STABILITY AND GEOMETRIC CONSERVATION LAWS FOR ALE FORMULATIONS

DANIELE BOFFI AND LUCIA GASTALDI

ABSTRACT. The aim of this paper is to investigate a model ALE scheme, with respect to various possible choices of time discretizations. For each time scheme, we investigate the relationships between stability and the so-called Geometric Conservation Laws (CGL). We shall see that GCL condition proves neither necessary nor sufficient for stability. In doing so, we review some known theoretical results and we prove some new stability results for space-time ALE discretizations. Several numerical experiments will be provided.

1. INTRODUCTION

In this paper we consider the finite element approximation of a parabolic problem in a moving two-dimensional domain. We assume that the motion of the domain boundary is given although in most applications arising from fluid-structure interaction problems it represents a further unknown of the problem. In such applications the fluid domain boundary may undergo a motion with large amplitude, which have to be taken into account using a dynamic mesh in the numerical computation. The Arbitrary Lagrangian Eulerian formulation has been introduced for this purpose, see [2, 1, 9] and it has been widely used in the numerical simulation of fluid-structure interaction systems, mainly associated with finite difference and finite volume schemes (see e.g. [3, 13, 12, 11, 7]). More recently, some contribution to the analysis of ALE numerical schemes based on finite element approximations has been given (see [4, 14, 15, 6]).

The literature on ALE formulation often refers to the so-called Geometric Conservation Laws (GCL) which are considered to be strictly related to the stability and the accuracy of the method. The GCL condition governs the geometric parameters of a given numerical scheme. In the case of spatial discretization based on finite volumes, the GCL condition requires that the numerical procedure reproduces exactly a constant solution. It has been stated that, in practice, the GCL can be violated if a sufficiently small time-step is selected for advancing the flow simulation [18]. But for unsteady flow simulation this is, in general, a major drawback since it dramatically increases the computational cost. Recently, in [7], it has been proved that satisfying an appropriate discrete GCL is a sufficient condition for a numerical scheme to be at least first-order time-accurate on moving meshes. In the case of finite element spatial discretizations, the relationships between GCL condition and stability and accuracy properties of the numerical scheme, have not been completely clarified yet. Some results in this direction can be found in [15]. In this paper we intend to review the known results on this subject and to state, if possible, some relations between GCL condition and convergence properties of various numerical schemes.

For this purpose, we shall analyze the application of some well-know time advancing schemes, like *implicit Euler, Crank–Nicolson, Bacward Differentiation Formulae*, to the ALE formulation. For each scheme we discuss the satisfaction of the GCL and its stability properties. We shall show that, in general, the GCL condition does not imply the stability of the method, except if we choose the time-step sufficiently small with respect to the domain velocity or the meshsize. In particular, in the case of the Crank–Nicolson case, we have to choose the time-step of the same order as h^2 (h being the meshsize), in order to obtain stability and convergence of the fully discrete scheme. A modification of the standard Crank–Nicolson method has been introduced to avoid such constraint, although a limitation on the time-step with respect to the domain velocity still remains. On the other hand, the violation of the GCL condition, as for example in the implicit Euler scheme, requires the use of sufficiently small time-steps with restrictions of the same type as above. We shall present some numerical experiments which confirm these issues for the various methods we have analyzed.

The outline of the paper is as follows. The next section is devoted to the presentation of the problem on moving domain and to its ALE formulation. In Sect. 3 the finite element semidiscretization is introduced. Sect. 4 is devoted to the space-time discretization. In particular, several time advancing schemes are proposed and their stability and accuracy properties are analyzed.

2. Statement of the problem and its ALE formulation

Let $T > 0$ be a real number and, for each $t \in [0,T]$, let Ω_t be a domain in $\mathbb{R}^2$ with a sufficiently smooth boundary. We shall use the following notation:

$$Q_T = \{(\mathbf{x}, t) \in \mathbb{R}^3 : \mathbf{x} \in \Omega_t,\ t \in\ ]0, T[\}. \tag{1}$$

We consider the linear advection-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + \operatorname{div}(\boldsymbol{\beta} u) = f & \text{for } (\mathbf{x}, t) \in Q_T \\ u = u_0 & \text{for } \mathbf{x} \in \Omega_0,\ t = 0 \\ u = 0 & \text{for } \mathbf{x} \in \partial\Omega_t,\ t \in I, \end{cases} \tag{2}$$

where $\boldsymbol{\beta}$ is a convection velocity, μ is a constant diffusivity, and Δ denotes the Laplace operator. We assume that Ω_0 is a bounded domain with Lipschitz continuous boundary and that the data satisfy the following regularity requirements:

$$\boldsymbol{\beta} \in W^{1,\infty}(Q_T), \quad f \in L^2(Q_T), \quad u_0 \in H_0^1(\Omega_0). \tag{3}$$

Let us introduce the Arbitrary Lagrangian Eulerian frame of reference and briefly recall the most used notation and assumptions. Let $\mathcal{A}_t$ be a family of mappings, which, at each $t \in\ ]0, T[$, maps the points $\boldsymbol{\xi}$ of a *reference domain* Ω_0, into the points $\mathbf{x}$ of the domain Ω_t at time t. Then for each $t \in [0, T[$

$$\mathcal{A}_t : \Omega_0 \to \Omega_t, \qquad \mathbf{x}(\boldsymbol{\xi}, t) = \mathcal{A}_t(\boldsymbol{\xi}). \tag{4}$$

We assume that $\mathcal{A}_t$ is surjective,

$$\Omega_t = \mathcal{A}_t(\Omega_0) \text{ is bounded and Lipschitz continuous,} \tag{5}$$

and satisfies the following regularity assumption

(6) $$\mathcal{A}_t \in W^{1,\infty}(\Omega_0)^2, \quad \mathcal{A}_t^{-1} \in W^{1,\infty}(\Omega_t)^2.$$

Moreover, for t_1 and t_2 in I, we use the following notation

(7) $$\mathcal{A}_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2} \quad \mathcal{A}_{t_1,t_2} = \mathcal{A}_{t_2} \circ \mathcal{A}_{t_1}^{-1}$$

to indicate the ALE mapping between two time levels. In the literature several techniques have been proposed in order to construct such mapping, assuming that the evolution of the boundary domain is known. For instance, one can use the harmonic extension of the boundary position, as in [10, 16, 15], or one can consider the reference domain as an elastic body which is deformed into the current domain, see [3, 6]. In [6] it has been proved that the second approach guarantees the fulfillment of the conditions (5) and (6).

Let us consider a function $g : Q_T \to \mathbb{R}$ defined on the Eulerian frame, then the time derivative in the ALE frame is defined as follows:

(8) $$\left.\frac{\partial g}{\partial t}\right|_{\boldsymbol{\xi}} : Q_T \to \mathbb{R}, \quad \left.\frac{\partial g}{\partial t}\right|_{\boldsymbol{\xi}}(\mathbf{x}, t) = \frac{\partial \hat{g}}{\partial t}(\boldsymbol{\xi}, t), \quad \boldsymbol{\xi} = \mathcal{A}_t^{-1}(\mathbf{x}),$$

where $\hat{g} : \Omega_0 \times]0, T[\to \mathbb{R}$ is the corresponding function of g in the ALE frame, that is $\hat{g}(\boldsymbol{\xi}, t) = g(\mathbf{x}(\boldsymbol{\xi}, t), t) = g(\mathcal{A}_t(\boldsymbol{\xi}), t)$.

The domain velocity $\mathbf{w}$ is defined by:

(9) $$\mathbf{w}(\mathbf{x}, t) = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\boldsymbol{\xi}}(\mathcal{A}_t^{-1}(\mathbf{x}), t),$$

while the *Jacobian matrix* of the ALE mapping $\mathbf{J}_t$ and its determinant J_t are given as:

(10) $$(\mathbf{J}_t)_{ij} = \frac{\partial x_i}{\partial \xi_j}, \quad J_t = \det(\mathbf{J}_t).$$

In the following, we shall assume that the Jacobian J_t, which is positive, is bounded away from zero by a constant κ independent of t, that is

(11) $$J_t \geq \kappa > 0 \quad \text{for all } t \in I.$$

Let $u : Q_T \to \mathbb{R}$ be regular enough, then applying the chain rule to the time derivative gives:

(12) $$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{\xi}} = \left.\frac{\partial u}{\partial t}\right|_{\mathbf{x}} + \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\boldsymbol{\xi}} \cdot \nabla_{\mathbf{x}} u = \left.\frac{\partial u}{\partial t}\right|_{\mathbf{x}} + \mathbf{w} \cdot \nabla_{\mathbf{x}} u.$$

Hence we obtain the ALE counterpart of (2), by substituting (12) in (2):

(13) $$\begin{cases} \left.\dfrac{\partial u}{\partial t}\right|_{\boldsymbol{\xi}} - \mu \Delta u + \operatorname{div}(\boldsymbol{\beta} u) - \mathbf{w} \cdot \nabla_{\mathbf{x}} u = f & \text{for } (\mathbf{x}, t) \in Q_T \\ u = u_0 & \text{for } \mathbf{x} \in \Omega_0, \ t = 0 \\ u = 0 & \text{for } \mathbf{x} \in \partial \Omega_t, \ t \in]0, T[. \end{cases}$$

Let us define a functional space compatible with the ALE mapping:

(14) $$\mathcal{H}(\Omega_t) = \{v : \Omega_t \to \mathbb{R} : v = \hat{v} \circ \mathcal{A}_t^{-1}, \ \hat{v} \in H_0^1(\Omega_0)\}, \text{ for each } t \in]0, T[.$$

In order to derive a variational formulation for (13), we recall the following identity, known as transport theorem or *Reynolds transport formula* (see e.g. [8]).

Let $\psi(\mathbf{x}, t)$ be a function defined on Ω_t for each t. Then for any arbitrary subdomain $V_t \subseteq \Omega_t$ such that $V_t = \mathcal{A}_t(V_0)$ with $V_0 \subseteq \Omega_0$ it holds:

$$(15) \quad \frac{d}{dt} \int_{V_t} \psi(\mathbf{x}, t) d\mathbf{x} = \int_{V_t} \left(\left. \frac{\partial \psi}{\partial t} \right|_{\boldsymbol{\xi}} + \psi \operatorname{div} \mathbf{w} \right) d\mathbf{x} = \int_{V_t} \left(\left. \frac{\partial \psi}{\partial t} \right|_{\mathbf{x}} + \nabla_{\mathbf{x}} \psi \cdot \mathbf{w} + \psi \operatorname{div} \mathbf{w} \right) d\mathbf{x}.$$

In particular, for any $v : Q_T \to \mathbb{R}$ such that $v = \hat{v} \circ \mathcal{A}_t^{-1}$, for some $\hat{v} : \Omega_0 \to \mathbb{R}$, we obtain from (15):

$$(16) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega_t} v d\mathbf{x} &= \int_{\Omega_t} v \operatorname{div} \mathbf{w} d\mathbf{x} \\ \frac{d}{dt} \int_{\Omega_t} v \psi d\mathbf{x} &= \int_{\Omega_t} v \left(\left. \frac{\partial \psi}{\partial t} \right|_{\boldsymbol{\xi}} + \psi \operatorname{div} \mathbf{w} \right) d\mathbf{x}. \end{aligned}$$

Then the ALE weak formulation in conservative form reads (see, e.g. [4]):

$$(17) \quad \begin{cases} \text{find } u : Q_T \to \mathbb{R} \text{ such that for each } t \in]0, T[, \ u(t) \in \mathcal{H}(\Omega_t), \text{ and} \\ \dfrac{d}{dt}(u(t), v)_t + a_t(u(t), v) + b_t(u(t), v; \mathbf{w}(t)) = (f(t), v)_t \ \forall v \in \mathcal{H}(\Omega_t), \\ u(0) = u_0 \qquad \text{in } \Omega_0, \end{cases}$$

where

$$(18) \quad \begin{aligned} (u, v)_t &= \int_{\Omega_t} u \, v \, d\mathbf{x}, \\ a_t(u, v) &= \int_{\Omega_t} \left(\mu \nabla_{\mathbf{x}} u \nabla_{\mathbf{x}} v + \operatorname{div}(\boldsymbol{\beta} u) v \right) d\mathbf{x}, \\ b_t(u, v; \mathbf{w}) &= -\int_{\Omega_t} \operatorname{div}(\mathbf{w} u) v \, d\mathbf{x}. \end{aligned}$$

Taking $v = u(t)$ in (17), the following a priori bound can be derived by means of (16), an integration by parts, and Gronwall's lemma:

$$(19) \quad \|u(t)\|_{L^2(\Omega_t)} + \mu \int_0^t \|\nabla_{\mathbf{x}} u\|_{L^2(\Omega_t)}^2 ds \leq \|u_0\|_{L^2(\Omega_0)} + C \int_0^t \|f\|_{H^{-1}(\Omega_t)}^2 ds.$$

3. Finite element semidiscretization of the ALE formulation

In this section we present briefly the finite element semidiscretization of (17). See [4, 15, 6] for the details. For simplicity, we assume that for each $t \in [0, T]$, Ω_t is a polygonal convex domain; we refer to [6] where the general case was considered.

