STABILITY AND CONVERGENCE OF COLLOCATION SCHEMES FOR RETARDED POTENTIAL INTEGRAL EQUATIONS

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Abstract. Time domain boundary integral formulations of transient scattering problems involve retarded potential integral equations. Solving such equations numerically is both complicated and computationally intensive, and numerical methods often prove to be unstable. Collocation schemes are easier to implement than full finite element formulations, but little appears to be known about their stability and convergence. Here we derive and analyse some new stable collocation schemes for the single layer equation for transient acoustic scattering, and use (spatial) Fourier and (temporal) Laplace transform techniques to demonstrate that such stable schemes are second order convergent.

Key words. convergence, stability, retarded potential, boundary integral.

AMS subject classifications. 65M12, 65R20, 78M05, 78M15

1. Introduction.

The scalar integral equation for $u(\boldsymbol{x},t)$ on $\Gamma \times (0,T)$

$$\int_\Gamma \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\boldsymbol{x}' = a(\boldsymbol{x},t) \tag{1.1}$$

is the single layer potential equation for transient acoustic scattering from the two-dimensional surface $\Gamma \subset \mathbb{R}^3$ [27, §2.3]. Here a is given on $\Gamma \times (0,T)$ for fixed $T > 0$, and u and a satisfy the causality condition

$$u \equiv 0, \quad a \equiv 0 \quad \text{for all } t \le 0. \tag{1.2}$$

Once the potential u has been calculated on Γ, the scattered field can be computed anywhere in $\mathbb{R}^3$. The time argument of the integrand in (1.1) is delayed, or retarded, and such equations are commonly called retarded potential integral equations (RPIEs). They also arise in boundary integral formulations of electromagnetic scattering problems [2, 21, 28, 29, 30, 31].

Existence, uniqueness and well–posedness results for (1.1) are given in [3, 19, 20, 27]. A similar argument to that used by Lubich [27, §2.3] in the case that Γ is a smooth, closed surface (based on results of Bamberger and Ha-Duong [3, Prop. 3]) can be used to deduce the following result from [19] when Γ is a flat plate. We use the notation

$$H_*^m(0,T) = \left\{ f\,\big|_{(0,T)} : f \in H^m(\mathbb{R}) \text{ with } f \equiv 0 \text{ on } (-\infty,0) \right\} ,$$

(this space is called H_0^m in [27, Ch. 2]) where $H^m(\mathbb{R})$ denotes the usual Sobolev space of order m [1, Ch. 6].

PROPOSITION 1.1. *(Ha-Duong [19, Thm. 3], Lubich [27, §2.3])*
For temporally smooth data $a(\cdot,\, t) \in H^{1/2}(\Gamma)$ which vanish near $t = 0$, the RPIE (1.1) has a unique smooth solution $u(\cdot,\, t) \in H^{-1/2}(\Gamma)$. Moreover there exists a constant C depending only on T and Γ such that

$$\|u\|_{H_*^m(0,T;H^{-1/2}(\Gamma))} \le C\, \|a\|_{H_*^{m+1}(0,T;H^{1/2}(\Gamma))} \quad (m \in \mathbb{R}) .$$

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The spaces $H_*^m(0,T;X)$ and their norms are as defined by Lions and Magenes [25, Chs. 1.1, 4.2]; namely

$$\|f\|^2_{H_*^m(0,T;X)} = \sum_{k=0}^{m} \|f^{(k)}\|^2_{L^2(0,T;X)}\,, \tag{1.3}$$

where $f^{(k)} = \partial^k f/\partial t^k$ and

$$\|f\|_{L^2(0,T;X)} = \left(\int_0^T \|f\|_X^2\,dt\right)^{1/2}.$$

Various numerical methods for computing u have been reported in the literature. Bamberger and Ha-Duong [3] describe a variational method for the problem when Γ is closed and smooth, that is based on the coercivity of a bilinear form corresponding to a full Galerkin approximation in time and space. This approach has been extended to deal with the case when Γ is a flat surface by Ha-Duong [19], who also gives a comprehensive survey of the numerical analysis of such schemes in [20]. However the variational method is complicated (and costly) to implement since it involves calculating five dimensional integrals over $\Gamma\times\Gamma\times(0,T)$, and collocation schemes are frequently used for RPIEs in electromagnetic scattering problems [28, 29, 31]. In both approaches it takes $O(N_T\,N_S^2)$ flops to compute the solution up to time $T = N_T\,\Delta t$, where N_S is the number of spatial degrees of freedom used in the approximation, so RPIE algorithms are highly computationally intensive. Recently Michielssen and co-workers [15, 16, 26] have introduced "fast methods" for time dependent boundary integral equations (BIEs) such as (1.1) that reduce the operation count to $O(N_T\,N_S^{3/2}\,\log N_S)$ (for a two-level scheme), or $O(N_T\,N_S\,\log^2 N_S)$ (multi-level). Although complicated to implement, these make the BIE approach for time-dependent scattering problems viable compared to methods based on solving PDEs in 3D space.

The usefulness of collocation methods is often limited by the fact that they tend to exhibit numerical instabilities (see e.g. [22, §5]). Fourier analysis [6, 7, 10] indicates that the most likely cause of instability is the inaccurate approximation of (1.1). Here we present two new *stable* collocation methods for the problem (1.1)–(1.2). Our other main result is a proof that these schemes converge. The proof relies on the spatial Fourier transform of (1.1) being a convolution equation in time, and we use the Laplace and Z transform techniques of Lubich [27] to bound the Fourier transform of the approximation error. We then use classical estimates derived by Bramble and Hilbert [4] and Thomée [33] to bound the discrete norm of the error as the mesh-size tends to zero. We believe that this is the first convergence proof for an actual collocation RPIE scheme.

2. Preliminaries. We now describe the notation and some basic results used in the manuscript. The stability and convergence analysis in §§4–5 is for the scalar RPIE (1.1) posed on an infinite flat surface, i.e. for

$$\int_{\mathbb{R}^2} \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,d\boldsymbol{x}' = a(\boldsymbol{x},t) \quad \text{on } \mathbb{R}^2\times(0,T)\,, \tag{2.1}$$

where u and a satisfy (1.2).

The singularity in the integrand can be removed by the polar coordinate transformation $\boldsymbol{x}' = \boldsymbol{x} + R\,\boldsymbol{e}_\theta$ where $\boldsymbol{e}_\theta = (\cos\theta,\ \sin\theta)$ (see also [5, 9]). When $\Gamma = \mathbb{R}^2$ causality (1.2) results in the RPIE

$$\int_0^t \int_0^{2\pi} u(\boldsymbol{x} + R\,\boldsymbol{e}_\theta,\ t - R)\ d\theta\ dR = a(\boldsymbol{x}, t)\ . \tag{2.2}$$

If Γ is finite then the integral is over the appropriate region of $(R, \theta)-$space (which depends on $\boldsymbol{x}$).

2.1. Continuous and discrete spatial Fourier transforms. The continuous Fourier transform (CFT) of a function $g \in L^2(\mathbb{R}^2)$ is $\widehat{g} \in L^2(\mathbb{R}^2)$ defined by

$$\widehat{g}(\boldsymbol{\omega}) \equiv \int_{\mathbb{R}^2} g(\boldsymbol{x})\, e^{-i\boldsymbol{x}\cdot\boldsymbol{\omega}}\, d\boldsymbol{x}\ ,$$

and the inverse transform is

$$g(\boldsymbol{x}) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \widehat{g}(\boldsymbol{\omega})\, e^{i\boldsymbol{x}\cdot\boldsymbol{\omega}}\, d\boldsymbol{\omega}\ .$$

