The de Rham-Witt complex and p-adic vanishing cycles[1]

Thomas Geisser[2] and Lars Hesselholt[2]

Introduction

We determine the structure of the de Rham-Witt complex of [**10, 9**] of a smooth scheme over a discrete valuation ring of mixed characteristic with log-poles along the special fiber and show that the sub-sheaf fixed by the Frobenius is isomorphic to the sheaf of p-adic vanishing cycles. We use this result together with the main results of *op. cit.* to evaluate the algebraic K-theory with coefficients of the quotient field K of the henselian local ring at a generic point of the special fiber. The result affirms the Lichtenbaum-Quillen conjecture for the field K.

In more detail, let V_0 be a henselian discrete valuation ring with quotient field K_0 of characteristic 0 and perfect residue field k_0 of odd characteristic p. Let X be a smooth V_0-scheme of relative dimension r, and let i and j denote the inclusion of the special and generic fiber, respectively, as in the cartesian diagram

$$\begin{array}{ccccc} Y & \xhookrightarrow{i} & X & \xhookleftarrow{j} & U \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k_0 & \hookrightarrow & \operatorname{Spec} V_0 & \hookleftarrow & \operatorname{Spec} K_0. \end{array}$$

The henselian local ring of X at a generic point of Y is a henselian discrete valuation ring V whose residue field k is the (non-perfect) function field of Y. A uniformizer of V_0 is also a uniformizer of V, and hence, V_0 and V have the same absolute ramification index e.

We consider the ring $\mathcal{O}_X$ with the log-structure $\alpha\colon M_X \to \mathcal{O}_X$ determined by the special fiber. The de Rham-Witt complex of [**10, 9**]

$$W.\,\Omega^*_{(X,M_X)} = W.\,\Omega^*_{(\mathcal{O}_X,M_X)}$$

is defined as the universal Witt complex over $(\mathcal{O}_X, M_X)$. The reduced sheaves

$$i^*\bar{W}_n\,\Omega^q_{(X,M_X)} = i^*(W_n\,\Omega_{(X,M_X)}/pW_n\,\Omega^q_{(X,M_X)})$$

[1] A previous version of this paper was entitled *On the K-theory of a henselian discrete valuation field with non-perfect residue field.*

[2] The authors were supported in part by grants from the National Science Foundation. The first author received additional support from the Japan Society for the Promotion of Science and the Alfred P. Sloan Foundation.

are quasi-coherent sheaves of $i^*\bar{W}_n(\mathcal{O}_X)$-modules on the small étale site of Y. We show that, Zariski locally on Y, the canonical projection $i^*\bar{W}_n(\mathcal{O}_X) \to \mathcal{O}_Y$ admits a section. Hence, we can view the sheaves in question, Zariski locally, as sheaves of quasi-coherent $\mathcal{O}_Y$-modules. We show in paragraph 1 that these are free of rank

$$\operatorname{rk}_{\mathcal{O}_Y} i^*\bar{W}_n\,\Omega^q_{(X,M_X)} = \binom{r+1}{q} e \sum_{s=0}^{n-1} p^{rs}$$

and give an explicit basis. This allows us, in paragraph 2, to calculate the kernel and cokernel of $1-F$. The result expresses the sheaf of p-adic vanishing cycles in terms of the de Rham-Witt sheaves.

THEOREM A. *Suppose $\mu_{p^v} \subset K_0$. Then there is a natural exact sequence*

$$0 \to i^*R^qj_*\mu_{p^v}^{\otimes q} \to i^*W.\,\Omega^q_{(X,M_X)}/p^v \xrightarrow{1-F} i^*W.\,\Omega^q_{(X,M_X)}/p^v \to 0$$

of sheaves of pro-abelian groups on the small étale site of Y.

We expect that theorem A is valid also if K_0 does not contain the p^vth roots of unity. More precisely, we expect that the terms in the sequence satisfy Galois descent for the extension $K_0(\mu_{p^v})/K_0$; compare [**13**, théorème 1(1)].

The algebraic K-theory of the field K_0 was determined in [**10**]. In paragraph 3, we combine theorem A and the main results of [**10, 9**] to extend this result to the field K. Indeed, we prove the following formula, predicted by the Beilinson-Lichtenbaum conjectures [**2, 16**].

THEOREM B. *Suppose that $\mu_{p^v} \subset K_0$. Then the canonical map*

$$K^M_*(K) \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \to K_*(K, \mathbb{Z}/p^v)$$

is an isomorphism.

The second tensor factor on the left is the symmetric algebra on the $\mathbb{Z}/p^v$-module μ_{p^v}, which is free of rank one, and the map of the statement takes a generator $\zeta \in \mu_{p^v}$ to the associated Bott element $b_\zeta \in K_2(K, \mathbb{Z}/p^v)$. The Milnor groups $K^M_q(K)/pK^M_q(K)$, which were evaluated by Kato in [**14**, theorem 2(1)], are concentrated in degrees $0 \le q \le r+2$. Hence, theorem B shows that the groups $K_*(K, \mathbb{Z}/p^v)$ are two-periodic above this range of degrees.

The results of this paper were reported in expository form in [**8**].

In this paper, a pro-object of a category $\mathcal{C}$ will be taken to mean a functor from the set of positive integers, viewed as a category with one arrow from $n+1$ to n, to $\mathcal{C}$, and a *strict* map between pro-objects a natural transformation. A general map between pro-objects X and Y of $\mathcal{C}$ is an element of

$$\operatorname{Hom}_{\text{pro}-\mathcal{C}}(X, Y) = \lim_n \operatorname*{colim}_m \operatorname{Hom}_{\mathcal{C}}(X_m, Y_n);$$

compare [**1**, appendix]. Occasionally, we view objects of $\mathcal{C}$ as constant pro-objects of $\mathcal{C}$. We abbreviate $e'' = e/(p-1)$ and $e' = pe/(p-1)$.

While this paper was written the authors visited the University of Tokyo. The second author also visited the Isaac Newton Institute of Mathematical Sciences and Stanford University. We would like to express our gratitude for the financial support and the hospitality that we received.

1. The de Rham-Witt complex

1.1. We consider the de Rham-Witt complex of log-$\mathbb{Z}_{(p)}$-algebras introduced in [**10**, §3]; see also [**9**]. It generalizes the de Rham-Witt complex of log-$\mathbb{F}_p$-algebras of Hyodo-Kato, [**11**]. Throughout, we assume that the prime p is odd.

A log-ring (A, M_A) in the sense of [**15**] is a ring A (in a topos) with a pre-log-structure, defined as a map of monoids $\alpha\colon M_A \to (A, \cdot)$ from a monoid M_A to the underlying multiplicative monoid of A. A log-differential graded ring (D, M_D) is a differential graded ring D, a pre-log-structure $\alpha\colon M_D \to (D^0, \cdot)$, and a map of monoids $d\log\colon M_D \to (D^1, +)$ which satisfies that $d \circ d\log = 0$ and that for all $x \in M_D$, $d\alpha(x) = \alpha(x) d\log x$. Maps of log-rings and log-differential graded rings are defined in the obvious way.

Let $W_n(A)$ be the ring of Witt vectors of length n in A. If $\alpha\colon M_A \to A$ is a pre-log-structure, then so is the composite

$$M_A \xrightarrow{\alpha} A \to W_n(A),$$

where the right hand map is the multiplicative section $a \mapsto [a]_n = (a, 0, \dots, 0)$. We denote this log-ring by $(W_n(A), M_A)$. By a Witt complex over (A, M_A) we mean the following structure:

(i) a pro-log-differential graded ring $(E^*_\cdot, M_E)$ and a strict map of pro-log-rings

$$\lambda\colon (W_\cdot(A), M_A) \to (E^0_\cdot, M_E);$$

(ii) a strict map of pro-log-graded rings

$$F\colon (E^*_\cdot, M_E) \to (E^*_{\cdot-1}, M_E)$$

such that $F\lambda = \lambda F$ and such that

$$\begin{aligned} F d\log_n \lambda(a) &= d\log_{n-1} \lambda(a), && \text{for all } a \in M_A, \\ F d\lambda([a]_n) &= \lambda([a]_{n-1})^{p-1} d\lambda([a]_{n-1}), && \text{for all } a \in A; \end{aligned}$$

(iii) a strict map of pro-graded modules over the pro-graded ring $E^*_\cdot$,

$$V\colon F_* E^*_{\cdot-1} \to E^*_\cdot,$$

such that $\lambda V = V\lambda$, $FV = p$ and $FdV = d$.

A map of Witt complexes over (A, M) is a strict map of pro-log differential graded rings which commutes with the maps λ, F and V. We write R for the structure map in the pro-system $E^*_\cdot$ and call it the restriction map. The definining relations imply that $dF = pFd$ and $Vd = pdV$, but in general there is no formula for VF; see [**9**, lemma 1.2.1]. The de Rham-Witt complex

$$W_\cdot\, \Omega^*_{(A, M_A)},$$

by definition, is the universal Witt complex over (A, M_A). The proof that it exists, which is given in [**9**, theorem A], also shows that the canonical map

$$\lambda\colon \Omega^q_{(W_n(A), M_A)} \twoheadrightarrow W_n\, \Omega^q_{(A, M_A)}$$

is surjective. Hence, every element on the right can be written non-uniquely as a differential q-form on $(W_n(A), M_A)$. The descending filtration of the de Rham-Witt complex by the differential graded ideals

$$\mathrm{Fil}^s\, W_n\, \Omega^*_{(A, M_A)} = V^s W_{n-s}\, \Omega^*_{(A, M_A)} + dV^s W_{n-s}\, \Omega^*_{(A, M_A)}$$

is called the standard filtration. It satisfies

$$F(\operatorname{Fil}^s W_n\,\Omega^q_{(A,M_A)}) \subset \operatorname{Fil}^{s-1} W_{n-1}\,\Omega^q_{(A,M_A)},$$
$$V(\operatorname{Fil}^s W_n\,\Omega^q_{(A,M_A)}) \subset \operatorname{Fil}^{s+1} W_{n+1}\,\Omega^q_{(A,M_A)},$$

but, in general, is not multiplicative. The restriction induces an isomorphism

$$W_n\,\Omega^q_{(A,M_A)}/\operatorname{Fil}^s W_n\,\Omega^q_{(A,M_A)} \xrightarrow{\sim} W_s\,\Omega^q_{(A,M_A)}.$$

1.2. Let X be as in the introduction. The canonical log-structure on X, we recall, is given by the cartesian square of sheaves of monoids

$$\begin{array}{ccc} M_X & \xrightarrow{\alpha} & \mathcal{O}_X \\ \downarrow & & \downarrow \\ j_*\mathcal{O}_U^* & \longrightarrow & j_*\mathcal{O}_U, \end{array}$$

so with a choice of uniformizer π of V_0 we have an isomorphism

$$\mathcal{O}_X^* \times \mathbb{N}_0 \xrightarrow{\sim} M_X,$$

which takes (u,i) to $\pi^i u$. In this case, the de Rham-Witt complex

$$W.\,\Omega^*_{(X,M_X)} = W.\,\Omega^*_{(\mathcal{O}_X,M_X)}$$

has an additional filtration by the differential graded ideals

$$U^m W_n\,\Omega^*_{(X,M_X)}$$

generated by $W_n(\mathfrak{m}_0^j\mathcal{O}_X)$, if $m = 2j$ is even, and by $W_n(\mathfrak{m}_0^j\mathcal{O}_X)\cdot d\log_n M_X$ and $W_n(\mathfrak{m}_0^{j+1}\mathcal{O}_X)$, if $m = 2j+1$ is odd. We call this the U-filtration.

LEMMA 1.2.1. *The U-filtration is multiplicative and preserved by the restriction, Frobenius, and Verschiebung. Moreover, if $X_j = X \times_{\operatorname{Spec} V_0} \operatorname{Spec}(V_0/\mathfrak{m}_0^j)$, if $i_j\colon X_j \to X$ is the closed immersion, and if $\alpha\colon M_{X_j} \to \mathcal{O}_{X_j}$ is the induced pre-log-structure, then the canonical projection induces an isomorphism*

$$i_j^*\big(W_n\,\Omega^q_{(X,M_X)}/U^{2j}W_n\,\Omega^q_{(X,M_X)}\big) \xrightarrow{\sim} W_n\,\Omega^q_{(X_j,M_{X_j})}.$$

PROOF. A functor, which has a right adjoint, preserves initial objects. Hence

$$i_j^*\big(W_n\,\Omega^q_{(X,M_X)}/U^{2j}W_n\,\Omega^q_{(X,M_X)}\big) = W_n\,\Omega^q_{(i_j^*\mathcal{O}_X,i_j^*M_X)}/U^{2j}W_n\,\Omega^q_{(i_j^*\mathcal{O}_X,i_j^*M_X)}.$$

Let (B,M) be a log-ring, let $J \subset B$ be an ideal, and let $(\bar{B},\bar{M})$ be the ring $\bar{B} = B/J$ with the induced pre-log-structure given by the composition

$$\bar{\alpha}\colon \bar{M} = M \xrightarrow{\alpha} B \xrightarrow{\mathrm{pr}} B/J.$$

One can show from the definitions that the map induced from the projection

$$W_n\,\Omega^*_{(B,M)} \to W_n\,\Omega^*_{(\bar{B},\bar{M})}$$

is surjective and that the kernel is equal to the differential graded ideal generated by the ideal $W_n(J) \subset W_n(B)$; see [**6**, lemma 2.2.1] for the proof. The lemma is a special case of this statement. $\square$

LEMMA 1.2.2. *Let e be the ramification index of V_0 and let $e'' = e/(p-1)$. Then*

$$\begin{aligned} p\,U^{2j} W_n\,\Omega^q_{(X,M_X)} &\subset U^{2\min\{j+e,pj\}} W_n\,\Omega^q_{(X,M_X)}, && \text{for } j \geq 0, \\ p\,U^{2j} W_n\,\Omega^q_{(X,M_X)} &= U^{2(j+e)} W_n\,\Omega^q_{(X,M_X)}, && \text{for } j \geq e''. \end{aligned}$$

PROOF. By the definition of the U-filtration, it suffices to show that

$$\begin{aligned} pW_n(\mathfrak{m}_0^j \mathcal{O}_X) &\subset W_n(\mathfrak{m}_0^{\min\{j+e,pj\}} \mathcal{O}_X), && \text{for } j \geq 0, \\ W_n(\mathfrak{m}_0^{j+e} \mathcal{O}_X) &\subset pW_n(\mathfrak{m}_0^j \mathcal{O}_X), && \text{for } j \geq e''. \end{aligned}$$

Let π be a uniformizer of V_0 with minimal polynomial $x^e + p\theta(x)$ and recall from the proof of [**10**, proposition 3.1.5] that $[\pi]^e + \theta([\pi])V(1)$ is contained in $pW_n(\mathcal{O}_X)$. The second inclusion follows by iterated use of this conguence. Finally, we recall from the proof of [**10**, lemma 3.1.1], p is congruent to $[p] + V(1)$ modulo $pVW_n(\mathcal{O}_X)$. The first inclusion follows by induction, since p has valuation e. □

The map $d\log_n \colon M_X \to W_n\,\Omega^1_{(X,M_X)}$ gives rise to a map of graded rings

$$\Lambda_{\mathbb{Z}}(M_X^{\mathrm{gp}}) \to W_n\,\Omega^*_{(X,M_X)},$$

and there is a descending filtration of the left hand side by graded ideals

$$U^m \Lambda_{\mathbb{Z}}(M_X^{\mathrm{gp}}),$$

which corresponds to the U-filtration on the right hand side. To define it, we first choose a uniformizer π of V_0 such that we have the isomorphism

$$\mathcal{O}_X^* \times \mathbb{Z} \xrightarrow{\sim} M_X^{\mathrm{gp}},$$

which takes (u, i) to $\pi^i u$. We define $U^m \Lambda_{\mathbb{Z}}(M_X^{\mathrm{gp}})$ to be $\Lambda_{\mathbb{Z}}(M_X^{\mathrm{gp}})$, if $m = 0$, and to be the graded ideal generated by $(1 + \mathfrak{m}_0^j \mathcal{O}_X) \times \{0\} \subset M_X^{\mathrm{gp}}$, if $m = 2j$ and $j > 0$, by $(1 + \mathfrak{m}_0 \mathcal{O}_X) \times \mathbb{Z} \subset M_X^{\mathrm{gp}}$, if $m = 1$, and by $(1 + \mathfrak{m}_0^j \mathcal{O}_X) \times \{0\} \wedge \{1\} \times \mathbb{Z} \subset M_X^{\mathrm{gp}} \wedge M_X^{\mathrm{gp}}$ and $(1 + \mathfrak{m}_0^{j+1} \mathcal{O}_X) \times \{0\} \subset M_X^{\mathrm{gp}}$, if $m = 2j+1$ and $j > 1$.

