ldots > p_c$. We introduce the following notation. Let $\log^{(0)} k = k$, and let

$$\log^{(j)} k = \log(\log^{(j-1)} k), \quad \text{for } j \geq 1, \text{ if the right hand side is well-defined.}$$

Here log denotes natural logarithm. For $k > 10$ we define

$$\log^* k = \min\{j > 0 : \log^{(j)} k \text{ is well-defined and } \log^{(j)} k \leq 10\}.$$

Our choice of the constant 10 is quite arbitrary. It is immediate that $\log^{(j)} k > 2$, for $j = 0, 1, \ldots, \log^* k$ and $k > 10$. Let

$$p_k(j) = \inf\left\{p > p_c : L(p) \leq \frac{2^k}{C_3 \log^{(j)} k}\right\}, \quad j = 0, 1, \ldots, \log^* k, \tag{2.14}$$

where the constant C_3 will be chosen later to be large. Since $L(p) \to \infty$, as $p \downarrow p_c$, $p_k(j)$ is well-defined, at least for $k \geq$ some $k_0 = k_0(C_3)$. We assume $k_0 > 10$. From the right continuity of $L(p)$ it follows that $2^k/L(p_k(j)) \geq C_3 \log^{(j)} k$. Together with (2.10) this implies that

$$C_3 \log^{(j)} k \leq \frac{2^k}{L(p_k(j))} \leq DC_3 \log^{(j)} k. \tag{2.15}$$

We define the events

$$H_k(j) = \left\{\text{there is a } p_k(j)\text{-open circuit } \mathcal{D} \text{ in } A_{k-1}, \text{ and } \mathcal{D} \overset{p_k(j)}{\longleftrightarrow} \infty\right\}, \tag{2.16}$$

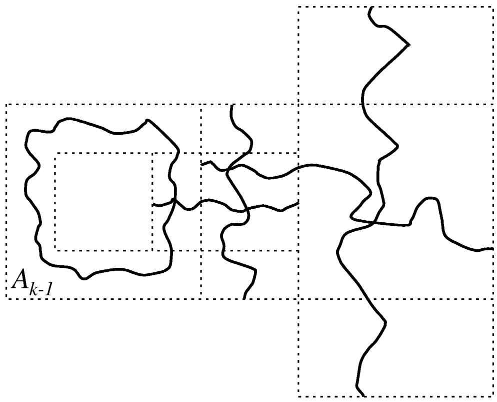

Fig. 1. A sketch of the event $J_k \cap J_k^{0,h} \cap J_k^{0,v} \cap J_k^{1,h} \cap J_k^{1,v}$.

where $k \geq k_0$, $0 \leq j \leq \log^* k$. Here, and later, we always understand that the circuit surrounds $B(2^{k-1})$. On the event $H_k(j)$ the invasion is already in the $p_k(j)$-open infinite cluster by the time it reaches A_k. We can find an upper bound for $P(H_k(j)^c)$ using standard 2D constructions [6, Figure 6], [7]. We have (see Figure 1)

$$H_k(j) \supset J_k(j) \cap \bigcap_{m=0}^{\infty} J_k^m(j), \tag{2.17}$$

where (dropping the index j, for convenience)

$$\begin{aligned}
J_k &= \{\text{there is a } p_k(j)\text{-open circuit in } A_{k-1}\}, \\
J_k^m &= J_k^{m,h} \cap J_k^{m,v}, \\
J_k^{m,h} &= \left\{ \begin{array}{l} \text{there is a } p_k(j)\text{-open horizontal crossing} \\ \text{of } [2^{k-2+m}, 2^{k+m}] \times [-2^{k-2+m}, 2^{k-2+m}] \end{array} \right\}, \quad m \geq 0 \\
J_k^{m,v} &= \left\{ \begin{array}{l} \text{there is a } p_k(j)\text{-open vertical crossing} \\ \text{of } [2^{k-1+m}, 2^{k+m}] \times [-2^{k-1+m}, 2^{k-1+m}] \end{array} \right\}, \quad m \geq 0.
\end{aligned}$$

We bound the probabilities of $J_k(j)^c$, $J_k^m(j)^c$ and $H_k(j)^c$ using (2.8). By RSW arguments

$$P(J_k(j)^c) \leq 4(1 - \sigma(2^k, 2^{k-2}, p_k(j))) \leq 16(1 - \sigma(2^{k-1}, 2^{k-2}, p_k(j))). \tag{2.18}$$

Therefore, by (2.8) and (2.15) we have

$$\begin{aligned}
P(J_k(j)^c) &\leq 16\, C_1 \exp(-C_2 2^{k-2}/L(p_k(j))) \\
&\leq 16\, C_1 \exp\Big\{ -\frac{1}{4} C_2 C_3 \log^{(j)} k \Big\}.
\end{aligned} \tag{2.19}$$

Similarly, we find that

$$
\begin{aligned}
P(J_k^m(j)^c) &\le 2(1-\sigma(2^{k+m},2^{k+m-1},p_k(j))) \\
&\le 2C_1\exp\{-C_2 2^{k+m-1}/L(p_k(j))\} \\
&\le 2C_1\exp\Big\{-\frac{1}{2}C_2C_3 2^m\log^{(j)} k\Big\}.
\end{aligned}
\tag{2.20}
$$

Summing over m and using (2.19) we get

$$
P(H_k(j)^c) \le P(J_k(j)^c)+\sum_{m=0}^{\infty} P(J_k^m(j)^c) \le (16\,C_1+C_4)\exp\Big\{-\frac{1}{4}C_2C_3\log^{(j)} k\Big\}.
$$

Since $\log^{(j)} k > 2$, the constant C_4 does not depend on C_3 as long as C_3 is larger than some fixed positive number. Writing $c = C_2C_3/4$ for short, we have

$$
P(H_k(j)^c) \le C_5\exp(-c\log^{(j)} k), \tag{2.21}
$$

where the constant c depends on C_3, and can be made large by choosing C_3 large.

On the event $H_k(j)$ we have

$$
X_k I[H_k(j)] \le Z_k(j) \stackrel{\text{def}}{=} \big|\{v\in A_k : v \stackrel{p_k(j)}{\longleftrightarrow} \infty\}\big|.
$$

Since $\log^{(0)} k > \ldots > \log^{(\log^* k)} k$, we have $p_k(0)\ge\ldots\ge p_k(\log^* k)$, and hence $H_k(0)\supset\ldots\supset H_k(\log^* k)$. Using the notation $H_k(\log^* k+1)=\emptyset$, for $k>k_0$ we have

$$
\begin{aligned}
EX_k &= E(X_k;H_k(0)^c) + \sum_{j=1}^{\log^* k+1} E(X_k;H_k(j-1)\cap H_k(j)^c) \\
&\le |A_k|P(H_k(0)^c) + \Bigg\{\sum_{j=1}^{\log^* k} E(Z_k(j-1);H_k(j)^c)\Bigg\} + EZ_k(\log^* k).
\end{aligned}
\tag{2.22}
$$

By (2.21), the first term on the right hand side is less than $|A_k|C_5e^{-ck}$. For the second term of (2.22), observe that $Z_k(j-1)$ is a decreasing variable as a function of the edge-values $\{\omega(e)\}_{e\in\mathbb{E}^2}$, and $H_k(j)^c$ is increasing. By the FKG inequality [12] and (2.21) we get

$$
\begin{aligned}
E(Z_k(j-1);H_k(j)^c) &\le EZ_k(j-1)\cdot P(H_k(j)^c) \\
&\le |A_k|\,\theta(p_k(j-1))\,C_5\exp\{-c\log^{(j)} k\}.
\end{aligned}
\tag{2.23}
$$

As for the last term in (2.22) we have $EZ_k(\log^* k)\le |A_k|\,\theta(p_k(\log^* k))$.