Note that this definition of the CFT is that used by Bramble and Hilbert [4] and differs from that of [1] by a factor of 2π. The CFT can be used to define the norm in $H^r(\mathbb{R}^2)$ when $r \ge 0$:

$$\|g\|_r = \|(1+\omega)^r\,\widehat{g}\|_{\mathcal{F}} \equiv \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} |(1+\omega)^r\,\widehat{g}(\boldsymbol{\omega})|^2\ d\boldsymbol{\omega}\right)^{1/2}, \tag{2.3}$$

where $\omega = |\boldsymbol{\omega}|$ (see [27, §2.1]). When $r = 0$ this is the Parseval–Plancherel identity. The discrete Fourier transform (DFT) of a function g evaluated at the nodes of a uniform $h \times h$ space mesh in $\mathbb{R}^2$ is denoted by $\tilde{g}$ and defined by

$$\tilde{g}(\boldsymbol{\omega}) = h^2 \sum_{j,k=-\infty}^{\infty} g(\boldsymbol{x}_{j,k}) e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}_{j,k}} \tag{2.4}$$

for $\boldsymbol{\omega} \in S_h = \{(\omega_1, \omega_2) : |\omega_1|,\ |\omega_2| \le \pi/h\}$, where $(j,k) \in \mathbb{Z}^2$ and $\boldsymbol{x}_{j,k} = (jh, kh)$. The function $\tilde{g}$ is $2\pi/h$ periodic in each component of $\boldsymbol{\omega}$. The DFT is defined for $g \in H^r(\mathbb{R}^2)$ with $r > 1$ [4, §4] and satisfies the discrete analogue of Parseval's identity:

$$\|\tilde{g}\|_{\mathcal{F}_h} = \|g\|_h\ , \tag{2.5}$$

where

$$\|\tilde{g}\|_{\mathcal{F}_h} = \left(\frac{1}{4\pi^2}\int_{S_h} |\tilde{g}(\boldsymbol{\omega})|^2\, d\boldsymbol{\omega}\right)^{1/2}$$

is the discrete Fourier norm and

$$\|g\|_h = \left(h^2 \sum_{j,k} |g(\boldsymbol{x}_{j,k})|^2\right)^{1/2}$$

is the discrete L^2 norm.

The following results due to Bramble and Hilbert [4] link the discrete and continuous Fourier transforms of a function.

PROPOSITION 2.1. *[4, Theorem 5]*
Let $g \in H^r(\mathbb{R}^2)$ for $r > 1$. Then there exists a constant C independent of h and g such that

$$\|\tilde{g} - \widehat{g}\|_{\mathcal{F}_h} \le Ch^r \|g\|_r. \tag{2.6}$$

PROPOSITION 2.2. *(Poisson sum formula [4, Theorem 6])*
Let $g \in H^r(\mathbb{R}^2)$ for $r > 1$. Then

$$\tilde{g}(\boldsymbol{\omega}) = \sum_{j,k} \widehat{g}(\boldsymbol{\omega} + 2\pi(j,k)/h) \quad a.e.. \tag{2.7}$$

2.2. Laplace and Z transforms in time. The Laplace transform of the causal function $f(t)$ (i.e. $f(t) \equiv 0$, $t < 0$) is

$$\bar{f}(s) = \int_0^\infty f(t) e^{-st}\, dt$$

where $s = \sigma + i\eta$ with $\sigma > 0$ and $\eta \in \mathbb{R}$. Throughout the paper σ is always assumed to be the **same** fixed positive constant. The Parseval Laplace identity is

$$\|e^{-\sigma t} f(t)\|_{L^2(\mathbb{R}^+)} = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} |\bar{f}(\sigma + i\eta)|^2\, d\eta \right)^{1/2}. \tag{2.8}$$

This is equivalent to the one dimensional version of (2.3) applied to the causal function $e^{-\sigma t} f(t)$ with $r = 0$. It follows that if $f \in H_*^m(\mathbb{R}^+)$, then

$$C \int_{-\infty}^{\infty} (1 + |s|)^{2m}\, |\bar{f}|^2\, d\eta \le 2\pi \sum_{k=0}^{m} \left\| \frac{\partial^k}{\partial t^k} \left(e^{-\sigma t} f(t)\right) \right\|_{L^2(\mathbb{R}^+)}^2 \le \int_{-\infty}^{\infty} (1 + |s|)^{2m}\, |\bar{f}|^2\, d\eta \tag{2.9}$$

where the constant C depends only on σ and m.

The Z transform is the discrete version of the Laplace transform defined by

$$Zf(s) = \sum_{n=0}^{\infty} f(n\Delta t) e^{-sn\Delta t} \tag{2.10}$$

where again $s = \sigma + i\eta$, but now $\eta \in [-\pi/\Delta t, \pi/\Delta t]$. The inversion formula is

$$f(n\Delta t) = \frac{\Delta t}{2\pi i} \int_{-\pi/\Delta t}^{\pi/\Delta t} e^{n\Delta t(\sigma + i\eta)} Zf(\sigma + i\eta)\, d\eta \tag{2.11}$$

for $n \in \mathbb{N}$. The Z and Laplace transforms are related by the Poisson sum formula

$$\Delta t\; Zf(s) = \sum_{k=-\infty}^{\infty} \bar{f}\left(s + i\,\frac{2\pi k}{\Delta t}\right), \tag{2.12}$$

a one dimensional version of (2.7), valid for $e^{-\sigma t} f(t) \in H^r(\mathbb{R}^+)$ with $r > 1/2$.

2.3. Fourier transformed RPIE. We suppose that $a(\cdot,t)$, $u(\cdot,t) \in L^2(\mathbb{R}^2)$ for $t \in (0,T)$ and take the CFT of the RPIE (2.2). This gives the first kind convolution Volterra integral equation

$$2\pi \int_0^t \widehat{u}(\boldsymbol{\omega}, t-R)\, J_0(\omega R)\, dR = \widehat{a}(\boldsymbol{\omega}, t) \quad \text{for } \boldsymbol{\omega} \in \mathbb{R}^2,\ t \in (0,T)\ , \tag{2.13}$$

where J_0 is the first kind Bessel function of order zero. We use the identity [18, §8.41]

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin\theta}\, d\theta \tag{2.14}$$

to obtain the integral equation.

The results of [27, §2.1] apply to give the following result for the infinite flat surface, analagous to Proposition 1.1 for the finite surface.

LEMMA 2.3. *Suppose that $a \in H_*^{m+1}\left(0,T; H^{r+1}(\mathbb{R}^2)\right)$ for integer $m \geq 0$ and $r \in [0,\infty)$. Then the solution $u(\boldsymbol{x},t)$ defined by (2.17) satisfies $u \in H_*^{m}\left(0,T; H^{r}(\mathbb{R}^2)\right)$.*

Proof. We essentially use the operator version of [27, Lemma 2.1] to obtain this result. We first extend the range of definition of a in time from $(0,T)$ to $(0,\infty)$ so that

$$\|a\|_{H_*^{m+1}(\mathbb{R}^+; H^{r+1}(\mathbb{R}^2))} \leq C \|a\|_{H_*^{m+1}(0,T; H^{r+1}(\mathbb{R}^2))} \tag{2.15}$$

(see e.g. [1, Thm. 6.3.5]). Then extending the definition of the convolution (2.13) to $\mathbb{R}^+$ and taking the Laplace transform in time gives

$$\bar{\widehat{a}}(\boldsymbol{\omega}, s) = \frac{2\pi}{\sqrt{\omega^2 + s^2}}\, \bar{\widehat{u}}(\boldsymbol{\omega}, s) \tag{2.16}$$

where the overbar denotes Laplace transform in t, and s is the Laplace transform parameter. Hence

$$\bar{\widehat{u}}(\boldsymbol{\omega}, s) = \frac{\sqrt{\omega^2 + s^2}}{2\pi}\, \bar{\widehat{a}}(\boldsymbol{\omega}, s) \tag{2.17}$$

and so

$$|\bar{\widehat{u}}(\boldsymbol{\omega}, s)|^2 \leq \frac{\omega^2 + |s|^2}{4\pi^2}\, |\bar{\widehat{a}}(\boldsymbol{\omega}, s)|^2 \leq \frac{(1+\omega)^2(1+|s|)^2}{4\pi^2}\, |\bar{\widehat{a}}(\boldsymbol{\omega}, s)|^2\, . \tag{2.18}$$

It follows from definition (1.3) that

$$\|u\|^2_{H_*^m(0,T;H^r(\mathbb{R}^2))} \leq e^{2\sigma T} \sum_{k=0}^{m} \int_0^\infty e^{-2\sigma t} \|u^{(k)}(\cdot,t)\|^2_{H^r(\mathbb{R}^2)}\, dt \equiv I_1\, .$$

The characterisation (2.3) of $H^r(\mathbb{R}^2)$ in terms of Fourier transforms gives

$$I_1 = \frac{e^{2\sigma T}}{4\pi^2} \sum_{k=0}^{m} \int_0^\infty \int_{\mathbb{R}^2} e^{-2\sigma t} (1+\omega)^{2r} |\widehat{u}^{(k)}(\boldsymbol{\omega}, t)|^2\, d\boldsymbol{\omega}\, dt$$

and reversing the order of integration and using the Laplace Parseval equality (2.8) results in

$$I_1 = \frac{e^{2\sigma T}}{8\pi^3} \sum_{k=0}^{m} \int_{\mathbb{R}^2} \int_{\mathbb{R}} (1+\omega)^{2r} |s|^{2k} |\bar{\widehat{u}}(\boldsymbol{\omega}, s)|^2\, d\eta\, d\boldsymbol{\omega}\, .$$