Let us consider a triangulation $\mathcal{T}_{h,0}$ of Ω_0 made up of triangles with straight sides, such that $\Omega_0 = \cup_{K \in \mathcal{T}_{h,0}} K$. Let us consider the Lagrangian finite element spaces

$$(20) \quad \begin{aligned} \mathcal{L}^k(\Omega_0) &= \{\hat{v}_h \in H^1(\Omega_0) : \ \hat{v}_h|_K \in \mathbf{P}_k(K), \text{ for all } K \in \mathcal{T}_{h,0}\} \\ \mathcal{L}_0^k(\Omega_0) &= \{v_h \in \mathcal{L}^k(\Omega_0) : v_h = 0 \text{ on } \partial\Omega_0\}. \end{aligned}$$

where $P_k(K)$ is the set of polynomials on K of degree less than or equal to k.

For each $t \in]0, T[$, we consider the discretization of $\mathcal{A}_t$ by means of piecewise linear Lagrangian finite elements as follows:

$$(21) \quad \mathcal{A}_{h,t} \in \mathcal{L}^1(\Omega_0)^2 \quad \text{for each } t \in]0, T[\text{ with } \mathbf{x}_h(\boldsymbol{\xi}, t) = \mathcal{A}_{h,t}(\boldsymbol{\xi}) = \sum_{i=1}^{\mathcal{N}_h} \mathbf{x}_{h,i}(t) \hat{\varphi}_i(\boldsymbol{\xi}),$$

where $\mathbf{x}_{h,i}(t) = \mathcal{A}_{h,t}(\boldsymbol{\xi}_i)$ denotes the position of the i-th node at time t and $\hat{\varphi}_i$ is the i-th basis function in $\mathcal{L}^1(\Omega_0)$.

Let $J_{h,t}$ be the determinant of the Jacobian matrix associated with $\mathcal{A}_{h,t}$. We assume that there exists a suitable positive number δ such that, for h small enough, it holds (see [6]):

$$(22) \qquad J_{h,t} > \delta\kappa.$$

For each $t \in]0,T[$, let $\mathcal{T}_{h,t}$ be the image of $\mathcal{T}_{h,0}$ under the discrete ALE mapping $\mathcal{A}_{h,t}$. Let us denote by K_t the image of a triangle $K \in \mathcal{T}_{h,0}$, that is $K_t = \mathcal{A}_{h,t}(K)$. Thanks to (21) K_t is a triangle with straight sides, too. Moreover, setting $\mathcal{T}_{h,t} = \{K_t = \mathcal{A}_{h,t}(K), \ K \in \mathcal{T}_{h,0}\}$, we have $\overline{\Omega_t} = \cap_{K_t \in \mathcal{T}_{h,t}} K_t$.

Let us set

$$(23) \qquad \begin{aligned} \mathcal{L}^k(\Omega_t) &= \{v_h : \Omega_t \to \mathbb{R} : v_h = \hat{v}_h \circ \mathcal{A}_{h,t}^{-1}, \ \hat{v}_h \in \mathcal{L}^k(\Omega_0)\} \\ \mathcal{H}_h(\Omega_t) &= \{v_h : \Omega_t \to \mathbb{R} : v_h = \hat{v}_h \circ \mathcal{A}_{h,t}^{-1}, \ \hat{v}_h \in \mathcal{L}_0^k(\Omega_0)\}. \end{aligned}$$

Since $\mathcal{A}_{h,t}$ is piecewise linear, we can characterize $\mathcal{L}^k(\Omega_t)$ and $\mathcal{H}_h(\Omega_t)$ as follows, see e.g. [4]:

$$(24) \qquad \begin{aligned} \mathcal{L}^k(\Omega_t) &= \{v_h \in H^1(\Omega_t) : \ v_h|_K \in \mathbf{P}_k(K), \text{ for all } K \in \mathcal{T}_{h,t}\} \\ \mathcal{H}_h(\Omega_t) &= \{v_h \in \mathcal{L}^k(\Omega_t) : \ v_h = 0 \text{ at the nodes along } \partial\Omega_t\}. \end{aligned}$$

Then the finite element spatial discretization of (17) is:

$$(25) \qquad \begin{cases} \text{find } u_h, \text{ such that for each } t \in]0,T[, \ u_h(t) \in \mathcal{H}_h(\Omega_t), \\ \dfrac{d}{dt}(u_h(t), v_h)_t + a_t(u_h(t), v_h) + b_t(u_h(t), v_h; \mathbf{w}_h) = (f(t), v_h)_t \qquad \forall v_h \in \mathcal{H}_h(\Omega_t), \\ u_h(0) = u_{h,0} \qquad \text{in } \Omega_0. \end{cases}$$

A stability inequality similar to (19) can be obtained also for the semidiscrete solution of (25); in particular, we observe that such inequality is independent of the domain velocity field.

Moreover, if the exact solution u of (17) belongs to $L^2(]0,T[; H^{k+1}(\Omega_t))$ for $k \geq 1$ with time derivative $\frac{\partial u}{\partial t} \in L^2(]0,T[; H^k(\Omega_t))$, then the following error estimate has been proved in [15]:

$$(26) \qquad \begin{aligned} \frac{1}{2}\|u(t) &- u_h(t)\|_{L^2(\Omega_t)}^2 + \frac{\mu}{4}\int_0^t \|\nabla_{\mathbf{x}}(u(s) - u_h(s))\|_{L^2(\Omega_t)}^2 ds \\ &\leq \tfrac{1}{2}\|u(0) - u_{h,0}\|_{L^2(\Omega_0)}^2 \\ &\quad + Ch^{2k}\left(\|u(t)\|_{H^k(\Omega_t)}^2 + \int_0^t \left(\left\|\frac{\partial u}{\partial t}\right\|_{H^k(\Omega_s)}^2 + \|u\|_{H^{k+1}(\Omega_s)}^2\right)ds\right), \end{aligned}$$

where the constant C, which is independent of h, might depend on $\|\mathbf{w}\|_{W^{2,\infty}(\Omega_t)}$.

4. Space-time discretization schemes

In order to obatin a full discretization of (17), we consider a uniform subdivision of the time interval $]0,T[$ and define $t^n = n\Delta t$ for $n = 0, \ldots, N$, $\Delta t > 0$ being the time step, and N equal to the integer part of $N/\Delta t$.

Since the semidiscrete problem (25) can be viewed as a system of ordinary differential equations, let us recall the time advancing schemes we are going to apply in the framework

of ordinary differential equations. Let us consider the following system of ordinary differential equations:

$$(27) \qquad \frac{d}{dt}\mathbf{Y} = \mathbf{G}(\mathbf{Y}, t) \quad t \in \,]0, T[\,; \qquad \mathbf{Y}(0) = \mathbf{Y}_0,$$

where $\mathbf{Y} : \,]0, T[\, \to \mathbb{R}^n$, is a differentiable vector valued function and $\mathbf{G} : \mathbb{R}^n \times \,]0, T[\, \to \mathbb{R}^n$ is continuous and Lipschiptz continuous with respect to $\mathbf{Y}$.

Here is a list of commonly used numerical methods ($\mathbf{Y}^n$ denotes the approximate solution at time t^n):

$$(28)$$

$$
\begin{array}{ll}
\mathbf{Y}^{n+1} - \mathbf{Y}^n = \Delta t \mathbf{G}(\mathbf{Y}^{n+1}, t^{n+1}) & \text{implicit (backward) Euler scheme} \\
\mathbf{Y}^{n+1} - \mathbf{Y}^n = \Delta t \mathbf{G}(\frac{\mathbf{Y}^{n+1} + \mathbf{Y}^n}{2}, t^{n+1/2}) & \text{mid point rule} \\
\mathbf{Y}^{n+1} - \mathbf{Y}^n = \frac{\Delta t}{2}\left(\mathbf{G}(\mathbf{Y}^{n+1}, t^{n+1}) + \mathbf{G}(\mathbf{Y}^n, t^n)\right) & \text{Crank–Nicolson scheme} \\
\frac{3}{2}\mathbf{Y}^{n+1} - 2\mathbf{Y}^n + \frac{1}{2}\mathbf{Y}^{n-1} = \Delta t \mathbf{G}(\mathbf{Y}^{n+1}, t^{n+1}) & \text{two-step BDF.}
\end{array}
$$

We observe that the first three methods can be obtained integrating system (27) from t^n to t^{n+1} and applying an appropriate quadrature rule, while the BDF methods are obtained through differentiation formulae.

More precisely, let us denote by $\mathcal{Q}$ a quadrature formula approximating the time integral from t^n to t^{n+1}, that is

$$(29) \qquad \mathcal{Q}(F) \approx \int_{t^n}^{t^{n+1}} F(t)\,dt.$$

In view of (29) the first three time advancing schemes in (28) can be written as follows:

$$(30) \qquad \mathbf{Y}^{n+1} - \mathbf{Y}^n = \mathcal{Q}(\mathbf{G}).$$

The two-step BDF scheme, instead, is obtained with the following procedure: one constructs the interpolant polynomial of second degree using the values of the discrete solution $\mathbf{Y}^{n-1}$, $\mathbf{Y}^n$ and $\mathbf{Y}^{n+1}$, then computes the derivative at t^{n+1} and enforces that it is equal to the right hand side of (27).

We shall treat separately the case of time-advancing schemes based on integration and the second order BDF scheme.

In order to introduce the time discretization of (25), we first discretize in time the ALE mapping using a linear interpolation in time. Thus we have that the discrete ALE mapping denoted by $\mathcal{A}_{h,\Delta t}$ has the following representation:

$$(31) \qquad \mathcal{A}_{h,\Delta t}(\boldsymbol{\xi}, t) = \frac{t - t^n}{\Delta t}\mathcal{A}_{h,t^{n+1}}(\boldsymbol{\xi}) + \frac{t^{n+1} - t}{\Delta t}\mathcal{A}_{h,t^n}(\boldsymbol{\xi}),$$

where $\mathcal{A}_{h,t}$ is the time continuous ALE mapping adopted in the semidiscrete problem (25).

As a consequence, the domain velocity is represented with a piecewise constant function with respect to time and it is obtained by:

$$(32) \qquad
\begin{aligned}
\tilde{\mathbf{w}}_{h,\Delta t}^{n+1}(\boldsymbol{\xi}) &= \frac{\mathcal{A}_{h,t^{n+1}}(\boldsymbol{\xi}) - \mathcal{A}_{h,t^n}(\boldsymbol{\xi})}{\Delta t}, \quad \text{for } t \in [t^n, t^{n+1}] \\
\mathbf{w}_{h,\Delta t}^{n+1}(\mathbf{x}, t) &= \tilde{\mathbf{w}}_{h,\Delta t}^{n+1}(\boldsymbol{\xi}) \circ \mathcal{A}_{h,\Delta t}^{-1}(\mathbf{x}).
\end{aligned}
$$

Let us introduce a notation which will be useful throughout the paper. For $\varphi : Q_T \to \mathbb{R}$ we set (see (7))

$$(33) \qquad \mathcal{M}_{t_1}(\varphi(t_2)) : \Omega_{t_1} \to \mathbb{R} \qquad \mathcal{M}_{t_1}(\varphi(t_2)) = \varphi(t_2) \circ \mathcal{A}_{h,t_1,t_2}.$$

Notice that, by definition of the finite element spaces (23), the test function v_h in (25), is given by $v_h = \hat{v}_h \circ \mathcal{A}_{h,t}^{-1}$, for some $\hat{v}_h$ definded over Ω_0. Then we set $\mathcal{M}_n(v_h) = \hat{v}_h \circ \mathcal{A}_{h,t_n}$.

The application of the first three schemes in (28) to system (25) gives the following full discretization of (17):

$$(34) \qquad \begin{cases} \text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\ (u_h^{n+1}, v_h)_{n+1} - (u_h^n, \mathcal{M}_n(v_h))_n + \mathcal{Q}_a(a_t(u_h, v_h)) + \mathcal{Q}_b(b_t(u_h, v_h; \mathbf{w}_{h,\Delta t}^{n+1})) \\ \qquad\qquad\qquad = \mathcal{Q}_a((f(t), v_h)_t) \qquad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\ u_h^0 = u_{h,0}, \end{cases}$$

where $\mathcal{Q}_a$ and Q_b are different quadrature rules. We point out that, in the quadrature terms, the function u_h is evaluated at the nodes of the quadrature formula using the available values u_h^n and u_h^{n+1}. These details will be made clear for each case.

