

Hyperbolicity of the non-linear models of Maxwell's equations

Denis Serre*

ENS Lyon, UMPA (UMR 5669 CNRS),
46, allée d'Italie, F-69364 Lyon Cedex 07.

April 17, 2003

Augmented systems of conservation laws

Consider a system of conservation laws

$$(1) \quad \partial_t u + \sum_{\alpha=1}^d \partial_\alpha f^\alpha(u) = 0, \quad t > 0, x \in \mathbb{R}^d,$$

where $u(x, t) \in \mathbb{R}^n$ is the unknown and f^α are given smooth fluxes. Recall that the existence of an additional conservation law

$$(2) \quad \partial_t \eta(u) + \sum_{\alpha} \partial_\alpha q^\alpha(u) = 0,$$

compatible with (1), and where η is a (scalar) function, strictly convex in the sense that $D^2\eta > 0$, ensures that (1) is hyperbolic in the direction of the time t . See [7, 9, 11] for instance. Actually, it allows to symmetrize (1) in the form

$$(3) \quad A^0(z) \partial_t z + \sum_{\alpha} A^\alpha(z) \partial_\alpha z = 0,$$

where $z := d\eta(u)$ is the “dual variable”. Symmetrization means that the matrices $A^0(z)$, $A^\alpha(z)$ are symmetric, the first one being positive definite. As is well-known, the symmetric form (3) has nice consequences for the Cauchy problem (see [7] for instance):

- i. Given an initial data of class $H^s(\mathbb{R}^d)$ (actually, uniformly locally in H^s is sufficient) with $s > 1 + d/2$ (which ensures that $H^s \subset C^1$), there exists a positive time T and a unique classical solution in the strip $(0, T) \times \mathbb{R}^d$.

*This research was funded in part by the EC as contract HPRN-CT-2002-00282. It was accomplished when the author was visiting the Isaac Newton Institute for Mathematical Sciences (Cambridge, UK) within the program “Nonlinear hyperbolic waves in phase dynamics and astrophysics”.

- ii. The uniqueness holds in the following stronger sense: The classical solution, when it exists, coincide with every weak entropy solutions. The latter are essentially bounded fields which satisfy (1) in the distributional sense, together with the “entropy inequality”

$$(4) \quad \partial_t \eta(u) + \sum_{\alpha} \partial_{\alpha} q^{\alpha}(u) \leq 0.$$

Obviously, the convexity of the entropy (which turns out to be the mechanical energy in isothermal models) is only a sufficient condition for hyperbolicity, but not a necessary one. It has been well-known for a long time that the mechanical energy of a hyperelastic material cannot be convex (see [4].) This observation led C. Dafermos [7] to the following procedure (see also [8].)

Assume that the system (1) is compatible with some special conservation laws, where the conserved quantities will be denoted by the vector P :

$$(5) \quad \partial_t P(u) + \sum_{\alpha} \partial_{\alpha} \pi^{\alpha}(u) = 0.$$

Although the components of P play a role very similar to that of η , we are reluctant to call them entropies. It turns out that the conservation laws (5) do not always depend on the equation of state of the underlying medium. On the contrary, it is common in thermodynamics that the knowledge of the entropy η determines completely the equation of state.

Assume now that the, *a priori* non convex, entropy can be rewritten as a strictly convex function of $(u_1, \dots, u_n, P_1, \dots, P_r)$:

$$\eta(u) = \phi(u_1, \dots, u_n, P_1, \dots, P_r), \quad D^2 \phi > 0.$$

Then we are tempted to increase the number of unknowns as well as of equations by writing a system

$$(6) \quad \partial_t u + \sum_{\alpha} \partial_{\alpha} f^{\alpha}(u) = 0,$$

$$(7) \quad \partial_t P + \sum_{\alpha} \partial_{\alpha} \pi^{\alpha}(u) = 0,$$

expecting that ϕ is an entropy of the resulting system. If it were the case, then the augmented system would be symmetrizable and the existence and uniqueness properties mentioned above would apply. Furthermore, the new system contains (1) in the sense that if an initial data (u^0, P^0) satisfies $P^0 \equiv P(u^0)$, then the classical solution will satisfy identically $P \equiv P(u)$, and therefore u will be a classical solution of (1) with data u^0 . Hence the existence and uniqueness properties hold also for (1).