Now using the inequality (2.18) and reversing the steps above we get

$$I_1 \le \frac{e^{2\sigma T}}{8\pi^5} \sum_{k=0}^{m+1} \int_{\mathbb{R}^2} \int_{\mathbb{R}} (1+\omega)^{2r+2} |s|^{2k} |\bar{\widehat{a}}(\boldsymbol{\omega}, s)|^2 \, d\eta \, d\boldsymbol{\omega}$$
$$= \frac{e^{2\sigma T}}{\pi^2} \sum_{k=0}^{m+1} \int_0^\infty e^{-2\sigma t} \|a^{(k)}(\cdot,t)\|^2_{H^{r+1}(\mathbb{R}^2)} \, dt \le \frac{e^{2\sigma T}}{\pi^2} \|a\|^2_{H_*^{m+1}(\mathbb{R}^+; H^{r+1}(\mathbb{R}^2))} .$$

Finally we use the extension result (2.15) to get

$$\|u\|_{H_*^m(0,T;H^r(\mathbb{R}^2))} \le \sqrt{I_1} \le C \, \|a\|_{H_*^{m+1}(0,T;H^{r+1}(\mathbb{R}^2))} ,$$

where C depends only on m, r, σ and T, and the result follows. ◻

We also require the following pointwise bound on $\widehat{u}$.

LEMMA 2.4. *Under the conditions of the previous lemma, there exists a constant C such that*

$$|\widehat{u}(\boldsymbol{\omega}, t)| \le \frac{e^{\sigma T}}{\sqrt{2\pi}} \|e^{-\sigma t} \widehat{u}(\boldsymbol{\omega}, \cdot)\|_{H^1(\mathbb{R}^+)} \le C \, (1+\omega) \, \|\widehat{a}(\boldsymbol{\omega}, \cdot)\|_{H^2(\mathbb{R}^+)}$$

for $t \in (0,T)$.

Proof. The first inequality follows from the standard result

$$|f(t)| \le \frac{1}{\sqrt{2\pi}} \|f\|_{H^1(\mathbb{R}^+)} \tag{2.19}$$

[1, Ex. 6.4.5] applied with $f(t) = e^{-\sigma t} \widehat{u}(\boldsymbol{\omega}, t)$. Multiplying (2.18) by $(1+|s|)^2$, where $s = \sigma + i\eta$, and using the norm equivalence (2.9), gives

$$\|e^{-\sigma t} \widehat{u}(\boldsymbol{\omega}, t)\|_{H^1(\mathbb{R}^+)} \le C_1 \, (1+\omega) \|e^{-\sigma t} \widehat{a}(\boldsymbol{\omega}, t)\|_{H^2(\mathbb{R}^+)} \le C_2 \, (1+\omega) \|\widehat{a}(\boldsymbol{\omega}, \cdot)\|_{H^2(\mathbb{R}^+)} ,$$

for constants C_1 and C_2, which results in the second inequality. ◻

3. Algorithms. Because we are primarily interested in the analysis of RPIE algorithms here, we concentrate on the case $\Gamma = \mathbb{R}^2$. The restriction to finite Γ should be obvious. The RPIE (2.1) is approximated on a square space grid of side h and uniformly spaced time levels $t^n = n\Delta t$ for $n \in \mathbb{Z}^+$ in terms of piecewise constant or linear space and time basis functions, i.e. the approximate solution is expanded as:

$$u(\boldsymbol{x}, t) \approx U(\boldsymbol{x}, t) = \sum_{m \ge 1} \sum_{j,k} U^m_{j,k} \, \phi^{[\alpha]}_j(x) \, \phi^{[\alpha]}_k(y) \, \psi^{[\beta]}_m(t)$$

for $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$, where $\alpha, \beta \in \{0, 1\}$ indicate the orders of the space and time basis functions respectively. The spatial basis functions are defined by

$$\phi^{[\alpha]}_j(x) = \phi^{[\alpha]}(x/h - j)$$

where

$$\phi^{[0]}(z) = \begin{cases} 1 & \text{if } |z| < 1/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \phi^{[1]}(z) = \begin{cases} 1 - |z| & \text{if } |z| < 1 \\ 0 & \text{otherwise} \end{cases}$$

are the standard constant "pulse" and linear "hat" basis functions. The basis functions in time are

$$\psi^{[0]}_m(t) = \phi^{[0]}(t/\Delta t - m + 1/2) \quad \text{and} \quad \psi^{[1]}_m(t) = \phi^{[1]}(t/\Delta t - m) .$$

When the temporal basis functions are piecewise linear (resp. constant) the approximate solution $U(\boldsymbol{x},t)$ is evaluated at time $t = t^n$ (resp. $t = t^{n-1/2}$), where n is an integer. Hence the coefficients $U^n_{j,k}$ correspond to the approximate solution at time $t = (n-(1-\beta)/2)\Delta t$ and

$$U(\boldsymbol{x}, t^{n-(1-\beta)/2}) = \sum_{j,k} U^n_{j,k}\, \phi^{[\alpha]}_j(x)\, \phi^{[\alpha]}_k(y)\,.$$

Note that the approximate solution automatically satisfies the causality condition $U(x,t) = 0$ for $t \le 0$.

We shall consider the four schemes denoted by SαTβ, for α, $\beta \in \{0,1\}$ to indicate the degree of the basis functions in space ("S") and time ("T"). They are obtained by substituting U for u in the RPIE (2.1), evaluating (collocating) at each space mesh node $\boldsymbol{x} = \boldsymbol{x}_{p,q}$ and time level $t = t^n$, and carrying out all the required integrations *exactly*. This can be written as

$$a(\boldsymbol{x}_{p,q}, t^n) = \int_{\mathbb{R}^2} \frac{U(\boldsymbol{x}' + \boldsymbol{x}_{p,q}, t^n - |\boldsymbol{x}'|)}{|\boldsymbol{x}'|}\, d\boldsymbol{x}' = \sum_{m=0}^{n-1} \sum_{j,k} C^m_{j,k}\, U^{n-m}_{p+j,q+k} \tag{3.1}$$

where the coefficients

$$C^m_{j,k} = \int_{\mathbb{R}^2} \frac{\phi^{[\alpha]}_j(x')\, \phi^{[\alpha]}_k(y')\, \psi^{[\beta]}_m(|\boldsymbol{x}'|)}{|\boldsymbol{x}'|}\, d\boldsymbol{x}' \tag{3.2}$$

are evaluated exactly. Because of the finite support of the spatial and temporal basis functions, $C^m_{j,k}$ is zero unless $\big||\boldsymbol{x}_{j,k}| - t^{m-(1-\beta)/2}\big| \le (1+\beta)\Delta t/2 + (1+\alpha)h/\sqrt{2}$. Also, it follows from the definition of the basis functions that

$$C^m_{j,k} = C^m_{k,j} = C^m_{-j,k} = C^m_{j,-k}\ .$$

The approximation scheme can hence be written as

$$\sum_{m=0}^{n-1} \mathbb{Q}^m U^{n-m}_{p,q} = a(\boldsymbol{x}_{p,q}, t^n) \tag{3.3}$$

where $\mathbb{Q}^m = \sum_{j,k} C^m_{j,k}\, S^j_x\, S^k_y$ for $m \ge 0$ are discrete operators written in terms of unit shift operators S_x and S_y defined by $S^j_x\, U_{p,q} = U_{p+j,q}$, $S^k_y\, U_{p,q} = U_{p,q+k}$.

The sum can be rearranged to give

$$\mathbb{Q}^0 U^n_{p,q} = a(\boldsymbol{x}_{p,q}, t^n) - \sum_{m=1}^{n-1} \mathbb{Q}^{n-m} U^m_{p,q} \qquad \text{for } n \ge 1$$

and solved at successive time-levels, provided the difference operator $\mathbb{Q}^0$ is invertible. We examine this and other aspects of these schemes in the next section.

4. Stability. We use Fourier methods developed in [6, 7, 10] to analyse the stability of each of the schemes of the previous section. The analysis is for the RPIE (2.1) on an infinite uniform space mesh with uniform time steps, and is analogous to a von Neumann stability analysis for a PDE approximation. Results for the more general RPIE (1.1) approximated on nonuniform grids cannot be obtained this way. However it is clear that infinite mesh stability is necessary for a scheme to be stable in more general circumstances as the mesh is refined [6, §4].