In the case of domains which do not depend on time, the analysis of the convergence of the fully discrete solution is based on a stability property of the fully discrete scheme, see e.g. [17, 19, 20, 5]. Here we shall try to extend such technique to the case of problems in moving domains.

Definition 1. We say that the scheme (34) is *stable* if there exist two real numbers α and β such that the following inequality holds for all $v_h^n \in \mathcal{H}_h(\Omega_{t^n})$ with $n = 0, \ldots, N$:

$$(35) \qquad \begin{aligned} & \frac{1}{2}\|v_h^{n+1}\|_{L^2(\Omega_{n+1})}^2 - \frac{1}{2}\|v_h^n\|_{L^2(\Omega_n)}^2 + \Delta t \mu \|\nabla_{\mathbf{x}}(\alpha \mathcal{M}_{\bar{t}^n}(v_h^{n+1}) + \beta \mathcal{M}_{\bar{t}^n}(v_h^n))\|_{L^2(\Omega_{\bar{t}^n})}^2 \\ & \qquad + \Delta t \mathcal{Q}_a((\text{div}(\beta(\alpha v_h^{n+1} + \beta v_h^n)), \alpha v_h^{n+1} + \beta v_h^n)_t) \\ & \qquad \leq (v_h^{n+1}, \alpha v_h^{n+1} + \beta \mathcal{M}_{n+1}(v_h^n))_{n+1} - (v_h^n, \alpha \mathcal{M}_n(v_h^{n+1}) + \beta v_h^n)_n \\ & \qquad + \mathcal{Q}_a(a_t(v_h, \alpha v_h^{n+1} + \beta v_h^n)) + \mathcal{Q}_b(b_t(v_h, \alpha v_h^{n+1} + \beta v_h^n; \mathbf{w}_{h,\Delta t}^{n+1})), \end{aligned}$$

where $\bar{t}^n$ is a properly chosen point in the interval $[t^n, t^{n+1}]$.

We observe that the above stability definition yields an a priori estimate similar to (19), that is

$$(36) \qquad \begin{aligned} & \frac{1}{2}\|u_h^{n+1}\|_{L^2(\Omega_{n+1})}^2 + \Delta t \mu \sum_{i=0}^{n} \|\nabla_{\mathbf{x}}(\alpha \mathcal{M}_{\bar{t}^i}(u_{h,i+1}) + \beta \mathcal{M}_{\bar{t}^n}(u_{h,i}))\|_{L^2(\Omega_{\bar{t}^i})}^2 \\ & \qquad\qquad \leq \frac{1}{2}\|u_{h,0}\|_{L^2(\Omega_0)}^2 + C \sum_{i=1}^{n+1} \|f(t^i)\|_{H^{-1}(\Omega_{\bar{t}^i})}. \end{aligned}$$

Indeed, if we take $v_h^n = u_h^n$ in (35) and use equation (34), we get, after an integration by parts:

$$\frac{1}{2}\|u_h^{n+1}\|_{L^2(\Omega_{n+1})}^2 - \frac{1}{2}\|u_h^n\|_{L^2(\Omega_n)}^2 + \Delta t\mu\|\nabla_{\mathbf{x}}(\alpha\mathcal{M}_{\bar{t}^n}(u_h^{n+1}) + \beta\mathcal{M}_{\bar{t}^n}(u_h^n))\|_{L^2(\Omega_{\bar{t}^n})}^2$$

$$+ \Delta t\mathcal{Q}_a\left(\int_{\Omega_t}\frac{1}{2}\operatorname{div}(\beta)(\alpha u_h^{n+1} + \beta u_h^n)^2\,d\mathbf{x}\right)$$

$$\leq (u_h^{n+1}, \alpha u_h^{n+1} + \beta\mathcal{M}_{n+1}(u_h^n))_{n+1} - (u_h^n, \alpha\mathcal{M}_n(u_h^{n+1}) + \beta u_h^n)_n$$

$$+ \mathcal{Q}_a(a_t(u_h^n, \alpha u_h^{n+1} + \beta u_h^n)) + \mathcal{Q}_b(b_t(u_h^n, \alpha u_h^{n+1} + \beta u_h^n; \mathbf{w}_{h,\Delta t}^{n+1}))$$

$$= \mathcal{Q}_a(f(t), \alpha u_h^{n+1} + \beta u_h^n)_t).$$

Then by Cauchy-Schwarz inequality and a discrete version of Gronwall's lemma (see e.g. [17], pag.14), we get (36).

Inequality (35) is also useful in order to estimate the difference between the solutions to the fully discrete problem (34) and to the semidiscrete problem (25) and, finally, to obtain the convergence of the fully discrete solution to the continuous one, thanks to (26).

Let us integrate (25) from t^n to t^{n+1} and subtract it from (34), then we obtain the following error equation:

$$(u_h^{n+1} - u_h(t^{n+1}), v_h)_{n+1} - (u_h^n - u_h(t^n), \mathcal{M}_n(v_h))_n$$

$$+ \mathcal{Q}_a(a_t(u_h^n - u_h(t), v_h)) + \mathcal{Q}_b(b_t(u_h^n - u_h(t), v_h; \mathbf{w}_{h,\Delta t}^{n+1}))$$

(37)
$$= \int_{t^n}^{t^{n+1}} a_t(u_h(t), v_h)\,dt - \mathcal{Q}_a(a_t(u_h(t), v_h))$$

$$+ \int_{t^n}^{t^{n+1}} b_t(u_h(t), v_h; \mathbf{w}_h(t))\,dt - \mathcal{Q}_b(b_t(u_h(t), v_h; \mathbf{w}_{h,\Delta t}^{n+1}))$$

$$+ \int_{t^n}^{t^{n+1}} (f(t), v_h)_t\,dt - \mathcal{Q}_a((f(t), v_h)_t).$$

Let α and β be given as in the definition 1. Taking $v_h = \alpha u_h^{n+1} + \beta u_h^n$ in (37), we can apply (35) and we obtain that, if a scheme (34) is stable in the sense of definition 1, the difference $u_h^n - u_h(t^n)$ can be bounded if we can estimate the error in the quadrature rules on the right hand side of (37). Therefore the convergence of the fully discrete solution is obtained if we can prove that the scheme is stable in the sense of definition 1 and if we can obtain convenient estimates for the error introduced by the quadrature rules.

One might think that the stability condition (35) can be related with a discrete version of the GCL condition to be used in the case of finite element spatial discretization. However, we shall see that such discrete GCL condition is neither sufficient nor necessary in order to have the stability.

For the finite element ALE formulation, the geometric conservation laws have been introduced in [13, 4] in the following form:

$$(38) \quad \int_{\Omega_{n+1}}\varphi_i(t^{n+1})\varphi_j(t^{n+1})\,d\mathbf{x} - \int_{\Omega_n}\varphi_i(t^n)\varphi_j(t^n))\,d\mathbf{x} = \mathcal{Q}_b\left(\int_{\Omega_t}\varphi_i(t)\varphi_j(t)\operatorname{div}\mathbf{w}_h(t)\,d\mathbf{x}\right)$$

$$\forall i, j = 1, \cdots, \mathcal{N}_h$$

Recall that in (38) the basis functions φ_i depend on the time through the discrete ALE mapping only. This condition can be derived taking in the first equation of (16) $v = \varphi_i \varphi_j$.

We observe that, thanks to (16), the left hand side in (38) is equal to

$$\int_{t^n}^{t^{n+1}} \int_{\Omega_s} \varphi_i(s) \varphi_j(s) \operatorname{div} \mathbf{w}_h(s) \, d\mathbf{x} \, ds.$$

Therefore a necessary and sufficient condition for the fulfillment of (38) is to use a time integration rule $\mathcal{Q}_b$ (for the term containing the domain velocity) with degree of precision $2s-1$, where s is the order of the polynomial used to represent the time evolution of the nodal displacement within each time step, see [4]. Since we choose piecewise linear polynomials for the mapping, the integration rule $\mathcal{Q}_b$ should be at least of degree of precision 1. For example, we can use the trapezoidal rule or the mid-point rule. Notice that implicit Euler scheme, which is associated with a quadrature formula $\mathcal{Q}_b$ of degree of precision 0, does not satisfy this sufficient condition.

In the following subsections we shall examine the possible schemes we can derive for (34) using different quadrature rules and discuss the fulfillment of GCL (38) and (35) and their convergence properties.

4.1. Implicit Euler scheme.
The use of implicit Euler method in scheme (34) gives rise to the following procedure:

$$(39) \quad \begin{cases} \text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\ (u_h^{n+1}, v_h)_{n+1} - (u_h^n, \mathcal{M}_n(v_h))_n + \Delta t a_{n+1}(u_h^{n+1}, v_h) + \Delta t b_{n+1}(u_h^{n+1}, v_h; \mathbf{w}_{h,\Delta t}^{n+1}) \\ \qquad\qquad\qquad = \Delta t(f(t^{n+1}), v_h)_{n+1} \qquad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\ u_h^0 = u_{h,0}. \end{cases}$$

Notice that in this case we set $\mathcal{Q}_a(F) = \mathcal{Q}_b(F) = \Delta t F(t^{n+1})$, which is a quadrature formula which is exact for constant polynomials only. Hence, the GCL condition (38) is not satisfied in this case. However, we can prove that, if we choose a time step sufficiently small, then the scheme is stable in the sense of definition 1.

For each $n = 0 \ldots, N$ let $v_h^n \in \mathcal{H}_h(\Omega_{t^n})$, we have (see the left hand side of (39), with $v_h = v_h^n$)

$$\left(v_h^{n+1}, v_h^{n+1}\right)_{n+1} - (v_h^n, \mathcal{M}_n(v_h^{n+1}))_n + \Delta t a_{n+1}(v_h^{n+1}, v_h^{n+1}) + \Delta t b_{n+1}(v_h^{n+1}, v_h^{n+1}; \mathbf{w}_{h,\Delta t}^{n+1})$$

$$(40) \qquad \begin{aligned} &= \frac{1}{2}\|v_h^{n+1}\|_{n+1}^2 - \frac{1}{2}\|v_h^n\|_n^2 + \frac{1}{2}\|\mathcal{M}_n(v_h^{n+1}) + v_h^n\|_n^2 \\ &\quad + \Delta t\mu\|\nabla_{\mathbf{x}} v_h^{n+1}\|_{n+1}^2 + \frac{1}{2}\Delta t \int_{\Omega_{n+1}} \operatorname{div} \boldsymbol{\beta}(v_h^{n+1})^2 \, d\mathbf{x} \\ &\quad + \frac{1}{2}\|v_h^{n+1}\|_{n+1}^2 - \frac{1}{2}\|\mathcal{M}_n(v_h^{n+1})\|_n^2 - \frac{\Delta t}{2} \int_{\Omega_{n+1}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2 \, d\mathbf{x}. \end{aligned}$$

For simplicity we used the notation $\|\cdot\|_n = \|\cdot\|_{L^2(\Omega_n)}$. Moreover, we integrated by parts the transports terms in the bilinear forms a and b. Thanks to (16) we have:

$$
(41) \quad
\begin{aligned}
\frac{1}{2}\|v_h^{n+1}\|_{n+1}^2 - \frac{1}{2}\|\mathcal{M}_n(v_h^{n+1})\|_n^2 &- -\frac{\Delta t}{2}\int_{\Omega_{n+1}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2\, d\mathbf{x} \\
&= \frac{1}{2}\int_{t^n}^{t^{n+1}}\int_{\Omega_t} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(\mathcal{M}_t(v_h^{n+1}))^2\, d\mathbf{x}\, dt - \frac{\Delta t}{2}\int_{\Omega_{n+1}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2\, d\mathbf{x},
\end{aligned}
$$

which can be read as a quadrature formula error.