We compare $\theta(p_k(j))$ to $\pi(p_c,2^k)$. An application of (2.11), (2.12) and (2.15) yields

$$
\begin{aligned}
\theta(p_k(j)) &\le C_6\pi(p_c,L(p_k(j))) = C_6\pi(p_c,2^k)\frac{\pi(p_c,L(p_k(j)))}{\pi(p_c,2^k)} \\
&\le \frac{C_6}{D_1}\pi(p_c,2^k)\left(\frac{2^k}{L(p_k(j))}\right)^{1/2} \le \frac{C_6}{D_1}\pi(p_c,2^k)(DC_3\log^{(j)} k)^{1/2}.
\end{aligned}
\tag{2.24}
$$

Here C_6 is the constant C_2/C_1 from (2.11). For $j = \log^* k$ we have $\log^{(j)} k \le 10$, which shows that $\theta(p_k(\log^* k)) = O(\pi(p_c, 2^k))$.

The bounds (2.24) and (2.23) imply that the right hand side of (2.22) is less than

$$C_7|A_k|\pi(p_c, 2^k)\left[\frac{\exp\{-ck\}}{\pi(p_c, 2^k)} + \left\{\sum_{j=1}^{\log^* k} (\log^{(j-1)} k)^{1/2-c}\right\} + 1\right], \tag{2.25}$$

where the constant C_7 depends on C_3. We show that the expression in the square brackets is less than a constant, if c is large enough (and therefore if C_3 is large enough). First, by (2.12) we have $\pi(p_c, 2^k) \ge C_8 2^{-k/2}$. If we choose $c \ge (\log 2)/2$, then $e^{-ck}/\pi(p_c, 2^k) \le 1/C_8$. In order to bound the sum over j, we require that $c \ge 3/2$. Then it is enough to show that

$$\sup_{k>10} \sum_{j=1}^{\log^* k} \left\{\log^{(j-1)} k\right\}^{-1} \le C_9 < \infty. \tag{2.26}$$

Recalling that $\log^{(j)} k > 2$, and applying this inequality with $j = \log^* k$, we see that the last term of the sum in (2.26) is at most $(e^2)^{-1}$. Similarly, the penultimate term is at most $(\exp\{e^2\})^{-1}$. By induction, this leads to the upper bound

$$\frac{1}{e^2} + \frac{1}{e^{e^2}} + \frac{1}{e^{e^{e^2}}} + \cdots = C_9$$

on the left hand side of (2.26). It follows that the expression in (2.25) is bounded above by

$$C_7|A_k|\pi(p_c, 2^k)[C_8^{-1} + C_9 + 1],$$

if $c \ge 3/2 = \max\{(\log 2)/2, 3/2\}$. Fix C_3 so that this holds.

Recalling that $|A_k|\pi(p_c, 2^k) = O(s(2^k))$, we see that $EX_k \le C_{10}s(2^k)$ for $k \ge k_0$. Increasing C_{10}, if necessary, we may assume this holds for all k. Summing over k and recalling that $n = 2^K$, (2.13) and (2.12) yield

$$\begin{aligned} E|\mathcal{S}_n| &\le C_{10}\sum_{k=1}^{K} s(2^k) \le C_{10}s(n)\sum_{k=1}^{K} \frac{2^{2k}\pi(p_c, 2^k)}{2^{2K}\pi(p_c, 2^K)} \\ &\le \frac{C_{10}}{D_1}s(n)\sum_{k=1}^{K} 2^{2(k-K)}2^{-\frac{1}{2}(k-K)} \le C_{11}s(n). \end{aligned} \tag{2.27}$$

This finishes the proof of the case $t = 1$ of Theorem 1.

For the extension to higher moments note that by Jensen's inequality we may restrict to integer t. We first prove a bound on EX_k^t. We have $X_k^t I[H_k(j)] \le Z_k(j)^t$. By the method of either [23] or [17] we obtain the bound

$$EZ_k(j)^t \le C_{12}(t)[|A_k|\pi(p_k(j), 2^k)]^t.$$

Using that $2^k \ge L(p_k(j))$ and recalling (2.11), (2.12) and (2.15), we have

$$\begin{aligned} \pi(p_k(j), 2^k) &\le \pi(p_k(j), L(p_k(j))) \le C_{13}\pi(p_c, L(p_k(j))) \\ &\le C_{14}\pi(p_c, 2^k)(DC_3 \log^{(j)} k)^{1/2}. \end{aligned}$$

This gives the following bound analogous to (2.25):

$$EX_k^t \le C_{15}(t,C_3)[s(2^k)]^t \left[\frac{\exp\{-ck\}}{\pi(p_c,2^k)^t} + \left\{ \sum_{j=1}^{\log^* k} \left(\log^{(j-1)} k \right)^{t/2-c} \right\} + 1 \right].$$

If $c \ge \max\{t\log 2/2, (t+2)/2\}$ the expression inside the square brackets is bounded by a constant, hence by choosing C_3 large enough we get

$$EX_k^t \le C_{16}(t)[s(2^k)]^t. \tag{2.28}$$

To turn this into a bound on $E|\mathcal{S}_n|^t$ we write

$$|\mathcal{S}_n|^t = \left(\sum_{k=1}^K X_k \right)^t = \sum_{1\le k_1,\dots,k_t\le K} \prod_{i=1}^t X_{k_i}.$$

By Hölder's inequality and (2.28) we have

$$E\prod_{i=1}^t X_{k_i} \le \prod_{i=1}^t \left(EX_{k_i}^t \right)^{1/t} \le C_{16}(t) \prod_{i=1}^t s(2^{k_i}).$$

Summing over $k_1,\dots,k_t$, by the calculation in (2.27) we obtain

$$E|\mathcal{S}_n|^t \le C_{16}(t) \left(\sum_{k=1}^K s(2^k) \right)^t \le C_{17}(t)[s(n)]^t,$$

which completes the proof of Theorem 1. □

2.4. Tightness. Proof of Theorem 2. From Markov's inequality and Theorem 1 it follows that

$$\sup_n P\left(\frac{|\mathcal{S}_n|}{n^2\pi_n} > \frac{1}{\varepsilon} \right) \le C_1\varepsilon \to 0, \quad \text{as } \varepsilon \to 0.$$