The situation is not so simple however, because ϕ is not in general an entropy of the augmented system ! It is clear that there is not a unique way (if any) to write η as a convex function of (u, P) , and one needs to find one such function that is an entropy. Also, the fluxes f^{α} and π^{α} may need to be rewritten as functions of (u, P) instead of u only,

and they do in general. For this reason, there does not exist yet a satisfactory theory of augmentation of systems of conservation laws. We content ourselves to treat each system of physical interest on a case-by-case basis. This is what Dafermos did for the system governing the motion of a hyperelastic material. The purpose of the present note is to give a convenient treatment of the non-linear models of electromagnetism. Our work has been partly influenced by that of Y. Brenier [3], who treated the special case of the Born–Infeld model. We emphasize that, thanks to the very special structure of the Born–Infeld model, Brenier could extend it to a rather simple system of ten equations/unknowns, with pretty accurate information such as the knowledge of the wave velocities, while in the general case, our extension consists in nine equations with a pretty involved nonlinear structure. In particular, our work does not contain that of Brenier.

The Coleman–Dill model for electromagnetism

We place ourselves in the context of an electromagnetic field (E, B) obeying to a non-linear system of conservation laws. The ambient space is \mathbb{R}^3 ($d = 3$.) The law of Faraday $\partial_t B + \text{curl} E = 0$ must be completed by the Ampère’s law $\partial_t D - \text{curl} H = 0$ (assuming for simplicity that there are neither charges nor currents.) In standard Maxwell’s equations, the equations of state are linear: $D = \epsilon E$ and $H = \mu B$, where ϵ and μ are constant symmetric tensors (often scalars.)

There are several reasons for dropping the standard, linear Maxwell’s equations. On the one hand, there are media where the equations of state become non-linear. On the other hand, the electric field in the vacuum grows like r^{-2} near a punctual charge, and this growth is responsible for an infinite total energy ! Several corrections of the electromagnetic theory have been made to resolve this contradiction, and one of them was to postulate a non-linear energy density which forces the electromagnetic field to remain finite, though behaving like r^{-2} in the far field, which is harmless. The most famous model in this respect is certainly that designed by M. Born & L. Infeld [2].

A general theory of non-linear electromagnetic models is due to B. D. Coleman & E. H. Dill. It assumes that the model be compatible with some energy conservation, where the stored energy has the form of a smooth function $W(B, D)$. Then, taking the conserved quantities B and D as primary variables, the equations of state read

$$E_i := \frac{\partial W}{\partial D_i}, \quad H_i := \frac{\partial W}{\partial B_i}.$$

hence the system

$$(8) \quad \partial_t B + \text{curl} \frac{\partial W}{\partial D} = 0, \quad \partial_t D - \text{curl} \frac{\partial W}{\partial B} = 0 \quad \text{div} B = \text{div} D = 0.$$

This system is compatible with the energy conservation law

$$(9) \quad \partial_t W + \text{div}(E \times H) = 0.$$

The energy flux $E \times H$ is often called the ‘‘Poynting vector’’. When W is a convex function of $u := (B, D)$, we conclude as usual that the system (8) is symmetrizable hyperbolic, and therefore that local existence and uniqueness properties hold.

However, it is not always the case that W is convex. For instance, in the Born–Infeld model, one has

$$W_{BI}(B, D) := \sqrt{1 + \|B\|^2 + \|D\|^2 + \|B \times D\|^2},$$

which fails to be convex far away from the origin, though (8) remains hyperbolic as is it well-known. When W is not convex, the theorems of Chapter 5 in [7] do not apply. Even Theorems 5.3.1 and 5.3.2, which deal with systems with linear differential constraints (called ‘‘involutions’’), do not apply, since they require that the entropy (here the energy) of the system be strictly convex in the directions of the ‘‘involution cone’’ \mathcal{C} . In our context, the involutions are $\operatorname{div} B = 0$ and $\operatorname{div} D = 0$, hence the cone is \mathbb{R}^6 , meaning that the convexity along the cone is the usual convexity. Hence the local existence and the stability of classical solutions remain open questions when W fails to be convex.

Remark: In the relativistic formalism, the Ampère’s law is viewed as the Euler–Lagrange equation of a Lagrangian $\mathcal{L}(B, E) = L(\|E\|^2 - \|B\|^2, E \cdot B)$, constrained by Faraday’s law. This gives the relations

$$D := \frac{\partial \mathcal{L}}{\partial E}, \quad H := -\frac{\partial \mathcal{L}}{\partial B}, \quad W = D \cdot E - \mathcal{L}.$$

More precisely,

$$(10) \quad W(B, D) = \sup_{e \in \mathbb{R}^3} \{D \cdot e - \mathcal{L}(B, e)\}.$$

The fact that \mathcal{L} depends only on two scalar quantities $\|E\|^2 - \|B\|^2$ and $E \cdot B$ is due to the invariance under Lorentz transformations: The electro-magnetic field must be viewed as the 2-differential form $\Omega_{EM} := dt \wedge (E \cdot dx) + dx \wedge (B \times dx)$. The invariants of differential forms of degree two under the action of the Lorentz group $\mathbf{O}(1, 3)$ turn out to be the above quantities. In this formulation, it is desirable to express the convexity of W in terms of a property of L .