LEMMA 1.2.3. *If x is a local section of $\mathfrak{m}_0^j \mathcal{O}_X$, then, modulo $U^{4j} W_n\,\Omega^1_{(X,M_X)}$,*

$$d\log_n(1+x) \equiv \sum_{0 \leq s < n} dV^s([x]_{n-s}).$$

PROOF. We first show that if R is a ring and $x \in R$, then

$$[1+x]_n - [1]_n \equiv \sum_{0 \leq s < n} V^s([x]_{n-s})$$

modulo the ideal $W_n((x^2)) \subset W_n(R)$. By naturality, we may assume that $R = \mathbb{Z}[x]$. If we write $[1+x]_n - [1]_n = (a_0, a_1, \dots, a_{n-1})$, then the statement we wish to show is that $a_s \equiv x$ modulo (x^2), for all $0 \leq s < n$. The statement for $s = 0$ is clear. We consider the ghost coordinate

$$(1+x)^{p^s} - 1 = a_0^{p^s} + p a_1^{p^{s-1}} + \dots + p^{s-1} a_{s-1}^p + p^s a_s.$$

The left hand side is equivalent to $p^s x$ modulo (x^2), and the right hand side, inductively, is equivalent to $p^s a_s$ modulo (x^2). It follows that a_s is equivalent to x modulo (x^2) as desired. If x is a local section of $\mathfrak{m}_0^j \mathcal{O}_X$, we may conclude that

$$[1+x]_n - [1]_n \equiv \sum_{0 \le s < n} V^s([x]_{n-s})$$

modulo $\bar{W}_n(\mathfrak{m}_0^{2j}\mathcal{O}_X)$. Differentiating this congruence we find that

$$d([1+x]_n) \equiv \sum_{0 \le s < n} dV^s([x]_{n-s})$$

modulo $U^{4j} W_n\,\Omega^1_{(X,M_X)}$. It remains to show that the left hand side is congruent to $d\log_n(1+x)$ modulo $U^{4j} W_n\,\Omega^1_{(X,M_X)}$. By definition, we have

$$[1+x]_n d\log_n(1+x) = d([1+x]_n),$$

and $[1+x]_n$ is a unit in $W_n(\mathcal{O}_X/\mathfrak{m}_0^{2j}\mathcal{O}_X)$. Therefore, it will suffice to show that the product $([1+x]_n - [1]_n) d([1+x]_n)$ is congruent to zero modulo $U^{4j} W_n\,\Omega^1_{(X,M_X)}$. But the two factors lie in $U^{2j} W_n\,\Omega^*_{(X,M_X)}$, and the U-filtration is multiplicative. □

The lemma determines $d\log_n$, modulo higher filtration, on $U^m \Lambda_{\mathbb{Z}}(M_X^{\mathrm{gp}})$, for $m \ge 2$. Moreover, there is a commutative square

$$\begin{array}{ccc} i^*(M_X^{\mathrm{gp}}/U^2(M_X^{\mathrm{gp}})) & \xrightarrow{d\log_n} & i^*(W_n\,\Omega^1_{(X,M_X)}/U^2 W_n\,\Omega^1_{(X,M_X)}) \\ \downarrow & & \downarrow \sim \\ M_Y^{\mathrm{gp}} & \xrightarrow{d\log_n} & W_n\,\Omega^1_{(Y,M_Y)}, \end{array}$$

and the right hand vertical map is an isomorphism by an argument similar to the proof of lemma 1.2.1.

1.3. We recall from [**9**, lemma 7.1.2] that the sheaf

$$\bar{W}_n\,\Omega^q_{(X,M_X)} = W_n\,\Omega^q_{(X,M_X)}/pW_n\,\Omega^q_{(X,M_X)}$$

is a quasi-coherent sheaf of $\bar{W}_n(\mathcal{O}_X)$-modules on the small étale site of X. The sheaf is supported on Y, so we may as well consider the quasi-coherent $i^*\bar{W}_n(\mathcal{O}_X)$-module $i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$. We first show that, Zariski locally on Y, we may view this as a quasi-coherent sheaf of $\mathcal{O}_Y$-modules. Let φ be the absolute Frobenius on Y.

LEMMA 1.3.1. *Let $x_1, \dots, x_r$ be local coordinates of an open neighborhood in X of a point of Y, and let $\bar{x}_1, \dots, \bar{x}_r$ be the corresponding local coordinates on Y. Then there is, in the corresponding open neighborhood of Y, a map of pro-rings*

$$\delta\colon \mathcal{O}_Y \to i^*\bar{W}_\cdot(\mathcal{O}_X)$$

such that $\delta(\bar{x}_i) = [x_i]$, $1 \le i \le r$, and such that $F\delta = \delta\varphi$.

PROOF. The ring homomorphism

$$f\colon W(k_0)[x_1, \dots, x_r] \to W(k_0)[x_1, \dots, x_r],$$

given by the Frobenius on $W(k_0)$ and with $f(x_i) = x_i^p$, $1 \le i \le r$, is a lifting of the Frobenius on $k_0[\bar{x}_1, \dots, \bar{x}_r]$. It determines a ring homomorphism

$$s_f\colon W(k_0)[x_1, \dots, x_r] \to W_n(W(k_0)[x_1, \dots, x_r]),$$

characterized by $w_j s_f = f^j$, $0 \leq j < n$, and, after reduction modulo p, a ring homomorphism

$$\bar{s}_f \colon k_0[x_1, \dots, x_r] \to \bar{W}_n(W(k_0)[x_1, \dots, x_r]).$$

We compose this map with the ring homomorphisms

$$\bar{W}_n(W(k_0)[x_1, \dots, x_r]) \to \bar{W}_n(V[x_1, \dots, x_r]) \to \bar{W}_n(\mathcal{O}_X)$$

induced from the canonical ring homomorphism $W(k_0) \to V$ and the chosen ring homomorphism $g \colon V[x_1, \dots, x_r] \to \mathcal{O}_X$ to get the top horizontal map in the following diagram.

$$\begin{array}{ccc} k_0[\bar{x}_1, \dots, \bar{x}_r] & \xrightarrow{t_g} & \bar{W}_n(\mathcal{O}_X) \\ \downarrow{\scriptstyle \bar{g}} & \nearrow & \downarrow{\scriptstyle p_n} \\ \mathcal{O}_Y & = & \mathcal{O}_Y. \end{array}$$

The right hand vertical map is the composition of the restriction of Witt vectors with kernel $V\bar{W}_n(\mathcal{O}_X)$ and the canonical projection $\bar{\mathcal{O}}_X \to \mathcal{O}_Y$ with kernel $\mathfrak{m}_0\mathcal{O}_X/p\mathcal{O}_X$. Since both ideals are nilpotent, and since the left hand vertical map is étale, there exists a unique ring homomorphism

$$\delta_n \colon \mathcal{O}_Y \to \bar{W}_n(\mathcal{O}_X)$$

making the above diagram commute. Moreover, one immediately verifies that $R\delta_n = \delta_{n-1}$ and that $\delta_n(\bar{x}_i) = [x_i]_n$, $1 \leq i \leq r$, as stated. It remains to show that $F\delta_n = \delta_{n-1}\varphi$, or equivalently, that the right hand square in the following diagram commutes.

$$\begin{array}{ccccc} k_0[\bar{x}_1, \dots, \bar{x}_r] & \xrightarrow{\bar{g}} & \mathcal{O}_Y & \xrightarrow{\delta_n} & \bar{W}_n(\mathcal{O}_X) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle F} \\ k_0[\bar{x}_1, \dots, \bar{x}_r] & \xrightarrow{\bar{g}} & \mathcal{O}_Y & \xrightarrow{\delta_{n-1}} & \bar{W}_{n-1}(\mathcal{O}_X). \end{array}$$

The outer square commutes, and the left hand square is cocartesian. It follows that there exists a map $\delta'_{n-1} \colon \mathcal{O}_Y \to \bar{W}_{n-1}(\mathcal{O}_X)$ making the right hand square commute. In order to show that $\delta'_{n-1} = \delta_{n-1}$, it will suffice to show that

$$\mathcal{O}_Y \xrightarrow{\delta'_{n-1}} \bar{W}_{n-1}(\mathcal{O}_X) \xrightarrow{p_{n-1}} \mathcal{O}_Y$$

is the identity map. This, in turn, follows from the calculation

$$p_{n-1}F\delta_n\bar{g} = \varphi p_n \delta_n \bar{g} = \varphi\bar{g},$$

since the left hand square of the diagram above is cocartesian. □

PROPOSITION 1.3.2. *The sheaf $i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ has, Zariski locally on Y, the structure of a free $\mathcal{O}_Y$-module of rank*

$$\mathrm{rk}_{\mathcal{O}_Y}\, i^*\bar{W}_n\,\Omega^q_{(X,M_X)} = \binom{r+1}{q} e \cdot \sum_{s=0}^{n-1} p^{rs}.$$

Proof. We give $i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ the $\mathcal{O}_Y$-module structure defined Zariski locally by lemma 1.3.1. The statement of the proposition is unchanged by étale extensions, so we may assume that $X = \mathbb{A}^r_{V_0}$. We proceed by induction on r; the basic case $r = 0$ follows from [**10**, proposition 3.4.1]. In the induction step we use [**9**, theorem B]. It shows that as an $\mathcal{O}_{\mathbb{A}^r_{k_0}}$-module, $i^*\bar{W}_n\,\Omega^q_{(\mathbb{A}^r_{V_0},M_{\mathbb{A}^r_{V_0}})}$ is the base-change along $\mathbb{A}^{r-1}_{k_0} \hookrightarrow \mathbb{A}^r_{k_0}$ of the $\mathcal{O}_{\mathbb{A}^{r-1}_{k_0}}$-module

$$i^*\bar{W}_n\,\Omega^q_{(\mathbb{A}^{r-1}_{V_0},M_{\mathbb{A}^{r-1}_{V_0}})} \oplus \bigoplus_{s=1}^{n-1} \bigoplus_{0\le j<p^s} F^s_*(i^*\bar{W}_{n-s}\,\Omega^q_{(\mathbb{A}^{r-1}_{V_0},M_{\mathbb{A}^{r-1}_{V_0}})})$$
$$\oplus\, i^*\bar{W}_n\,\Omega^{q-1}_{(\mathbb{A}^{r-1}_{V_0},M_{\mathbb{A}^{r-1}_{V_0}})} \oplus \bigoplus_{s=1}^{n-1} \bigoplus_{0\le j<p^s} F^s_*(i^*\bar{W}_{n-s}\,\Omega^{q-1}_{(\mathbb{A}^{r-1}_{V_0},M_{\mathbb{A}^{r-1}_{V_0}})}),$$

where the index j is an integer prime to p. By induction and by lemma 1.3.1, this module has rank

$$\binom{r}{q} e \cdot \Big(\sum_{t=0}^{n-1} p^{(r-1)t} + \sum_{s=1}^{n-1}(p^s - p^{s-1})p^{(r-1)s}\sum_{t=0}^{n-1-s} p^{(r-1)t}\Big)$$
$$+ \binom{r}{q-1} e \cdot \Big(\sum_{t=0}^{n-1} p^{(r-1)t} + \sum_{s=1}^{n-1}(p^s - p^{s-1})p^{(r-1)s}\sum_{t=0}^{n-1-s} p^{(r-1)t}\Big),$$

and since

$$\binom{r}{q-1} e + \binom{r}{q} e = \binom{r+1}{q} e,$$

it remains to show that

$$\sum_{t=0}^{n-1} p^{(r-1)t} + \sum_{s=1}^{n-1}(p^s - p^{s-1})p^{(r-1)s}\sum_{t=0}^{n-1-s} p^{(r-1)t} = \sum_{s=0}^{n-1} p^{rs}.$$

To see this, we rewrite the first summand on the left as

$$1 + p^{r-1}\sum_{t=0}^{n-2} p^{(r-1)t},$$

and rewrite the sth summand on the left as

$$p^{rs} + p^{r(s+1)-1}\sum_{t=0}^{n-(s+1)-1} p^{(r-1)t} - p^{rs-1}\sum_{t=0}^{n-s-1} p^{(r-1)t}.$$

The statement follows. □

In the statements and proofs of theorem 1.3.3 and theorem 1.3.4 below, we will write $[x]$ for $[x]_n$ and $d\log x$ for $d\log_n x$. Let π be a uniformizer of V_0.

Theorem 1.3.3. *Let $0 \le j < e'$, let v be the unique integer such that*

$$e\Big(\frac{p^{-v}-1}{p^{-1}-1}\Big) \le j < e\Big(\frac{p^{-(v+1)}-1}{p^{-1}-1}\Big),$$

and let $x_1,\dots,x_r$ be local coordinates of a neighborhood in X of a point of Y. In the corresponding open neighborhood of Y, the sheaf $E^q_n = i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ has the following structure.