We can show the required lower bound on $|\mathcal{S}_n|$ based on the idea of the lower bound of [29, Theorem 1] and the method of [17, Theorem 8]. Again, we may assume $n = 2^K$. For $k \le K$ let

$$Y_k = \left| \left\{ v \in B(3\cdot 2^{k-2}) \setminus B(2^{k-1}) : v \xleftrightarrow{p_c} \partial B(2^k) \text{ inside } A_k \right\} \right|.$$

Define the event

$$G_k = \left\{ \text{there is a } p_c\text{-open circuit } \mathcal{D} \text{ in } B(2^k)\setminus B(3\cdot 2^{k-2}) \text{ and } Y_k \ge \frac{1}{2}EY_k \right\}.$$

By (B) of Section 1.4, on the event G_k we have $|\mathcal{S}_n| \ge Y_k \ge \frac{1}{2}EY_k$. As in [17, Theorem 8] one can show that for $k \le K$ we have $EY_k \ge C_1 s(2^k) \ge C_1 4^{k-K} s(n)$, and $P(G_k) \ge C_3 > 0$. Then for a fixed integer $\ell > 0$ and $\varepsilon < \frac{1}{2}C_2 4^{-\ell}$ we have

$$P\left(\frac{|\mathcal{S}_n|}{s(n)} \le \varepsilon \right) \le P\left(\bigcap_{k=K-\ell}^K G_k^c \right) \le (1-C_3)^{\ell+1},$$

since the G_k are independent. This proves the second part of the claim. □

3. The invasion cluster looks like the IIC

In Section 3.1 we describe the idea of the proof of Theorem 3. The proof of Theorem 4, the random site case, requires additional arguments and is given in Section 3.2. We do not give the proof of Theorem 3 in detail, since it is essentially a simplification of the argument for the random site case. The necessary changes are indicated in Section 3.3.

3.1. Idea for the fixed site case. Let $E \in \mathcal{F}_0$, and consider the first statement of Theorem 3. To analyze $P(\tau_v E, v \in \mathcal{S})$, let $B(N, v)$ be a box centered at v such that $1 \ll N \ll |v|$. Suppose we know that by the time the invasion reaches $B(N, v)$, it is in a p-open infinite cluster with $p - p_c$ very small. Then with large probability all bonds in $B(N, v)$ satisfy $\omega(e) \notin [p_c, p]$. In this case the event $v \in \mathcal{S}$ implies $v \xleftrightarrow{p_c} \partial B(N, v)$. The latter is the conditioning in Kesten's theorem, so we hope to apply (1.5) for the configuration inside $B(N, v)$, with v replacing the origin, to get

$$P\big(\tau_v E, v \xleftrightarrow{p_c} \partial B(N, v)\big) \approx \nu(E)\, P\big(v \xleftrightarrow{p_c} \partial B(N, v)\big).$$

To make this work we need to decouple the box from the configuration outside. For this we put an annulus $B(M, v) \setminus B(N, v)$ with $N \ll M \ll |v|$ around the box. With large probability, the annulus will contain a p_c-open circuit, that will be used to prevent information from outside from influencing the configuration in $B(N, v)$. It is a well-known consequence of the RSW technology that there are constants μ and C such that for all $M > N \geq 1$ we have

$$P(\text{there is no } p_c\text{-open circuit in } B(M) \setminus B(N)) \leq C \left(\frac{N}{M}\right)^{\mu}. \tag{3.1}$$

Therefore our main goal will be to show that $p - p_c$ can be made small enough so that the invasion process inside the circuit mimics critical percolation.

3.2. Random site case. We clarify the extra argument necessary when v is random. We can write

$$P(\tau_{I_n} E) = \sum_{v \in B(n)} E\left(\frac{I[\tau_v E,\, v \in \mathcal{S}]}{|\mathcal{S}_n|}\right). \tag{3.2}$$

If $|\mathcal{S}_n|$ was concentrated around its mean, we could easily apply the result of Theorem 3, and obtain Theorem 4 by averaging over v. However, one expects the fluctuations of $|\mathcal{S}_n|$ to be of the same order as $E|\mathcal{S}_n|$. In fact, we can expect that $|\mathcal{S}_n|/E|\mathcal{S}_n|$ has a non-trivial limit distribution as $n \to \infty$. Nevertheless, considering the box $B(N, v)$ as before, it is natural to believe that for each fixed v the denominator inside the expectation in (3.2) decouples from the event $\tau_v E, v \xleftrightarrow{p_c} \partial B(N, v)$. Thus we hope to apply the same argument as in the fixed site case. However, to make it work, we need to ensure that there is a p_c-open circuit in the annulus $B(M, v) \setminus B(N, v)$ even when v is random. For this we will need to use the tightness of $|\mathcal{S}_n|$.

Proof. (Theorem 4) We start by proving the first statement of the theorem, that is, when $E \in \mathcal{F}_0$. The second statement will only require a little bit of extra argument. We use the notation of the proof of Theorem 1.

We start with the argument that the annulus centered at I_n contains a p_c-open circuit with large probability. Recall that $s(n) = n^2\pi_n$. Let $2^K \le n < 2^{K+1}$. Let $\varepsilon > 0$ be given, which will be used to control errors. By Theorem 2 there is an $x = x(\varepsilon) > 0$, such that

$$P(|\mathcal{S}_n| < xs(n)) \le \varepsilon, \quad \text{if } n \text{ is large enough.} \tag{3.3}$$

Let $\mathrm{An}(a,b) = B(a) \setminus B(b)$, and define

$$\begin{aligned} F = F_{M,N} &= \{\text{there is a } p_c\text{-open circuit in } \mathrm{An}(M,N)\} \\ F(\mathcal{D}) &= \{\mathcal{D} \text{ is the outermost } p_c\text{-open circuit in } \mathrm{An}(M,N)\}. \end{aligned}$$