The extra dependent variable

The key observation is that, for physically relevant solutions, that is those satisfying the natural constraints $\operatorname{div} B = \operatorname{div} D = 0$, the vector $P := B \times D$ obeys to some conservation law

$$(11) \quad \partial_i P_i + \operatorname{div}(E_i D + H_i B) + \partial_i(W - E \cdot D - H \cdot B) = 0.$$

It is amazing that equation (11) is not any more in conservation form when B or D fails to be solenoidal. This fact resembles much the case of a hyperelastic material, where extra conservation laws hold only when the tensor part of the unknown is a deformation tensor. In both situations, the constraints have the form $Lu = 0$ where L is a linear differential

operator in the space variable, which are compatible with the evolution in the sense that, if they are satisfied at initial time, then they persist when time increases.

We point out an important difference however, in that (11) does involve W itself. Therefore, its Rankine–Hugoniot conditions are usually not compatible with those of (8) once shock waves develop. An exception to this flaw is the Born–Infeld model, since its characteristic fields are linearly degenerate.

The advantage of augmenting the system becomes clear in the Born–Infeld case. Then the energy density W becomes a convex function of (B, D, P) , when written in the form

$$\sqrt{1 + \|B\|^2 + \|D\|^2 + \|P\|^2}.$$

There remains however to find a new way to write the fluxes in (8, 11), in such a way that the above function be an entropy of the augmented system. More precisely, what we need is the following. Given a convex function $\phi(B, D, P)$ that coincides with W on the “equilibrium” submanifold

$$\Sigma := \{(B, D, B \times D); B, D \in \mathbb{R}^3\},$$

find a system

$$(12) \quad \partial_t B + \operatorname{curl} E = 0, \quad \operatorname{div} B = 0,$$

$$(13) \quad \partial_t D - \operatorname{curl} H = 0, \quad \operatorname{div} D = 0,$$

$$(14) \quad \partial_t P + \operatorname{Div} T = 0,$$

where

- i. $E = E(B, D, P)$, $H = H(B, D, P)$ and $T = T(B, D, P)$ coincide, on Σ , with $\partial W / \partial D$, $\partial W / \partial B$ and $E \otimes D + H \otimes B + (W - E \cdot D - H \cdot B)I_3$ respectively,

- ii. ϕ is an entropy of the resulting system.

Of course, the second point is the difficult one. Once this program is achieved, we may apply the local existence and uniqueness properties to (12, 13, 14) and, whenever $P \equiv B \times D$ holds at initial time, this remains true for every time. In the latter situation, (B, D) is a classical solution to (8).

The augmented system

As mentioned above, there is not yet a systematic method for solving the above program. Thus we give the system that fits the above requirements, without convincing explanations. To begin with, the chain rule suggests natural equations of state for E and H :

$$(15) \quad E = \frac{\partial \phi}{\partial D} - B \times \frac{\partial \phi}{\partial P}, \quad H = \frac{\partial \phi}{\partial B} + D \times \frac{\partial \phi}{\partial P}.$$

There remains to choose $T(B, D, P)$ in an appropriate way and this is the less clear point. The following choice works:

$$(16) \quad T(B, D, P) := \frac{\partial\phi}{\partial B} \otimes B + \frac{\partial\phi}{\partial D} \otimes D - P \otimes \frac{\partial\phi}{\partial P} + \left(\phi - B \cdot \frac{\partial\phi}{\partial B} - D \cdot \frac{\partial\phi}{\partial D} - P \cdot \frac{\partial\phi}{\partial P} \right) I_3.$$

The fact that T coincides with

$$T_0 := E \otimes D + H \otimes B + (W - E \cdot D - H \cdot B) I_3$$

on the equilibrium manifold $P = B \times D$ is tricky. It involves the following crucial identity for vectors $X, Y, Z \in \mathbb{R}^3$:

$$(X \times Y) \otimes Z + (Y \times Z) \otimes X + (Z \times X) \otimes Y = \det(X, Y, Z) I_3.$$

Last but not least, one obtains the following entropy identity:

$$(17) \quad \partial_t \phi(B, D, P) + \operatorname{div}(E \times H) = \operatorname{div} \left(\left((P - B \times D) \cdot \frac{\partial\phi}{\partial P} \right) \frac{\partial\phi}{\partial P} \right).$$

Notice that the right-hand side in (17) vanishes identically when the solution comes from a solution of (8). Then (17) reduces to (9) as expected.