(i) *If $n \le v$, then $\mathrm{gr}^{2j}_U E^q_n = \mathrm{gr}^{2j+1}_U E^q_n = 0$.*

(ii) *If $0 \le v < n$, then $\mathrm{gr}_U^{2j}\, i^* E_n^q$ is a free $\mathcal{O}_Y$-module with a basis as follows: if p does not divide j, let $0 < s < n - v$, and let $0 \le i_1, \dots, i_r < p^s$; let m be maximal with i_m prime to p, if such an m exists; then the local sections*

$$V^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_q}),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}),$$

where $1 \le m_1 < \dots < m_q \le r$ (resp. $1 \le m_1 < \dots < m_{q-1} \le r$), and where all $m_i \ne m$, are basis elements; in addition, the local sections

$$[\pi]^j d\log x_{m_1} \dots d\log x_{m_q},$$

where $1 \le m_1 < \dots < m_q \le r$, are basis elements; if p divides j, let $0 < s < n - v$, and let $0 \le i_1, \dots, i_r < p^s$ not all divisible by p; let m be maximal with i_m prime to p; then the local sections

$$V^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_q}),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}),$$

where $1 \le m_1 < \dots < m_q \le r$ (resp. $1 \le m_1 < \dots < m_{q-1} \le r$), and where all $m_i \ne m$, are basis elements; in addition, the local sections

$$[\pi]^j d\log x_{m_1} \dots d\log x_{m_q},$$

where $1 \le m_1 < \dots < m_q \le r$, are basis elements.

(iii) *If $0 \le v < n$, then $\mathrm{gr}_U^{2j+1} E_n^q$ is a free $\mathcal{O}_Y$-module with a basis as follows: let $0 < s < n - v$, and let $0 \le i_1, \dots, i_r < p^s$ not all divisible by p; let m be maximal with i_m not divisible by p; then the local sections*

$$V^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}} d\log \pi),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-2}} d\log \pi),$$

where $1 \le m_1 < \dots < m_{q-1} \le r$ (resp. $1 \le m_1 < \dots < m_{q-2} \le r$), and where all $m_i \ne m$, are basis elements; in addition, the local sections

$$[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}} d\log \pi,$$

where $1 \le m_1 < \dots < m_{q-1} \le r$, are basis elements.

(iv) *The sheaf $U^{2e'} E_n^q$ is equal to zero.*

PROOF. We recall from proposition 1.3.2 that, Zariski locally, E_n^q is a free $\mathcal{O}_Y$-module. It is generated by local sections

$$[\pi]^j d\log x_{m_1} \dots d\log x_{m_q},$$
$$[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}} d\log \pi,$$

where $0 \le j < e'$ and $1 \le m_1, \dots, m_q \le r$, and by the local sections

$$V^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_q}),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}),$$
$$V^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}} d\log \pi),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-2}} d\log \pi),$$

where $0 < s < n$, $0 \le j < e'$, $0 \le i_1, \dots, i_r < p^s$, and $1 \le m_1, \dots, m_q \le r$, and where not all of $i_1, \dots, i_r$ and j are divisible by p. Suppose, for instance, that

$0 < s < n$, $0 \leq j < e'$, and $0 \leq i_1, \ldots, i_r < p^s$ and that all of $i_1, \ldots, i_r$ and j were divisible by p. Then

$$V^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_q})$$
$$= V^{s-1}([x_1]^{p^{-1}i_1} \ldots [x_r]^{p^{-1}i_r}[\pi]^{p^{-1}j} d\log x_{m_1} \ldots d\log x_{m_q} V(1)),$$

which, since $V(1) = -\theta([\pi])^{-1}[\pi]^e$, can be rewritten in terms of the generators listed. (Note that $p^{-1}j + e \geq j$ if and only if $e' \geq j$.) There are some obvious relations between the second set of the generators. Indeed, if $s > 0$, we have

$$V^s d([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_2} \ldots d\log x_{m_q}) = 0,$$

which gives the relation

$$\sum_{m_1} i_{m_1} V^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_q})$$
$$+ jV^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log \pi d\log x_{m_2} \ldots d\log x_{m_{q-1}}) = 0,$$

and analogously, we find that

$$\sum_{m_1} i_{m_1} dV^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_{q-1}})$$
$$+ jdV^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log \pi d\log x_{m_2} \ldots d\log x_{m_{q-1}}) = 0,$$

$$\sum_{m_1} i_{m_1} V^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_{q-1}} d\log \pi) = 0,$$

$$\sum_{m_1} i_{m_1} dV^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_{q-2}} d\log \pi) = 0.$$

We proceed to show that the local sections listed in the statement of the theorem will suffice as generators.

Let $0 \leq j < e'$ and suppose that p does not divide j. We consider the generators

$$[\pi]^j d\log x_{m_1} \ldots d\log x_{m_q}, \qquad 1 \leq m_1 < \cdots < m_q \leq r$$

and

$$V^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_q}), \qquad 1 \leq m_1 < \cdots < m_q \leq r,$$
$$dV^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_{q-1}}), \quad 1 \leq m_1 < \cdots < m_{q-1} \leq r,$$

where $0 < s < n$ and $0 \leq i_1, \ldots, i_r < p^s$. Suppose that not all of $i_1, \ldots, i_r$ are divisible by p, and let $1 \leq m \leq r$ be the largest integer such that i_m is prime to p. Then the above relations show that the generators

$$V^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_q}), \qquad 1 \leq m_1 < \cdots < m_q \leq r,$$
$$dV^s([x_1]^{i_1} \ldots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \ldots d\log x_{m_{q-1}}), \quad 1 \leq m_1 < \cdots < m_{q-1} \leq r,$$

where some $m_i = m$, can be rewritten in terms of generators, where all $m_i \neq m$. Therefore, the former generators may be omitted. Let v be the integer defined in the statement of the theorem.

The minimal polynomial is of π is of the form $x^e + p\theta(x)$, where $\theta(x)$ is a polynomial over $W(k_0)$ of degree $< e$ such that $\theta(0)$ is a unit. We recall from the proof of [**10**, proposition 3.1.5] that in $\bar{W}_n(\mathcal{O}_X)$,

$$[\pi]_n^e + \theta([\pi]_n)V(1) = 0.$$

By iterated use of this relation, we find that

$$[\pi]^j = \pm V^v([\pi]^{p^v(j-e(\frac{p^{-v}-1}{p^{-1}-1}))}\theta([\pi])^{\frac{p^{v+1}-1}{p-1}-1}),$$

and hence, the generators with $s \geq n - v$ are zero in E_n^q. By this kind of reasoning, we see that the local sections listed in the statement of the theorem generate E_n^q as an $\mathcal{O}_Y$-module. To see that they form a basis, we must show that the number of generators is equal to the rank of E_n^q, which is known from proposition 1.3.2. We postpone the argument to the proof of theorem 1.3.4 below.

It remains to show that the filtration of the basis elements is as stated. Let $A^{m,q} \subset E_n^q$ be the sub-$\mathcal{O}_Y$-module generated by those basis elements that are listed in the statement of the theorem as having filtration greater than or equal to m. Then $A^{m,q} \subset U^{m,q} = i^* U^m E_n^q$, and we must show that also $U^{m,q} \subset A^{m,q}$. We recall that if $m = 2j$ (resp. if $m = 2j+1$), then $U^{m,*}$ is the differential graded ideal generated by $i^*\bar{W}_n(\mathfrak{m}_0^j\mathcal{O}_X) \subset i^*\bar{W}_n(\mathcal{O}_X)$ (resp. by $i^*\bar{W}_n(\mathfrak{m}_0^j\mathcal{O}_X) \cdot d\log M_X \subset i^*\bar{W}_n\,\Omega^1_{(X,M_X)}$ and $i^*\bar{W}_n(\mathfrak{m}_0^{j+1}\mathcal{O}_X) \subset i^*\bar{W}_n(\mathcal{O}_X)$). So it suffices to show that $i^*\bar{W}_n(\mathfrak{m}_0^j\mathcal{O}_X) \subset A^{2j,0}$, that the product maps $A^{m,q} \otimes A^{m',q'}$ to $A^{m+m',q+q'}$, and that the differential takes $A^{m,q}$ to $A^{m,q+1}$. The first is a statement about ideals of the ring of Witt vectors and is straightforward to verify. The second is verified by explicitly calculating the products of basis elements of $A^{m,q}$ and $A^{m',q'}$; compare [**9**, §4]. The last statement is immediate. □

The basis of $i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ given by theorem 1.3.3 has the property that the basis elements are not equivalent to elements of higher U-filtration. We proceed to give a basis with the property that the basis elements are not equivalent to elements of higher filtration with respect to the standard filtration.

THEOREM 1.3.4. *Let $x_1, \dots, x_r$ be local coordinates of an open neighborhood of X around a point of Y. Then in the corresponding open neighborhood of Y, the sheaf $E_n^q = i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ has the following structure. For all $0 \leq s < n$, $\mathrm{gr}^s E_n^q$ is a free $\mathcal{O}_Y$-module with a basis as follows. Let $0 \leq i_1, \dots, i_r < p^s$.*

(i) If $(i_1, \dots, i_r) \neq (0, \dots, 0)$, let $v = \min\{v_p(i_1), \dots, v_p(i_r), v_p(j - e')\}$; if $v < v_p(i_m)$, for all $1 \leq m \leq r$, then the local sections

$$V^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_q}),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}),$$

where $1 \leq m_1 < \dots < m_q \leq r$ (resp. $1 \leq m_1 < \dots < m_{q-1} \leq r$), are basis elements; if $v = v_p(i_m)$, for some $1 \leq m \leq r$, let m be maximal with this property; then the local sections

$$V^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_q}),$$
$$V^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}} d\log \pi),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}),$$
$$dV^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-2}} d\log \pi),$$

where $1 \leq m_1 < \dots < m_q \leq r$ (resp. $1 \leq m_1 < \dots < m_{q-1} \leq r$, resp. $1 \leq m_1 < \dots < m_{q-2} \leq r$), and where all $m_i \neq m$ are basis elements.

(ii) *If* $(i_1,\dots,i_r)=(0,\dots,0)$, *the local sections*

$$V^s([\pi]^j d\log x_{m_1}\dots d\log x_{m_q}),$$

where $1\le m_1<\dots<m_q\le r$, *and if* $s>v_p(j-e')$, *the local sections*

$$dV^s([\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-1}}),$$

where $1\le m_1<\dots<m_{q-1}\le r$, *and if* $s\le v_p(j-e')$, *the local sections*

$$V^s([\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-1}} d\log\pi),$$

where $1\le m_1<\dots<m_{q-1}\le r$, *are basis elements.*

PROOF. For every element of the basis given by theorem 1.3.3, we find an equivalent element of $\mathrm{Fil}^s\, i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ with $0\le s<n$ as large as possible. We first explain the way in which the indices change. Let $0\le j<e'$ be given, and let v be the unique integer such that

$$e(\frac{p^{-v}-1}{p^{-1}-1})\le j<e(\frac{p^{-(v+1)}-1}{p^{-1}-1});$$

let further $0\le s<n-v$ and $0\le i_1,\dots,i_r<p^s$ be given. We define $s'=s+v$ and $i'_k=p^v i_k$, $1\le k\le r$, and let

$$j'=p^v(j-e(\frac{p^{-v}-1}{p^{-1}-1}))=p^v j-pe(\frac{p^v-1}{p-1}).$$

Then $v<s'<n$ and $0\le i'_1,\dots,i'_r<p^{s'}$, and j' is an integer with $0\le j'<e$. In addition, $v_p(j'-e')=v_p(j)+v$ and $v_p(i'_k)=v_p(i_k)+v$.

Conversely, if $0\le j'<e$, $0\le s'<n$, and $0\le i'_1,\dots,i'_r<p^s$ are given, let v be the minimum of $v_p(i'_1),\dots,v_p(i'_r)$, $v_p(j'-e')$, and s'. We define $s=s'-v$ and $i_k=p^{-v}i'_k$, $1\le k\le r$, and let

$$j=p^{-v}j'+e(\frac{p^{-v}-1}{p^{-1}-1})=p^{-v}(j'+pe\frac{p^v-1}{p-1}).$$

Then $0\le s<n-v$ and $0\le i_1,\dots,i_r<p^s$. Moreover, since $j\in\mathbb{Z}[\frac{1}{p}]$ and $v_p(j)\ge 0$, j is an integer and satisfies

$$e(\frac{p^{-v}-1}{p^{-1}-1})\le j<e(\frac{p^{-(v+1)}-1}{p^{-1}-1}).$$

By iterated use of the relation $[\pi]^e=-\theta([\pi])V(1)$, we find that

$$V^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j)=\pm V^{s'}([x_1]^{i'_1}\dots[x_r]^{i'_r}[\pi]^{j'}\bar{\theta}([\pi])^{\frac{p^{v+1}-1}{p-1}}).$$

The left hand side, modulo elements of higher U-filtration, is equal to a local section of $\mathcal{O}_Y^*$ times the local section $V^{s'}([x_1]^{i'_1}\dots[x_r]^{i'_r}[\pi]^{j'})$. It follows that if we replace the local sections

$$\begin{aligned}&V^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m_1}\dots d\log x_{m_q}),\\&V^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-1}} d\log\pi),\\&dV^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-1}}),\\&dV^s([x_1]^{i_1}\dots[x_r]^{i_r}[\pi]^j d\log x_{m_1}\dots d\log x_{m_{q-2}} d\log\pi),\end{aligned}$$

which constitute the basis of theorem 1.3.3, by the local sections

$$V^{s'}([x_1]^{i'_1}\dots[x_r]^{i'_r}[\pi]^{j'}d\log x_{m_1}\dots d\log x_{m_q}),$$
$$V^{s'}([x_1]^{i'_1}\dots[x_r]^{i'_r}[\pi]^{j'}d\log x_{m_1}\dots d\log x_{m_{q-1}}d\log\pi),$$
$$dV^{s'}([x_1]^{i'_1}\dots[x_r]^{i'_r}[\pi]^{j'}d\log x_{m_1}\dots d\log x_{m_{q-1}}),$$
$$dV^{s'}([x_1]^{i'_1}\dots[x_r]^{i'_r}[\pi]^{j'}d\log x_{m_1}\dots d\log x_{m_{q-2}}d\log\pi),$$

then we still have a basis. We leave it to the reader to check that the latter elements constitute the basis of the statement. It is clear that, for every $0 \leq j < e$ and $0 \leq i_1, \dots, i_r < p^s$, there are $\binom{r}{q}+\binom{r}{q-1}$ elements in this basis. These elements form a set of generators, and proposition 1.3.2 shows that they are linearly independent. □

ADDENDUM 1.3.5. *There is a natural exact sequence*

$$0 \to i^*\bar{W}_n\,\Omega^q_X \xrightarrow{j_*} i^*\bar{W}_n\,\Omega^q_{(X,M_X)} \xrightarrow{\partial} \bar{W}_n\,\Omega^{q-1}_Y \to 0.$$

PROOF. The map j_* is induced by the canonical map from X with the trivial log-structure to X with the canonical log-structure. To construct the map ∂ we show that the map

$$W_n\,\Omega^q_Y \oplus W_n\,\Omega^{q-1}_Y \xrightarrow{\sim} W_n\,\Omega^q_{(Y,M_Y)},$$

which to (ω,ω') assigns $\omega + \omega' d\log_n \pi$, is an isomorphism. The statement is étale local, so it suffices to consider the case $Y = \mathbb{A}^r_{k_0}$. We proceed by induction on $r \geq 0$. The case $r = 0$ follows from definitions, and the induction step from the fact that the domain and target for $Y = \mathbb{A}^r_{k_0}$ is given by the *same* formula [**9**, theorem B] in terms of the domain and target for $Y = \mathbb{A}^{r-1}_{k_0}$. We define ∂ as the composite

$$i^*W_n\,\Omega^q_{(X,M_X)} \to W_n\,\Omega^q_{(Y,M_Y)} \xleftarrow{\sim} W_n\,\Omega^q_Y \oplus W_n\,\Omega^{q-1}_Y \to W_n\,\Omega^{q-1}_Y,$$

where the left hand map is the projection onto the quotient by $U^2 i^*W_n\,\Omega^q_{(X,M_X)}$, and where the right hand map is the projection onto the second summand.