We are going to choose M, N in the course of the proof so that $1 \ll N \ll M \ll n$. In any case, we assume that $B(N)$ contains all edges on which E depends. From (3.3) we obtain

$$\begin{aligned} &P(I_n \in B(\lfloor\sqrt{n}\rfloor)) \\ &\le P(|\mathcal{S}_n| < xs(n)) + E\left(\frac{\sum_{v\in B(\lfloor\sqrt{n}\rfloor)} I[v \in \mathcal{S}]}{|\mathcal{S}_n|}; |\mathcal{S}_n| \ge xs(n)\right) \\ &\le \varepsilon + \frac{1}{xs(n)} E|\mathcal{S} \cap B(\lfloor\sqrt{n}\rfloor)| \\ &\le \varepsilon + \frac{C_2}{x}\frac{n}{s(n)} \le 2\varepsilon, \end{aligned} \tag{3.4}$$

provided n is large enough, since $s(n) \ge Cn^2 n^{-1/2}$ by (2.12). Recall the definition of the event $H_k(0)$ in (2.16). By (2.21) we have for some $c > 0$

$$P\left(\bigcup_{k=\lfloor K/2\rfloor+1}^{\infty} H_k(0)^c\right) \le C_3 \exp\{-c(\lfloor K/2\rfloor + 1)\} \le \varepsilon, \tag{3.5}$$

if K, and therefore if n is large enough. The choice of c will not play a role this time. We will drop the argument 0, and write H_k instead of $H_k(0)$, and p_k instead of $p_k(0)$ in the rest of the proof. We also introduce the notation $k_n = \lfloor K/2\rfloor$. We want to bound the probability of $\tau_{I_n}F^c$, the event that the required p_c-open circuit surrounding I_n does not exist. We have

$$\begin{aligned} P(\tau_{I_n}F^c) &= E\left(\frac{\sum_{v\in B(n)} I[v \in \mathcal{S}, \tau_v F^c]}{|\mathcal{S}_n|}\right) \\ &\le P(I_n \in B(\lfloor\sqrt{n}\rfloor)) + P(|\mathcal{S}_n| < xs(n)) + P\left(\bigcup_{k=k_n+1}^{\infty} H_k^c\right) \\ &\quad + \frac{1}{xs(n)} \sum_{v\in \mathrm{An}(n,\lfloor\sqrt{n}\rfloor)} E\left(I[v \in \mathcal{S}, \tau_v F^c]; \bigcap_{k=k_n+1}^{\infty} H_k\right). \end{aligned} \tag{3.6}$$

By (3.3), (3.4) and (3.5) the sum of the first three terms on the right hand side is less than 4ε. By the observation $2^{k_n} \le \lfloor \sqrt{n} \rfloor$, the sum over v on the right hand side of (3.6) is less than

$$\sum_{k=k_n+1}^{K+1} \sum_{v \in A_k} E(I[v \in \mathcal{S},\, \tau_v F^c]; H_k) \le \sum_{k=k_n+1}^{K+1} \sum_{v \in A_k} E\big(I\big[v \overset{p_k}{\longleftrightarrow} \infty\big] I[\tau_v F^c]\big).$$

To explain the last inequality, consider the first time t_k that an edge of the p_k-open infinite cluster is invaded. On the event H_k this cannot happen later than first time the invasion reaches the circuit $\mathcal{D}$ in the definition of H_k. In particular, at time t_k the invasion will not have reached A_k. Hence the vertex v is invaded *after* time t_k, and therefore it is in the p_k-open infinite cluster. Altogether this implies that

$$P(\tau_{I_n} F^c) \le 4\varepsilon + \frac{1}{x s(n)} \sum_{k=k_n+1}^{K+1} \sum_{v \in A_k} P\big(v \overset{p_k}{\longleftrightarrow} \infty, \tau_v F^c\big) \tag{3.7}$$

Applying the FKG inequality and (2.24) we get

$$P\big(v \overset{p_k}{\longleftrightarrow} \infty, \tau_v F^c\big) \le \theta(p_k) P(F^c) \le C_4 P(F^c) \pi(p_c, 2^k) \sqrt{k}. \tag{3.8}$$

Summing (3.8) over $v \in A_k$ it follows that the right hand side of (3.7) is less than

$$4\varepsilon + \frac{C_4 P(F^c)}{x s(n)} \sum_{k=1}^{K+1} s(2^k)\sqrt{k} \le 4\varepsilon + \frac{C_5\, P(F^c)}{x} \sqrt{\log n}. \tag{3.9}$$

From (3.1) it follows that for some $C_6 = C_6(\varepsilon, x)$ if $M = C_6 N (\log n)^{1/(2\mu)}$, then the second term on the right hand side of (3.9) is less than ε. With this choice of M, we have

$$P(\tau_{I_n} F^c) \le 5\varepsilon, \tag{3.10}$$

for any fixed N, provided n is large enough.

The bound (3.10) shows that up to a small additive error we can write (3.2) in the form

$$P(\tau_{I_n} E) \approx P(\tau_{I_n} E,\, \tau_{I_n} F) = \sum_{v \in B(n)} E\left(\frac{I[\tau_v E,\, v \in \mathcal{S},\, \tau_v F]}{|\mathcal{S}_n|} \right). \tag{3.11}$$

The next step is to use the disjoint decomposition $F = \cup_{\mathcal{D}} F(\mathcal{D})$ to write the expectation in (3.11) as a sum over $\mathcal{D}$. There is an additional technicality. Later we need that v is typically sufficiently far away from the origin, so we use (3.4) again, to restrict the sum in (3.11) to $v \in \mathrm{An}(n, \lfloor \sqrt{n} \rfloor)$. Equations (3.10), (3.4) and (2.21) yield

$$\begin{aligned} P(\tau_{I_n} E) &\le 8\varepsilon + P(\tau_{I_n} E,\, \tau_{I_n} F,\, \{I_n \notin B(\lfloor \sqrt{n} \rfloor)\},\, H_{k_n - 1}) \\ &= 8\varepsilon + \sum_{v \in \mathrm{An}(n, \lfloor \sqrt{n} \rfloor)} \sum_{\mathcal{D}} E\left(\frac{I[\tau_v E,\, v \in \mathcal{S},\, \tau_v F(\mathcal{D})]}{|\mathcal{S}_n|}; H_{k_n - 1} \right) \\ &\le 8\varepsilon + P(\tau_{I_n} E). \end{aligned} \tag{3.12}$$

Here the second sum is over all circuits $\mathcal{D}$ in $\mathrm{An}(M, N)$.

For decoupling we want to replace $|\mathcal{S}_n|$ by the quantity

$$W_n(\tau_v \mathcal{D}) = |\mathrm{ext}(\tau_v \mathcal{D}) \cap \mathcal{S}_n|,$$

where $\mathrm{ext}(\tau_v \mathcal{D})$ denotes the graph exterior to $\tau_v \mathcal{D}$ (the edges and vertices of $\tau_v \mathcal{D}$ belong to $\mathrm{ext}(\tau_v \mathcal{D})$). We denote by $\mathrm{int}(\tau_v \mathcal{D})$ the interior of $\tau_v \mathcal{D}$. We have $|\mathrm{int}(\tau_v \mathcal{D})| \leq (2M+1)^2$. From the choice of N at the end of the proof it will be clear that $M^2 = o(n)$, which implies that for large n we have the (deterministic) inequalities:

$$W_n(\tau_v \mathcal{D}) \leq |\mathcal{S}_n| \leq (1+\varepsilon) W_n(\tau_v \mathcal{D}).$$