Let us remind that $\mathbf{w}_{h,\Delta t}^{n+1}$ is constant in the interval $]t^n, t^{n+1}]$ and observe that $\mathcal{M}_t(v_h^{n+1})$ and v_h^{n+1} attain the same values at the corresponding nodes of the moving mesh; hence we can write

$$
\begin{aligned}
\frac{1}{2}\int_{t^n}^{t^{n+1}}\int_{\Omega_t} &\operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(\mathcal{M}_t(v_h^{n+1}))^2\, d\mathbf{x}\, dt - \frac{\Delta t}{2}\int_{\Omega_{n+1}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2\, d\mathbf{x} \\
&= \frac{1}{2}\int_{t^n}^{t^{n+1}} \left(\int_{\Omega_t} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(\mathcal{M}_t(v_h^{n+1}))^2\, d\mathbf{x} - \int_{\Omega_{n+1}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2\, d\mathbf{x} \right) dt \\
&= \frac{1}{2}\int_{t^n}^{t^{n+1}} \int_{\Omega_0} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(\mathcal{M}_0(v_h^{n+1}))^2 \left(J_t - J_{t^{n+1}} \right) d\mathbf{x}\, dt.
\end{aligned}
$$

Therefore, it remains to estimate the difference between the two Jacobians. Thanks to (31) and (32) we have:

$$
(42) \qquad |J_t - J_{t^{n+1}}| \leq C\Delta t \|D_{\boldsymbol{\xi}}\tilde{\mathbf{w}}_{h,\Delta t}\|_{L^\infty(\Omega_0)}\|D_{\boldsymbol{\xi}}\mathcal{A}_{h,\Delta t}\|_{L^\infty(\Omega_0)},
$$

where C is independent of h, Δt, and the mapping, and $D_{\boldsymbol{\xi}}$ denotes the spatial derivative on the reference domain.

Inserting (42) in the previous inequality and going back to the current domain we have the following estimate for the quadrature error term, thanks to (22):

$$
\begin{aligned}
\frac{1}{2}\int_{t^n}^{t^{n+1}}\int_{\Omega_t} &\operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(\mathcal{M}_t(v_h^{n+1}))^2\, d\mathbf{x}\, dt - \frac{\Delta t}{2}\int_{\Omega_{n+1}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2\, d\mathbf{x} \\
&\leq \bar{C}\Delta t^2 \int_{\Omega_{n+1}} |\operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}|(v_h^{n+1})^2\, d\mathbf{x}
\end{aligned}
$$

where $\bar{C} = C\|D_{\boldsymbol{\xi}}\tilde{\mathbf{w}}_{h,\Delta t}\|_{L^\infty(\Omega_0)}\|D_{\boldsymbol{\xi}}\mathcal{A}_{h,\Delta t}\|_{L^\infty(\Omega_0)}$.

Hence we have proved that the implicit Euler scheme satisfies the following stability condition:

$$
(43) \quad
\begin{aligned}
\frac{1}{2}\|v_h^{n+1}\|_{n+1}^2 - \frac{1}{2}\|v_h^n\|_n^2 & \\
+ \Delta t \mu \|\nabla_{\mathbf{x}} v_h^{n+1}\|_{n+1}^2 &+ \frac{1}{2}\Delta t \int_{\Omega_{n+1}} \operatorname{div} \boldsymbol{\beta}(v_h^{n+1})^2\, d\mathbf{x} \\
- \bar{C}\Delta t^2 \int_{\Omega_{n+1}} &|\operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}|(v_h^{n+1})^2\, d\mathbf{x} \\
&\leq (v_h^{n+1}, v_h^{n+1})_{n+1} - (v_h^n, \mathcal{M}_n(v_h^{n+1}))_n \\
&+ \Delta t\, a_{n+1}(v_h^{n+1}, v_h^{n+1}) + \Delta t\, b_{n+1}(v_h^{n+1}, v_h^{n+1}; \mathbf{w}_{h,\Delta t}^{n+1}),
\end{aligned}
$$

which differs from (35) because of the last term before the inequality sign.

We observe that we can still derive an a priori estimate like (36) if we choose Δt sufficiently small. Moreover, one could exploit the sign of the right hand side of (41) in the case of dilatation of the domain, but this is beyond the aims of this paper.

Lemma 1. *For $n = 0, \ldots, N$, let $u_h^n \in \mathcal{H}_h(\Omega_{t^n})$ be the solution of (39). Then there exists $\overline{\Delta t}$ small enough, so that the following a priori estimate holds true for all $\Delta t \leq \overline{\Delta t}$:*

$$
(44) \quad
\begin{aligned}
&\|u_h^{n+1}\|_{n+1}^2 + \Delta t \mu \sum_{i=0}^{n} \|\nabla_{\mathbf{x}} u_{h,i+1}\|_{i+1}^2 \\
&\leq C \left(\|u_{h,0}\|_{L^2(\Omega_0)}^2 + \sum_{i=1}^{n+1} \|f(t^i)\|_{H^{-1}(\Omega_{t^i})}^2 \right).
\end{aligned}
$$

The proof is based on (43) and the discrete Gronwall's lemma with Δt such that

$$
\bar{C} \Delta t^2 \| \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1} \|_{L^\infty(\Omega_{n+1})} + \frac{\Delta t}{2} \| \operatorname{div} \boldsymbol{\beta} \|_{L^\infty(\Omega_{n+1})} \leq \frac{1}{2}.
$$

Notice that in the limit as Δt goes to zero, this constraint is not too restrictive; however, in practical computation this can increase dramatically the already high computational cost of an unsteady flow simulation and estimating a priori the maximum time step could be a cumbersome task.

We end this subsection devoted to the implicit Euler scheme with an error estimate. Since we already have an estimate of the difference between the continuous solution of (17) and that of the semidiscrete problem (25), it is enough to bound $u_h^n - u_h(t^n)$. Using the stability inequality (43) and the error equation (37) with $v_h = u_h^{n+1} - u_h(t^{n+1})$, we obtain:

$$
\begin{aligned}
&\frac{1}{2} \|u_h^{n+1} - u_h(t^{n+1})\|_{n+1}^2 - \frac{1}{2} \|u_h^n - u_h(t^n)\|_n^2 \\
&\quad + \Delta t \mu \|\nabla_{\mathbf{x}} u_h^{n+1} - u_h(t^{n+1})\|_{n+1}^2 + \frac{1}{2} \Delta t \int_{\Omega_{n+1}} \operatorname{div} \boldsymbol{\beta} (u_h^{n+1} - u_h(t^{n+1}))^2 \, d\mathbf{x} \\
&\quad - \bar{C} \Delta t^2 \int_{\Omega_{n+1}} |\operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1}| (u_h^{n+1} - u_h(t^{n+1}))^2 \, d\mathbf{x} \\
&\leq (u_h^{n+1} - u_h(t^{n+1}), u_h^{n+1} - u_h(t^{n+1}))_{t^{n+1}} - (u_h^n - u_h(t^n), \mathcal{M}_n(u_h^{n+1} - u_h(t^{n+1})))_n \\
&\quad + a_{n+1}(u_h^{n+1} - u_h(t^{n+1}), u_h^{n+1} - u_h(t^{n+1})) + b_{n+1}(u_h^{n+1} - u_h(t^{n+1}), u_h^{n+1} - u_h(t^{n+1}); \mathbf{w}_{h,\Delta t}^{n+1}) \\
&= \int_{t^n}^{t^{n+1}} a_t(u_h(t), \mathcal{M}_t(u_h^{n+1} - u_h(t^{n+1}))) \, dt - a_{n+1}(u_h(t^{n+1}), u_h^{n+1} - u_h(t^{n+1})) \\
&\quad + \int_{t^n}^{t^{n+1}} b_t(u_h(t), \mathcal{M}_t(u_h^{n+1} - u_h(t^{n+1})); \mathbf{w}_h(t)) \, dt - b_{n+1}(u_h(t^{n+1}), u_h^{n+1} - u_h(t^{n+1}); \mathbf{w}_{h,\Delta t}^{n+1}) \\
&\quad + \int_{t^n}^{t^{n+1}} (f(t), \mathcal{M}_t(u_h^{n+1} - u_h(t^{n+1})))_t \, dt - (f(t^{n+1}), u_h^{n+1} - u_h(t^{n+1}))_{n+1}.
\end{aligned}
$$

The three terms on the right hand side can be estimated as we have done for (41) with the appropriate changes (see [6] for the details). Hence after some work using also (26), we

obtain:

$$\|u_h^{n+1} - u(t^{n+1})\|_{n+1}^2 + \Delta t\mu \sum_{i=0}^{n} \|\nabla_{\mathbf{x}}(u_{h,i+1} - u(t^{i+1}))\|_{i+1}^2$$

$$\leq C\|u_{h,0} - u_0\|_{L^2(\Omega_0)}^2 + Ch^{2k}\left(\|u(t)\|_{H^k(\Omega_t)}^2 + \int_0^{t^{n+1}}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^k(\Omega_s)}^2 + \|u\|_{H^{k+1}(\Omega_s)}^2\right)ds\right)$$

$$+ C\Delta t^2 \int_0^{t^{n+1}}\left(\left\|\nabla_{\mathbf{x}}\frac{\partial u_h}{\partial t}\Big|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)} + \left\|\frac{\partial u_h}{\partial t}\Big|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)}^2 + \|\nabla_{\mathbf{x}}u_h\|_{L^2(\Omega_t)}^2\right.$$

$$\left.+ \left\|\frac{\partial f}{\partial t}\Big|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)}^2 + \|f\|_{L^2(\Omega_t)}^2\right)dt + C\Delta t^3 \sum_{i=0}^{n}\left\|\nabla_{\mathbf{x}}\frac{\partial u_h}{\partial t}\Big|_{\boldsymbol{\xi}}\right\|_{i+1}^2,$$

where the constants C might depend on the mapping and the velocity domain. We refer also to [15] for a different proof.

4.2. First order method satisfying GCL condition.

We have seen in the previous subsection that the scheme (39) does not satisfy the GCL. In this subsection we present a modification of the implicit Euler scheme which satisfies the GCL. The idea consists in using a midpoint rule in (38), so that the integration scheme reads as follows:

$$(45)\quad\begin{cases} \text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\ (u_h^{n+1}, v_h)_{n+1} - (u_h^n, \mathcal{M}_n(v_h))_n + \Delta t a_{t^{n+1/2}}(\mathcal{M}_{n+1/2}(u_h^{n+1}), \mathcal{M}_{n+1/2}(v_h)) \\ \quad + \Delta t b_{t^{n+1/2}}(\mathcal{M}_{n+1/2}(u_h^{n+1}), \mathcal{M}_{n+1/2}(v_h); \mathbf{w}_{h,\Delta t}) \\ \quad\quad = \Delta t(f(t^{n+1/2}), \mathcal{M}_{n+1/2}(v_h))_{t^{n+1/2}} \quad\quad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\ u_h^0 = u_{h,0}. \end{cases}$$

This method was proposed in [4], where the stability analysis was performed. More precisely, it is shown that this method fulfills the stability condition (35) we have introduced here, without any restriction on the time step. In [6] this fact was the starting point to prove the error estimate with the same techniques we used in this paper.

4.3. Crank–Nicolson scheme.

The application of Crank–Nicolson method in (34) can be written in a straightforward way as follows:

$$(46)\quad\begin{cases} \text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\ (u_h^{n+1}, v_h)_{n+1} - (u_h^n, \mathcal{M}_n(v_h))_n + \frac{\Delta t}{2}(a_{n+1}(u_h^{n+1}, v_h) + a_n(u_h^n, \mathcal{M}_n(v_h))) \\ \quad + \frac{\Delta t}{2}(b_{n+1}(u_h^{n+1}, v_h; \mathbf{w}_{h,\Delta t}^{n+1}) + b_n(u_h^n, \mathcal{M}_n(v_h); \mathbf{w}_{h,\Delta t}^{n+1})) \\ \quad\quad = \frac{\Delta t}{2}((f(t^{n+1}), v_h)_{n+1} + (f(t^n), \mathcal{M}_n(v_h))_n) \quad\quad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\ u_h^0 = u_{h,0}, \end{cases}$$