To summarize, we have built a system (12, 13, 14) of nine conservation laws in nine unknowns, where the equations of state are (15, 16). We call it the ‘‘augmented system’’. It is endowed with the entropy ϕ , meaning that it is formally compatible with (17). We therefore may apply Theorem 5.1.1 of [7]:

Theorem 1 *Assume that the function $U := (B, D, P) \mapsto \phi$ is strictly convex, that is $D^2\phi > 0$, and smooth enough. Assume a $\mathcal{C}^1(\mathbb{R}^3)$ -initial data U^0 that takes values in some compact subset \mathcal{O} of \mathbb{R}^9 , and such that $\nabla U^0 \in H^s$ for some $s > 3/2$. Then there exists $\tau > 0$ and a unique \mathcal{C}^1 -solution U of the initial-value problem of the augmented system for $0 \leq t < \tau$. Furthermore,*

$$\nabla_{x,t} U \in \mathcal{C}^0([0, \tau); H^s(\mathbb{R}^3)).$$

Since the equation of state coincide with that of the Maxwell’s equation on the equilibrium manifold, we have the following corollary, which we prove by choosing $P^0 := B^0 \times D^0$.

Theorem 2 *Assume that the function $(B, D, P) \mapsto \phi$ is strictly convex, that is $D^2\phi > 0$, and smooth enough. Assume a $\mathcal{C}^1(\mathbb{R}^3)$ -initial data $V^0 = (B^0, D^0)$ that takes values in some compact subset \mathcal{O} of \mathbb{R}^6 , and such that $\nabla V^0 \in H^s$ for some $s > 3/2$. Then there exists $\tau > 0$ and a unique \mathcal{C}^1 -solution V of the initial-value problem of the Maxwell’s equations (8) for $0 \leq t < \tau$. Furthermore,*

$$\nabla_{x,t} V \in \mathcal{C}^0([0, \tau); H^s(\mathbb{R}^3)).$$

We warn the reader that weak entropy solutions of (8) do not solve (12, 13, 14) in general, because the Rankine–Hugoniot relations of (11) are not compatible with those of (8). This phenomenon is studied in greater details below. For the moment, let us say that it prevents to transfer the weak-strong uniqueness property (Theorem 5.2.1 in [7]) from (12, 13, 14) to (8). Hence the augmentation of Maxwell’s system resolves the local existence question and the uniqueness within classical solutions, but not the weak-strong uniqueness. For classical solutions, we have:

Theorem 3 *Assume that the function $(B, D, P) \mapsto \phi$ is strictly convex, that is $D^2\phi > 0$, and smooth enough.*

Suppose V and \bar{V} are classical solutions of Maxwell’s equations (8) on $[0, \tau)$, taking values in a compact subset \mathcal{O} of \mathbb{R}^6 , with initial data V^0 and \bar{V}^0 . Then

$$\int_{|x|<R} \|V(x, t) - \bar{V}(x, t)\|^2 dx \leq ae^{bt} \int_{|x|<R+Mt} \|V^0(x) - \bar{V}^0(x)\|^2 dx$$

holds for any $R > 0$ and $t \in [0, \tau)$, with positive constants a, b and M that depend only on \mathcal{O} , except for b , which depends also on the Lipschitz constants of the solutions.

Evolution of $P - B \times D$. One checks easily that $\delta := P - B \times D$ satisfies the evolution equation

$$(18) \quad \partial_t \delta = \left(\frac{\partial \phi}{\partial P} \cdot \nabla \right) \delta + \left(\operatorname{div} \frac{\partial \phi}{\partial P} \right) \delta + \left(\nabla \frac{\partial \phi}{\partial P} \right) \delta,$$

where in the last term $(\nabla X)\delta$ stands for the vector of components $(\partial_\alpha X) \cdot \delta$. Equation (18) confirms that the augmented system is compatible with the nonlinear Maxwell’s system, in the following sense:

- i. Given a classical solution (B, D) of (8), then $(B, D, B \times D)$ is a classical solution of the augmented system,
- ii. Given a classical solution (B, D, P) of the augmented system that satisfies $P = B \times D$ at initial time (Maxwell-type initial data), then $P \equiv B \times D$ remains true for positive time and (B, D) is a solution of (8).

Compatibility of the Rankine–Hugoniot relations

We prove here what we claimed in the previous sections.