4.1. DFT of the schemes. Using definition (2.4), the DFT of the difference equation (3.3) over the space mesh node points is

$$\sum_{m=0}^{n-1} q_m(\boldsymbol{\omega})\,\tilde{U}^{n-m}(\boldsymbol{\omega}) = \tilde{a}(\boldsymbol{\omega}, t^n) \tag{4.1}$$

for all $\boldsymbol{\omega} \in S_h$ and $n \geq 1$, where the functions $q_m(\boldsymbol{\omega})$ are the discrete transforms of the difference operators $\mathbb{Q}^m$ and are given by

$$q_m(\boldsymbol{\omega}) = \sum_{j,k} C^m_{j,k}\, e^{-ih(j\omega_1 + k\omega_2)} \quad \text{for } m \geq 0, \tag{4.2}$$

where the $C^m_{j,k}$ are defined in (3.2). If $q_0(\boldsymbol{\omega}) \neq 0$ then the solution of the scalar convolution sum equation (4.1) is

$$\tilde{U}^n(\boldsymbol{\omega}) = \frac{1}{q_0(\boldsymbol{\omega})} \sum_{m=1}^{n} p_m(\boldsymbol{\omega})\,\tilde{a}(\boldsymbol{\omega}, t^{n-m+1}) \tag{4.3}$$

where the coefficients p_n are defined recursively for all $\boldsymbol{\omega} \in S_h$ by

$$p_1(\boldsymbol{\omega}) = 1\,, \quad p_n(\boldsymbol{\omega}) = \frac{-1}{q_0(\boldsymbol{\omega})} \sum_{m=1}^{n-1} q_m(\boldsymbol{\omega})\,p_{n-m}(\boldsymbol{\omega}) \quad \text{for } n \geq 2. \tag{4.4}$$

The assumption that $q_0(\boldsymbol{\omega}) \neq 0$ for all $\boldsymbol{\omega} \in S_h$ is equivalent to the invertibility of the difference operator $\mathbb{Q}^0$ [8, 14]. The following two lemmas provide more information about q_0 and the other q_m.

LEMMA 4.1. *The coefficients q_m for scheme $S\alpha T\beta$ defined in (4.2) satisfy*

$$q_m(\boldsymbol{\omega}) = 2\pi \sum_{j,k} \Phi^{[\alpha]}(h\omega_1 + 2\pi j)\,\Phi^{[\alpha]}(h\omega_2 + 2\pi k)\,I^m_{j,k}(\boldsymbol{\omega}), \tag{4.5}$$

where

$$\Phi^{[0]}(z) = 2\sin(z/2)/z\,, \quad \Phi^{[1]}(z) = 2(1-\cos z)/z^2 \tag{4.6}$$

are the Fourier transforms of the basis functions $\phi^{[\alpha]}(x)$ defined in §3 and

$$I^m_{j,k}(\boldsymbol{\omega}) = \int_0^\infty \psi^{[\beta]}_{m^*}(R)\,J_0\left(R\,|\boldsymbol{\omega} + 2\pi(j,k)/h|\right)\,dR \quad \textit{for } m \geq 0\,, \tag{4.7}$$

where $m^* = m + 1 - \beta$.

Proof. Recall the labelling of the schemes used in §3: $\alpha, \beta \in \{0, 1\}$ indicate the order of the space and time basis functions respectively. We first substitute the approximate solution U for u in the left hand side of the RPIE (2.2) at time $t = t^n$, and use this to define

$$A(\boldsymbol{x}, t^n) = \sum_{m=0}^{n-1} \sum_{j,k} U^{n-m}_{j,k} \int_0^\infty \psi^{[\beta]}_{m^*}(R) \int_0^{2\pi} \phi^{[\alpha]}_j(x + R\cos\theta)\,\phi^{[\alpha]}_k(y + R\sin\theta)\,d\theta\,dR$$

Note that it follows from the numerical scheme (3.1) that $A(\boldsymbol{x}_{p,q}, t^n) = a(\boldsymbol{x}_{p,q}, t^n)$ on the grid, resulting in

$$\tilde{A} = \tilde{a}\,. \tag{4.8}$$

Taking the *continuous* Fourier transform of A with respect to $\boldsymbol{x}$ gives

$$\sum_{m=0}^{n-1}\sum_{j,k} U_{j,k}^{n-m}\int_0^\infty \psi_{m^*}^{[\beta]}(R)\int_0^{2\pi}\Phi^{[\alpha]}(h\omega_1)\,\Phi^{[\alpha]}(h\omega_2)\,e^{i\boldsymbol{\omega}\cdot(h(j,k)+R\boldsymbol{e}_\theta)}\,d\theta\,dR = \widehat{A}(\boldsymbol{\omega},t^n)\,,$$

which can be rearranged as

$$2\pi\sum_{m=0}^{n-1}\tilde{U}^{n-m}(\boldsymbol{\omega})\Phi^{[\alpha]}(h\omega_1)\,\Phi^{[\alpha]}(h\omega_2)\int_0^\infty \psi_{m^*}^{[\beta]}(R)J_0(\omega R)dR = \widehat{A}(\boldsymbol{\omega},t^n)$$

using the definition (2.4) and the Bessel function identity (2.14). Finally we apply the Poisson sum formula (2.7) to both sides of this equation and use (4.8) and the periodicity of the DFT $\tilde{U}^m(\boldsymbol{\omega}+2\pi(j,k)/h)=\tilde{U}^m(\boldsymbol{\omega})$ to obtain

$$2\pi\sum_{j,k}\sum_{m=0}^{n-1}\tilde{U}^{n-m}(\boldsymbol{\omega})\Phi^{[\alpha]}(h\omega_1+2\pi j)\,\Phi^{[\alpha]}(h\omega_2+2\pi k)I_{j,k}^m(\boldsymbol{\omega}) = \tilde{a}(\boldsymbol{\omega},t^n)$$

The result follows by comparing this with (4.1). ∎

LEMMA 4.2. *The coefficient $q_0(\boldsymbol{\omega})\geq C\Delta t$ for all $\boldsymbol{\omega}\in S_h$ where $C>0$ depends only on the mesh ratio $\Delta t/h$ (which is a fixed number in the scheme).*

Proof. We first show that each term in the summation (4.5) for q_0 is non-negative for each of the four schemes under consideration. Clearly $\Phi^{[\alpha]}(h\omega_1+2\pi j)\geq 0$ for all $j\in\mathbb{Z}$, $\boldsymbol{\omega}\in S_h$ by definition (4.6). Also, (4.7) with $m=0$ can be written as $I_{j,k}^0=\omega_{j,k}^{-1}F^{[\beta]}(\Delta t\omega_{j,k})$, where $\omega_{j,k}=|\boldsymbol{\omega}+2\pi(j,k)/h|$,

$$F^{[0]}(t)=\int_0^t J_0(s)\,ds \quad\text{and}\quad F^{[1]}(t)=\int_0^t(1-s/t)J_0(s)ds=\int_0^t s^{-1}J_1(s)\,ds\,.$$

It follows from results in [32, §5] that $F^{[\beta]}(t)>0$ for $\beta\in\{0,1\}$ and all $t>0$, and hence each term in the summation (4.5) for q_0 is non-negative.

Pulling out the term with $j=k=0$ and using the definition (4.6) then gives

$$q_0(\boldsymbol{\omega})\geq 2\pi\Phi^{[\alpha]}(h\omega_1)\,\Phi^{[\alpha]}(h\omega_2)\,I_{0,0}^0(\boldsymbol{\omega})\geq 2\pi(2/\pi)^{2(\alpha+1)}\,I_{0,0}^0(\boldsymbol{\omega})$$

for $\boldsymbol{\omega}\in S_h$, $\beta\in\{0,1\}$ where $I_{0,0}^0(\boldsymbol{\omega})=\omega^{-1}F^{[\beta]}(\omega\Delta t)$. The turning points of the functions $F^{[\beta]}$ occur at the zeros $z_{\beta,l}$ of the Bessel function J_β, and following [32, §5], it can be shown that $F^{[\beta]}(t)\geq F^{[\beta]}(z_{\beta,2})$ for all $t\geq z_{\beta,1}$. After a little manipulation we have $I_{0,0}^0(\boldsymbol{\omega})\geq\Delta tF^{[\beta]}(z_{\beta,2})/\max(z_{\beta,1},\sqrt{2}\,\pi\,\Delta t/h)$ where $\Delta t\omega\leq\sqrt{2}\,\pi\,\Delta t/h$ for all $\boldsymbol{\omega}\in S_h$. ∎

4.2. Stability results. To define stability we follow [6] and investigate the growth of perturbations in the solution of the homogeneous problem for which $a\equiv 0$. Because of linearity, it is enough to consider the propagation of non-zero initial data $U^1\neq 0$. The homogeneous stability problem is thus (3.3) with $a\equiv 0$ and U^1 a given, non-zero mesh function, i.e.

$$\mathbb{Q}^0U_{p,q}^n=-\sum_{m=1}^{n-1}\mathbb{Q}^mU_{p,q}^{n-m} \tag{4.9}$$

for $n\geq 2$ with $U_{p,q}^1\not\equiv 0$.