Hence, denoting the value of the expectation in (3.12) by $E(v, \mathcal{D}, n)$, we have

$$E(v, \mathcal{D}, n) \leq E\left(\frac{I[\tau_v E,\, v \in \mathcal{S},\, \tau_v F(\mathcal{D})]}{W_n(\tau_v \mathcal{D})}; H_{k_n-1}\right) \leq (1+\varepsilon) E(v, \mathcal{D}, n). \quad (3.13)$$

We continue by showing that on the event $\tau_v F(\mathcal{D})$, invasion inside $\tau_v \mathcal{D}$ can be decoupled from invasion outside, and it can be approximated by critical percolation. Let us write the configuration $\omega(\cdot) \in [0,1]^{\mathbb{E}^2}$ as $\omega = \eta \oplus \xi$, where ξ is the configuration in $\mathrm{int}(\tau_v \mathcal{D})$ and η is the configuration in $\mathrm{ext}(\tau_v \mathcal{D})$. (In particular, the states of the edges of $\tau_v \mathcal{D}$ are represented by η.) We want to rewrite the expectation in the middle of (3.13) by first conditioning on η. We claim that

(i) the event $\tau_v F(\mathcal{D}) \cap H_{k_n-1}$ only depends on η, given that n is large enough and $v \in \mathrm{An}(n, \lfloor \sqrt{n} \rfloor)$;
(ii) the random variable $I[\tau_v F(\mathcal{D})](W_n(\tau_v \mathcal{D}))^{-1}$ only depends on η, given that n is large enough and $v \in \mathrm{An}(n, \lfloor \sqrt{n} \rfloor)$.

To prove (i) first note that the event $\tau_v F(\mathcal{D})$ only depends on η. Moreover, we show that if $\tau_v F(\mathcal{D})$ occurs, then (at least for n large) the occurrence of H_{k_n-1} is equivalent to the occurrence of

$$\widetilde{H} = \left\{ \begin{array}{c} \text{there is a } p_{k_n-1}\text{-open circuit } \mathcal{E} \text{ in } A_{k_n-2}, \\ \text{and } \mathcal{E} \overset{p_{k_n-1}}{\longleftrightarrow} \infty \text{ outside } \mathrm{int}(\tau_v \mathcal{D}) \end{array} \right\}. \quad (3.14)$$

Assume that $\tau_v F(\mathcal{D})$ occurs. Then the occurrence of $\widetilde{H}$ implies the occurrence of H_{k_n-1}. For the converse we need to show that if H_{k_n-1} occurs, then the infinite path in its definition can be chosen to lie outside $\tau_v \mathcal{D}$. For this fix a p_{k_n-1}-open circuit $\mathcal{E}$ in A_{k_n-2} whose existence is implied by H_{k_n-1}. For n large, the interior of $\tau_v \mathcal{D}$ is disjoint from $B(2^{k_n-2})$, and hence $\mathcal{E}$ lies in $\mathrm{ext}(\tau_v \mathcal{D})$. Now let ρ be a p_{k_n-1}-open path connecting $\mathcal{E}$ to infinity; this path starts in $\mathrm{ext}(\tau_v \mathcal{D})$. Since $p_{k_n-1} > p_c$, the edges of $\tau_v \mathcal{D}$ are p_{k_n-1}-open. Therefore, if some pieces of ρ happen to be inside $\mathrm{int}(\tau_v \mathcal{D})$, we can replace them by arcs of $\tau_v \mathcal{D}$, and still have a p_{k_n-1}-open path. This shows that $\widetilde{H}$ occurs, and thus (i) is established.

To establish (ii) we first note that since $v \in B(\lfloor \sqrt{n} \rfloor)^c$, for large n we have $0 \notin \mathrm{int}(\tau_v \mathcal{D})$. Statement (ii) now follows from Lemma 1 below. Since the validity of the lemma is intuitively clear, we defer its proof to the end of this section.

Lemma 1. *Let $e_1, e_2, \dots$ be the history of the invasion process, i.e., the sequence of edges invaded. Let $\mathcal{E}$ be a circuit for which $0 \notin \operatorname{int} \mathcal{E}$. Let $\omega = \eta \oplus \xi$, where η is the configuration in $\operatorname{ext} \mathcal{E}$, and ξ is the configuration in $\operatorname{int} \mathcal{E}$. Given that $\mathcal{E}$ is p_c-open, the set*

$$\widetilde{\mathcal{H}} = \{e_i : i \geq 1\} \cap \operatorname{ext} \mathcal{E} \tag{3.15}$$

only depends on η.

Conditioning on η and using statements (i) and (ii) we can rewrite the middle expression in (3.13) as

$$\begin{aligned} EE & \left(\frac{I[\tau_v E,\, v \in \mathcal{S}] I[\tau_v F(\mathcal{D})] I[H_{k_n - 1}]}{W_n(\tau_v \mathcal{D})} \,\middle|\, \eta \right) \\ & = E \left(\frac{I[\tau_v F(\mathcal{D}),\, H_{k_n-1}]}{W_n(\tau_v \mathcal{D})} P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta) \right). \end{aligned} \tag{3.16}$$

We claim that

$$I[v \in \mathcal{S}] = I[v \in \mathcal{S}] I[\tau_v \mathcal{D} \subset \mathcal{S}] \quad \text{a.s. on } \tau_v F(\mathcal{D}). \tag{3.17}$$

To see this note that $0 \notin \operatorname{int}(\tau_v \mathcal{D})$ implies that if $v \in \mathcal{S}$ then the invasion has to cross $\tau_v \mathcal{D}$. By (B) of Section 1.4 we have

$$I[\tau_v \mathcal{D} \text{ is reached}] = I[\tau_v \mathcal{D} \subset \mathcal{S}] \quad \text{almost surely on } \tau_v F(\mathcal{D}). \tag{3.18}$$

This proves (3.17).

From (3.18) it is apparent that the event $\{\tau_v \mathcal{D} \subset \mathcal{S}\} \cap \tau_v F(\mathcal{D})$ only depends on η. This fact and (3.17) allow us to write the right hand side of (3.16) in the form

$$E \left(\frac{I[\tau_v F(\mathcal{D}),\, H_{k_n-1},\, \tau_v \mathcal{D} \subset \mathcal{S}]}{W_n(\tau_v \mathcal{D})} P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta) \right). \tag{3.19}$$

The rest of the proof is concerned with analyzing $P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta)$. The idea is that given the event $\tau_v \mathcal{D} \subset \mathcal{S}$, the condition $v \in \mathcal{S}$ can be replaced by $v \overset{p_c}{\longleftrightarrow} \tau_v \mathcal{D}$. Recall that ξ denotes the configuration in $\operatorname{int}(\tau_v \mathcal{D})$, and let

$$Q = Q(\tau_v \mathcal{D}, n) = \{\text{there is no edge } e \in \operatorname{int}(\tau_v \mathcal{D}) \text{ for which } \xi(e) \in [p_c, p_{k_n-1}]\}.$$