Theorem 4 *Let (B, D) be a piecewise smooth solution of the Maxwell’s system (8). Hence $(B, D, P := B \times D)$ is a solution of the augmented system, except perhaps across discontinuities.*

Assume moreover that $(B, D, P = B \times D)$ satisfies the jump relation for (14) (this means that it is a weak solution of the augmented system.) Then (B, D) also satisfies the Poynting equation (9)

In other words, a field that satisfies the augmented system and that keeps $P \equiv B \times D$ does not have dissipative shocks. We expect that its discontinuities are contacts.

Proof.

Let $(B, D, B \times D)$ be a discontinuous solution of the augmented system. Obviously, (B, D) is a weak solution of the Maxwell system. Consider a discontinuity across a smooth hypersurface. We denote by ν the unit normal to the surface, and σ its normal velocity. The Rankine–Hugoniot relations for (12, 13, 14) are

$$\begin{aligned}\sigma[B] &= -[E \times \nu], \\ \sigma[D] &= [H \times \nu], \\ \sigma[P] &= [(D \cdot \nu)\phi_D + (B \cdot \nu)\phi_B - (\phi_P \cdot \nu)P] \\ &\quad + [\phi - B \cdot \phi_B - D \cdot \phi_D - P \cdot \phi_P]\nu.\end{aligned}$$

Starting from these identities, plus the fact that $P \equiv B \times D$, we have to show that the jump condition $[E \times H] \cdot \nu = \sigma[\phi]$ associated to (9) holds true.

As usual, $[g] := g^+ - g^-$ is the jump of a quantity g . We shall also use the notation

$$\langle g \rangle := \frac{1}{2}(g^+ + g^-).$$

We point out that, for every bilinear map Q , there holds

$$[Q(g, h)] = Q([g], \langle h \rangle) + Q(\langle g \rangle, [h]).$$

To begin with, we eliminate the derivatives ϕ_B and ϕ_D by using the equations of state, and the vector P by using the assumption. This yields to the following form of the third jump relation:

$$\begin{aligned}\sigma[B \times D] &= [(D \cdot \nu)E + (B \cdot \nu)H] + [\phi - B \cdot H - D \cdot E + (B \times D) \cdot \phi_P]\nu \\ &\quad + [(D \cdot \nu)B \times \phi_P + (B \cdot \nu)\phi_P \times D + (\phi_P \cdot \nu)D \times B].\end{aligned}$$

Because of circular symmetry, the brackets in the last line equals $[\det(D, B, \phi_P)]\nu$. Hence there remains

$$\sigma[B \times D] = [(D \cdot \nu)E + (B \cdot \nu)H] + [\phi - B \cdot H - D \cdot E]\nu.$$

Let us develop the bilinear terms. First of all:

$$\sigma[B \times D] = \sigma([B] \times \langle D \rangle + \langle B \rangle \times [D]).$$

Together with the Rankine–Hugoniot relations, that gives

$$\sigma[B \times D] = \langle B \rangle \times [H \times \nu] + \langle D \rangle \times [E \times \nu].$$

Next,

$$[(D \cdot \nu)E] = \langle D \cdot \nu \rangle [E] + [D \cdot \nu] \langle E \rangle,$$

and similarly

$$[(B \cdot \nu)H] = \langle B \cdot \nu \rangle [H] + [B \cdot \nu] \langle H \rangle.$$

Using then the formula

$$(19) \quad X \times (Y \times Z) = (X \cdot Z)Y - (X \cdot Y)Z,$$

there comes

$$[\phi - B \cdot H - D \cdot E]\nu = -(\langle B \rangle \cdot [H])\nu - (\langle D \rangle \cdot [E])\nu - [B \cdot \nu] \langle H \rangle - [D \cdot \nu] \langle E \rangle.$$

Developing again, we obtain

$$[\phi]\nu = ([B] \cdot \langle H \rangle)\nu + ([D] \cdot \langle E \rangle)\nu - [B \cdot \nu] \langle H \rangle - [D \cdot \nu] \langle E \rangle.$$

Then using again Formula (19), we have

$$[\phi]\nu = [B] \times (\nu \times \langle H \rangle) + [D] \times (\nu \times \langle E \rangle).$$

We multiply by σ and use again the Rankine–Hugoniot relation, to end with the equivalent relation

$$\sigma[\phi]\nu = [E \times \nu] \times \langle H \times \nu \rangle + \langle E \times \nu \rangle \times [H \times \nu].$$

This exactly means that

$$\sigma[\phi]\nu = [(E \times \nu) \times (H \times \nu)],$$

or in other words

$$(\sigma[\phi] + [H \times E] \cdot \nu)\nu = 0,$$

which implies the desired identity.