Let $x_1, \dots, x_r$ be local coordinates of an open neighborhood on X around a point of Y, and let $\bar{x}_1, \dots, \bar{x}_r$ be the corresponding local coordinates on Y. Lemma 1.3.1 allows us to view the sequence of the statement as a sequence of $\mathcal{O}_Y$-modules, and theorem 1.3.3 gives a basis of the middle term. The map ∂ takes

$$\partial(V^s([x_1]^{i_1}\dots[x_r]^{i_r}d\log x_{m_1}\dots d\log x_{m_{q-1}}d\log\pi))$$
$$= V^s([\bar{x}_1]^{i_1}\dots[\bar{x}_r]^{i_r}d\log\bar{x}_{m_1}\dots d\log\bar{x}_{m_{q-1}}),$$
$$\partial(dV^s([x_1]^{i_1}\dots[x_r]^{i_r}d\log x_{m_1}\dots d\log_{x_{m_{q-2}}} d\log\pi))$$
$$= dV^s([\bar{x}_1]^{i_1}\dots[\bar{x}_r]^{i_r}d\log\bar{x}_{m_1}\dots d\log\bar{x}_{m_{q-2}}),$$

and annihilates all remaining basis elements. It follows that the composite $\partial \circ j_*$ is equal to zero. One shows, in a manner similar to the proof of proposition 1.3.2 that, Zariski locally on Y, the sheaves $i^*\bar{W}_n\,\Omega^q_X$ and $\bar{W}_n\,\Omega^{q-1}_Y$ with the $\mathcal{O}_Y$-module structure given by lemma 1.3.1 are free and that their ranks satisfy the equation

$$\mathrm{rk}_{\mathcal{O}_Y}\, i^*\bar{W}_n\,\Omega^q_X + \mathrm{rk}_{\mathcal{O}_Y}\,\bar{W}_n\,\Omega^{q-1}_Y = \mathrm{rk}_{\mathcal{O}_Y}\, i^*\bar{W}_n\,\Omega^q_{(X,M_X)}.$$

This completes the proof. □

1.4. We end this paragraph with the following result on the structure of the higher torsion in the de Rham-Witt complex. The proof we give here uses the cyclotomic trace; see [**8**]. It would be desirable to have a purely algebraic proof.

PROPOSITION 1.4.1. *If $\mu_{p^v} \subset K_0$, then for all $0 \leq m < v$ and all $q \geq 0$, multiplication by p^m induces an isomorphism of sheaves of pro-abelian groups*

$$\bar{W}.\,\Omega^q_{(X,M_X)} = \mathrm{gr}^0_p W.\,\Omega^q_{(X,M_X)} \xrightarrow{\sim} \mathrm{gr}^m_p W.\,\Omega^q_{(X,M_X)}.$$

PROOF. We must show that for all $0 \leq m < v$ and all $q \geq 0$, the following sequence of sheaves of pro-abelian groups on the small étale site of X is exact.

$$0 \to W.\,\Omega^q_{(X,M_X)}/p \xrightarrow{p^m} W.\,\Omega^q_{(X,M_X)}/p^{m+1} \xrightarrow{\mathrm{pr}} W.\,\Omega^q_{(X,M_X)}/p^m \to 0.$$

This, in turn, is equivalent to the statement that for all $0 \leq m < v$ and all $q, s \geq 0$, the following sequence is exact.

$$0 \to W.\,\Omega^{q-2s}_{(X,M_X)} \otimes \mu_p^{\otimes s} \to W.\,\Omega^{q-2s}_{(X,M_X)} \otimes \mu_{p^{m+1}}^{\otimes s} \to W.\,\Omega^{q-2s}_{(X,M_X)} \otimes \mu_{p^m}^{\otimes s} \to 0.$$

Clearly only the injectivity of the left hand map is at issue. We recall that for all $0 \leq m < v$ and all $q \geq 0$, [**9**, theorem E] gives an isomorphism

$$\bigoplus_{s \geq 0} W.\,\Omega^{q-2s}_{(X,M_X)} \otimes \mu_{p^m}^{\otimes s} \xrightarrow{\sim} \mathrm{TR}^{\cdot}_q(X|X_K; p, \mathbb{Z}/p^m).$$

We conclude that for all $0 \leq m < v$ and $q \geq 0$, the map induced from the reduction

$$\mathrm{TR}^{\cdot}_q(X|X_K; p, \mathbb{Z}/p^{m+1}) \to \mathrm{TR}^{\cdot}_q(X|X_K; p, \mathbb{Z}/p^m)$$

is a surjection. It follows that the long-exact coefficient sequence breaks up into short-exact sequences of the form

$$0 \to \mathrm{TR}^{\cdot}_q(X|X_K; p, \mathbb{Z}/p) \to \mathrm{TR}^{\cdot}_q(X|X_K; p, \mathbb{Z}/p^{m+1}) \to \mathrm{TR}^{\cdot}_q(X|X_K; p, \mathbb{Z}/p^m) \to 0.$$

The proposition follows. □

2. p-adic vanishing cycles

2.1. Let $(i^*\bar{W}.\,\Omega^q_{(X,M_X)})^{F=1}$ and $(i^*\bar{W}.\,\Omega^q_{(X,M_X)})_{F=1}$ denote the kernel and cokernel, respectively, of the map

$$1 - F\colon i^*\bar{W}.\,\Omega^q_{(X,M_X)} \to i^*\bar{W}.\,\Omega^q_{(X,M_X)}$$

of sheaves of pro-abelian groups on the small étale site of Y. (We consider these sheaves both in the Nisnevich and étale topology.) The U-filtration is preserved by $1-F$, and hence, induces filtrations of kernel and cokernel sheaves. We begin with the following observation.

LEMMA 2.1.1. *Suppose that $m \geq 2$. Then the map*

$$R - F\colon U^m\bar{W}_n\Omega^q_{(X,M_X)} \twoheadrightarrow U^m\bar{W}_{n-1}\Omega^q_{(X,M_X)}$$

is surjective, for all integers n and q.

PROOF. We consider the case $m = 2j$; the case $m = 2j+1$ is similar. It suffices to show that if $a_0, \dots, a_q$ and $a'_0, \dots, a'_{q-1}$ are local sections of $\mathcal{O}_X$ such that $\operatorname{ord}_Y(a_i), \operatorname{ord}_Y(a'_{i'}) \geq j$, for some $0 \leq i \leq q$ and $0 \leq i' \leq q-1$, then the following elements are in the image of $R - F$.

$$V^{s_0}[a_0]_{n-1} dV^{s_1}[a_1]_{n-1} \dots dV^{s_q}[a_q]_{n-1},$$
$$V^{s_0}[a_0]_{n-1} dV^{s_1}[a_1]_{n-1} \dots dV^{s_{q-1}}[a_{q-1}]_{n-1} d\log_{n-1}\pi.$$

Indeed, every element of $U^m \bar{W}_{n-1}\Omega^q_{(X,M_X)}$ is a sum of such elements. We now use that since $j \geq 1$, the following series converges.

$$\sum_{t\geq 0} F^t(V^{s_0}[a_0]_{n+t} dV^{s_1}[a_1]_{n+t} \dots dV^{s_q}[a_q]_{n+t}),$$
$$\sum_{t\geq 0} F^t(V^{s_0}[a_0]_{n+t} dV^{s_1}[a_1]_{n+t} \dots dV^{s_{q-1}}[a_{q-1}]_{n+t} d\log_{n+t}\pi).$$

The images by $R-F$ of the sums of these series are equal to the given elements. $\square$

We evaluate the filtration quotients of the kernel of $1 - F$. Let $\Omega^q_{Y,\log} \subset \Omega^q_Y$ be the subsheaf generated locally for the étale topology on Y by the local sections of the form $d\log a_1 \dots d\log a_q$. If y is a local section of $\mathcal{O}_Y$, we denote by $\tilde{y}$ any lifting of y to a local section of $i^*\mathcal{O}_X$.

Theorem 2.1.2. *The sheaf* $M^q_\cdot = (i^*\bar{W}_\cdot\,\Omega^q_{(X,M_X)})^{F=1}$ *of pro-abelian groups has the following structure:*

(i) *There is an isomorphism*

$$\Omega^q_{Y,\log} \xrightarrow{\sim} \operatorname{gr}^0_U M^q_\cdot \qquad (\textit{resp. } \Omega^{q-1}_{Y,\log} \xrightarrow{\sim} \operatorname{gr}^1_U M^q_\cdot),$$

which to $d\log y_1 \dots d\log y_q$ (*resp.* $d\log y_1 \dots d\log y_{q-1}$) *assigns* $d\log \tilde{y}_1 \dots d\log \tilde{y}_q$ (*resp.* $d\log \tilde{y}_1 \dots d\log \tilde{y}_{q-1} d\log \pi$).

(ii) *If* $0 < j < e'$, *and if* p *does not divide* j (*resp. if* p *divides* j), *there is an isomorphism*

$$\Omega^{q-1}_Y / B\Omega^{q-1}_Y \xrightarrow{\sim} \operatorname{gr}^{2j}_U M^q_\cdot \qquad (\textit{resp. } \Omega^{q-1}_Y / Z\Omega^{q-1}_Y \xrightarrow{\sim} \operatorname{gr}^{2j}_U M^q_\cdot),$$

which to $a\,d\log y_1 \dots d\log y_{q-1}$ *assigns* $d\log(1+\pi^j\tilde{a}) d\log\tilde{y}_1 \dots d\log \tilde{y}_{q-1}$.

(iii) *If* $0 < j < e'$, *there is an isomorphism*

$$\Omega^{q-2}_Y / Z\Omega^{q-2}_Y \xrightarrow{\sim} \operatorname{gr}^{2j+1}_U M^q_\cdot,$$

which takes $a\,d\log y_1 \dots d\log y_{q-2}$ *to* $d\log(1+\pi^j\tilde{a}) d\log \tilde{y}_1 \dots d\log\tilde{y}_{q-2} d\log\pi$.

(iv) *If* $e' \leq j$, *then* $U^{2j}M^q_\cdot$ *is equal to zero.*

Proof. We first evaluate the kernel of $1 - F$ in terms of the basis of theorem 1.3.3. A local section ω of $E^q_n = i^*\bar{W}_n\,\Omega^q_{(X,M_X)}$ can be written uniquely

$$\begin{aligned}\omega = &\sum_{s,j,i,m} a^{(s)}_{j,m,i}\, dV^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}) \\ &+ \sum_{s,j,i,m} c^{(s)}_{j,m,i}\, V^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_q}) \\ &+ \sum_{s,j,i,m} b^{(s)}_{j,m,i}\, dV^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-2}} d\log\pi) \\ &+ \sum_{s,j,i,m} d^{(s)}_{j,m,i}\, V^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}} d\log\pi),\end{aligned}$$

where the coefficients $a^{(s)}_{j,m,i}$, $b^{(s)}_{j,m,i}$, $c^{(s)}_{j,m,i}$, and $d^{(s)}_{j,m,i}$ are local sections of $\mathcal{O}_Y$, where $0 \leq s < n$, and where the indices j, i, and m vary as in the statement of theorem 1.3.3. The restriction map (the structure map in the inverse system) annihilates the summands with $s = n-1$ and leaves the remaining summands unchanged. The Frobenius annihilates the summands with $s > 0$ in the second and fourth lines above,

$$F\big(c^{(0)}_{j,m}[\pi]^j d\log x_{m_1} \dots d\log x_{m_q}\big) = (c^{(0)}_{j,m})^p [\pi]^{pj} d\log x_{m_1} \dots d\log x_{m_q},$$

and similarly for the summand $s = 0$ in the fourth line. We evaluate the Frobenius on the summands in the first line; the summands in the third line are treated completely analogously. Since $FdV = d$,

$$\begin{aligned} &F\big(a^{(s)}_{j,m,i} dV^s([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}})\big) \\ &= (a^{(s)}_{j,m,i})^p dV^{s-1}([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}), \end{aligned}$$

which we must express as a linear combination of the basis elements. To this end we write $i_m = k_m p^{s-1} + i'_m$ with $0 \leq i'_1, \dots, i'_r < p^{s-1}$. Then

$$\begin{aligned} &dV^{s-1}([x_1]^{i_1} \dots [x_r]^{i_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}) \\ &= \bar{x}_1^{k_1} \dots \bar{x}_r^{k_r} dV^{s-1}([x_1]^{i'_1} \dots [x_r]^{i'_r} [\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}) \\ &+ \sum_{1 \leq m \leq r} k_{m_0} \bar{x}_1^{k_1} \dots \bar{x}_r^{k_r} V^{s-1}([x_1]^{i'_1} \dots [x_r]^{i'_r} [\pi]^j d\log x_{m_0} \dots d\log x_{m_{q-1}}) \end{aligned}$$

is a linear combination of basis elements (except that $m_0, m_1, \dots, m_q$ may not be written in increasing order). We now give the equations that the coefficients must satisfy in order for ω to lie in the kernel of $1 - F \colon E^q_n \to E^q_{n-1}$. Most importantly, we have that for all $0 \leq s < n-1$,

$$\begin{aligned} a^{(s)}_{j,m,i} &= \sum_{0 \leq k_1, \dots, k_r < p} (a^{(s+1)}_{j,m,kp^s+i})^p \bar{x}_1^{k_1} \dots \bar{x}_r^{k_r}, \\ b^{(s)}_{j,m,i} &= \sum_{0 \leq k_1, \dots, k_r < p} (b^{(s+1)}_{j,m,kp^s+i})^p \bar{x}_1^{k_1} \dots \bar{x}_r^{k_r}. \end{aligned} \tag{2.1.3}$$

In addition, there are equations that express the coefficients $c^{(s)}_{j,m,i}$ and $d^{(s)}_{j,m,i}$, where either $1 \leq s < n-1$ or $s = 0$ and p does not divide j, as functions of the coefficients $a^{(s+1)}_{j,m,i}$ and $b^{(s+1)}_{j,m,i}$. Similarly, the coefficients $c^{(0)}_{pj',m}$ and $d^{(0)}_{pj',m}$ are functions of the coefficients $a^{(1)}_{j,m}$ and $c^{(0)}_{j',m}$ and $b^{(1)}_{j,m}$ and $d^{(0)}_{j',m}$, respectively. Hence, the $c^{(s)}_{j,m,i}$ and $d^{(s)}_{j,m,i}$, where $0 \leq s < n-1$, are uniquely determined by the $a^{(s+1)}_{j,m,i}$ and $d^{(s+1)}_{j,m,i}$, and therefore, we can disregard the former when solving for ω in the kernel of $1 - F$. Finally, the coefficients $c^{(n-1)}_{j,m,i}$ and $d^{(n-1)}_{j,m,i}$ do not appear in any equations and hence are unrestricted variables.