Since Crank–Nicolson scheme is associated with the trapezoidal rule, we have that the GCL (38) is obviously satisfied. However, it is not clear whether the present scheme is stable in the sense of definition 1. Namely, let us take $\alpha = \beta = 1$ in the right hand side of (35), then with some computations, using the GCL condition (38), integration by parts,

and the identity $(u, u + v) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u + v\|^2 - \frac{1}{2}\|v\|^2$, we get:

$$
(v_h^{n+1}, v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))_{n+1} - (v_h^n, \mathcal{M}_n(v_h^{n+1}) + v_h^n)_n
$$

$$
+ \frac{\Delta t}{2}(a_{n+1}(v_h^{n+1}, v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n)) + a_n(v_h^n, \mathcal{M}_n(v_h^{n+1}) + v_h^n))
$$

$$
+ \frac{\Delta t}{2}(b_{n+1}(v_h^{n+1}, v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n); \mathbf{w}_{h,\Delta t}^{n+1}) + b_n(v_h^n, \mathcal{M}_n(v_h^{n+1}) + v_h^n; \mathbf{w}_{h,\Delta t}^{n+1}))
$$

$$
= \|v_h^{n+1}\|_{n+1}^2 - \|v_h^n\|_n^2 + \frac{\Delta t}{4}\mu\left(\|\nabla_{\mathbf{x}}(v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))\|_{n+1}^2 + \|\nabla_{\mathbf{x}}(\mathcal{M}_n(v_h^{n+1}) + v_h^n)\|_n^2\right)
$$

$$
+ \frac{\Delta t}{8}\left(\int_{\Omega_{n+1}} \operatorname{div}\boldsymbol{\beta}(t^{n+1})(v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))^2 d\mathbf{x} + \int_{\Omega_n} \operatorname{div}\boldsymbol{\beta}(t^n)(\mathcal{M}_n(v_h^{n+1}) + v_h^n)^2 d\mathbf{x}\right)
$$

$$
- \frac{\Delta t}{4}\left(\int_{\Omega_{n+1}} \operatorname{div}\mathbf{w}_{h,\Delta t}^{n+1}(v_h^{n+1})^2 d\mathbf{x} + \int_{\Omega_n} \operatorname{div}\mathbf{w}_{h,\Delta t}^{n+1}(v_h^n)^2 d\mathbf{x}\right.
$$

$$
\left. - \int_{\Omega_{n+1}} \operatorname{div}\mathbf{w}_{h,\Delta t}^{n+1}v_h^{n+1}\mathcal{M}_{n+1}(v_h^n) d\mathbf{x} - \int_{\Omega_n} \operatorname{div}\mathbf{w}_{h,\Delta t}^{n+1}\mathcal{M}_n(v_h^{n+1})v_h^n d\mathbf{x}\right)
$$

$$
+ \frac{\Delta t}{4}(a_{n+1}(v_h^{n+1}, v_h^{n+1}) - a_n(\mathcal{M}_n(v_h^{n+1}), \mathcal{M}_n(v_h^{n+1}))
$$

$$
- a_{n+1}(\mathcal{M}_{n+1}(v_h^n), \mathcal{M}_{n+1}(v_h^n)) + a_n(v_h^n, v_h^n)).
$$

Let us consider the terms in the last two lines which contain v_h^{n+1} (the others can be treated similarly); we have after integration by parts:

$$
a_{n+1}(v_h^{n+1}, v_h^{n+1}) - a_n(\mathcal{M}_n(v_h^{n+1}), \mathcal{M}_n(v_h^{n+1})) =
$$

$$
\mu\left(\int_{\Omega_{n+1}} (\nabla_{\mathbf{x}} v_h^{n+1})^2 d\mathbf{x} - \int_{\Omega_n} (\nabla_{\mathbf{x}}\mathcal{M}_n(v_h^{n+1}))^2 d\mathbf{x}\right)
$$

$$
- \frac{1}{2}\left(\int_{\Omega_{n+1}} \operatorname{div}\boldsymbol{\beta}(t^{n+1})(v_h^{n+1})^2 d\mathbf{x} - \int_{\Omega_n} \operatorname{div}\boldsymbol{\beta}(t^n)(\mathcal{M}_n(v_h^{n+1}))^2 d\mathbf{x}\right).
$$

As done in Subsection 4.1, we can write the integrals on the reference domain and estimate the difference of the mapping at two successive times. Hence we have:

$$
\left|\int_{\Omega_{n+1}} (\nabla_{\mathbf{x}} v_h^{n+1})^2 d\mathbf{x} - \int_{\Omega_n} (\nabla_{\mathbf{x}}\mathcal{M}_n(v_h^{n+1}))^2 d\mathbf{x}\right| \le C\Delta t\|\nabla_{\mathbf{x}} v_h^{n+1}\|_{n+1}^2 \le C\Delta t h^{-2}\|v_h^{n+1}\|_{n+1}^2,
$$

$$
\left|\int_{\Omega_{n+1}} \operatorname{div}\boldsymbol{\beta}(t^{n+1})(v_h^{n+1})^2 d\mathbf{x} - \int_{\Omega_n} \operatorname{div}\boldsymbol{\beta}(t^n)(\mathcal{M}_n(v_h^{n+1}))^2 d\mathbf{x}\right| \le C\Delta t\|v_h^{n+1}\|_{n+1}^2,
$$

where the constants C depend on $\|D_{\boldsymbol{\xi}}\tilde{\mathbf{w}}_{h,\Delta t}\|_{L^\infty(\Omega_0)}$ and $\|D_{\boldsymbol{\xi}}\mathcal{A}_{h,\Delta t}\|_{L^\infty(\Omega_0)}$. We have used also an inverse inequality which is valid for quasiuniform meshes on the reference domain.

Dealing with the other terms in a similar way we obtain the following inequality:
(47)

$$\|v_h^{n+1}\|_{n+1}^2 - \|v_h^n\|_n^2 + \frac{\Delta t}{4}\mu\left(\|\nabla_{\mathbf{x}}(v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))\|_{n+1}^2 + \|\nabla_{\mathbf{x}}(\mathcal{M}_n(v_h^{n+1}) + v_h^n)\|_n^2\right)$$

$$+ \frac{\Delta t}{8}\left(\int_{\Omega_{n+1}} \operatorname{div}\boldsymbol{\beta}(t^{n+1})(v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))^2 d\mathbf{x} + \int_{\Omega_n} \operatorname{div}\boldsymbol{\beta}(t^n)(\mathcal{M}_n(v_h^{n+1}) + v_h^n)^2 d\mathbf{x}\right)$$

$$- \bar{K}\Delta t(\|v_h^{n+1}\|_{n+1}^2 + \|v_h^n\|_n^2)$$

$$= (v_h^{n+1}, v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))_{n+1} - (v_h^n, \mathcal{M}_n(v_h^{n+1}) + v_h^n)_n$$

$$+ \frac{\Delta t}{2}(a_{n+1}(v_h^{n+1}, v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n)) + a_n(v_h^n, \mathcal{M}_n(v_h^{n+1}) + v_h^n))$$

$$+ \frac{\Delta t}{2}(b_{n+1}(v_h^{n+1}, v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n); \mathbf{w}_{h,\Delta t}^{n+1}) + b_n(v_h^n, \mathcal{M}_n(v_h^{n+1}) + v_h^n; \mathbf{w}_{h,\Delta t}^{n+1})),$$

where

(48) $$\bar{K} = C(1 + \Delta t\|D_{\boldsymbol{\xi}}\tilde{\mathbf{w}}_{h,\Delta t}\|_{L^\infty(\Omega_0)}\|D_{\boldsymbol{\xi}}\mathcal{A}_{h,\Delta t}\|_{L^\infty(\Omega_0)}\|div\,\mathbf{w}_{h,\Delta t}^{n+1}\|_{L^\infty(\Omega_0)} + \Delta t h^{-2}).$$

The inequality (47) still represents a stability inequality under some restriction on the time step. Hence we have the following a priori estimate:

Lemma 2. *For $n = 0, \ldots, N$, let $u_h^n \in \mathcal{H}_h(\Omega_{t^n})$ be the solution of (46). There exists $\overline{\Delta t}$ satisfying:*

$$\overline{\Delta t}h^{-2} \leq 1 \qquad and \qquad \bar{K}\overline{\Delta t} \leq 1/2,$$

$\bar{K}$ given by (48), such that, for all $\Delta t \leq \overline{\Delta t}$, the following a priori estimate holds true:

$$\|u_h^{n+1}\|_{n+1}^2 + \frac{\Delta t}{4}\mu\left(\|\nabla_{\mathbf{x}}(v_h^{n+1} + \mathcal{M}_{n+1}(v_h^n))\|_{n+1}^2 + \|\nabla_{\mathbf{x}}(\mathcal{M}_n(v_h^{n+1}) + v_h^n)\|_n^2\right)$$

(49)

$$\leq C\left(\|u_{h,0}\|_{L^2(\Omega_0)}^2 + \sum_{i=1}^{n+1}\|f(t^i)\|_{H^{-1}(\Omega_{t^i})}^2\right).$$

We observe that, while Crank–Nicolson method is unconditionally stable for parabolic problems on fixed domain (see, e.g. [20]), in the case of moving domain we have obtained a constraint on Δt which relates Δt both to the meshsize h and the velocity domain.

Lemma 3. *Under the same restriction on Δt as in Lemma 2, and suitable regularity assumptions on the solution to (17) and (25), we obtain the following second order error estimate:*

$$\|u_h^{n+1} - u(t^{n+1})\|_{n+1}^2 + \Delta t\mu\sum_{i=0}^n\|\nabla_{\mathbf{x}}(u_{h,i+1} - u(t^{i+1}))\|_{i+1}^2$$

$$\leq C\|u_{h,0} - u_0\|_{L^2(\Omega_0)}^2 + Ch^{2k}\left(\|u(t)\|_{H^k(\Omega_t)}^2 + \int_0^{t^{n+1}}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^k(\Omega_s)}^2 + \|u\|_{H^{k+1}(\Omega_s)}^2\right)ds\right)$$

$$+ C\Delta t^4\int_0^{t^{n+1}}\sum_{i=1}^2\left(\left\|\nabla_{\mathbf{x}}\frac{\partial^i u_h}{\partial t^i}\Big|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)} + \left\|\frac{\partial^i u_h}{\partial t^i}\Big|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)}^2 + \left\|\frac{\partial^i f}{\partial t^i}\Big|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)}^2\right)dt$$

$$+ C\Delta t^4\int_0^{t^{n+1}}\left(\|\nabla_{\mathbf{x}}u_h\|_{L^2(\Omega_t)}^2 + \|f\|_{L^2(\Omega_t)}^2\right)dt + C\Delta t^5\sum_{i=0}^n\left\|\nabla_{\mathbf{x}}\frac{\partial^2 u_h}{\partial t^2}\Big|_{\boldsymbol{\xi}}\right\|_{i+1}^2.$$

The proof of this error estimate is based on the error equation (37), with $v_h = (u_h^{n+1} - u_h(t^{n+1})) + \mathcal{M}_{n+1}(u_h^n - u_h(t^n))$, and on (47). As an example, we show the details of the proof of the last term in (37). One easily verifies that

$$\int_{t^n}^{t^{n+1}} (f(t), v_h)_t \, dt - \frac{\Delta t}{2}(f(t^{n+1}), v_h)_{n+1} + (f(t^n), v_h)_n$$

$$= \int_{t^n}^{t^{n+1}} (t^{n+1} - s)(t^n - s)\frac{d^2}{dt^2}\int_{\Omega_t} f(t)v_h \, d\mathbf{x}dt.$$

Applying (16) twice (recall that v_h is the mapping through $\mathcal{A}_{h,t^{n+1}}$ of a certain $\hat{v}_h$ defined on Ω_0), we have

$$\frac{d^2}{dt^2}\int_{\Omega_t} f(t)v_h \, d\mathbf{x} = \int_{\Omega_t}\left(\left.\frac{\partial^2 f}{\partial t^2}\right|_{\boldsymbol{\xi}} + \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\xi}} \operatorname{div} \mathbf{w} + f \operatorname{div}\left(\left.\frac{\partial \mathbf{w}}{\partial t}\right|_{\boldsymbol{\xi}}\right) + f(\operatorname{div} \mathbf{w})^2\right)v_h \, d\mathbf{x}$$

$$\leq C\left(\left\|\left.\frac{\partial^2 f}{\partial t^2}\right|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)} + \left\|\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)} + \|f\|_{L^2(\Omega_t)}\right)\|v_h\|_{n+1}.$$

Therefore we have

$$\int_{t^n}^{t^{n+1}} (f(t), v_h)_t \, dt - \frac{\Delta t}{2}(f(t^{n+1}), v_h)_{n+1} + (f(t^n), v_h)_n$$

$$\leq C\Delta t^{5/2}\int_{t^n}^{t^{n+1}}\left(\left\|\left.\frac{\partial^2 f}{\partial t^2}\right|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)} + \left\|\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\xi}}\right\|_{L^2(\Omega_t)} + \|f\|_{L^2(\Omega_t)}\right) dt \, \|v_h\|_{n+1}.$$

A possible way to circumvent the restriction on Δt to be of the same order as h^2 is to use the following modification of the Crank–Nicolson method as suggested in [15]:

$$(50) \quad \begin{cases} \text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\ (u_h^{n+1}, v_h)_{n+1} - (u_h^n, v_h)_n + \Delta t a_{t^{n+1/2}}\left(\dfrac{\mathcal{M}_{n+1/2}(u_h^{n+1} + u_h^n)}{2}, v_h\right) \\ \quad + \Delta t b_{t^{n+1/2}}\left(\dfrac{\mathcal{M}_{n+1/2}(u_h^{n+1} + u_h^n)}{2}, v_h; \mathbf{w}_{h,\Delta t}^{n+1/2}\right) \\ \qquad = \Delta t(f(t^{n+1/2}), v_h)_{t^{n+1/2}} \qquad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\ u_h^0 = u_{h,0}. \end{cases}$$

In [15] it has been shown the following inequality:

$$\|u_h^{n+1}\|_{n+1}^2 + \frac{\Delta t}{2}\mu\|\nabla_{\mathbf{x}}\mathcal{M}_{n+1/2}(u_h^{n+1} + u_h^n)\|_{n+1/2}^2$$

$$(51) \quad - \frac{\Delta t}{4}\int_{\Omega_{t^{n+1/2}}} \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1/2}|\mathcal{M}_{n+1/2}(u_h^{n+1}) - \mathcal{M}_{n+1/2}(u_h^n)|^2 d\mathbf{x}$$

$$\leq \|u_h^n\|_n^2 + C\Delta t\|f(t^{n+1/2})\|_{H^{-1}(\Omega_{t^{n+1/2}})}^2,$$

which yields an a priori estimate under the constraint that Δt is sufficiently small with respect to the domain velocity. Working as above one could prove also an error estimate similar to that of Lemma 3.