QED

Examining these calculations, we see that we have proved the following. For every discontinuous field that satisfies

$$(20) \quad \sigma[B] = -[E \times \nu], \quad \sigma[D] = [H \times \nu]$$

plus the equations of state, there holds

$$\begin{aligned} \sigma^2[B \times D] &= \sigma[(D \cdot \nu)\phi_D + (B \cdot \nu)\phi_B - (\phi_P \cdot \nu)B \times D] \\ &\quad - \sigma[B \cdot \phi_B + D \cdot \phi_D + (B \times D) \cdot \phi_P]\nu + ([E \times H] \cdot \nu)\nu. \end{aligned}$$

Assume now that (B, D) is an *admissible* weak solution of Maxwell's system, meaning that it satisfies (20), together with the “entropy” inequality

$$\epsilon := \sigma[\phi] - [E \times H] \cdot \nu \geq 0.$$

Then we derive, denoting $P := B \times D$,

$$(21) \quad \begin{aligned} \sigma^2[P] &= \sigma[(D \cdot \nu)\phi_D + (B \cdot \nu)\phi_B - (\phi_P \cdot \nu)P] \\ &\quad + \sigma[\phi - B \cdot \phi_B - D \cdot \phi_D - P \cdot \phi_P]\nu - \epsilon\nu. \end{aligned}$$

This identity may be converted into an integral formula. Assume that (B, D) is smooth away from a smooth hypersurface $\Sigma \subset (0, \tau) \times \Omega$. Then let $\theta \in \mathcal{D}((0, \tau) \times \Omega)^3$ be a test field. Then there holds

$$(22) \quad \int_0^\tau dt \int_\Omega (P \cdot \partial_t \theta + T : \nabla_x \theta) dx = \int_0^\tau dt \int_{\Sigma(t)} \frac{\epsilon}{\sigma} \theta \cdot \nu dS(x).$$

In particular, we have a new kind of entropy inequality:

$$(23) \quad \left(\theta \cdot \frac{\nu}{\sigma} \geq 0 \text{ on } \Sigma \right) \implies \left(\int_0^\tau dt \int_\Omega (P \partial_t \theta + T : \nabla_x \theta) dx \geq 0 \right).$$

Open questions

We list now a few open questions that seem of some mathematical interest.

- i. What is the physical meaning of (23) ? Does it always make sense, or can the normal velocity σ vanish ?
- ii. What are the wave speeds in either the Maxwell's equations or the augmented system ? So long as we restrict to the equilibrium manifold, the Maxwell's velocities are part of the "augmented" velocities. The three extra velocities can be computed by linearizing (18) around a constant solution that is at equilibrium. We obtain

$$\partial_t \delta' = \left(\frac{\partial \phi}{\partial P} \cdot \nabla \right) \delta',$$

where δ' stands for the infinitesimal perturbation of δ , namely $\delta' = P' - B' \times D - B \times D'$, with obvious notations. Therefore the extra velocities merge into a unique one with multiplicity 3,

$$\lambda(U; \xi) := -\frac{\partial \phi}{\partial P} \cdot \xi.$$

Does this multiplicity persist away from the equilibrium manifold ? If it did, the corresponding characteristic field would be linearly degenerate and "integrable", according to a theorem of G. Boillat [1] (see also [11] vol I, page 81.) This does not seem to be the case.

- iii. Identify, among the energies W that come from an invariant Lagrangian $L(\|E\|^2 - \|B\|^2, E \cdot B)$, those which can be written as convex functions of $(B, D, B \times D)$. We know that the Born-Infeld energy works. Presumably, a small and localized disturbance of L_{BI} yields an admissible energy. The difficulty here is that, given an energy, there is a lot of freedom when writing it as a function of $(B, D, B \times D)$, since we are completely free outside the equilibrium manifold $P = B \times D$, a non-convex set.

Given the Lagrangian $L(\gamma, \delta)$, with $\gamma := (\|E\|^2 - \|B\|^2)/2$ and $\delta := E \cdot B$, there is however a "natural" (although non unique) way to define $\phi(B, D, P)$ such that