We now solve these equations and determine the kernel of $1 - F$. It follows from lemma 2.1.1 that the sequence

$$0 \to M^q_\cdot / U^2 M^q_\cdot \to E^q_\cdot / U^2 E^q_\cdot \xrightarrow{1-F} E^q_\cdot / U^2 E^q_\cdot$$

is exact, or equivalently, that the canonical projection induces an isomorphism

$$M^q_\cdot / U^2 M^q_\cdot \xrightarrow{\sim} (\bar{W}_\cdot \Omega^q_{(Y, M_Y)})^{F=1}.$$

The structure of the pro-sheaf on the right is well-understood; see [**21**, proposition 2.4.1]. The statement for $\mathrm{gr}^0_U M^q_{\cdot}$ and $\mathrm{gr}^1_U M^q_{\cdot}$ follows.

We proceed to show that $\mathrm{gr}^m_U M^q_{\cdot}$ for $m = 2j$ and $1 \le j < e'$ is as stated. The argument for $m = 2j+1$ and $1 \le j < e'$ is entirely similar. We have a map of sheaves of pro-abelian groups

$$\Omega^{q-1}_Y \to \mathrm{gr}^m_U M^q_{\cdot},$$

which takes $a d\log y_1 \dots d\log y_{q-1}$ to $d\log(1+\tilde{a}\pi^j) d\log \tilde{y}_1 \dots d\log \tilde{y}_{q-1}$, and we wish to show that this induces the isomorphism of the statement. By lemma 1.2.3, the image of $a_{j,m} d\log \bar{x}_{m_1} \dots d\log \bar{x}_{m_{q-1}}$ in $\mathrm{gr}^{2j}_U M^q_n$ is equal to the class of the sum

$$\tau = \sum_{0 \le s < n} dV^s([\tilde{a}_{j,m}][x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}).$$

In order to rewrite this sum in terms of the basis of theorem 1.3.3, we write $a_{j,m}$ for $0 \le s < n$ uniquely as

$$a_{j,m} = \sum_{0 \le i_1, \dots, i_r < p^s} (a^{(s)}_{j,m,i})^{p^s} \bar{x}_1^{i_1} \dots \bar{x}_r^{i_r}.$$

The coefficients $a^{(s)}_{j,m,i}$ satisfy (2.1.3), and conversely, if $a^{(s)}_{j,m,i}$, $0 \le s < n$, satisfy (2.1.3), then $a_{j,m}$ defined by the equations above is independent of $0 \le s < n$. We can now rewrite the class of τ as

$$\begin{aligned}\tau = &\sum_{s,j,i,m} a^{(s)}_{j,m,i} dV^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_{q-1}}) \\ &+ \sum_{s,j,i,m} c^{(s)}_{j,m,i} V^s([x_1]^{i_1} \dots [x_r]^{i_r}[\pi]^j d\log x_{m_1} \dots d\log x_{m_q})\end{aligned}$$

for certain uniquely determined coefficients $c^{(s)}_{j,m,i}$, $0 \le s < n-1$. The theorem follows from the observation that a basis of $\Omega^{q-1}_Y / B\Omega^{q-1}_Y$ (resp. $\Omega^{q-1}_Y / Z\Omega^{q-1}_Y$) as an $\mathcal{O}^{p^s}_Y$-module is given by the local sections

$$\bar{x}_1^{i_1} \dots \bar{x}_r^{i_r} d\log \bar{x}_{m_1} \dots d\log \bar{x}_{m_{q-1}}$$

with $0 \le i_1, \dots, i_r < p^s$ (resp. $0 \le i_1, \dots, i_r < p^s$, not all divisible by p), and $1 \le m_1 < \dots < m_{q-1} \le r$ such that if m is largest with i_m prime to p, then $m_i \ne m$, for all $1 \le i \le q-1$. □

ADDENDUM 2.1.4. *The canonical projection induces an isomorphism of sheaves of pro-abelian groups on the small étale site of Y in the Nisnevich or étale topology,*

$$i^*(\bar{W}_{\cdot}\Omega^q_{(X,M_X)})_{F=1} \xrightarrow{\sim} (\bar{W}_{\cdot}\Omega^q_{(Y,M_Y)})_{F=1},$$

and in the étale topology, the common sheaf is zero.

PROOF. We recall from lemma 2.1.1 that the map

$$U^2 i^* \bar{W}_{\cdot}\Omega^q_{(X,M_X)} \xrightarrow{1-F} U^2 i^* \bar{W}_{\cdot}\Omega^q_{(X,M_X)}$$

is surjective. The isomorphism of the statement follows in view of the isomorphism of lemma 1.2.1. Finally, by the proof of addendum 1.3.5, there is split-exact sequence

$$0 \to \bar{W}_{\cdot}\Omega^q_Y \to \bar{W}_{\cdot}\Omega^q_{(Y,M_Y)} \to \bar{W}_{\cdot}\Omega^{q-1}_Y \to 0,$$

and by [**12**, proposition I.3.26], the map $1 - F$ induces surjections of the left and right hand terms in the étale topology. $\square$

Theorem 2.1.5. *There is a natural exact sequence*

$$0 \to i^* R^q j_* \mu_p^{\otimes q} \to i^* \bar{W}.\, \Omega^q_{(X,M_X)} \xrightarrow{1-F} i^* \bar{W}.\, \Omega^q_{(X,M_X)} \to 0$$

of sheaves of pro-abelian groups on the small étale site of Y.

Proof. In order to construct the left hand map of the statement, we consider the symbol maps

$$i^* R^q j_* \mu_p^{\otimes q} \leftarrow i^* (M_X^{\mathrm{gp}})^{\wedge q} \to i^* (\bar{W}.\, \Omega^q_{(X,M)})^{F=1}$$

The right hand map takes a local section $a_1 \wedge \dots \wedge a_q$ to $d \log a_1 \dots d \log a_q$, and the left hand map takes the same local section to the symbol $\{a_1, \dots, a_q\}$; we recall the definition of the latter. By Hilbert's theorem 90, the Kummer sequence

$$0 \to \mu_p \to \mathcal{O}_U^* \xrightarrow{p} \mathcal{O}_U^* \to 0$$

gives rise to an exact sequence

$$0 \to i^* j_* \mu_p \to i^* j_* \mathcal{O}_U^* \xrightarrow{p} i^* j_* \mathcal{O}_U^* \xrightarrow{\partial} i^* R^1 j_* \mu_p \to 0$$

of sheaves of abelian groups on the small étale site of Y. The symbol $\{a\}$ is defined as the image of the local section a by the composite

$$i^* M_X^{\mathrm{gp}} \xrightarrow{\sim} i^* j_* \mathcal{O}_U^* \xrightarrow{\partial} i^* R^1 j_* \mu_p,$$

and $\{a_1, \dots, a_q\}$ as the product of $\{a_1\}, \dots, \{a_q\}$.

Let V' be the strictly henselian local ring of X at the generic point of Y, and let K' be the quotient field of V'. Let $\tau\colon \operatorname{Spec} k \to Y$ be the inclusion of the generic point and consider the maps induced by the symbol maps

$$\tau^* i^* R^q j_* \mu_p^{\otimes q} \leftarrow \tau^* i^* (M_X^{\mathrm{gp}})^{\wedge q}/p \to \tau^* (i^* \bar{W}.\, \Omega^q_{(X,M_X)})^{F=1}.$$

The left and right hand terms are canonically isomorphic to the skyscraper sheafs given by the (pro-abelian) groups $\bar{K}_q^M(K')$ and $(\bar{W}.\, \Omega^q_{(V',M_{V'})})^{F=1}$, respectively. It follows that the left hand map is a surjection whose kernel is equal to the subsheaf generated by the sections $a_1 \wedge \dots \wedge a_q$ with some $a_i + a_j = 1$. These sections are annihilated by the right hand symbol map, and hence, we have an induced map

$$\tau^* i^* R^q j_* \mu_p^{\otimes q} \to \tau^* (i^* \bar{W}.\, \Omega^q_{(X,M_X)})^{F=1}. \tag{2.1.6}$$

This map preserves U-filtrations, and [**15**, theorem 2(1)] and theorem 2.1.2 show that the induced map of filtration quotients is an isomorphism. It follows that the map is an isomorphism.

We consider the following commutative diagram.

$$\begin{array}{ccccc}
i^* R^q j_* \mu_p^{\otimes q} & \twoheadleftarrow & i^* (M_X^{\mathrm{gp}})^{\wedge q} & \longrightarrow & (i^* \bar{W}.\, \Omega^q_{(X,M_X)})^{F=1} \\
\big\downarrow & & & & \big\downarrow \\
\tau_* \tau^* i^* R^q j_* \mu_p^{\otimes q} & & \xrightarrow[\sim]{(2.1.6)} & & \tau_* \tau^* (i^* \bar{W}.\, \Omega^q_{(X,M_X)})^{F=1}.
\end{array}$$

It is proved in [**3**, proposition 6.1(i)] that the left hand vertical map is injective and in *loc. cit.*, corollary 6.1.1, that the upper left hand horizontal map is surjective. Moreover, the right hand vertical map is injective, since, locally on Y, the sheaf

$i^*\bar{W}.\,\Omega^q_{(X,M_X)}$ is a quasi-coherent $\mathcal{O}_Y$-module. It follows that the upper horizontal maps have the same kernel, and hence, the symbol maps gives rise to a map

$$i^*R^qj_*\mu_p^{\otimes q} \to (i^*\bar{W}.\,\Omega^q_{(X,M_X)})^{F=1}.$$

Again this map preserves U-filtration, and [**3**, theorem 1.4] (see also [**21**, proposition 2.4.1]) and theorem 2.1.2 show that it is an isomorphism. □

REMARK 2.1.7. It is possible from the proof of theorem 2.1.2 to derive the following more precise statement about the injectivity of the map

$$i^*R^qj_*\mu_p^{\otimes q} \to i^*\bar{W}_n\,\Omega^q_{(X,M_X)}.$$

As in the statement of theorem 1.3.3, let $v = v(j)$ be the unique integer such that

$$e\big(\frac{p^{-v}-1}{p^{-1}-1}\big) \leq j < e\big(\frac{p^{-(v+1)}-1}{p^{-1}-1}\big).$$

Then the map is injective as soon as $n \geq v+2$, for all $0 \leq j < e'$.

PROOF OF THEOREM A. The surjectivity of $1-F$ is an immediate consequence of addendum 2.1.4. We show by induction on $v \geq 1$ that the symbol maps

$$i^*R^qj_*\mu_{p^v}^{\otimes q} \leftarrow i^*(M_X^{\mathrm{gp}})^{\wedge q}/p^v \to (i^*W.\,\Omega^q_{(X,M_X)}/p^v)^{F=1}$$

are surjective and have the same kernel. The case $v=1$ is theorem 2.1.5. In the induction step we consider the following diagram, where $E^q_. = i^*W.\,\Omega^q_{(X,M_X)}$.

$$\begin{array}{ccccccccc}
0 & \to & i^*R^qj_*\mu_p^{\otimes q} & \to & i^*R^qj_*\mu_{p^v}^{\otimes q} & \to & i^*R^qj_*\mu_{p^{v-1}}^{\otimes q} & \to & 0 \\
 & & \uparrow\!\!\uparrow & & \uparrow & & \uparrow\!\!\uparrow & & \\
 & & i^*(M_X^{gp})^{\wedge q}/p & \to & i^*(M_X^{gp})^{\wedge q}/p^v & \to & i^*(M_X^{gp})^{\wedge q}/p^{v-1} & \to & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow & & \downarrow\!\!\downarrow & & \\
0 & \to & (E^q_./p)^{F=1} & \to & (E^q_./p^v)^{F=1} & \to & (E^q_./p^{v-1})^{F=1} & \to & 0
\end{array}$$

The upper horizontal sequence is exact in the middle, and the middle and lower horizontal sequences are both exact. The statement for the latter follows from proposition 1.4.1 and addendum 2.1.4. By induction, the right hand vertical maps are surjective and have the same kernel. The same is true for the left hand vertical maps. It follows that the middle vertical maps are surjective. This, in turn, implies that the upper right hand horizontal map is surjective. We claim that the upper left horizontal map is injective. Indeed, this is equivalent, by the long-exact cohomology sequence, to the statement that in the sequence

$$i^*R^{q-1}j_*\mathbb{Z}/p(q) \to i^*R^{q-1}j_*\mathbb{Z}/p^v(q) \to i^*R^{q-1}j_*\mathbb{Z}/p^{v-1}(q) \to 0$$

the right hand map is surjective. But cup product by a primitive p^vth root of unity induces an isomorphism of the sheaves $\mathbb{Z}/p^v(q-1)$ and $\mathbb{Z}/p^v(q)$, and we have already proved that the following sequence is exact.

$$i^*R^{q-1}j_*\mathbb{Z}/p(q-1) \to i^*R^{q-1}j_*\mathbb{Z}/p^v(q-1) \to i^*R^{q-1}j_*\mathbb{Z}/p^{v-1}(q-1) \to 0.$$

It remains to show that the middle vertical maps in the diagram above have the same kernel. To this end, let, as in the proof of theorem 2.1.5, $\tau\colon \operatorname{Spec} k \to Y$ be the inclusion of the generic point and consider the symbol maps

$$\tau^* i^* R^q j_* \mu_{p^v}^{\otimes q} \leftarrow \tau^* i^* (M_X^{\mathrm{gp}})^{\wedge q}/p^v \to \tau^* (i^* W.\,\Omega^q_{(X,M_X)}/p^v)^{F=1}.$$

The kernel of the left hand map is generated by the symbols $\{a_1, \dots, a_q\}$ with some $a_i + a_j = 1$, and these sections are contained in the kernel of the right hand map. Hence, we have an induced map

$$\tau^* i^* R^q j_* \mu_{p^v}^{\otimes q} \to \tau^* (i^* W.\,\Omega^q_{(X,M_X)}/p^v)^{F=1}.$$

It is an isomorphism by induction and by the exactness of the upper and lower horizontal rows in the diagram above. We consider the following diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & i^* R^q j_* \mu_p^{\otimes q} & \longrightarrow & i^* R^q j_* \mu_{p^v}^{\otimes q} & \longrightarrow & i^* R^q j_* \mu_{p^{v-1}}^{\otimes q} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \tau_*\tau^* i^* R^q j_* \mu_p^{\otimes q} & \longrightarrow & \tau_*\tau^* i^* R^q j_* \mu_{p^v}^{\otimes q} & \longrightarrow & \tau_*\tau^* i^* R^q j_* \mu_{p^{v-1}}^{\otimes q}. & &
\end{array}$$

By induction and by [**3**, proposition 6.1(i)], the right and left hand vertical maps are injective. Hence, also the middle vertical map is injective. A similar argument shows that the right hand vertical map in the following diagram is injective.

$$\begin{array}{ccccc}
i^* R^q j_* \mu_{p^v}^{\otimes q} & \twoheadleftarrow & i^* (M_X^{\mathrm{gp}})^{\wedge q}/p^v & \twoheadrightarrow & (i^* \bar{W}.\,\Omega^q_{(X,M_X)}/p^v)^{F=1} \\
\big\downarrow & & & & \big\downarrow \\
\tau_*\tau^* i^* R^q j_* \mu_{p^v}^{\otimes q} & & \xrightarrow{\sim} & & \tau_*\tau^* (i^* \bar{W}.\,\Omega^q_{(X,M_X)}/p^v)^{F=1}.
\end{array}$$

It follows that the symbol maps induce an isomorphism of the upper left and right hand terms as desired. □

3. Henselian discrete valuation rings

3.1. In this paragraph we prove theorem B of the introduction. The proof uses the following commutative diagram of pro-abelian groups in which the right hand vertical map is the cyclotomic trace of [**4**]. We refer the reader to [**8**] for an introduction and a comprehensive list of references to this construction.

$$\begin{array}{ccc}
K_*^M(K) \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) & \longrightarrow & K_*(K, \mathbb{Z}/p^v) \\
\big\downarrow & & \big\downarrow \\
W.\,\Omega^*_{(V,M_V)} \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) & \xrightarrow{\sim} & \mathrm{TR}^{\boldsymbol{\cdot}}_*(V|K; p, \mathbb{Z}/p^v).
\end{array} \tag{3.1.1}$$

The isomorphism of theorem B is derived from the fact, proved in [**10**, theorem C] and [**9**, theorem E], that the lower horizontal map is an isomorphism.