4.4. **Second order BDF scheme.** In this subsection we briefly discuss the properties of the second order BDF scheme. Since the second order BDF method is a two–step method, we introduce the following discretization of the mapping which uses quadratic interpolation in time in the interval $[t^n, t^{n+1}]$:

$$(52) \quad \begin{aligned} \mathcal{A}_{h,\Delta t}(\boldsymbol{\xi}, t) &= \frac{(t - t^n)(t - t^{n-1})}{2\Delta t^2} \mathcal{A}_{h,t^{n+1}}(\boldsymbol{\xi}) \\ &\quad - \frac{(t - t^{n+1})(t - t^{n-1})}{\Delta t^2} \mathcal{A}_{h,t^n}(\boldsymbol{\xi}) + \frac{(t - t^{n+1})(t - t^n)}{2\Delta t^2} \mathcal{A}_{h,t^{n-1}}(\boldsymbol{\xi}), \end{aligned}$$

so that the domain velocity has the following representation:

$$(53)$$
$$\tilde{\mathbf{w}}_{h,\Delta t}^{n+1}(\boldsymbol{\xi}, t) = \frac{2t - t^n - t^{n-1}}{2\Delta t^2} \mathcal{A}_{h,t^{n+1}}(\boldsymbol{\xi}) - \frac{2t - t^{n+1} - t^{n-1}}{\Delta t^2} \mathcal{A}_{h,t^n}(\boldsymbol{\xi}) + \frac{2t - t^{n+1} - t^n}{2\Delta t^2} \mathcal{A}_{h,t^{n-1}}(\boldsymbol{\xi})$$
$$\mathbf{w}_{h,\Delta t}^{n+1}(\mathbf{x}, t) = \tilde{\mathbf{w}}_{h,\Delta t}^{n+1}(\boldsymbol{\xi}, t) \circ \mathcal{A}_{h,\Delta t}^{-1}(\boldsymbol{\xi}, t).$$

Then the two step BDF method applies to (25) as follows:

$$(54) \quad \begin{cases} \text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\ \frac{3}{2}(u_h^{n+1}, v_h)_{n+1} - 2(u_h^n, \mathcal{M}_n(v_h))_n + \frac{1}{2}(u_h^{n-1}, \mathcal{M}_{n-1}(v_h))_{n-1} + \Delta t a_{n+1}(u_h^{n+1}, v_h) \\ \quad + \Delta t b_{n+1}(u_h^{n+1}, v_h; \mathbf{w}_{h,\Delta t}^{n+1}) = \Delta t(f(t^{n+1}), v_h)_{n+1} \qquad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\ u_h^0 = u_{h,0}, \quad u_h^1 \text{ given by Crank-Nicolson method.} \end{cases}$$

First af all, we observe that for this method we need a different approach than for the previous ones, because the time advancing scheme is based on differentiation instead of integration. A first consequence of this fact is that the discrete GCL has to be written in a different way, that is:

$$(55) \quad \begin{aligned} \frac{3}{2} \int_{\Omega_{n+1}} \varphi_i(t^{n+1})\varphi_j(t^{n+1})d\mathbf{x} &- 2 \int_{\Omega_n} \varphi_i(t^n)\varphi_j(t^n)d\mathbf{x} + \frac{1}{2} \int_{\Omega_{n-1}} \varphi_i(t^{n-1})\varphi_j(t^{n-1})d\mathbf{x} \\ &= \Delta t \int_{\Omega_{n+1}} \varphi_i(t^{n+1})\varphi_j(t^{n+1}) \operatorname{div} \mathbf{w}_{h,\Delta t}^{n+1} d\mathbf{x} \end{aligned}$$

Notice that, thanks to (16), the integral on the right hand side is equal to

$$\frac{d}{dt} \int_{\Omega_{n+1}} \varphi_i(t^{n+1})\varphi_j(t^{n+1})d\mathbf{x}.$$

Writing the above integral on the reference domain, we have

$$\int_{\Omega_{n+1}} \varphi_i(t^{n+1})\varphi_j(t^{n+1})d\mathbf{x} = \int_{\Omega_0} \hat{\varphi}_i \hat{\varphi}_j J_{t^{n+1}} d\boldsymbol{\xi}$$

and we see that we are led to a polynomial of degree 4 in time. Since the differentiation formula is exact for polynomials of second degree in time, we conclude that the GCL condition (55) cannot be satisfied.

In [11, 15], a modified version of (54) has been proposed in order to satisfy a GCL condition, which comes from the following integral identity (see (27)):

$$\frac{3}{2}\mathbf{Y}^{n+1} - 2\mathbf{Y}^n + \frac{1}{2}\mathbf{Y}^{n-1} = \frac{3}{2} \int_{t^n}^{t^{n+1}} \mathbf{G}(\mathbf{Y}(t), t)dt - \frac{1}{2} \int_{t^{n-1}}^{t^n} \mathbf{G}(\mathbf{Y}(t), t)dt.$$

One can modify the GCL condition (55) in the following integral form:

$$
(56) \quad
\begin{aligned}
&\frac{3}{2}\int_{\Omega_{n+1}} \varphi_i(t^{n+1})\varphi_j(t^{n+1})d\mathbf{x} - 2\int_{\Omega_n}\varphi_i(t^n)\varphi_j(t^n)d\mathbf{x} + \frac{1}{2}\int_{\Omega_{n-1}}\varphi_i(t^{n-1})\varphi_j(t^{n-1})d\mathbf{x} \\
&\qquad = \frac{3}{2}\mathcal{Q}_b^n\left(\int_{\Omega_t}\varphi_i(t)\varphi_j(t)\operatorname{div}\mathbf{w}(t)d\mathbf{x}\right) - \frac{1}{2}\mathcal{Q}_b^{n-1}\left(\int_{\Omega_t}\varphi_i(t)\varphi_j(t)\operatorname{div}\mathbf{w}(t)d\mathbf{x}\right),
\end{aligned}
$$

where $\mathcal{Q}_b^i$ denotes the quadrature formula approximating the integral from t^i to t^{i+1}.

Then, following [15], the modified formulation of BDF scheme satisfying the GCL condition (56) reads:

$$
(57) \quad
\begin{cases}
\text{for } n = 1, \cdots, N \text{ find } u_h^n \in \mathcal{H}_h(\Omega_{t^n}) \text{ such that} \\
\frac{3}{2}(u_h^{n+1}, v_h)_\nu - 2(u_h^n, \mathcal{M}_n(v_h))_n + \frac{1}{2}(u_h^{n-1}, \mathcal{M}_{n-1}(v_h))_{t-1} + \Delta t a_{n+1}(u_h^{n+1}, v_h) \\
\quad + \frac{3}{2}\Delta t b_{n+1/2}(\mathcal{M}_{n+1/2}(u_h^{n+1}), \mathcal{M}_{n+1/2}(v_h); \mathbf{w}_{h,\Delta t}^{n+1}(t^{n+1/2})) \\
\quad - \frac{\Delta t}{2} b_{n-1/2}(\mathcal{M}_{n-1/2}(u_h^{n+1}), \mathcal{M}_{n-1/2}(v_h); \mathbf{w}_{h,\Delta t}^{n+1}(t^{n-1/2})) \\
\qquad\qquad\qquad\qquad\qquad = \Delta t(f(t^{n+1}), v_h)_{n+1} \qquad \forall v_h \in \mathcal{H}_h(\Omega_{t^{n+1}}), \\
u_h^0 = u_{h,0}, \quad u_h^1 \text{ given by Crank-Nicolson method.}
\end{cases}
$$

In [15] an inequality similar to (51) has been proved, from which, if Δt is chosen sufficiently small with respect to the domain velocity, an a priori estimate like in Lemma 2 and the error estimate can be deduced.

We end this section collecting all the properties of the space-time schemes introduced above, in the following table:

TABLE 1. Properties of the space-time schemes.

	GCL condition	Stability property	Rate of convergence
Implicit Euler IE (39)	NO	Δt small with respect to the domain velocity	$\Delta t + h^{2k}$
modified Implici Euler by midpoint rule mIE (45)	YES	incondiotionally stable $\forall \Delta t$	$\Delta t + h^{2k}$
Crank–Nicolson CN (46)	YES	$\Delta t \approx h^2$ and Δt small with respect to the domain velocity	$\Delta t^2 + h^{2k}$
modified Crank–Nicolson mCN (50)	YES	$\Delta t \approx h^2$ and Δt small with respect to the domain velocity	$\Delta t^2 + h^{2k}$
second order BDF BDF2 (54)	NO	no results	no results
modified BDF2 mBDF (57)	YES	Δt small with respect to the domain velocity	$\Delta t^2 + h^{2k}$

5. NUMERICAL RESULTS

In this section we report some numerical results which confirm the analysis performed in the previous section. We first test the stability and then the accuracy properties of the considered schemes. All our numerical experiments are based on piecewise linear finite elements.

Let us consider the following problem:

$$
(58) \qquad \begin{cases} u_t - .01\Delta u = 0 & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t \\ u_0 = 1600x(1-x)y(1-y) & \text{in } \Omega_0, \end{cases}
$$

where Ω_0 is the unit square $[0,1]^2$ and Ω_t is a dilation of Ω_0 according to the follwing rule:

$$
\mathcal{A}_t(\boldsymbol{\xi}) = \mathbf{x}(\boldsymbol{\xi},t) = (2 - \cos(20\pi t))\boldsymbol{\xi}.
$$

Notice that, in this case, $\mathbf{x}$ is linear with respect to $\boldsymbol{\xi}$, so that no space interpolation is needed. We have interpolated in time the domain motion and velocity according to (31) and (32).

Thanks to (19), the norm $\|u(t)\|_{L^2(\Omega_t)}$ decreases with t. Hence, if the time discretization is stable, we expect the same behavior for the computed solution also.

We have used a triangular mesh generated by dividing each side of Ω_0 into 16 parts and four different values of dt, ranging from .02 up to .0001. Fig. 1 reports the plot of the norm $\|u_h^n\|_{L^2(\Omega_n)}$ for $n = 0, \ldots, N$. First of all, we notice that as Δt gets sufficiently small, all the methods produce a solution with a decreasing norm as espected in the continuous case. When Δt is big with respect to the velocity domain only the implicit Euler scheme preserves the decreasing behavior of the norm of the exact solution, while the Crank–Nicolson method produces high oscillations and the second order BDF scheme give rise to some wiggles. In particular, the oscillations of the modified Crank–Nicolson scheme where predicted by the theoretical result (51).