In order to bound the probability of Q^c, we estimate the number of edges e in the interior of $\tau_v \mathcal{D}$ for which $\xi(e) \in [p_c, p_{k_n-1}]$. It is known that $\theta(p)$ grows at least linearly for $p > p_c$ near p_c [12, Theorem 5.8]. Therefore (2.11) and (2.15) imply

$$\begin{aligned} p_{k_n-1} - p_c & \leq C_7(\theta(p_{k_n-1}) - \theta(p_c)) = C_7 \theta(p_{k_n-1}) \leq C_8 \pi(p_c, L(p_{k_n-1})) \\ & \leq C_8 \pi \left(p_c, \frac{2^{k_n-1}}{D C_9 (k_n - 1)} \right), \end{aligned} \tag{3.20}$$

where C_9 denotes the constant C_3 from (2.15). Here $2^{k_n-1}/(k_n - 1) \asymp \sqrt{n}/(\log n)$. It is known [16, Lemma 8.5], that there are constants C_{10} and $\zeta > 0$, such that

$$\pi(p_c, m) \leq C_{10} m^{-\zeta}, \quad \text{for } m \geq 1. \tag{3.21}$$

From (3.21) and (3.20) it follows that

$$p_{k_n-1} - p_c \le C_{11} \frac{(\log n)^{\zeta}}{n^{\zeta/2}}. \tag{3.22}$$

Recall that the circuit $\tau_v \mathcal{D}$ lies in the annulus $B(M,v) \setminus B(N,v)$. The relation between M and N is $M = C_6 N (\log n)^{1/(2\mu)}$, for some constant μ, and C_6 only depending on ε, not on n. The number of edges in $\mathrm{int}(\tau_v \mathcal{D})$ is at most $O(M^2) = O(N^2 (\log n)^{1/\mu})$. Hence

$$\begin{aligned} P(Q^c \mid \eta) &\le E(|\{e \in \mathrm{int}(\tau_v \mathcal{D}) : \xi(e) \in [p_c, p_{k_n-1}]\}|) \\ &\le C_{12} M^2 (p_{k_n-1} - p_c) \\ &\le C_{12} C_{11} C_6^2 N^2 \frac{(\log n)^{\zeta + 1/\mu}}{n^{\zeta/2}} \\ &\le C_{13}(N) \cdot n^{-\zeta/4}. \end{aligned} \tag{3.23}$$

For arbitrary fixed N this bound is uniform in $\mathcal{D}$.

Recall that our aim is to analyze $P(\tau_v E,\ v \in \mathcal{S} \mid \eta)$ inside the expression in (3.19). Fix an η such that

$$\tau_v F(\mathcal{D}),\ H_{k_n-1} \text{ and } \{\tau_v \mathcal{D} \subset \mathcal{S}\} \text{ occur.} \tag{3.24}$$

Then on the event Q exactly those edges of $\mathrm{int}(\tau_v \mathcal{D})$ will be invaded that are p_c-open, and have a p_c-open connection to $\tau_v \mathcal{D}$. This means that

$$I[Q] I[\tau_v E,\ v \in \mathcal{S}] = I[Q] I\big[\tau_v E,\ v \xleftrightarrow{p_c} \tau_v \mathcal{D}\big], \tag{3.25}$$

for any η satisfying (3.24).

From (3.25) we have

$$\begin{aligned} P(\tau_v E,\ v \in \mathcal{S} \mid \eta) - P(Q^c \mid \eta) &\le P\big(\tau_v E,\ v \xleftrightarrow{p_c} \tau_v \mathcal{D} \bigm| \eta\big) \\ &= P\big(E,\ 0 \xleftrightarrow{p_c} \mathcal{D} \bigm| \tau_{-v}\eta\big) \\ &= P\big(E,\ 0 \xleftrightarrow{p_c} \mathcal{D}\big) \\ &\le P(\tau_v E,\ v \in \mathcal{S} \mid \eta) + P(Q^c \mid \eta), \end{aligned} \tag{3.26}$$

where at the second equality we have used that the event $E,\ 0 \xleftrightarrow{p_c} \mathcal{D}$ is independent of $\tau_{-v}\eta$. Now we are in a position to apply (1.5) with a small modification. We claim that

$$\lim_{\substack{N \to \infty \\ \mathcal{D} \text{ surrounds } B(N)}} P\big(E \bigm| 0 \xleftrightarrow{p_c} \mathcal{D}\big) = \nu(E). \tag{3.27}$$

In order to prove this, note that by the remark after Theorem 3 in [17], the conclusion of (1.5) holds even in the case when $B(n)$ is replaced by an arbitrary increasing sequence of sets whose union is $\mathbb{Z}^2$. Given any sequence of circuits for which the limit in (3.27) is different from $\nu(E)$, we get a contradiction. Hence for N large enough we have

$$(1-\varepsilon) P\big(E,\ 0 \xleftrightarrow{p_c} \mathcal{D}\big) \le \nu(E) P\big(0 \xleftrightarrow{p_c} \mathcal{D}\big) \le (1+\varepsilon) P\big(E,\ 0 \xleftrightarrow{p_c} \mathcal{D}\big). \tag{3.28}$$

We note that (3.28) also holds when $\nu(E) = 0$. In fact, by RSW considerations $P(E, 0 \overset{p_c}{\longleftrightarrow} \partial B(N)) = 0$ for all large enough N when $\nu(E) = 0$. We fix N such that (3.28) is satisfied.

The last step is to show that $P(Q^c \,|\, \eta)$ is an error term in (3.26). Recalling that $\mathcal{D}$ lies inside $B(M)$ and that $M = C_6 N (\log n)^{1/(2\mu)}$, by (2.12) we have

$$P\big(0 \overset{p_c}{\longleftrightarrow} \mathcal{D}\big) \geq \pi_M \geq \frac{D_1}{\sqrt{C_6 N}(\log n)^{1/(4\mu)}}.$$

Hence recalling (3.23) we conclude that for n large enough we have

$$P(Q^c \,|\, \eta) \leq \varepsilon \nu(E) P\big(0 \overset{p_c}{\longleftrightarrow} \mathcal{D}\big). \tag{3.29}$$

This bound fails when $\nu(E) = 0$. However, it is simple to adapt what follows to this situation. By (3.26), (3.28) and (3.29) we have

$$\begin{aligned} \frac{1}{1+2\varepsilon} P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta) &\leq \nu(E) P\big(0 \overset{p_c}{\longleftrightarrow} \mathcal{D}\big) \\ &\leq \frac{1}{1-2\varepsilon} P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta). \end{aligned} \tag{3.30}$$

We can replace E by the sure event in the arguments above. Then (3.30) and its version for the sure event imply that for n large enough we have

$$\frac{1-2\varepsilon}{1+2\varepsilon} \nu(E) P(v \in \mathcal{S} \,|\, \eta) \leq P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta) \leq \frac{1+2\varepsilon}{1-2\varepsilon} \nu(E) P(v \in \mathcal{S} \,|\, \eta). \tag{3.31}$$

Similarly, the inequalities (3.12) and (3.13) hold with E replaced by the sure event. We plug the estimate of (3.31) back into the expression in (3.19). Recall that (3.19) also equals the middle expression in (3.13). Then it is a simple matter to deduce from (3.12), (3.13), their versions for the sure event and (3.31) that for all large n we have

$$\begin{aligned} P(\tau_{I_n} E) &\leq 8\varepsilon + \frac{(1+2\varepsilon)(1+\varepsilon)}{1-2\varepsilon} \nu(E), \\ P(\tau_{I_n} E) &\geq -8\varepsilon + \frac{1-2\varepsilon}{(1+2\varepsilon)(1+\varepsilon)} \nu(E). \end{aligned} \tag{3.32}$$

Since ε was arbitrary, this implies the first statement of Theorem 4.