Definition 4.3. *The numerical scheme (4.9) is said to be stable on* $(0, T)$ *if there exists a constant* C *independent of* n *and* h *such that*

$$\|U^n\|_h \leq C\,\|U^1\|_h$$

whenever $t^n < T$*, for all functions* U^1 *for which* $\|U^1\|_h < \infty$.

It is straightforward to show that stability corresponds to the existence of a constant C such that $|p_n(\boldsymbol{\omega})| \leq C$ for all n and all $\boldsymbol{\omega} \in S_h$ (details are given in [6]). Unfortunately there appears to be no obvious way to check this condition by analysis, and we resort to testing it numerically for many individual frequencies $\boldsymbol{\omega} \in S_h$ to determine the stability of the four schemes. Results are shown in Figure 4.1 and indicate that the two schemes based on piecewise constant spatial basis functions (S0T0 and S0T1) are unstable for many values of mesh ratio, whereas the two schemes based on piecewise linear spatial basis functions (S1T0 and S1T1) appear stable over the range of mesh ratios tested. Stability over a wide range of mesh ratios is very important, since practical calculations over general surfaces may involve space mesh elements of vastly different sizes. Hence we do not consider schemes S0T0 and S0T1 further here.

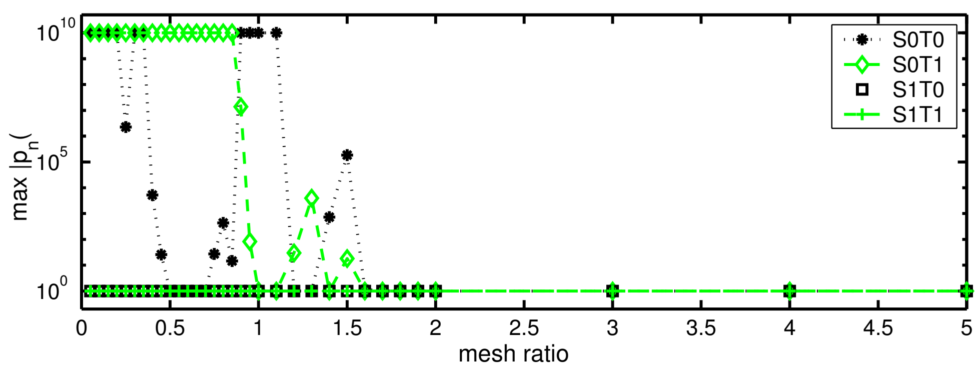

Fig. 4.1. *Stability plot for each of the four schemes* $S\alpha T\beta$. *The graph shows* $\min(\max_{n,j,k}\{|p_n(\boldsymbol{\omega}_{j,k})|\},\ 1e10)$ *plotted against the mesh ratio* $\rho = \Delta t/h$*, where the maximum is taken over timesteps* $n \leq \min(1000, 1000/\rho)$ *and frequencies* $\boldsymbol{\omega}_{j,k} = 0.1\,\pi\,(j,k)/h$ *for* $0 \leq j,\ k \leq 10$.

It is shown in [11] that removing the singularity in the RPIE integrals (1.1) by using local polar coordinates (see also [5, 9]) can also lead to stable collocation schemes. The polar approximation based on the trapezoidal rule in R and arbitrarily accurate integration in θ for which the temporal and spatial basis functions are piecewise linear also appears stable over all values of mesh ratio considered [11]. The disadvantage of this scheme is that the transformed region of integration has a complicated shape that depends on $\boldsymbol{x}$ when Γ is finite and so the scheme is not straightforward to implement in practice. We note also that the collocation RPIE scheme due to Rynne and Smith [31] (which uses piecewise constant basis functions in space, piecewise linears in time, and the midpoint quadrature rule to evaluate the coefficients $C^m_{j,k}$) can be made stable at any value of mesh ratio by averaging in time [7, 10, 31] (which filters out high frequency instabilities). However this is not entirely satisfactory because, for example, electromagnetic scattering problems involve more complicated RPIEs and hence are harder to stabilise [8]. We believe that a minimum requirement for a scalar RPIE scheme to be generally useful is that it should be stable over a wide range of mesh ratio when applied on an infinite flat plate without recourse to any filtering.

4.3. Further properties of the Fourier transformed schemes. This subsection lays the groundwork for the convergence analysis of S1T0 and S1T1, which appear stable for a wide range of mesh ratio values. We make precise the relationship between the stability coefficients q_m for the schemes and appropriate quadrature approximations of the Fourier transformed RPIE (2.13). The connection between q_m for piecewise linear in time RPIE schemes (like S1T1) and the trapezoidal rule approximation of (2.13) was first described in [10]. The q_m for the piecewise constant in time scheme S1T0 are similarly connected to the midpoint rule approximation of (2.13).

Letting $\widehat{u}_{\Delta t}^{m+1/2}(\boldsymbol{\omega})$ denote the approximation of $\widehat{u}(\boldsymbol{\omega}, t^{m+1/2})$ obtained by using the composite midpoint rule for (2.13) with spacing Δt, we have

$$2\pi\Delta t \sum_{m=0}^{n-1} J_0(\omega t^{n-m-1/2})\, \widehat{u}_{\Delta t}^{m+1/2}(\boldsymbol{\omega}) = \widehat{a}(\boldsymbol{\omega}, t^n)\,.$$

Comparing this with the DFT equation (4.1) and matching the q_m and Bessel function terms gives $q_m(\boldsymbol{\omega}) \sim 2\pi\Delta t J_0(\omega t^{m+1/2})$ for $m \geq 0$. Similarly, comparing the coefficients for S1T1 with the trapezoidal rule approximation of (2.13) gives $q_0(\boldsymbol{\omega}) \sim \pi\Delta t J_0(0)$ and $q_m(\boldsymbol{\omega}) \sim 2\pi\Delta t J_0(\omega t^m)$ for $m \geq 1$ [10]. To see just how close this match is we define $\alpha_m(\boldsymbol{\omega})$ for each scheme by

$$\alpha_0(\boldsymbol{\omega}) \equiv J_0(\omega\, t^{(1-\beta)/2})/(\beta+1) - q_0(\boldsymbol{\omega})/(2\pi\Delta t)\,, \tag{4.10a}$$

$$\alpha_m(\boldsymbol{\omega}) \equiv J_0(\omega\, t^{m+(1-\beta)/2}) - q_m(\boldsymbol{\omega})/(2\pi\Delta t) \quad \text{for } m \geq 1\,, \tag{4.10b}$$

where recall that $\beta = 0$ for S1T0 and $\beta = 1$ for S1T1. The following result states the small $h\omega$ behaviour of the α_m.

LEMMA 4.4. *There exists a constant C independent of h, $\boldsymbol{\omega}$, and m such that the coefficients α_m for S1T0 and S1T1 satisfy*

$$|\alpha_m(\boldsymbol{\omega})| \leq C(h\omega)^2 \tag{4.11}$$

for all $m \geq 0$ and $\boldsymbol{\omega} \in S_h$.