$W(B, D) \equiv \phi(B, D, B \times D)$. We shall assume that L is even with respect to δ , which means that it is Lorentz- and orientation-invariant. We start from Definition (10). Given B and D , we write

$$\sup_{e \in \mathbb{R}^3} = \sup_{\gamma, \delta} \sup_{e \in \gamma, \delta},$$

where $\sup_{e \in \gamma, \delta}$ is a supremum over e , constrained by $(\|e\|^2 - \|B\|^2)/2 = \gamma$ and $e \cdot B = \delta$. To begin with, we solve this sub-problem, where L remains constant. The maximum of $D \cdot e$ is achieved at some point e that belong to the plane spanned by B and D . With the two constraints, the possible points are the intersections of a sphere with a line,

$$e = \frac{\delta}{\|B\|^2} B + a(B \times D) \times B,$$

where a obeys to

$$\frac{\delta^2}{\|B\|^2} + a^2 \|B\|^2 \|B \times D\|^2 = \|B\|^2 + 2\gamma.$$

The supremum is achieved when a is positive. We obtain

$$\sup_{e \in \gamma, \delta} \{D \cdot e - L(\gamma, \delta)\} = \frac{\delta B \cdot D + \|B \times D\| \sqrt{\|B\|^4 + 2\gamma \|B\|^2 - \delta^2}}{\|B\|^2} - L(\gamma, \delta).$$

Since L is even with respect to δ , we may replace $B \cdot D$ by its absolute value. Then the expression $|B \cdot D|$ equals $(\|B\|^2 \|D\|^2 - \|B \times D\|^2)^{1/2}$. Finally, we may write

$$(24) \quad W(B, D) = h(\|B\|, \|D\|, \|B \times D\|),$$

with

$$(25) \quad h(b, d, p) := \sup_{\gamma, \delta} \left\{ \frac{\delta \sqrt{b^2 d^2 - p^2} + p \sqrt{b^4 + 2\gamma b^2 - \delta^2}}{b^2} - L(\gamma, \delta) \right\}.$$

The convexity of $\phi(B, D, P) := h(\|B\|, \|D\|, \|P\|)$ is equivalent to that of h . Hence we obtained a sufficient condition (a rather obscure one, indeed) in order that an augmented system with a convex energy exist. Can we make this condition more explicit? Is this condition necessary? We leave these questions open. Remark that formula (25) can be used to find H in the Born-Infeld model:

$$L_{BI} = -\sqrt{1 + \|B\|^2 - \|E\|^2 - (E \cdot B)^2}, \quad W_{BI} = \sqrt{1 + \|B\|^2 + \|D\|^2 + \|P\|^2}.$$

Notice that Formula (25) also reads

$$h(b, d, p) := \frac{1}{b} \sup \left\{ \rho \sqrt{b^2 d^2 - p^2} \cos \theta + p \rho \sin \theta - bL \left(\frac{\rho^2 - b^2}{2}, \rho b \cos \theta \right) \right\},$$

where the supremum is taken over $\rho \geq 0$ and $\theta \in [0, \pi/2]$.

Planar waves

We consider now solutions of Maxwell's equations that depend only on time and a single space variable, say x_1 . Then B_1 and D_1 depend only on time and may be considered as prescribed data. To avoid complications due to inhomogeneity, we assume that B_1 and D_1 are constant initially, hence constant forever. Without loss of generality, we may assume that $B_1 \equiv D_1 \equiv 0$, up to a composition of the energy by a translation. Denoting $x := x_1$, Maxwell's equations reduce to

$$\begin{aligned}\partial_t B_2 - \partial_x(\partial W/\partial D_3) &= 0, & \partial_t B_3 + \partial_x(\partial W/\partial D_2) &= 0, \\ \partial_t D_2 + \partial_x(\partial W/\partial B_3) &= 0, & \partial_t D_3 - \partial_x(\partial W/\partial B_2) &= 0.\end{aligned}$$

We assume a form (24) for the energy. Then the above system rewrites, in complex variables $w := B_2 + iB_3$ and $z := D_3 - iD_2$,

$$\begin{aligned}\partial_t w - \partial_x(h_p w) - \partial_x\left(\frac{1}{d}h_d z\right) &= 0, \\ \partial_t z - \partial_x(h_p z) - \partial_x\left(\frac{1}{b}h_b w\right) &= 0.\end{aligned}$$

In general, this system of four equations does not decouple into closed proper sub-systems. We remind the reader that the analysis done in [10] concluded to the existence of a weak entropy solution for any bounded initial data, provided W has the form of a function of $(\|B\|^2 + \|D\|^2)^{1/2}$, with suitable convexity properties. But such an assumption fits hardly with the requirement that W comes from an invariant Lagrangian. Therefore, we wish to relax it. The main property that we wish to preserve is the decoupling of the system. It turns out that an energy of the form

$$W(B, D) = h(r, p), \quad r = \sqrt{b^2 + d^2}$$

is convenient. For then the system rewrites

$$\begin{aligned}\partial_t(w + z) - \partial_x(h_p(w + z)) - \partial_x\left(\frac{1}{r}h_r(w + z)\right) &= 0, \\ \partial_t(w - z) - \partial_x(h_p(w - z)) + \partial_x\left(\frac{1}{r}h_r(w - z)\right) &= 0.\end{aligned}$$