To conclude we describe the modifications necessary for the second statement. Recall that $T_v E_{\mathcal{K}}$ is the event that the edges in the set $\tau_v \mathcal{K}$ are invaded, and $\tau_v E'_{\mathcal{K}}$ is the event that these edges belong to the p_c-open cluster of v. It is not hard to check that the manipulations leading to (3.12), (3.13), (3.16) and (3.19) are still valid when $\tau_{I_n} E$ is replaced by $T_{I_n} E_{\mathcal{K}}$, and $\tau_v E$ is replaced by $T_v E_{\mathcal{K}}$. The first difference arises when we approximate invasion by critical percolation on the event Q. This time (3.25) is replaced by

$$I[Q] I[T_v E_{\mathcal{K}},\, v \in \mathcal{S}] = I[Q] I[\tau_v E'_{\mathcal{K}},\, v \overset{p_c}{\longleftrightarrow} \tau_v \mathcal{D}], \tag{3.33}$$

for any η satisfying (3.24). This holds for the same reason as (3.25), namely that on Q exactly those edges of $\mathrm{int}(\tau_v \mathcal{D})$ will be invaded that have a p_c-open connection to $\mathcal{D}$. In the case when $\mathcal{K}$ is connected and contains 0, the event $E'_{\mathcal{K}}$

is a cylinder event. In this case, Kesten's theorem applies, and (3.28) holds with E replaced by $E'_{\mathcal{K}}$. It is not hard to see that in this case the rest of the argument applies without change to show that $\lim P(T_{I_n} E_{\mathcal{K}}) = \nu(E'_{\mathcal{K}})$. We have to work a little bit more for a general $\mathcal{K}$ by approximating $E'_{\mathcal{K}}$ by cylinder events.

Denote the event $\left\{0 \overset{p_c}{\longleftrightarrow} \mathcal{D}\right\}$ by $A_{\mathcal{D}}$. Then it is enough to show that we still have

$$\lim_{\substack{N\to\infty \\ \mathcal{D} \text{ surrounds } B(N)}} P(E'_{\mathcal{K}} \mid A_{\mathcal{D}}) = \nu(E'_{\mathcal{K}}). \tag{3.34}$$

Fix ℓ with the property $\mathcal{K} \subset B(\ell)$. For $m > \ell$ let

$$\mathcal{C}_m(0) = \text{the } p_c\text{-open cluster of 0 inside } B(m).$$

We approximate $E'_{\mathcal{K}}$ by the event $E'_{\mathcal{K},m} = \{\mathcal{K} \subset \mathcal{C}_m(0)\} \subset E'_{\mathcal{K}}$. On the event $(E'_{\mathcal{K}} \setminus E'_{\mathcal{K},m}) \cap A_{\mathcal{D}}$ the event $A_{\mathcal{D}}$ and the event $O_m = \{B(\ell) \overset{p_c}{\longleftrightarrow} \partial B(m)\}$ occur disjointly. By the BK inequality [3] we have

$$P(E'_{\mathcal{K},m},\, A_{\mathcal{D}}) \le P(E'_{\mathcal{K}},\, A_{\mathcal{D}}) \le P(E'_{\mathcal{K},m},\, A_{\mathcal{D}}) + P(O_m)P(A_{\mathcal{D}}). \tag{3.35}$$

We claim that the right hand side is $(1 + o(1))P(E'_{\mathcal{K},m},\, A_{\mathcal{D}})$ as $m \to \infty$. This follows from three facts: $P(O_m) \to 0$ as $m \to \infty$,

$$P(E'_{\mathcal{K},m},\, A_{\mathcal{D}}) \ge P(E'_{\mathcal{K},m})P(A_{\mathcal{D}})$$

(by the FKG inequality), and $P(E'_{\mathcal{K},m}) \ge P(E'_{\mathcal{K},\ell}) > 0$. Since $E'_{\mathcal{K},m}$ is a cylinder event, the conclusion of (3.34) holds for $E'_{\mathcal{K},m}$. Putting this together with (3.35) and the fact that $\lim_{m\to\infty} \nu(E'_{\mathcal{K},m}) = \nu(E'_{\mathcal{K}})$ we get (3.34). $\square$

Proof. (Lemma 1) Recall that G_i $(i \ge 0)$ denotes the invasion cluster at time i.

We separate three phases in the invasion process. The first phase starts at time 0. Let R_1 be the hitting time of the circuit $\mathcal{E}$, i.e., the first time that an edge with an end-vertex on $\mathcal{E}$ is invaded. It may happen that the invasion never reaches $\mathcal{E}$; in this case we let $R_1 = \infty$, and there are no other phases. Otherwise the second phase starts at time $R_1 + 1$. Let R_2 be the time when all edges with a p_c-open connection to G_{R_1}, that are themselves p_c-open, have been invaded. We have $R_2 < \infty$ almost surely on the event $\{R_1 < \infty\}$. Since G_{R_1} contains a vertex of $\mathcal{E}$, the following holds:

$$\begin{aligned}&\text{almost surely on } \{\mathcal{E} \text{ is } p_c\text{-open}\} \cap \{R_1 < \infty\} \text{ all edges}\\ &\text{of the circuit } \mathcal{E} \text{ are invaded during the second phase.}\end{aligned} \tag{3.36}$$

The third phase is the rest of the process.

The set of edges $\mathcal{H}_1$ that are invaded during the first phase only depends on η, since we do not look at any ω-values inside $\mathcal{E}$. In the case $R_1 = \infty$ we are done, since $\widetilde{\mathcal{H}} = \mathcal{H}_1$.