Proof. We prove the result for scheme S1T0 and note that the details for S1T1 are similar. Substituting the q_m equation (4.5) for S1T0 into the definition (4.10) of α_m gives

$$\begin{aligned}\alpha_m(\boldsymbol{\omega}) &= J_0(\omega\, t^{m+1/2}) - \frac{1}{\Delta t}\sum_{j,k} \Phi^{[1]}(h\omega_1 + 2\pi j)\, \Phi^{[1]}(h\omega_2 + 2\pi k)\, I_{j,k}^m(\boldsymbol{\omega}) \\ &= (T_1 - T_2 - T_3 - T_4)/\Delta t\end{aligned}$$

where, using the definition (4.6) of $\Phi^{[1]}$,

$$T_1 = \Big(1 - \Phi^{[1]}(h\omega_1)\, \Phi^{[1]}(h\omega_2)\Big)\; I_{0,0}^m(\boldsymbol{\omega}) + 2\Delta t J_0(t^{m+1/2}\omega) - I_{0,0}^m(\boldsymbol{\omega})\,,$$

$$T_2 = 2\, \Phi^{[1]}(h\omega_1)\; (1 - \cos(h\omega_2)) \sum_{k\neq 0} \frac{I_{0,k}^m(\boldsymbol{\omega})}{(h\omega_2 + 2\pi k)^2}\,,$$

$$T_3 = 2\, \Phi^{[1]}(h\omega_2)\; (1 - \cos(h\omega_1)) \sum_{j\neq 0} \frac{I_{j,0}^m(\boldsymbol{\omega})}{(h\omega_1 + 2\pi j)^2}\,, \quad \text{and}$$

$$T_4 = 4\; (1 - \cos(h\omega_1))\, (1 - \cos(h\omega_2)) \sum_{j,k\neq 0} \frac{I_{j,k}^m(\boldsymbol{\omega})}{(h\omega_1 + 2\pi j)^2 (h\omega_2 + 2\pi k)^2}\,.$$

Since $\left|\psi_m^{[0]}(z)\right| \le 1$ and $|J_0(z)| \le 1$ for all $z \in \mathbb{R}$ it follows from definition (4.7) that $\left|I_{j,k}^m(\boldsymbol{\omega})\right| \le \Delta t$ for all $(j,k) \in \mathbb{Z}^2$ and $m \ge 0$. It then follows from the inequalities $|1-\cos z| \le z^2/2$ and $|\Phi^{[1]}(z)| \le 1$, and the boundedness of the sum $\sum_{k\neq 0} k^{-2}$ that $|T_2|$, $|T_3| \le C\,h^2\omega^2\Delta t$ and $|T_4| \le C\,(h\omega_1)^2(h\omega_2)^2\Delta t$, and hence is also bounded by $C\,h^2\omega^2\Delta t$ for $\boldsymbol{\omega} \in S_h$.

Using standard results for the midpoint quadrature rule [12] gives

$$I_{0,0}^m(\boldsymbol{\omega}) = \Delta t J_0(t^{m+1/2}\omega) + \omega^2\Delta t^3 J_0''(\omega R_m)/24$$

for some $R_m \in (t^m, t^{m+1})$, and hence

$$\left|I_{0,0}^m(\boldsymbol{\omega}) - \Delta t J_0(t^{m+1/2}\omega)\right| \le C\Delta t(h\omega)^2$$

since $|J_0''(z)|$ is bounded for all $z \in \mathbb{R}$. It thus follows from the triangle inequality and the additional bound $|1-\Phi^{[1]}(z)| \le z^2/12$ that $|T_1| \le C\,h^2\omega^2\Delta t$. □

This result means that the scaled coefficients $q_m(\boldsymbol{\omega})$ (which are DFTs of the difference operators $\mathbb{Q}^m$) are second order accurate approximations of the Bessel functions in the Fourier transformed RPIE (2.13). We use this to establish convergence of the schemes S1T0 and S1T1 in the next section.

The midpoint and trapezoidal quadrature rules are both known to give stable schemes for Volterra equations like (2.13), although the leading error term for the trapezoidal rule solution is oscillatory [17, 23, 24]. It is also known [17, 24] that all higher order Newton–Cotes quadrature rules give rise to unstable approximations of (2.13). Hence one would need to be careful in constructing approximations of (2.1) that use temporal basis functions of higher degree, incase they give rise to the same instabilities.

5. Convergence. In this section we demonstrate that the schemes S1T0 and S1T1 for the infinite flat plate problem (2.1) are convergent for values of the mesh ratio $\Delta t/h$ at which they are stable. We work with spatially Fourier transformed quantities and also use Laplace and Z transforms in time to obtain the results. The proof relies on the Fourier transformed RPIE (2.13) being a convolution equation in time and we use techniques due originally to Lubich [27] to obtain bounds for the Fourier transform of the approximation error. We then use arguments similar to those used to prove convergence of approximation schemes for a linear PDE by Thomée [33] (similar techniques are used for hyperbolic equations in [13]). This type of convergence analysis relies crucially on estimates given by Bramble and Hilbert [4] and Thomée [33]. The analysis of schemes for retarded potential integrals is much more complicated than those for PDEs, and much of this section is devoted to formulating the problem in such a way so as to use these estimates.

Throughout this section C will denote a generic constant, that can depend upon the mesh ratio, σ, T, and the norm exponents m and r, but is independent of u, a and h.

5.1. Hypotheses and definitions. We make the following assumptions on the problem and numerical solution.

HYPOTHESES. *Suppose*

(H1) that the incident field $a \in H_*^{5+\beta}\left(0,T;H^{6+\beta}(\mathbb{R}^2)\right)$*;*

(H2) that numerical scheme S1Tβ *for (2.1) is stable at the mesh ratio* $\rho = \Delta t/h \in (0,\infty)$*, and the mesh ratio remains fixed as* Δt *and* h *go to zero.*

As in §3 the approximate solution corresponding to S1Tβ for $\beta \in \{0,1\}$ is denoted by $U(\boldsymbol{x},t)$. We explicitly need to make assumption (H2) because stability for these schemes has only been verified numerically and not proved rigorously. Note that it follows from (H1) and Lemma 2.3 that $u \in H_*^{4+\beta}\big(0,T;H^{5+\beta}(\mathbb{R}^2)\big)$.

We now define convergence for an RPIE scheme, and in the subsequent lemma we show what quantities need to be bounded in order to prove that the schemes converge.

DEFINITION 5.1. *A scheme for the RPIE (2.1) is convergent on* $(0,T)$ *if the difference between the exact and approximate solutions* $\|u(\cdot,t)-U(\cdot,t)\|_h \to 0$ *as* $h \to 0$ *whenever* $t<T$.

LEMMA 5.2. *For RPIE (2.1) with* $a(\boldsymbol{x},t)$ *satisfying (H1), schemes S1Tβ for* $\beta \in \{0,1\}$ *satisfy*

$$\|u(\cdot,t^{n*})-U(\cdot,t^{n*})\|_h \le Ch^r\|a\|_{H_*^2(0,T;H^{r+2}(\mathbb{R}^2))} + \|\varepsilon_n\|_{\mathcal{F}_h}$$

for $1<r\le 4+\beta$, *where* $t^{n*}=t^{n-(1-\beta)/2}$ *and* ε_n *satisfies the convolution equation*

$$\sum_{m=1}^{n} q_{n-m}(\boldsymbol{\omega})\,\varepsilon_m(\boldsymbol{\omega}) = \mathcal{E}_n(\boldsymbol{\omega}) \tag{5.1}$$

with

$$\mathcal{E}_n(\boldsymbol{\omega}) = 2\pi\int_0^{t^n} J_0(\omega(t^n-R))\,\widehat{u}(\boldsymbol{\omega},R)\,dR - \sum_{m=1}^{n} q_{n-m}(\boldsymbol{\omega})\,\widehat{u}(\boldsymbol{\omega},t^{m-(1-\beta)/2}) \tag{5.2}$$

and the q_m *given by (4.5). Hence they are convergent if* $\|\varepsilon_n\|_{\mathcal{F}_h}\to 0$ *as* $h\to 0$.

Proof. It follows from the discrete Parseval identity (2.5) and the triangle inequality that

$$\|u(\cdot,t^{n*})-U(\cdot,t^{n*})\|_h \le \|\tilde{u}(\cdot,t^{n*})-\widehat{u}(\cdot,t^{n*})\|_{\mathcal{F}_h} + \|\tilde{U}^n-\widehat{u}(\cdot,t^{n*})\|_{\mathcal{F}_h}\,.$$

The first term on the right hand side above can be bounded using Proposition 2.1. When $r>1$ this gives

$$\begin{aligned}\|\widehat{u}(\cdot,t)-\tilde{u}(\cdot,t)\|_{\mathcal{F}_h} &\le Ch^r\|u(\cdot,t)\|_{H^r(\mathbb{R}^2)}\\ &\le Ch^r\|u\|_{H_*^1(0,T;H^r(\mathbb{R}^2))}\\ &\le Ch^r\|a\|_{H_*^2(0,T;H^{r+1}(\mathbb{R}^2))}\,,\end{aligned} \tag{5.3}$$

from Lemma 2.3.