Hence, writing $w + z =: \rho \exp(i\theta)$ and $w - z =: \sigma \exp(i\alpha)$ (polar decomposition of complex numbers), we obtain a 2×2 system in (ρ, σ) :

$$(26) \quad \partial_t \rho - \partial_x(h_p \rho) - \partial_x\left(\frac{1}{r}h_r \rho\right) = 0,$$

$$(27) \quad \partial_t \sigma - \partial_x(h_p \sigma) + \partial_x\left(\frac{1}{r}h_r \sigma\right) = 0.$$

The fact that the above system is closed follows from the identities

$$2r^2 = \rho^2 + \sigma^2, \quad 4p = \rho^2 - \sigma^2.$$

We emphasize that the energy (in)equality (9) reads in terms of (ρ, σ) only and hence can be used as an entropy criterion for the system (26, 27):

$$\partial_t h(r, p) + \partial_x \left((r^{-2} h_r^2 + h_p^2) p - 2r h_r h_p \right) \leq 0.$$

In particular, our analysis above gives us the non-trivial fact that the strict convexity of h implies the hyperbolicity of the sub-system.

We postpone the study of the Cauchy problem for this 2×2 system to a future work. For the moment, let us just say that, given a weak entropy solution of (26, 27) that is non-negative ($\rho \geq 0, \sigma \geq 0$), we may build a weak entropy solution of the plane wave system by solving the following transport equations

$$\left(\partial_t - h_p \partial_x - \frac{1}{r} h_r \partial_x \right) \theta = 0, \quad \left(\partial_t - h_p \partial_x + \frac{1}{r} h_r \partial_x \right) \alpha = 0.$$

We recall that, following the procedure in [10], we actually may solve the conservation laws

$$\begin{aligned} \partial_t(\rho f(\theta)) - \partial_x(h_p \rho f(\theta)) - \partial_x \left(\frac{1}{r} h_r \rho f(\theta) \right) &= 0, \\ \partial_t(\sigma g(\alpha)) - \partial_x(h_p \sigma g(\alpha)) + \partial_x \left(\frac{1}{r} h_r \sigma g(\alpha) \right) &= 0, \end{aligned}$$

for every smooth functions f and g simultaneously. The choices of the sine and cosine functions give exactly the Maxwell's equations for planar waves.

An interesting problem in the relativistic context is to characterize those energies of the form $h(r, p)$ that derive from an invariant Lagrangian $L(\gamma, \delta)$. We leave this question for a future work too.

Acknowledgements. The author thanks C. Dafermos and Y. Brenier for fruitful discussions, and the later for providing a preliminary version of [3].

References

- [1] G. Boillat. Chocs caractéristiques. *C. R. Acad. Sci. Paris, Série I* **274** (1972), pp 1018–21.
- [2] M. Born, L. Infeld. Foundations of a new field theory. *Proc. Roy. Soc. London, A* **144** (1934), pp 425–451.
- [3] Y. Brenier. Hydrodynamic structure of the augmented Born–Infeld equations. Preprint (2003).
- [4] Ph. Ciarlet. *Mathematical elasticity*. North–Holland, Amsterdam (1988).
- [5] B. D. Coleman, E. H. Dill. Thermodynamic restrictions on the constitutive equations of electromagnetic theory. *Z. Angew. Math. Phys.*, **22** (1971), pp 691–702.

- [6] C. Dafermos. Quasilinear hyperbolic systems with involutions. *Arch. Rational Mech. Anal.*, **94** (1986), pp 373–389.
- [7] C. Dafermos. *Hyperbolic conservation laws in continuum physics*. Grundlehren der mathematischen Wissenschaften, **325**. Springer-Verlag, Heidelberg (2000).
- [8] S. Demoulini, D. Stuart, A. Tzavaras. A variational approximation scheme for three-dimensional elastodynamics with polyconvex energy. *Arch. Rational Mech. Anal.*, **157** (2001), pp 325–344.
- [9] S. Godunov. Lois de conservation et intégrales d'énergie. *Lecture Notes in Math.* **1270**, pp 135–149. Springer-Verlag, Heidelberg (1987).
- [10] D. Serre. Les ondes planes en électromagnétisme non-linéaire. *Physica D***31** (1988), pp 227–251.
- [11] D. Serre. *Systems of conservation laws*. Cambridge Univ. Press, Cambridge (vol I: 1999, vol II: 2000).