The next numerical experiments are devoted to check the accuracy of the space-time schemes. We present three examples of domain motion: the initial domain Ω_0 is always the unit square which translates with a rigid motion in the first case, dilates into a rectangle in the second one and , deformation into a trapezoid in the last example. For each test case, we have the following expression for the mapping ($\boldsymbol{\xi} = (\xi, \eta)$):

$$
\mathcal{A}_t^1(\boldsymbol{\xi}) = \mathbf{x}^1(\boldsymbol{\xi},t) = \begin{pmatrix} \xi + \sin(t) \\ \eta + 1 - \cos(t) \end{pmatrix} \quad t \in [0,\pi] \quad \text{rigid motion}
$$

$$
\mathcal{A}_t^2(\boldsymbol{\xi}) = \mathbf{x}^2(\boldsymbol{\xi},t) = (2 - \cos(\pi t))\boldsymbol{\xi} \quad t \in [0,\pi] \qquad \text{dilation}
$$

$$
\mathcal{A}_t^3(\boldsymbol{\xi}) = \mathbf{x}^3(\boldsymbol{\xi},t) = \begin{pmatrix} (1 + t(t-1)\eta)\xi \\ \eta \end{pmatrix} \quad t \in [0,\pi] \quad \text{deformation into a trapezoid}
$$

We shall denote by Ω_t^i the domain obtained by means of the mapping $\mathcal{A}_t^i$, for $i = 1, 2, 3$.

As an example, Fig. 2 reports the pictures of the domain Ω_t at some time t, for each choice of the mapping. For $t \in]0, T[$, we solve on the domain Ω_t the following problem:
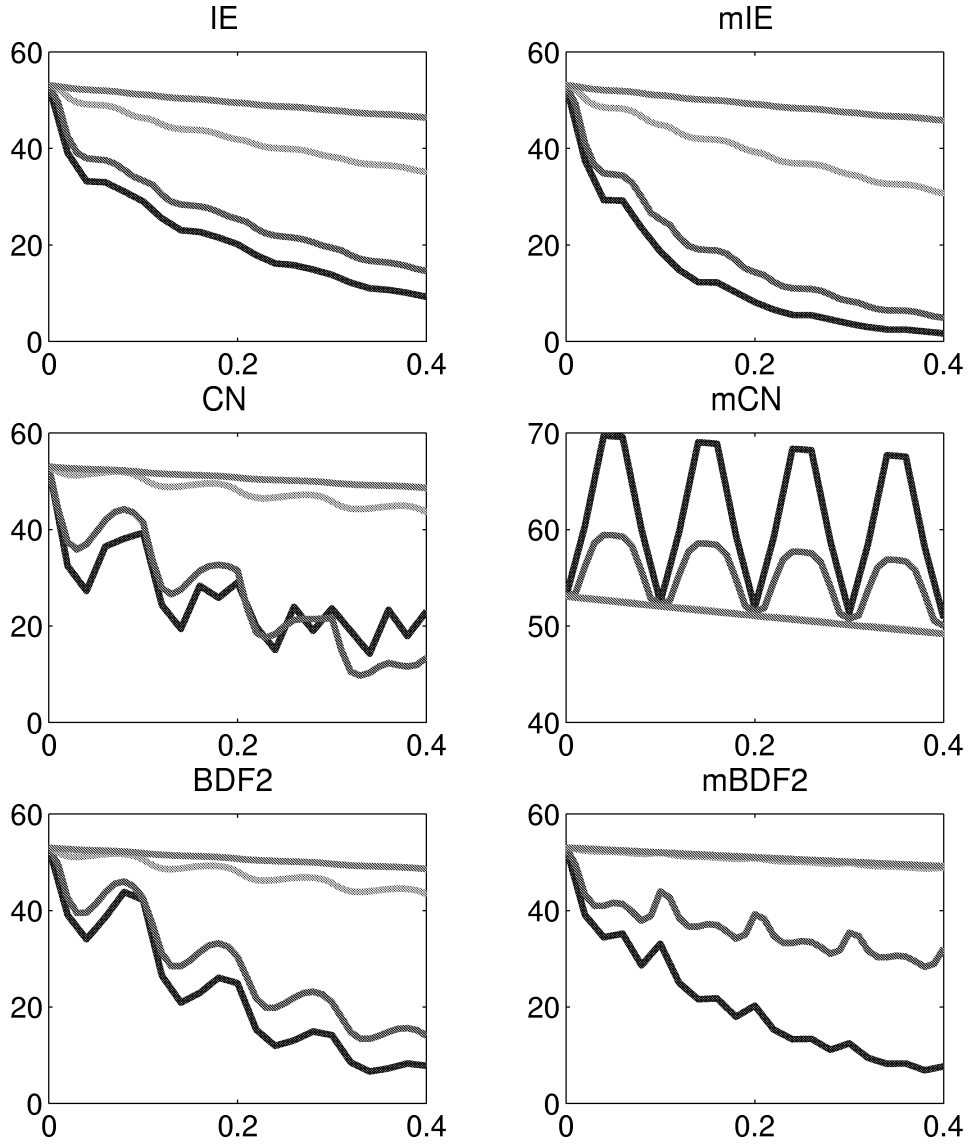
FIGURE 1. L^2-norm of the computed solution of (58) as a function of time. $\Delta t = .02$: blue line. $\Delta t = .01$: red line. $\Delta t = .001$: green line. $\Delta t = .0001$: magenta line.

$$(59) \qquad \begin{cases} u_t - \Delta u = f & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t \\ u(0) = u_0 & \text{in } \Omega_0, \end{cases}$$

with f and u_0 chosen in such a way that the exact solution has the following expressions $(\mathbf{x} = (x, y))$:

$$u^1(\mathbf{x}, t) = (1 + t(1-t)(2-t))(x - \sin t)(1 - x + \sin t)(y - 1 + \cos t)(2 - y - \cos t)$$
$$(x, y) \in \Omega_t^1, \ t \in [0, \pi]$$
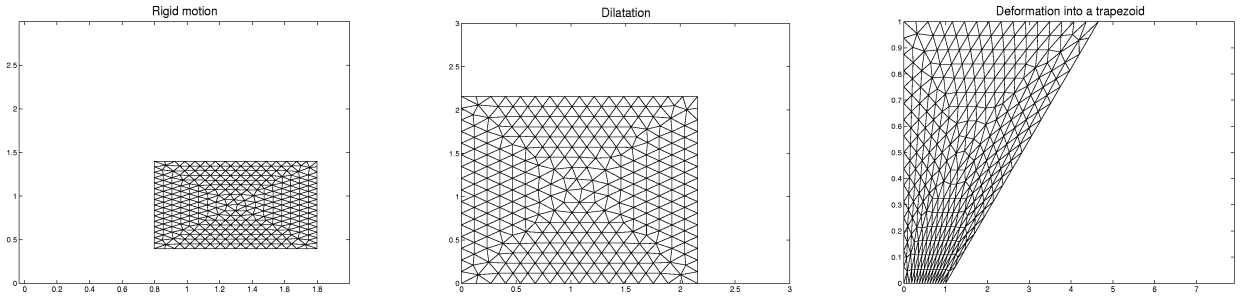
FIGURE 2.   Domain deformations.

$$u^2(\mathbf{x}, t) = (1 - \sin t) \sin \frac{\pi x}{2 - \cos(\pi t)} \sin \frac{\pi y}{2 - \cos(\pi t)} + \sin t \sin \frac{2\pi x}{2 - \cos(\pi t)} \sin \frac{2\pi y}{2 - \cos(\pi t)}$$
$$(x, y) \in \Omega_t^2, \ t \in [0, \pi]$$

$$u^3(\mathbf{x}, t) = (1 + 2\sin t) \sin \frac{\pi x}{1 + t(t-1)y} \sin(\pi y) \qquad (x, y) \in \Omega_t^3, \ t \in [0, \pi]$$

Fig.s 3, 4 and 5 collect the $L^\infty(0, T; L^2(\Omega_t))$-errors, obtained with the various space-time schemes which are plotted in a loglog scale versus $1/h$. More precisely, the blue lines correspond to the choice $\Delta t \approx h^2$ for schemes based on implicit Euler method, and $\Delta t \approx h$ for second order methods based on Crank–Nicolson and BDF. We see that the rates of convergence are the expected ones in all our experiments, with the exception of Crank–Nicolson scheme (46) and second order BDF (54) in Fig. 4. For these two cases we performed the same experiments with $\Delta t \approx h^2$ (the results are plotted with the red line) and we see that in this case we recover the rate of convergence as before. The motivation of this result can be found in the convergence estimate, which is conditioned to choice $\Delta t \approx h^2$, see Lemma 3.

Finally, in Fig.s 6, 7 and 8, we plot the computed solution evaluated at different time.

REFERENCES

1. J. Donea, *Computational methods for transient analysis*, Computational Methods in Mechanics, vol. 1, ch. Arbitrary Lagrangian Eulerian methods, North-Holland, Elsevier, 1983.
2. J. Donea, S. Giuliani, and J.P. Halleux, *An Arbitrary Lagrangian–Eulerian finite element method for transient dynamic fluid-structure interactions*, Comp. Meth. Appl. Mech. Engrg **33** (1982), 689–723.
3. C. Farhat, M. Lesoinne, and N. Maman, *Mixed explicit/implicit time integration of coupled aeroelastic problems: three-field formulation, geometry conservation and distributed solution*, Int. J. Num. Meth. Fluids **21** (1995), no. 10, 807–835.
4. L. Formaggia and F. Nobile, *A stability analysis for the Arbitrary Lagrangian Eulerian formulation with finite elements*, East-West J. Numer. Math. **7** (1999), 105–132.
5. H. Fujita and T. Suzuki, *Evolution problems*, Handbook of numerical analysis. Vol. II (P.G. Ciarlet and J.L. Lions, eds.), North-Holland Publishing Co., 1991, pp. 789–928.
6. L. Gastaldi, *A priori error estimates for the arbitrary Lagrangian Eulerian formulation with finite elements*, East-West J. Numer. Math. **9** (2001), no. 2, 123–156. MR **2002m:**65094
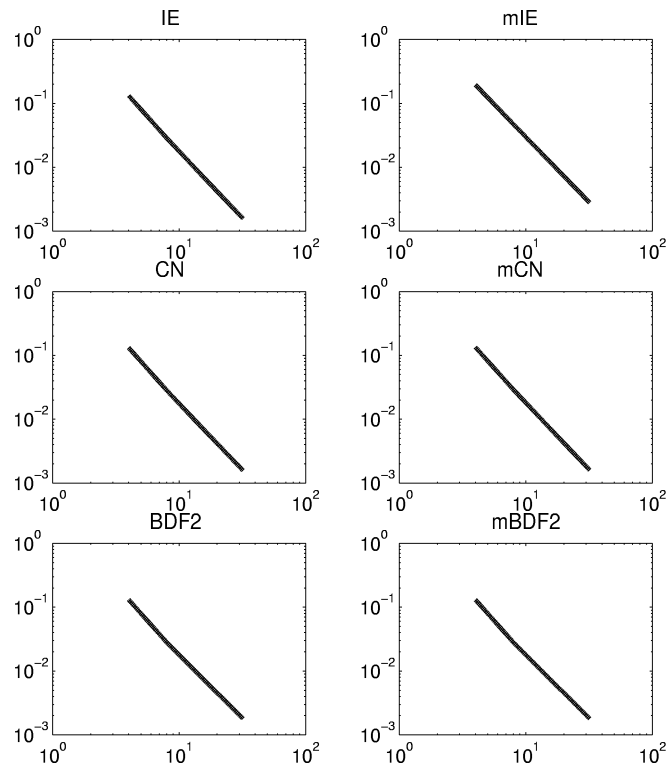
FIGURE 3. Rate of convergence with respect to Δt and h

7. H. Guillard and C. Farhat, *On the significance of the geometric conservation law for flow computational on moving meshes*, Tech. Report 99-0793, AIAA, 1999.

8. Thomas J.R. Hughes and Jerrold E. Marsden, *A short course in fluid mechanics*, Mathematics Lecture Series, Publish or Perish, inc, University of California, Berkeley, 1976.

9. T.J.R. Hughes, W.K. Liu, and T.K. Zimmermann, *Lagrangian–Eulerian finite element formulation for incompressible viscous flows*, Comp. Meth. Appl. Mech. Engrg. **29** (1981), 329–349.

10. D. Jaubert and V. Moreau, *Fluid-structure interaction: applications in the context of a flow in arterial medium*, Tech. report, CSR4, 1997.

11. B. Koobus and Charbel Farhat, *Second-order time-accurate and geometrically conservative implicit schemes for flow computations on unstructured dynamic meshes*, Comput. Meths. Appl. Mech. Engrg. **170** (1999), 103–130.

12. P.L. Le Tallec and J. Mouro, *Structures en grand déplacements couplées à des fluides en mouvement*, Tech. Report 2961, INRIA, 1996.

13. M. Lesoinne and C. Farhat, *Geometric conservation laws for aeroelastic computations using unstructured dynamics meshes*, Tech. Report 95-1709, AIAA, 1995.

14. F. Nobile, *Problemi di interazione fluido-struttura in emodinamica*, Master's thesis, Politecnico di Milano, 1998.

15. _____, *Numerical approximation of fluid-structure interaction problems with application to haemodynamics*, Ph.D. thesis, Department of Mathematics, cole Polytechnique Fdrale de Lausanne, Switzerland, 2001.