We now examine the second term. Comparing the Fourier transformed RPIE (2.13) at $t=t^n$ with the DFT of the numerical scheme (4.1) gives

$$\sum_{m=1}^{n} q_{n-m}(\boldsymbol{\omega})\,\Big(\tilde{U}^m(\boldsymbol{\omega})-\widehat{u}(\boldsymbol{\omega},t^{m*})\Big) = \tilde{a}^n(\boldsymbol{\omega})-\widehat{a}^n(\boldsymbol{\omega})+\mathcal{E}_n(\boldsymbol{\omega})\,.$$

Setting

$$\beta_m(\boldsymbol{\omega}) = \tilde{U}^m(\boldsymbol{\omega})-\widehat{u}(\boldsymbol{\omega},t^{m*})-\varepsilon_m(\boldsymbol{\omega})\,,$$

it follows from the definition (5.1) of ε_m that

$$\sum_{m=1}^{n} q_{n-m}(\boldsymbol{\omega})\,\beta_m(\boldsymbol{\omega}) = \tilde{a}^n(\boldsymbol{\omega})-\widehat{a}^n(\boldsymbol{\omega})\,. \tag{5.4}$$

The triangle inequality gives

$$\|\tilde{U}^n - \widehat{u}(\cdot, t^{n*})\|_{\mathcal{F}_h} \le \|\varepsilon_n\|_{\mathcal{F}_h} + \|\beta_n\|_{\mathcal{F}_h}$$

and so it remains only to show that $\|\beta_n\|_{\mathcal{F}_h} \to 0$ as $h \to 0$.

Inverting the convolution sum (5.4) using the formula (4.3) gives

$$\beta_n = q_0^{-1} \sum_{m=1}^{n} p_{n+1-m} \left(\tilde{a}^m - \widehat{a}^m\right)$$

where the p_m are defined by (4.4). The scheme is stable by hypothesis (H2), which means that the p_n are bounded, and hence it follows from the triangle inequality and the lower bound on q_0 given in Lemma 4.2 that

$$\|\beta_n\|_{\mathcal{F}_h} \le C\, h^{-1} \sum_{m=1}^{n} \|\tilde{a}^m - \widehat{a}^m\|_{\mathcal{F}_h}$$

for some constant C. Hypothesis (H1) and Proposition 2.1 together give

$$\|\tilde{a}(\cdot,t) - \widehat{a}(\cdot,t)\|_{\mathcal{F}_h} \le C\, h^{r+2}\, \|a(\cdot,t)\|_{H^{r+2}(\mathbb{R}^2)} \le C\, h^{r+2}\, \|a\|_{H^1_*(0,T;H^{r+2}(\mathbb{R}^2))}$$

when $t < T$, for any $r > -1$. Thus

$$\|\beta_n\|_{\mathcal{F}_h} \le C\, h^{-1} \sum_{m=1}^{n} \|\tilde{a}^m - \widehat{a}^m\|_{\mathcal{F}_h} \le C\, h^{r}\, \|a\|_{H^1_*(0,T;H^{r+2}(\mathbb{R}^2))}$$

since $n \le T/(\rho h)$ (where ρ is the mesh ratio). Combining this with inequality (5.3) completes the proof. ∎

The rest of this section is devoted to deriving two different bounds on ε_n; the first bound is valid for all $\boldsymbol{\omega}$ in S_h and the second when $h\omega$ is small. These bounds are then combined to show that $\|\varepsilon_n\|_{\mathcal{F}_h} = O(h^2)$ as $h \to 0$, and hence that we can use the previous lemma with $r = 2$ to prove second order convergence for the schemes S1Tβ.

5.2. Bound on ε_n for all $\boldsymbol{\omega} \in S_h$. Here we combine a bound on the size of the error term $\mathcal{E}_n(\boldsymbol{\omega})$ defined in (5.2) with the stability hypothesis (H2) in order to bound ε_n.

LEMMA 5.3. *Under hypotheses (H1) and (H2), there exists $\zeta \in H^2(\mathbb{R}^2)$ such that*

$$|\varepsilon_n(\boldsymbol{\omega})| \le \zeta(\boldsymbol{\omega}) \tag{5.5}$$

when $t^n < T$.

Proof. Using (4.10) to replace the q_m terms in (5.2) gives

$$\frac{\mathcal{E}_n}{2\pi} = \int_0^{t^n} J_0(\omega(t^n - R))\, \widehat{u}(\boldsymbol{\omega}, R)\, dR - \Delta t \sum_{m=1}^{n} \left[J_0(\omega t^{n-m*}) - \alpha_{n-m}(\boldsymbol{\omega})\right] \widehat{u}(\boldsymbol{\omega}, t^{m*})\,.$$

This is the error in the midpoint (resp. trapezoidal) rule approximation of the integral when $\beta = 0$ (resp. 1), with additional terms involving the α's. It follows from standard results for these quadrature rules that if $t^n < T$ then

$$\left|\frac{\mathcal{E}_n}{2\pi}\right| \le Ch^2 \left|\frac{\partial^2}{\partial R^2} J_0(\omega(t^n - R))\, \widehat{u}(\boldsymbol{\omega}, R)\right|_{R=\mu} + \Delta t \sum_{m=1}^{n} |\alpha_{n-m}(\boldsymbol{\omega})\, \widehat{u}(\boldsymbol{\omega},^{m*})|$$

for some $\mu \in (0, t^n)$. Hence, using the bound (4.11) on the size of the $|\alpha_m|$, and the fact that $J_0(z)$, $J_0'(z)$ and $J_0''(z)$ are all bounded it follows that

$$|\mathcal{E}_n| \leq Ch^2 \left(\omega^2 |\widehat{u}(\boldsymbol{\omega}, \mu)| + \omega |\widehat{u}^{(1)}(\boldsymbol{\omega}, \mu)| + |\widehat{u}^{(2)}(\boldsymbol{\omega}, \mu)|\right)$$

and the pointwise bound from Lemma 2.4 then gives

$$|\mathcal{E}_n| \leq Ch^2 (1+\omega)^3 \|\widehat{a}(\boldsymbol{\omega}, \cdot)\|_{H^4(0,T)} \,. \tag{5.6}$$

Now inverting the convolution sum (5.1) and using an identical argument to Lemma 5.2, we get

$$|\varepsilon_n| \leq C\, h^{-1} \sum_{m=1}^{n} |\mathcal{E}_m| \leq C\,(1+\omega)^3 \|\widehat{a}(\boldsymbol{\omega}, \cdot)\|_{H^4(0,T)} \equiv \zeta(\boldsymbol{\omega})$$

for $t_n \leq T$. Hypothesis (H1) guarantees that $(1+\omega)^2 \zeta(\boldsymbol{\omega}) \in L^2(\mathbb{R}^2)$ as required. □

5.3. Bound on ε_n for small $h\boldsymbol{\omega}$. This is the most technical part of the convergence proof. We need to get an $O(h^2)$ bound on $\|\varepsilon_n\|$ when $h\omega$ is sufficiently small. Taking the Z transform (2.10) of the convolution sum (5.1) gives

$$Zq(\boldsymbol{\omega}, s)\, Z\varepsilon(\boldsymbol{\omega}, s) = Z\mathcal{E}(\boldsymbol{\omega}, s)$$

and hence

$$|Z\varepsilon(\boldsymbol{\omega}, s)| = |Zq(\boldsymbol{\omega}, s)|^{-1} |Z\mathcal{E}(\boldsymbol{\omega}, s)|$$

(for $Zq \neq 0$) where $s = \sigma + i\eta$ and $\eta \in [-\pi/\Delta t, \pi/\Delta t]$. Ideally, we would obtain upper bounds on $1/|Zq|$ and $|Z\mathcal{E}|$, use them to bound $|Z\varepsilon|$ and use the inverse Z transform to bound $|\varepsilon_n|$. Unfortunately this is not straigtforward, but we can make progress by a less direct route. We first use (4.10) and Lemma 4.4 to obtain the following information on the Z transform of the q_m.