Suppose that the residue field k is separably closed. It follows from theorem A that the left hand vertical map is injective and induces an isomorphism onto the Frobenius fixed set of the target. Similarly, the right hand vertical map is injective and induces an isomorphism onto the Frobenius fixed set of the target; the argument will be given below. Hence, theorem B follows in this case.

In general, the vertical maps in (3.1.1) are not injective. But they induce surjections onto the Frobenius fixed sets of the respective targets, and the two kernels can be expressed in terms of de Rham-Witt groups. To prove theorem B, one must show that the map of kernels induced from the upper horizontal map is an isomorphism. The proof of this occupies most of the paragraph.

3.2. Let the field K be as in the statement of theorem B. We first consider the left hand vertical map in (3.1.1).

PROPOSITION 3.2.1. *Suppose that $\mu_p \subset K_0$. Then there is a natural exact sequence of pro-abelian groups*

$$0 \to (\bar{W}.\,\Omega^{q-1}_{(V,M_V)} \otimes \mu_p)_{F=1} \to \bar{K}^M_q(K) \to (\bar{W}.\,\Omega^q_{(V,M_V)})^{F=1} \to 0,$$

where the left hand map takes the class of $[a]d\log x_1 \dots d\log x_{q-1} \otimes \zeta$ to the class of the symbol $\{1 + a(1-\zeta)^p, x_1, \dots, x_{q-1}\}$.

PROOF. It follows from [**14**, theorem 2(1)] and from theorem 2.1.2 above that the map, which to $\{a_1, \dots, a_q\}$ associates $d\log a_1 \dots d\log a_q$, induces an isomorphism of pro-abelian groups

$$\bar{K}^M_q(K)/U^{2e'}\bar{K}_q(K) \xrightarrow{\sim} (\bar{W}.\,\Omega^q_{(V,M_V)})^{F=1}.$$

Indeed, the right hand side is the stalk at the generic point of Y of the sheaf of pro-abelian groups $(i^*\bar{W}.\,\Omega^q_{(X,M_X)})^{F=1}$ on the small étale site of Y in the Nisnevich topology. Similarly, [**14**, theorem 2(1)] and addendum 2.1.4 shows that the left hand map of the statement induces an isomorphism of pro-abelian groups

$$(\bar{W}.\,\Omega^{q-1}_{(V,M_V)} \otimes \mu_p)_{F=1} \xrightarrow{\sim} U^{2e'}\bar{K}^M_q(K).$$

This completes the proof. □

REMARK 3.2.2. Suppose that $\mu_{p^v} \subset K_0$. One can deduce from proposition 3.2.1 that there exists a natural exact sequence of pro-abelian groups

$$0 \to (W.\,\Omega^{q-1}_{(V,M_V)} \otimes \mu_{p^v})_{F=1} \to K^M_q(K)/p^v \to (W.\,\Omega^q_{(V,M_V)}/p^v)^{F=1} \to 0.$$

However, at this time, we do not have a purely algebraic proof of this deduction. We also do not have an explicit description of the left hand map.

We now turn our attention to the right hand vertical map in (3.1.1). To this end, we consider the cyclotomic trace

$$\mathrm{tr}\colon K_q(K, \mathbb{Z}/p) \to \mathrm{TC}^{\boldsymbol{\cdot}}_q(V|K; p, \mathbb{Z}/p)$$

from K-theory to topological cyclic homology; see [**10**, §1]. The right hand side is related to $\mathrm{TR}^{\boldsymbol{\cdot}}_*(V|K; p, \mathbb{Z}/p)$ by a natural exact sequence of pro-abelian groups

$$0 \to \mathrm{TR}^{\boldsymbol{\cdot}}_{q+1}(V|K; p, \mathbb{Z}/p)_{F=1} \xrightarrow{\delta} \mathrm{TC}^{\boldsymbol{\cdot}}_q(V|K; p, \mathbb{Z}/p) \to \mathrm{TR}^{\boldsymbol{\cdot}}_q(V|K; p, \mathbb{Z}/p)^{F=1} \to 0.$$

We consider the composition of the left hand map by the canonical map

$$(\bar{W}.\,\Omega^{q+1}_{(V,M_V)})_{F=1} \to \mathrm{TR}^{\boldsymbol{\cdot}}_{q+1}(V|K; p, \mathbb{Z}/p)_{F=1}.$$

Proposition 3.2.3. *For all integers q, the cyclotomic trace and the map δ give rise to a natural isomorphism of pro-abelian groups*

$$K_q(K,\mathbb{Z}/p) \oplus (\bar{W}.\,\Omega^{q+1}_{(V,M_V)})_{F=1} \xrightarrow{\sim} \mathrm{TC}^{\cdot}_q(V|K;p,\mathbb{Z}/p).$$

Proof. We consider the following diagram of pro-abelian groups, where the horizontal maps are given by the cyclotomic trace on the first summand and the boundary map on the second summand, and where the vertical maps are induced by the canonical projection.

$$\begin{array}{ccc} K_q(V,\mathbb{Z}/p) \oplus (\bar{W}.\,\Omega^{q+1}_V)_{F=1} & \longrightarrow & \mathrm{TC}^{\cdot}_q(V;p,\mathbb{Z}/p) \\ \downarrow & & \downarrow \\ K_q(k,\mathbb{Z}/p) \oplus (\bar{W}.\,\Omega^{q+1}_k)_{F=1} & \longrightarrow & \mathrm{TC}^{\cdot}_q(k;p,\mathbb{Z}/p). \end{array}$$

The lower horizontal map is an isomorphism by [**7**, theorem 4.2.2], and we claim that also the top horizontal map is an isomorphism. Indeed, by addendum 2.1.4, the left hand vertical map induces an isomorphism of the second summand of the domain onto the second summand of the target, so the claim follows from the fact that the cyclotomic induces an isomorphism of the relative groups

$$K_q(V,\mathfrak{m},\mathbb{Z}/p) \xrightarrow{\sim} \mathrm{TC}^{\cdot}_q(V,\mathfrak{m};p,\mathbb{Z}/p);$$

this follows from [**17**], [**20, 19**], and [**5**, theorem 2.1.1].

We recall that [**10**, addendum 1.5.7] gives a map of localization sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & K_q(V,\mathbb{Z}/p) & \xrightarrow{j_*} & K_q(K,\mathbb{Z}/p) & \xrightarrow{\partial} & K_{q-1}(k,\mathbb{Z}/p) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \mathrm{TC}^{\cdot}_q(V;p,\mathbb{Z}/p) & \xrightarrow{j_*} & \mathrm{TC}^{\cdot}_q(V|K;p,\mathbb{Z}/p) & \xrightarrow{\partial} & \mathrm{TC}^{\cdot}_{q-1}(k;p,\mathbb{Z}/p) & \longrightarrow \cdots . \end{array}$$

Moreover, addendum 1.3.5 gives rise to a map of exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & (\bar{W}.\,\Omega^{q+1}_V)_{F=1} & \xrightarrow{j_*} & (\bar{W}.\,\Omega^{q+1}_{(V,M_V)})_{F=1} & \xrightarrow{\partial} & (\bar{W}.\,\Omega^{q}_k)_{F=1} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \mathrm{TC}^{\cdot}_q(V;p,\mathbb{Z}/p) & \xrightarrow{j_*} & \mathrm{TC}^{\cdot}_q(V|K;p,\mathbb{Z}/p) & \xrightarrow{\partial} & \mathrm{TC}^{\cdot}_{q-1}(k;p,\mathbb{Z}/p) & \longrightarrow \cdots \end{array}$$

The injectivity of the upper horizontal map j_* follows from theorem 2.1.2(i). The left hand square commutes by the universal property of the de Rham-Witt complex. Finally, it follows from the description of the upper horizontal map ∂ in terms of local coordinates that, in order to show that the right hand square commutes, it will suffice to show that the lower horizontal map ∂ takes $d\log\pi$ to 1. But this is follows from the definition of $d\log\pi$ and from the commutativity of the right hand square in the previous diagram. This completes the proof. □

Proof of theorem B. We first note that if the statement is proved in the basic case $v = 1$, then the general case $v \geq 1$ follows inductively by using that the coefficient sequence breaks into short-exact sequences

$$0 \to K_q(K,\mathbb{Z}/p) \to K_q(K,\mathbb{Z}/p^v) \to K_q(K,\mathbb{Z}/p^{v-1}) \to 0.$$

Hence, it suffices to consider the case $v = 1$. It follows from propositions 3.2.1 and 3.2.3 that the left and right hand vertical maps in (3.1.1) are surjections onto the domain and target, respectively, of the canonical map

$$(\bar{W}.\,\Omega^{*}_{(V,M_V)} \otimes S_{\mathbb{Z}/p}(\mu_p))^{F=1} \xrightarrow{\sim} \mathrm{TR}^{\boldsymbol{\cdot}}_{*}(V|K;p,\mathbb{Z}/p)^{F=1}.$$

The two propositions further identify the kernel of both of the vertical maps in (3.1.1) with the direct sum

$$\bigoplus_{s\geq 1} (\bar{W}.\,\Omega^{q+1-2s}_{(V,M_V)} \otimes \mu_p^{\otimes s})_{F=1}.$$

It remains to show that the map between the two kernels induced by the upper horizontal map in (3.1.1) is an isomorphism. This, in turn, is equivalent to showing that the following diagram (3.2.4) of pro-abelian groups commutes. The three unmarked maps are as follows: The upper horizontal map is induced by the lower horizontal map in (3.1.1); the lower horizontal map is the composition of the canonical map from Milnor K-theory to algebraic K-theory followed by the cyclotomic trace; and the left hand vertical map is the left hand map in proposition 3.2.1.

$$(3.2.4)\qquad \begin{array}{ccc} (\bar{W}.\,\Omega^{q-1}_{(V,M_V)} \otimes \mu_p)_{F=1} & \longrightarrow & \mathrm{TR}^{\boldsymbol{\cdot}}_{q+1}(V|K;p,\mathbb{Z}/p)_{F=1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ \bar{K}^M_q(K) & \longrightarrow & \mathrm{TC}^{\boldsymbol{\cdot}}_q(V|K;p,\mathbb{Z}/p)/\delta(\bar{W}.\,\Omega^{q+1}_{(V,M_V)})_{F=1} \end{array}$$

It follows from addendum 2.1.4 that every element of the upper left hand term can be written in the form $[a]d\log x_1 \dots d\log x_{q-1} \otimes \zeta$. And since all maps in (3.2.4) are $\bar{K}^M_*(K)$-linear, we can further assume that $q = 1$. Hence, it suffices to prove the following proposition 3.2.5. $\square$

Proposition 3.2.5. *The image of* $[a] \otimes \zeta$ *by the composite map*

$$W.(V) \otimes \mu_p \to \mathrm{TR}^{\boldsymbol{\cdot}}_2(V|K;p,\mathbb{Z}/p) \xrightarrow{\delta} \mathrm{TC}^{\boldsymbol{\cdot}}_1(V|K;p,\mathbb{Z}/p)$$

is congruent, modulo $\delta(\bar{W}.\,\Omega^2_{(V,M_V)})_{F=1}$, *to* $d\log(1 + a(1-\zeta)^p)$.

Proof. We can assume that V is $\mathfrak{m}$-adically complete. Indeed, the completion map $V \to V^{\wedge}$ induces an isomorphism of all three terms in the statement. The line of proof is similar to that of [**10**, addendum 3.3.9]. We apply *loc. cit.*, lemma 3.3.10, to the 3×3-diagram of cofibration sequences

$$\begin{array}{ccccccc} E_{11} & \xrightarrow{f_{11}} & E_{12} & \xrightarrow{f_{12}} & E_{13} & \xrightarrow{f_{13}} & \Sigma E_{11} \\ \big\downarrow{\scriptstyle g_{11}} & & \big\downarrow{\scriptstyle g_{12}} & & \big\downarrow{\scriptstyle g_{13}} & & \big\downarrow{\scriptstyle \Sigma g_{11}} \\ E_{21} & \xrightarrow{f_{21}} & E_{22} & \xrightarrow{f_{22}} & E_{23} & \xrightarrow{f_{23}} & \Sigma E_{21} \\ \big\downarrow{\scriptstyle g_{21}} & & \big\downarrow{\scriptstyle g_{22}} & & \big\downarrow{\scriptstyle g_{23}} & & \big\downarrow{\scriptstyle \Sigma g_{21}} \\ E_{31} & \xrightarrow{f_{31}} & E_{32} & \xrightarrow{f_{32}} & E_{33} & \xrightarrow{f_{33}} & \Sigma E_{31} \\ \big\downarrow{\scriptstyle g_{31}} & & \big\downarrow{\scriptstyle g_{32}} & & \big\downarrow{\scriptstyle g_{33}}\ (-1) & & \big\downarrow{\scriptstyle -\Sigma g_{11}} \\ \Sigma E_{11} & \xrightarrow{\Sigma f_{11}} & \Sigma E_{12} & \xrightarrow{\Sigma f_{12}} & \Sigma E_{13} & \xrightarrow{-\Sigma f_{13}} & \Sigma^2 E_{11} \end{array}$$

obtained as the smash product of the coefficient sequence

$$S^0 \xrightarrow{p} S^0 \to M_p \xrightarrow{\beta} S^1$$

and the fundamental cofibration sequence

$$\mathrm{TC}^n(V|K;p) \to \mathrm{TR}^n(V|K;p) \xrightarrow{R-F} \mathrm{TR}^{n-1}(V|K;p) \xrightarrow{\delta} \Sigma\,\mathrm{TC}^n(V|K;p).$$

We recall from *loc. cit.*, lemma 3.3.10, that if $e_{ij} \in \pi_*(E_{ij})$ are classes such that $g_{33}(e_{33}) = f_{12}(e_{12})$ and $f_{33}(e_{33}) = g_{21}(e_{21})$, then the sum $f_{21}(e_{21}) + g_{12}(e_{12})$ is in the image of $\pi_*(E_{11}) \to \pi_*(E_{22})$. In the case at hand, we consider the class

$$e_{33} = [a]_{n-1}^p \cdot b_\zeta \in \pi_2(E_{33}) = \pi_2(M_p \wedge \mathrm{TR}^{n-1}(V|K;p)).$$

We wish to show that the image e_{31} of e_{33} by the map

$$f_{33*} = (\mathrm{id} \wedge \delta)_* \colon \pi_2(M_p \wedge \mathrm{TR}^{n-1}(V|K;p)) \to \pi_1(M_p \wedge \mathrm{TC}^n(V|K;p))$$

is congruent, modulo the image of $\pi_2(E_{23}) \to \pi_1(E_{31})$, to the class

$$e_{31}' = d\log_n(1 + a(1-\zeta)^p).$$

We shall use, repeatedly, that the canonical map

$$W_n\,\Omega^q_{(V,M_V)} \to \mathrm{TR}^n_q(V|K;p) = \pi_q(S^0 \wedge \mathrm{TR}^n(V|K;p))$$

is an isomorphism, if $q \leq 2$. This was proved in [**10**, theorem 3.3.8] for $V = V_0$. The general case follows from this by [**9**, theorems B and C].