During the second phase all invaded edges have ω-value less than p_c, and the set of edges invaded is

$$\mathcal{H}_2 \overset{\text{def}}{=} \left\{e \in E(G_{R_1})^c : \omega(e) < p_c,\, e \overset{p_c}{\longleftrightarrow} G_{R_1}\right\}.$$

We show that the set $\widetilde{\mathcal{H}}_2 \stackrel{\text{def}}{=} \mathcal{H}_2 \cap \operatorname{ext}\mathcal{E}$ only depends on η. Since $\mathcal{E}$ is p_c-open, any $e \in \widetilde{\mathcal{H}}_2$ in fact has a p_c-open connection to G_{R_1} outside $\operatorname{int}\mathcal{E}$. (See the discussion following (3.14).) Therefore,

$$\widetilde{\mathcal{H}}_2 = \left\{e \in E(G_{R_1})^c \cap \operatorname{ext}\mathcal{E} : \omega(e) < p_c,\ e \stackrel{p_c}{\longleftrightarrow} G_{R_1} \text{ outside } \operatorname{int}\mathcal{E}\right\}.$$

Since G_{R_1} only depends on η, this shows that $\widetilde{\mathcal{H}}_2$ only depends on η, as well.

To discuss the third phase, we need to introduce some more notation. Let Σ be the set of times when an edge in $\operatorname{int}\mathcal{E}$ is invaded during the third phase. For any graph G let $\widetilde{\Delta}G = \Delta G \cap E(\operatorname{ext}\mathcal{E})$. Also, define the graph $\widetilde{G}_i$ by $E(\widetilde{G}_i) = E(G_i) \cap E(\operatorname{ext}\mathcal{E})$. As a result of the previous paragraph, we have:

$$\text{the graph } \widetilde{G}_{R_2} \text{ only depends on } \eta. \tag{3.37}$$

Consider the step at some time $i > R_2$, and first assume that $i \notin \Sigma$. Then, by the definition of the invasion process, e_i minimizes ω on ΔG_{i-1}, furthermore, since $e_i \in \operatorname{ext}\mathcal{E}$, e_i minimizes ω on $\widetilde{\Delta}G_{i-1}$. We noted before, that by the end of the second phase all edges of $\mathcal{E}$ are invaded, hence the edges of $\mathcal{E}$ do not belong to ΔG_{i-1}. This implies that $\widetilde{\Delta}G_{i-1} = \widetilde{\Delta}\widetilde{G}_{i-1}$. Thus

$$\begin{aligned}&\text{for } i > R_2 \text{ and } i \notin \Sigma \text{ we have } E(\widetilde{G}_i) = E(\widetilde{G}_{i-1}) \cup \{f\},\\ &\text{where } f \text{ minimizes } \omega \text{ on the set } \widetilde{\Delta}\widetilde{G}_{i-1}.\end{aligned} \tag{3.38}$$

On the other hand

$$\text{for } i > R_2 \text{ and } i \in \Sigma \text{ we have } \widetilde{G}_i = \widetilde{G}_{i-1}. \tag{3.39}$$

From (3.38) and (3.39) we see that whenever the set $\widetilde{G}_i$ changes, it changes in a fashion determined only by η. Using (3.37), (3.38) and (3.39) we get by induction that the sequence

$$\{\widetilde{G}_i : i \geq R_2,\ i \notin \Sigma\}$$

only depends on η. In particular, the set

$$\begin{aligned}\widetilde{\mathcal{H}}_3 &\stackrel{\text{def}}{=} \{e_i : i > R_2,\ i \notin \Sigma\}\\ &= \{\text{edges outside } \operatorname{int}\mathcal{E} \text{ that are invaded in the third phase}\}\end{aligned}$$

only depends on η. Since $\widetilde{\mathcal{H}} = \mathcal{H}_1 \cup \widetilde{\mathcal{H}}_2 \cup \widetilde{\mathcal{H}}_3$, this shows that $\widetilde{\mathcal{H}}$ only depends on η, and the proof of the lemma is complete. □

3.3. Single site case. In this section we indicate the necessary changes for the proof of Theorem 3.

Proof. (Theorem 3) Let $n = |v|$ and $2^K \leq n < 2^{K+1}$. Other notation will have the same meaning as in the proof of Theorem 4. Let $\varepsilon > 0$ be given. By the FKG inequality and (2.24) we have

$$\begin{aligned}P(\tau_v F^c,\ v \in \mathcal{S}) &\leq P(H_K^c) + P\left(\tau_v F^c,\ v \stackrel{p_K}{\longleftrightarrow} \infty\right)\\ &\leq P(H_K^c) + P(F^c)\theta(p_K)\\ &\leq P(H_K^c) + C_1\pi_n\sqrt{K}P(F^c).\end{aligned} \tag{3.40}$$

By choosing the constant C_3 in the definition of p_K large, we can achieve

$$\frac{P(H_K^c)}{\pi_n} \to 0, \quad \text{as } n \to \infty. \tag{3.41}$$

As in the argument preceding (3.10), we can find constants $\mu > 0$ and $C_3 = C_3(\varepsilon)$, such that if $M = C_3 N (\log n)^{1/(2\mu)}$, then for large n we have

$$\text{RHS of (3.40)} \le \varepsilon \pi_n. \tag{3.42}$$

On the other hand, by (B) of Section 1.4, the FKG inequality, the RSW Lemma, and (2.12), we have

$$\begin{aligned} &P(v \in \mathcal{S}) \\ &\quad \ge P\big(\text{there is a } p_c\text{-open circuit in } A_{K+2} \text{ and } v \stackrel{p_c}{\longleftrightarrow} \partial B(2^{K+2})\big) \\ &\quad \ge C_4 P\big(v \stackrel{p_c}{\longleftrightarrow} \partial B(2^{K+2})\big) \ge C_5 \pi_n. \end{aligned} \tag{3.43}$$

This implies that

$$P(\tau_v F^c \,|\, v \in \mathcal{S}) \le \frac{\varepsilon}{C_5}. \tag{3.44}$$

By (3.41) and (3.43) we also obtain

$$P(H_K^c \,|\, v \in \mathcal{S}) \to 0, \qquad \text{as } n \to \infty. \tag{3.45}$$

From (3.44) and (3.45) it follows that

$$\begin{aligned} P(\tau_v E \,|\, v \in \mathcal{S}) - \frac{2\varepsilon}{C_5} &\le \sum_{\mathcal{D}} P(\tau_v E,\, H_K,\, \tau_v F(\mathcal{D}) \,|\, v \in \mathcal{S}) \\ &\le P(\tau_v E \,|\, v \in \mathcal{S}) \end{aligned} \tag{3.46}$$

By conditioning on the configuration η in $\text{ext}(\tau_v \mathcal{D})$, and using Lemma 1, we can rewrite the summand in (3.46) as

$$\frac{1}{P(v \in \mathcal{S})} E\big\{I[\tau_v F(\mathcal{D}),\, H_K,\, \tau_v \mathcal{D} \subset \mathcal{S}]\, P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta)\big\}.$$

The quantity $P(\tau_v E,\, v \in \mathcal{S} \,|\, \eta)$ can be analyzed by the method of Theorem 4. The rest of the proof is analogous to the random site case.

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