16. A. Quarteroni, M. Tuveri, and A. Veneziani, *Computational vascular fluid dynamics: problems, models and methods*, Comput. Visual. Sci. **2** (2000), 163–197.

17. Alfio Quarteroni and Alberto Valli, *Numerical approximation of partial differential equations*, Springer Verlag, 1994.
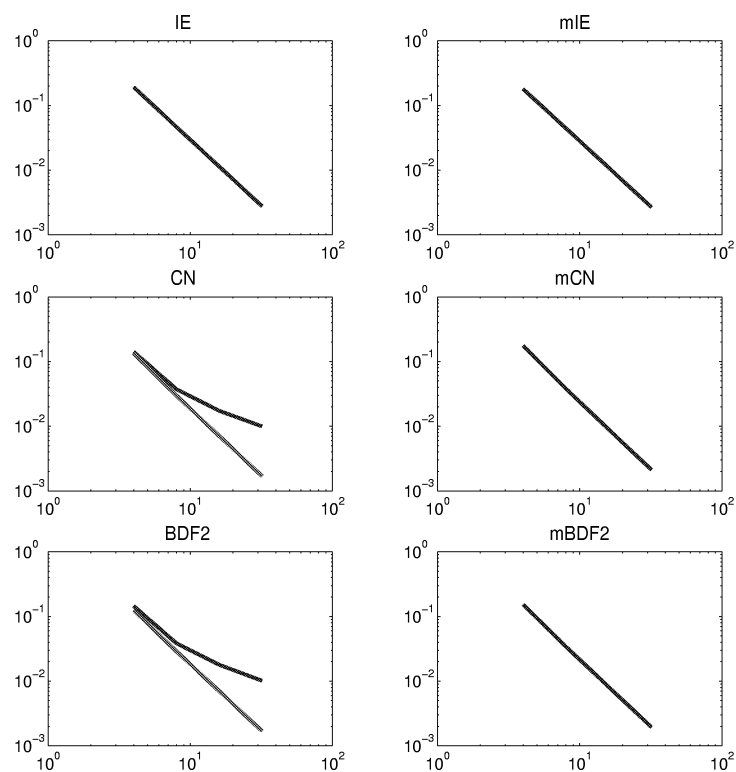
FIGURE 4.   Rate of convergence with respect to Δt and h

18. L. Sakell and D.D. Knight (eds.), *The first AFOSR conference on dynamic motion CFD*, Rutgers University, NJ, 1996.
19. Giuseppe Savaré, *A(Θ)-stable approximations of abstract Cauchy problems*, Numer. Math. **65** (1993), no. 3, 319–335. MR **94h:65062**
20. Vidar Thomée, *Galerkin finite element methods for parabolic problems*, Springer Series in Computational Mathematics, vol. 25, Springer-Verlag, Berlin, 1997. MR **98m:65007**

DIPARTIMENTO DI MATEMATICA "F. CASORATI", UNIVERSITÀ DI PAVIA, ITALY
E-mail address: boffi@dimat.unipv.it
URL: http://www-dimat.unipv.it/~boffi/

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BRESCIA, ITALY
E-mail address: gastaldi@ing.unibs.it
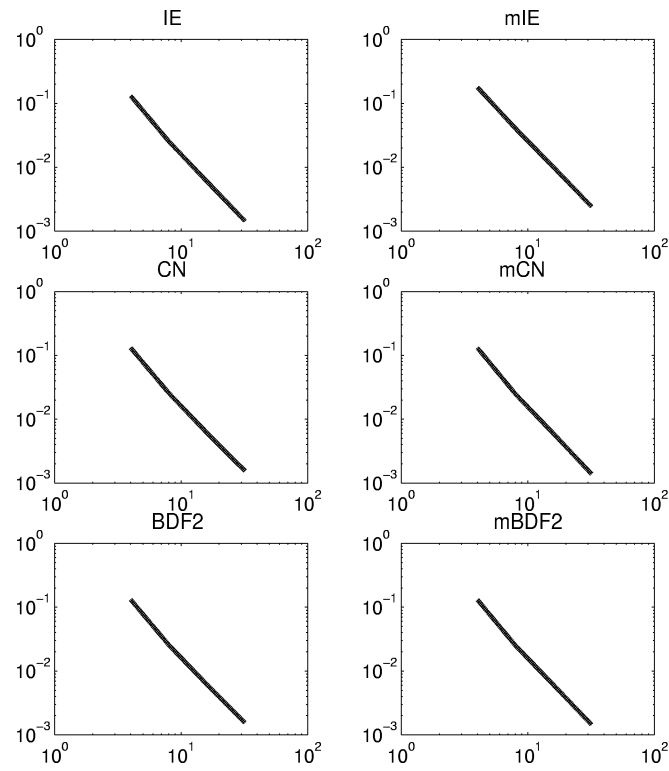*URL*: http://dm.ing.unibs.it/gastaldi/

FIGURE 5.   Rate of convergence with respect to Δt and h
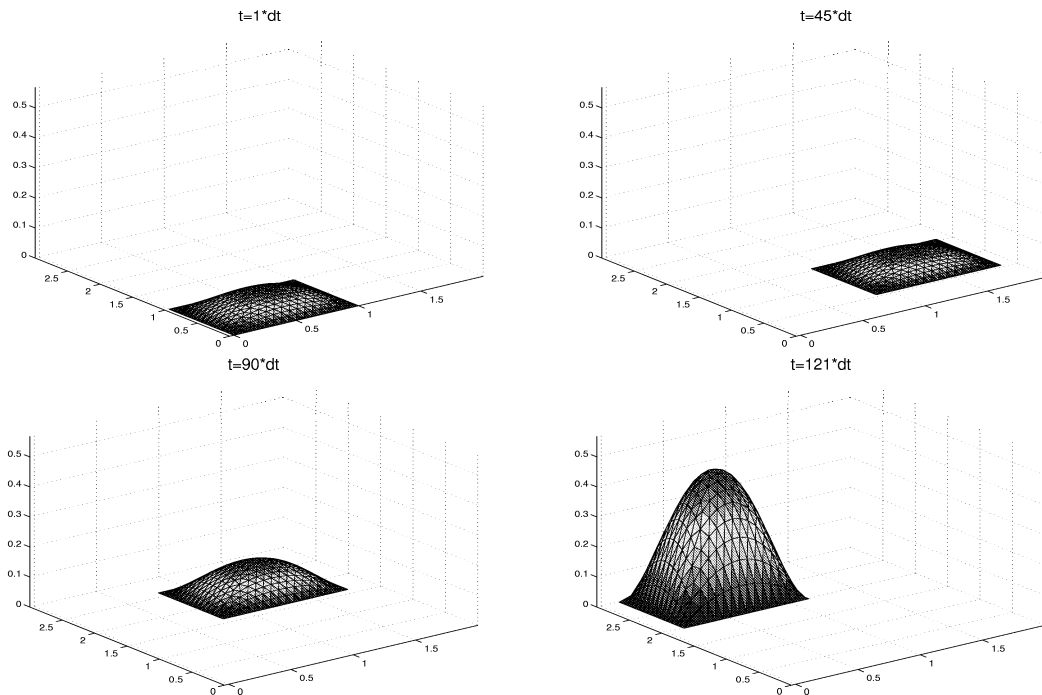

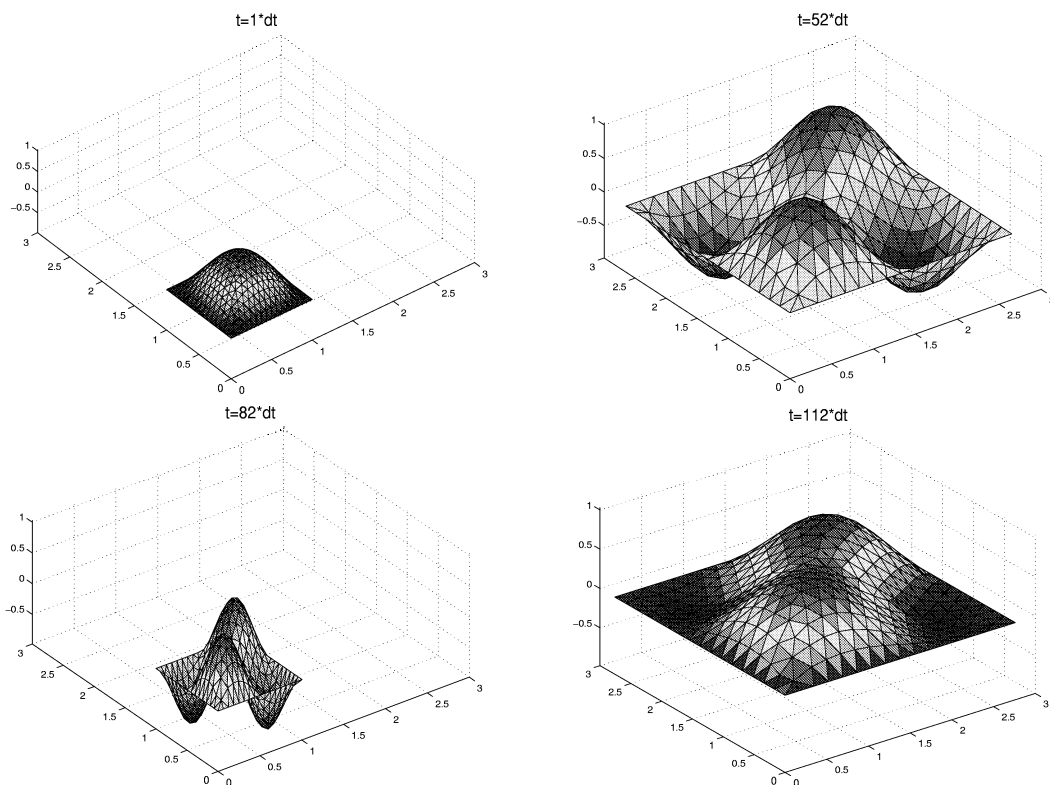FIGURE 6.   Computed solution corresponding to the exact solution u^1

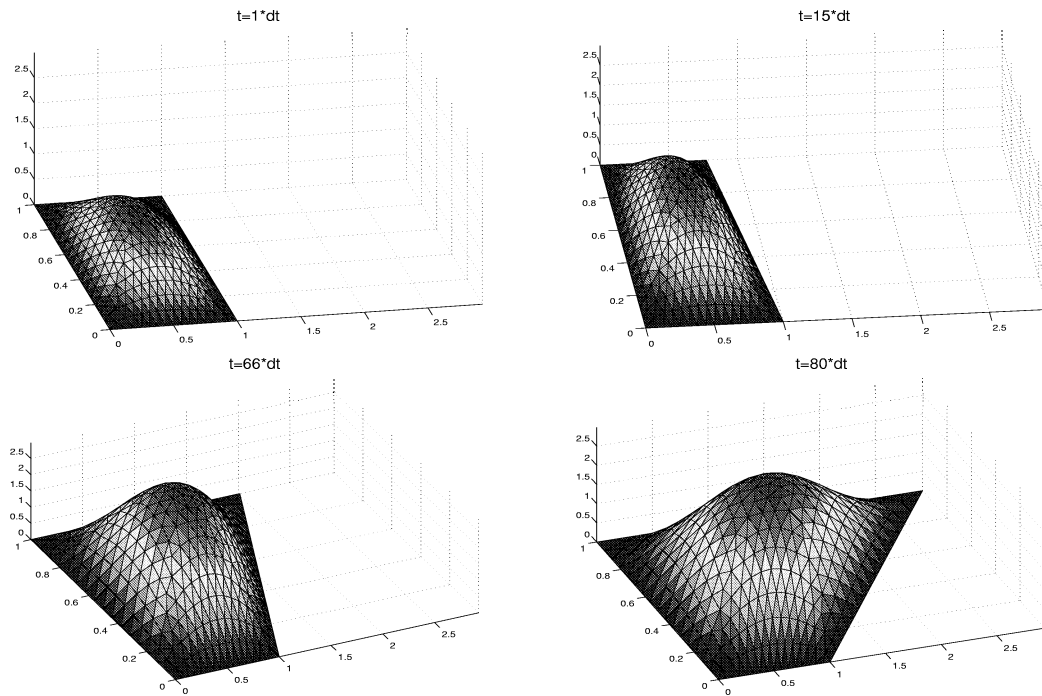FIGURE 7. Computed solution corresponding to the exact solution u^2

FIGURE 8. Computed solution corresponding to the exact solution u^3