By the definition of the Bott element, the image of e_{33} by the map

$$g_{33*} = (\beta \wedge \mathrm{id})_* \colon \pi_2(M_p \wedge \mathrm{TR}^{n-1}(V|K;p)) \to \pi_1(S^0 \wedge \mathrm{TR}^{n-1}(V|K;p))$$

is equal to the class

$$e_{13} = [a]_{n-1}^p d\log_{n-1}\zeta.$$

Since we assume that V is $\mathfrak{m}$-adically complete, the proof of lemma 2.1.1 shows that this class is in the image of

$$f_{12*} = R - F \colon W_n\,\Omega^1_{(V,M_V)} \to W_{n-1}\,\Omega^1_{(V,M_V)}.$$

Indeed, the class e_{13}, lies in $U^{2e''}W_{n-1}\Omega^1_{(V,M_V)}$, and $e'' \geq 1$. This also shows that the class e_{31}, which we wish to determine, is contained in the image of the map $g_{21*} \colon \pi_1(E_{21}) \to \pi_1(E_{31})$.

We write $\zeta = 1 + u\pi^{e''}$, where $u \in V^*$ is a unit, and consider the class

$$e_{12} = -\sum_{s=0}^{n-1}\sum_{t=0}^{s} dV^s([a]_{n-s}^{p^t}[u]_{n-s}[\pi]_{n-s}^{e''}).$$

SUBLEMMA 3.2.6. $f_{12*}(e_{12}) \in e_{13} + U^{4e''}W_{n-1}\Omega^1_{(V,M_V)}$.

Proof. We assume that $u = 1$ (the general case is only notationally more complicated) and calculate

$$\begin{aligned}
(R-F)e_{12} &= -\sum_{s=0}^{n-2}\sum_{t=0}^{s} dV^s([a]^{p^t}_{n-1-s}[\pi]^{e''}_{n-1-s}) \\
&\quad + \sum_{s=1}^{n-1}\sum_{t=0}^{s} dV^{s-1}([a]^{p^t}_{n-s}[\pi]^{e''}_{n-s}) + Fd([a]_n[\pi]^{e''}_n) \\
&= \sum_{s=0}^{n-2} dV^s([a]^{p^{s+1}}_{n-1-s}[\pi]^{e''}_{n-1-s}) + Fd([a]_n[\pi]^{e''}_n) \\
&= \sum_{s=0}^{n-2} d([a]^p_{n-1}V^s([\pi]^{e''}_{n-1-s})) + Fd([a]_n[\pi]^{e''}_n) \\
&= \sum_{s=0}^{n-2}[a]^p_{n-1}dV^s([\pi]^{e''}_{n-1-s}) \\
&\quad + \sum_{s=0}^{n-2} p[a]^{p-1}_{n-1}d[a]_{n-1}\cdot V^s([\pi]^{e''}_{n-1-s}) \\
&\quad + [a]^{p-1}_{n-1}[\pi]^e_{n-1}d([a]_{n-1}[\pi]^{e''}_{n-1}).
\end{aligned}$$

The summands in the last two lines lie in $pU^{2e''} = U^{2(e+e'')}$, and by lemma 1.2.3, the sum in the third last line is congruent to $[a]^p_{n-1}d\log_{n-1}(1+\pi^{e''})$ modulo $U^{4e''}$. □

It follows from lemma 2.1.1 that for $m \geq 2$,

$$(3.2.7)\qquad (R-F)^{-1}(U^m W_{n-1}\,\Omega^q_{(V,M_V)}) = U^m W_n\,\Omega^q_{(V,M_V)} + \ker(R-F),$$

and hence, sublemma 3.2.6 implies that

$$f^{-1}_{12*}(e_{13} + U^{4e''}W_{n-1}\Omega^1_{(V,M_V)}) = e_{12} + U^{4e''}W_n\,\Omega^1_{(V,M_V)} + \mathrm{im}(f_{11*}).$$

We next consider the image of this subset by the map

$$g_{12*} = p\colon W_n\,\Omega^1_{(V,M_V)} \to W_n\,\Omega^1_{(V,M_V)},$$

Sublemma 3.2.8. *The subset $g_{12*}(f^{-1}_{12*}(e_{13}+U^{4e''}W_{n-1}\Omega^1_{(V,M_V)}))$ is equal to the subset $d\log_n(1+a(1-\zeta)^p) + U^{2(e'+e'')}W_n\,\Omega^1_{(V,M_V)} + \mathrm{im}(g_{12*}f_{11*})$.*

Proof. We again assume $u = 1$ and recall from lemma 1.2.2 that

$$pU^{4e''}W_n\,\Omega^1_{(V,M_V)} = U^{2(e'+e'')}W_n\,\Omega^1_{(V,M_V)}.$$

Hence, in view of the equation (3.2.7), it will suffice to prove that

$$g_{12*}(e_{12}) \in d\log_n(1+a(1-\zeta)^p) + U^{2(e'+e'')}W_n\,\Omega^1_{(V,M_V)}.$$

To this end, we use that in $W_m(V)$,

$$[\pi]^{e'}_m \equiv p(-\sum_{v=0}^{m-1} V^v([\pi]^{e''}_{m-v})),$$

modulo $W_m(\mathfrak{m}^{2e'})$. If we rewrite

$$e_{12} = \sum_{s=0}^{n-1} dV^s([a]_{n-s}(-\sum_{v=0}^{n-s-1} V^v([\pi]^{e''}_{n-s-v}))),$$

this implies that $g_{12*}(e_{12})$ is congruent, modulo $U^{4e'} W_n\, \Omega^1_{(V,M_V)}$, to the sum

$$\sum_{s=0} dV^s([a]_{n-s}[\pi]^{e'}_{n-s}).$$

Finally, lemma 1.2.3 shows that this sum is congruent, modulo $U^{4e'} W_n\, \Omega^1_{(V,M_V)}$ to the class $e_{22} = d\log_n(a(1-\zeta)^p)$. □

Recall that the map

$$\bar{f}_{21*} \colon \pi_1(E_{21})/\operatorname{im}(f_{23*}) \hookrightarrow \pi_1(E_{22})$$

induced by f_{21*} is identified with the canonical inclusion

$$(W_n\, \Omega^1_{(V,M_V)})^{F=1} \hookrightarrow W_n\, \Omega^1_{(V,M_V)}.$$

We can now conclude that $\bar{f}^{-1}_{21*}(g_{12*}(f^{-1}_{12*}(e_{13})))$ is contained in

$$d\log_n(1 + a(1-\zeta)^p) + U^{2(e'+e'')}(W_n\, \Omega^1_{(V,M_V)})^{F=1} + \operatorname{im}(\bar{g}_{11*}).$$

The image of this set by the map

$$\bar{g}_{21*} \colon \pi_1(E_{21})/\operatorname{im}(f_{23*}) \to \pi_1(E_{31})/\operatorname{im}(g_{21*}f_{23*})$$

is equal to the class of $d\log_n(1 + a(1-\zeta)^p)$, provided that

$$U^{2(e'+e'')}(W_n\, \Omega^1_{(V,M_V)})^{F=1} \subset p(W_n\, \Omega^1_{(V,M_V)})^{F=1}.$$

We shall prove in corollary 3.2.10 below that this is almost true. More precisely, we will show that given $n \geq 1$, there exists $m \geq n$ such that the left hand side is contained in the image of the composite

$$p(W_m\, \Omega^1_{(V,M_V)})^{F=1} \hookrightarrow W_m\, \Omega^1_{(V,M_V)} \xrightarrow{R^{m-n}} W_n\, \Omega^1_{(V,M_V)}.$$

We may then conclude that given $n \geq 1$, there exists $m \geq n$ such that the map of the statement takes $[a]_m \otimes \zeta$ to $d\log_n(1 + a(1-\zeta)^p)$. The proposition follows. □

LEMMA 3.2.9. *The map, which to $x \otimes \zeta$ assigns $xd\log\zeta$, gives an isomorphism of pro-abelian groups*

$$W.(\mathfrak{m}) \otimes \mu_p \xrightarrow{\sim} {}_pU^{2(e''+1)} W.\, \Omega^1_{(V,M_V)}.$$

PROOF. It follows from [**9**, theorem E] that the map

$$W.(V) \otimes \mu_p \xrightarrow{\sim} {}_pW.\, \Omega^1_{(V,M_V)},$$

which to $x \otimes \zeta$ associates $xd\log\zeta$, is an isomorphism of pro-abelian groups. This map factors as the composite

$$W.(V) \otimes \mu_p \xrightarrow{\sim} {}_pU^{2e''} W.\, \Omega^1_{(V,M_V)} \hookrightarrow {}_pW.\, \Omega^1_{(V,M_V)},$$

and since the right hand map is injective, the two maps are necessarily isomorphisms. We wish to conclude that the map of the statement is an isomorphism. To this end, we consider the following diagram with exact collumns.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
W.(\mathfrak{m}) \otimes \mu & \longrightarrow & {}_pU^{2(e''+1)}W.\,\Omega^1_{(V,M_V)} \\
\downarrow & & \downarrow \\
W.(V) \otimes \mu & \xrightarrow{\sim} & {}_pU^{2e''}W.\,\Omega^1_{(V,M_V)} \\
\downarrow & & \downarrow \\
W.(k) \otimes \mu & \longrightarrow & {}_p(U^{2e''}W.\,\Omega^1_{(V,M_V)}/U^{2(e''+1)}W.\,\Omega^1_{(V,M_V)}) \\
\downarrow & & \\
0 & &
\end{array}$$

The top left hand vertical map is injective since for every regular $\mathbb{F}_p$-algebra A and for all $n \geq 1$, the map induced by the restriction ${}_pW_n(A) \to {}_pW_{n-1}(A)$ is equal to zero. It will suffice to show that the lower horizontal map is injective. To prove this, we compose with the inclusion

$${}_p(U^{2e''}W.\,\Omega^1_{(V,M_V)}/U^{2(e''+1)}W.\,\Omega^1_{(V,M_V)}) \hookrightarrow W.\,\Omega^1_{(V,M_V)}/U^{2(e''+1)}W.\,\Omega^1_{(V,M_V)}$$

and use that by lemma 1.2.1,

$$W.\,\Omega^1_{(V,M_V)}/U^{2(e''+1)}W.\,\Omega^1_{(V,M_V)} \xrightarrow{\sim} W.\,\Omega^1_{(V_{e''+1},M_{e''+1})}.$$

The composite map

$$W_n(k) \otimes \mu_p \to W.\,\Omega^1_{(V_{e''+1},M_{e''+1})}$$

takes $a \otimes \zeta$ to $\tilde{a} d\log_n \bar{\zeta}$, where $\tilde{a} \in W_n(V_{e''+1})$ is any lifting of $a \in W_n(k)$. The ring $V_{e''+1}$ is isomorphic to the truncated polynomial ring $k[t]/(t^{e''+1})$, and we can choose the isomorphism such that the induced map of residue fields is the identity map. The image of $\bar{\zeta}$ by this isomorphism has the form $1 + ut^{e''}$, where $u \in k^*$ is a unit. (Since $\zeta \in V_0$, we can even assume that $u \in k_0^*$.) Hence, it follows from lemma 1.2.3 that the composition

$$W_n(k) \otimes \mu_p \to W_n\,\Omega^1_{(V_{e''+1},M_{e''+1})} \xrightarrow{\sim} W_n\,\Omega^1_{(k[t]/(t^{e''+1}),\mathbb{N}_0)}$$

is equal to the $W_n(k)$-linear map that takes $1 \otimes \zeta$ to the sum

$$d\log_n \bar{\zeta} = \sum_{s=0}^{n-1} dV^s([u]_{n-s}[t]^{e''}_{n-s}).$$

The domain of this map is given by

$$W_n(k)/pW_n(k) \otimes \mu_p = W_n(k)/VFW_n(k) \otimes \mu_p = W_n(k)/VW_n(k^p) \otimes \mu_p,$$

and the target is given by proposition A.1.1 below. We must show that if n is sufficiently large, then for all $a \in W_n(k)$, the product

$$\Theta = a d\log_n \bar{\zeta} = (\sum_{r=0}^{n-1} V^r([a_r]_{n-r})) \cdot (\sum_{s=0}^{n-1} dV^s([u]_{n-s}[t]_{n-s}^{e''}))$$

is equal to zero if and only if $a_0 = 0$ and $a_r \in k^p$, for all $1 \leq r < n$. We write $e'' = p^v i$ with i prime to p and proceed to rewrite the summands of $\Theta = \sum \Theta_{r,s}$ in the form of proposition A.1.1. We first note that by the Leibniz rule,

$$\Theta_{r,s} = d(V^r([a_r]_{n-r}) \cdot V^s([u]_{n-s}[t]_{n-s}^{p^v i})) - dV^r([a_r]_{n-r}) \cdot V^s([u]_{n-s}[t]_{n-s}^{p^v i}),$$

for all $0 \leq r, s < n$.

Suppose first that $r > 0$. If $0 \leq s \leq v$ and $s \geq r$, we get

$$\begin{aligned}\Theta_{r,s} &= p^r dV^s([a_r]_{n-s}^{p^{s-r}}[u]_{n-s}) \cdot [t]_n^{p^{v-s}i} \\ &\quad + p^{r+v-s} i V^s([a_r]_{n-s}^{p^{s-r}}[u]_{n-s}) \cdot [t]_n^{p^{v-s}i} d\log_n t \\ &\quad - V^s([a_r]_{n-s}^{p^{s-r}} d\log_{n-s} a_r \cdot [u]_{n-s}) \cdot [t]_n^{p^{v-s}i}.\end{aligned}$$

The first summand on the right is zero, since

$$p^r dV^s([a_r]_{n-s}^{p^{s-r}}[u]_{n-s}) \in \mathrm{Fil}^{s+r} W_n \Omega_k^1$$

and $p^{r+s} \cdot p^{s-v} i \geq e'' + 1$, and the second term is zero for similar reasons. The third term is zero if and only if

$$V^s([a_r]_{n-s}^{p^{s-r}} d\log_{n-s} a_r \cdot [u]_{n-s}) \in \mathrm{Fil}^{s+1} W_n \Omega_k^1,$$

and this happens if and only if $a_r \in k^p$. Indeed, the filtration of the groups $W_n\Omega_k^q$ is known completely by [**12**, proposition I.2.12]. If $0 \leq s \leq v$ and $s < r$, we have

$$\begin{aligned}\Theta_{r,s} &= p^s dV^s(V^{r-s}([a_r]_{n-r})[u]_{n-s}) \cdot [t]_n^{p^{v-s}i} \\ &= p^v i V^s(V^{r-s}([a_r]_{n-r})[u]_{n-s}) \cdot [t]_n^{p^{v-s}i} d\log_n t \\ &\quad - V^s(dV^{r-s}([a_r]_{n-r})[u]_{n-s}) \cdot [t]_n^{p^{v-s}i},\end{aligned}$$

and all three terms are zero, since $p^r \cdot p^{v-s} i \geq e'' + 1$. If $0 \leq v < s$ and $s \geq r$,

$$\begin{aligned}\Theta_{r,s} &= p^r dV^{s-v}(V^v([a_r]_{n-s}^{p^{s-r}}[u]_{n-s}) \cdot [t]_{n-s+v}^i) \\ &\quad - V^{s-v}(V^v([a_r]_{n-s}^{p^{s-r}} d\log_{n-s} a_r \cdot [u]_{n-s}) \cdot [t]_{n-s+v}^i).\end{aligned}$$

The first term is zero, since $p^{v+r} i \geq e'' + 1$, and the second term is equal to zero, if and only if $a_r \in k^p$. Finally, if $0 \leq v < s$ and $s < r$, we have

$$\begin{aligned}\Theta_{r,s} &= p^s dV^{s-v}(V^v(V^{r-s}([a_r]_{n-r})[u]_{n-s}) \cdot [t]_{n-s+v}^i) \\ &\quad - V^{s-v}(V^v(dV^{r-s}([a_r]_{n-r})[u]_{n-s}) \cdot [t]_{n-s+v}^i),\end{aligned}$$

and both terms are zero, since $p^{r-s+v} i \geq e'' + 1$.

We next evaluate the remaining summands $\Theta_{0,s}$. If $0 \leq s \leq v$, we have

$$\begin{aligned}\Theta_{0,s} &= dV^s([a_0]_{n-s}^{p^s}[u]_{n-s}) \cdot [t]_n^{p^{v-s}i} \\ &\quad + p^{v-s} i V^s([a_0]_{n-s}^{p^s}[u]_{n-s}) \cdot [t]_n^{p^{v-s}i} d\log_n t \\ &\quad - V^s([a_0]_{n-s}^{p^s} d\log_{n-s} a_0 \cdot [u]_{n-s}) \cdot [t]_n^{p^{v-s}i}.\end{aligned}$$

The first term is zero if and only if $s = 0$ and $a_0 \in k^p$, the second term is zero if and only if $s < v$, and the last term is zero if and only if $a_0 \in k^p$. Finally, if $s > v$,

$$\begin{aligned}\Theta_{0,s} &= dV^{s-v}(V^v([a_0]_{n-s}^{p^s}[u]_{n-s}) \cdot [t]_{n-s+v}^i) \\ &\quad - V^{s-v}(V^v([a_0]_{n-s}^{p^s} d\log_{n-s} \cdot [u]_{n-s}) \cdot [t]_{n-s+v}^i).\end{aligned}$$

The first term is zero if and only if a_0 is zero, and the second term is zero if and only if $a_0 \in k^p$.

We can now show that for $n > v$, the product

$$\Theta = a d\log_n \bar{\zeta} \in W_n\,\Omega^1_{(k[t]/(t^{e''+1}),\mathbb{N}_0)}$$

is equal to zero if and only if $a_0 = 0$ and $a_r \in k^p$, for all $1 \le r < n$, as desired. To this end we use the direct sum decomposition of the de Rham-Witt group on the left exhibited by proposition A.1.1 below. Suppose first that $1 \le r < n$. Then $\Theta_{r,s} = 0$ if and only if $r > s$ or $r \le s$ and $a_r \in k^p$. Suppose that $r \le s$. Then the element $\Theta_{r,s}$ belongs to the direct summand $V^s(W_{n-s}\Omega_k^1) \cdot [t]_n^{p^{v-s}i}$, if $s \le v$, and to the direct summand $V^{s-v}(W_{n-s+v}\Omega_k^q \cdot [t]_{n-s+v}^i)$, if $s > v$. In particular, two non-zero elements $\Theta_{r,s}$ and $\Theta_{r',s'}$ belong to the same summand if and only if $s = s'$. It follows that the sum

$$\sum_{r=1}^{n-1}\sum_{s=0}^{n-1}\Theta_{r,s} = \sum_{r=1}^{n-1}\sum_{s=r}^{n-1}\Theta_{r,s}$$

is equal to zero if and only if $a_r \in k^p$, for all $1 \le r < n$. A similar argument shows that no cancellation can occur between the elements $\Theta_{r,s}$, $1 \le r \le s < n$, and the elements $\Theta_{0,s'}$, $0 \le s' < n$. Finally, $\Theta_{0,s}$ is non-zero, if a_0 is non-zero and $s \ge v$. This completes the proof. □

COROLLARY 3.2.10. *The map induced from multiplication by p,*

$$p\colon U^{2(e''+1)}(W.\,\Omega^1_{(V,M_V)})^{F=1} \xrightarrow{\sim} U^{2(e'+1)}(W.\,\Omega^1_{(V,M_V)})^{F=1},$$

is an isomorphism of pro-abelian groups.

PROOF. We abbreviate $E^q_\cdot = W.\,\Omega^q_{(V,M_V)}$ and consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^{2(e''+1)}(E^1_\cdot)^{F=1} & \longrightarrow & U^{2(e''+1)}E^1_\cdot & \xrightarrow{1-F} & U^{2(e''+1)}E^1_\cdot & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & U^{2(e'+1)}(E^1_\cdot)^{F=1} & \longrightarrow & U^{2(e'+1)}E^1_\cdot & \xrightarrow{1-F} & U^{2(e'+1)}E^1_\cdot & \longrightarrow & 0 \end{array}$$

It follows from lemma 1.2.2 that the middle and right hand vertical maps, which are induced by multiplication by p, are well-defined and surjective. The left hand vertical map is the induced map of kernels of the maps $1 - F$. This is the map of the statement. The horizontal maps $1-F$ induce a map between the kernels of the middle and right hand vertical maps,

$$1 - F\colon {}_pU^{2(e''+1)}E^1_\cdot \xrightarrow{\sim} {}_pU^{2(e''+1)}E^1_\cdot,$$

and lemma 3.2.9 shows that this map is an isomorphism of pro-abelian groups. Indeed, the map $1 - F\colon \bar{W}.(\mathfrak{m}) \to \bar{W}.(\mathfrak{m})$ is an isomorphism, since the geometric series $1 + F + F^2 + \dots$ converges. The corollary follows. □

Appendix A. Truncated polynomial algebras

A.1. In this appendix, we give an explicit formula for the de Rham-Witt complex of a truncated polynomial algebra in terms of the de Rham-Witt complex of the coefficient ring. The formula is derived from the corresponding formula, proved in [**9**, theorem B], for the de Rham-Witt complex of a polynomial algebra, and it generalizes the formula of the thesis of Kåre Nielsen [**18**], where the case with coefficient ring $\mathbb{F}_p$ was considered.

Let A be a $\mathbb{Z}_{(p)}$-algebra with $p \neq 2$, and let $A[t]$ be the polynomial algebra in one variable with the pre-log structure $\alpha\colon \mathbb{N}_0 \to A$, $\alpha(i) = t^i$. One can show as in [**9**, theorem B] that every element $\omega^{(n)} \in W_n\,\Omega^q_{(A[t],\mathbb{N}_0)}$ can be written uniquely

$$\omega^{(n)} = \sum_{i\in\mathbb{N}_0} (a^{(n)}_{0,i}[t]^i_n + b^{(n)}_{0,i}[t]^i_n d\log_n t) + \sum_{s=1}^{n-1}\sum_{i\in I_p} (V^s(a^{(n-s)}_{s,i}[t]^i_{n-s}) + dV^s(b^{(n-s)}_{s,i}[t]^i_{n-s}))$$

where $a^{(m)}_{s,i} \in W_m\Omega^q_A$ and $b^{(m)}_{s,i} \in W_m\Omega^{q-1}_A$, and where I_p denotes the set of positive integers prime to p. The formulas for the product, differential, and Frobenius and Verschiebung operators may be found in *op. cit.*, §4.2. We now fix an integer $N \geq 1$ and consider the subgroup

$$I^q_n \subset W_n\,\Omega^q_{(A[t],\mathbb{N}_0)}$$

of those elements $\omega^{(n)}$ such that $a^{(m)}_{s,i} \in \mathrm{Fil}^v\, W_m\Omega^q_A$ and $b^{(m)}_{s,i} \in \mathrm{Fil}^v\, W_m\Omega^{q-1}_A$, for some $0 \leq v < m$ with $p^v i \geq N$. We consider the ring $A[t]/(t^N)$ with the induced pre-log structure. The following result expresses $W_n\,\Omega^q_{(A[t]/(t^N),\mathbb{N}_0)}$ as a direct sum of groups $W_{m-v}\,\Omega^q_A$ and $W_{m-v}\,\Omega^{q-1}_A$.

Proposition A.1.1. *The canonical projection induces an isomorphism*

$$W_n\,\Omega^q_{(A[t],\mathbb{N}_0)}/I^q_n \xrightarrow{\sim} W_n\,\Omega^q_{(A[t]/(t^N),\mathbb{N}_0)}.$$

Proof. We see as in the proof of lemma 1.2.1 above that it suffices to show that I^*_n is a differential graded ideal with $W_n((t^N)) \subset I^0_n$ and that if J^*_n is another differential graded ideal with $W_n((t^N)) \subset J^0_n$, then $I^*_n \subset J^*_n$. We leave the former statement to the reader and prove the latter. We first show that elements of the form $V^s(a^{(n-s)}_{s,i}[t]^i_{n-s})$, where $a \in \mathrm{Fil}^v\, W_{n-s}\Omega^q_A$, for some $0 \leq v < n-s$ with $p^v i \geq N$, are contained in J^q_n. By definition of the standard filtration,

$$a^{(n-s)}_{s,i} = V^v(\omega) + dV^v(\omega')$$

for some $\omega \in W_{n-s-v}\Omega^q_A$ and $\omega' \in W_{n-s-v}\Omega^{q-1}_A$, and hence $V^s(a^{(n-s)}_{s,i}[t]^i_{n-s})$ is equal to the sum

$$V^{s+v}(\omega[t]^{p^v i}_{n-s-v}) + p^s dV^{s+v}(\omega'[t]^{p^v i}_{n-s-v}) - iV^{s+v}(\omega'[t]^{p^v i}_{n-s-v}\, d\log t).$$

We consider the left hand term. By [**9**, theorem A], the canonical map

$$\Omega^q_{W_n(A)} \to W_n\,\Omega^q_A$$

is surjective. This shows that ω can be written as a sum of elements of the form $x_0 dx_1 \dots dx_q$, where $x_0, \dots, x_q \in W_{n-s-v}(A)$. But

$$V^{s+v}(x_0 dx_1 \dots dx_q [t]_{n-s-v}^{p^v i}) = V^{s+v}(x_0 [t]_{n-s-v}^{p^i v}) dV^{s+v}(x_1) \dots dV^{s+v}(x_q),$$

which is contained in J_n^q, and hence $V^{s+v}(\omega[t]_{n-s-v}^{p^i v})$, too, is contained in J_n^q. One shows in a similar manner that $dV^{s+v}(\omega'[t]_{n-s-v}^{p^v i})$ and $V^{s+v}(\omega'[t]_{n-s-v}^{p^v i}\, d\log t)$ are contained in J_n^q. Hence $V^s(a_{s,i}^{(n-s)}[t]_{n-s}^i)$ is contained in J_n^q. The remaining cases are treated in a completely analogous manner. □

References

[1] M. Artin and B. Mazur, *Étale homotopy*, Lecture Notes in Mathematics, vol. 100, Springer-Verlag, 1969.

[2] A. A. Beilinson, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer-Verlag, 1987, pp. 1–25.

[3] S. Bloch and K. Kato, *p-adic etale cohomology*, Publ. Math. IHES **63** (1986), 107–152.

[4] M. Bökstedt, W.-C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 465–540.

[5] T. Geisser and L. Hesselholt, *On the K-theory and topological cyclic homology of smooth schemes over a discrete valuation ring*, Preprint 2001.

[6] ———, *On the K-theory of complete regular local $\mathbb{F}_p$-algebras*, Preprint 2000.

[7] ———, *Topological cyclic homology of schemes*, K-theory (Seattle, 1997), Proc. Symp. Pure Math., vol. 67, 1999, pp. 41–87.

[8] L. Hesselholt, *Algebraic K-theory and trace invariants*, Proceedings of the International Congress of Mathematicians (Beijing, 2002), World Scientific, Beijing, China, 2002.

[9] L. Hesselholt and I. Madsen, *On the de Rham-Witt complex in mixed characteristic*, Preprint 2002.

[10] ———, *On the K-theory of local fields*, Ann. Math. (to appear).

[11] O. Hyodo and K. Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, Périodes p-adiques, Asterisque, vol. 223, 1994, pp. 221–268.

[12] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. Éc. Norm. Sup. (4) **12** (1979), 501–661.

[13] B. Kahn, *Deux théorèmes de comparaison en cohomologie étale: applications*, Duke Math. J. **69** (1993), 137–165.

[14] K. Kato, *Galois cohomology of complete discrete valuation fields*, Algebraic K-theory, Part II (Oberwolfach, 1980), Lecture Notes in Math., vol. 967, Springer, Berlin-New York, 1982, pp. 215–238.

[15] ———, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory, Proceedings of the JAMI Inaugural Conference (Baltimore, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224.

[16] S. Lichtenbaum, *Values of zeta-functions at non-negative integers*, Number theory, Lecture Notes in Math., vol. 1068, Springer-Verlag, 1983, pp. 127–138.

[17] R. McCarthy, *Relative algebraic K-theory and topological cyclic homology*, Acta Math. **179** (1997), 197–222.

[18] K. Nielsen, *On trace homology and algebraic K-theory of truncated polynomial algebras*, Thesis, Aarhus University, 2001.

[19] I. A. Panin, *On a theorem of Hurewicz and K-theory of complete discrete valuation rings*, Math. USSR Izvestiya **29** (1987), 119–131.

[20] A. A. Suslin, *On the K-theory of local fields*, J. Pure Appl. Alg. **34** (1984), 304–318.

[21] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), 233–411.

University of Southern California, Los Angeles, California
E-mail address: geisser@math.usc.edu

Massachusetts Institute of Technology, Cambridge, Massachusetts
E-mail address: larsh@math.mit.edu