Lemma 5.4. *We can write $q_m = q_m^a + q_m^b$ for all $0 \leq m\Delta t \leq T$, where*

$$|q_m^b| \leq C\Delta t (h\omega)^2$$

and the sequence q_m^a is defined through its Z transform

$$Zq^a(\boldsymbol{\omega}, s) = 2\pi \begin{cases} e^{s\Delta t/2} \left(\dfrac{1}{\sqrt{s^2+\omega^2}} - \dfrac{1}{s} + \dfrac{\Delta t}{e^{s\Delta t/2} - e^{-s\Delta t/2}} \right), & \beta = 0 \\ \dfrac{1}{\sqrt{s^2+\omega^2}} - \dfrac{1}{s} + \dfrac{\Delta t}{2} \left(\dfrac{e^{s\Delta t}+1}{e^{s\Delta t}-1} \right), & \beta = 1\,. \end{cases}$$

Proof. The two cases are very similar so we just consider $\beta = 0$. From Lemma 4.4 we have $q_m = 2\pi \Delta t J_0(\omega t^{m+1/2}) - 2\pi \Delta t \alpha_m$ with $|\alpha_m| \leq C(h\omega)^2$. We write the Bessel function term as $J_0(\omega t) = f(t) + 1$, where $f(t) \equiv J_0(\omega t) - 1$ and take its Z transform to get

$$\sum_{m=0}^{\infty} J_0(\omega t^{m+1/2})\, e^{-sm\Delta t} = \sum_{m=0}^{\infty} f(t^{m+1/2})\, e^{-sm\Delta t} + \frac{1}{1 - e^{-s\Delta t}} \,. \tag{5.7}$$

(The reason for working with f rather than directly with $J_0(\omega t)$ is that $f(0) = 0$.) Splitting the sum $\sum_{m=0}^{\infty} f_{m/2}\, e^{-sm\Delta t/2}$ into odd and even terms and rearranging gives

$$\sum_{m=0}^{\infty} f(\omega t^{m+1/2})\, e^{-sm\Delta t} = e^{s\Delta t/2} \left(\sum_{m=0}^{\infty} f(\omega t^{m/2})\, e^{-sm\Delta t/2} - \sum_{m=0}^{\infty} f(\omega t^{m})\, e^{-sm\Delta t} \right) .$$

The Laplace Poisson sum formula (2.12) with spacing Δt is

$$\Delta t \sum_{m=0}^{\infty} f(\omega t^m) e^{-sm\Delta t} = \frac{1}{\sqrt{s^2+\omega^2}} - \frac{1}{s} + \sum_{l\neq 0} \theta_l(s,\omega,\Delta t)$$

where

$$\theta_l(s,\omega,\Delta t) = \frac{1}{\sqrt{s_l^2+\omega^2}} - \frac{1}{s_l} \quad \text{for} \quad s_l = s + i\,\frac{2\pi l}{\Delta t} .$$

Substituting this and the similar Poisson sum formula with spacing $\Delta t/2$ into the above identity for f gives

$$\Delta t \sum_{m=0}^{\infty} f(\omega t^{m+1/2})\, e^{-sm\Delta t} = e^{s\Delta t/2} \left(\frac{1}{\sqrt{s^2+\omega^2}} - \frac{1}{s} + \Theta \right)$$

where

$$\Theta = \sum_{l\neq 0} \left\{ 2\theta_l(s,\omega,\Delta t/2) - \theta_l(s,\omega,\Delta t) \right\} .$$

It then follows from (5.7) that

$$2\pi\Delta t \sum_{m=0}^{\infty} J_0(\omega t^{m+1/2}) e^{-sm\Delta t} = Zq^a(\boldsymbol{\omega}, s) + Z\kappa \,, \tag{5.8}$$

where $\{\kappa_m\}$ is the inverse Z transform of $2\pi e^{s\Delta t/2}\Theta$.

It can be shown that if $hw < 1/(\rho\sqrt{2})$ and h is sufficiently small, then $|\Theta| \leq C\Delta t\,(h\omega)^2$. Hence it follows from the inverse transform formula (2.11) that

$$|\kappa_n| \leq \Delta t\, e^{\sigma(n+1/2)\Delta t} \int_{-\pi/\Delta t}^{\pi/\Delta t} |\Theta| d\eta \leq C e^{T\sigma} \Delta t (h\omega)^2$$

for $n\Delta t \leq T$. The result then follows upon comparing (5.8) with (4.10) and using the bound on $|\alpha_m|$ given in Lemma 4.4. ∎

We next obtain upper bounds on $1/|Zq^a|$.

LEMMA 5.5. *If $\omega\Delta t \leq \pi/\sqrt{2}$ and Δt is small enough, then the Z transforms defined in the previous lemma satisfy*

$$\frac{1}{|Zq^a|} \leq \begin{cases} \frac{2}{\pi}\,|\sqrt{s^2+\omega^2}|\,, & \beta = 0 \\ (2\pi\sigma)^{-1}|s^2+\omega^2|\,, & \beta = 1 \end{cases}$$

where $s = \sigma + i\eta$ and $\eta \in [-\pi/\Delta t, \pi/\Delta t]$.

Proof. The two cases work quite differently, and a great deal of algebraic manipulation (the details are omitted) is required to obtain the results.

Case $\beta = 0$**.** Set

$$P = \frac{1}{\sqrt{s^2+\omega^2}} \quad \text{and} \quad Q = \frac{\Delta t}{(e^{s\Delta t/2} - e^{-s\Delta t/2})} - \frac{1}{s}\,.$$

Then

$$|Zq^a| = 2\pi\, e^{\sigma\Delta t/2}\, \{|P+Q|\} \geq 2\pi\, e^{\sigma\Delta t/2}\, \{|P| - |Q|\}\,.$$

It can be shown that $|Q|$ is monotonic increasing in $\eta\Delta t$ for $\eta\Delta t \in [0,\pi]$, and hence

$$|Q| \leq |Q|_{\eta\Delta t=\pi} = \frac{\Delta t}{\pi}\sqrt{1-\pi+\pi^2/4} + O(\Delta t^2)\,.$$

So if Δt is sufficiently small, then $|Q| \leq 3\Delta t/(5\pi)$. It can also be shown that $|P| \geq 4\Delta t/(5\pi)$ if Δt is sufficiently small and $\omega\Delta t \leq \pi/\sqrt{2}$. Hence under these conditions we have $|Q| \leq 3|P|/4$ and so

$$|Zq^a| \geq \pi\, e^{\sigma\Delta t/2}\, |P|/2 \geq \pi|P|/2\,,$$

and the result follows from the definition of P.

Case $\beta = 1$**.** We use P as above and define

$$R = \frac{\Delta t}{2}\frac{(e^{s\Delta t}+1)}{(e^{s\Delta t}-1)} - \frac{1}{s}\,.$$

Then

$$|Zq^a| = 2\pi\,|P+R| \geq 2\pi\,\Re(P+R) = 2\pi\,\{\Re(P) + \Re(R)\}\,.$$

It can be shown that

$$\Re(P) \geq \frac{\sigma}{|s^2+\omega^2|} > 0 \quad \text{and} \quad \Re(R) \geq 0\,,$$

from which the result follows immediately. ◻

We split the error from (5.1) into two parts, $\varepsilon_n = \varepsilon_n^a + \varepsilon_n^b$, satisfying

$$\sum_{m=0}^{n} q^a_{n-m}(\boldsymbol{\omega})\varepsilon^a_m(\boldsymbol{\omega}) = \mathcal{E}_n(\boldsymbol{\omega}) \quad \text{and} \quad \sum_{m=0}^{n} q_{n-m}(\boldsymbol{\omega})\varepsilon^b_m(\boldsymbol{\omega}) = -\sum_{m=0}^{n} q^b_{n-m}(\boldsymbol{\omega})\varepsilon^a_m(\boldsymbol{\omega})$$

where $q_m = q_m^a + q_m^b$ as defined in Lemma 5.4, and we have taken all sums to start from $m=0$ rather than $m=1$ for ease of manipulation (the $m=0$ terms are zero by causality). We first bound $|\varepsilon_m^b|$ in terms of $|\varepsilon_m^a|$, so that the problem reduces to finding a bound on $|\varepsilon_m^a|$. Inverting the second convolution sum gives

$$\varepsilon_n^b = \frac{-1}{q_0}\sum_{m=0}^{n} p_{n-m}\sum_{k=0}^{m} q^b_{m-k}\varepsilon^a_m$$

where $|p_m| \leq C$ by the stability hypothesis (H2), and $q_0 \geq C\Delta t$ from Lemma 4.2. If $n\Delta t \leq T$ then it follows from Lemma 5.4 that

$$|\varepsilon_n^b| \leq \frac{C}{\Delta t}\sum_{m=0}^{n}\sum_{k=0}^{m}|q^b_{m-k}|\,|\varepsilon^a_m| \leq \frac{CT^2h^2\omega^2}{\Delta t^2}\max_{m\leq n}|\varepsilon^a_m| \leq C\omega^2\max_{m\leq n}|\varepsilon^a_m|\,. \tag{5.9}$$

It remains to bound $|\varepsilon_n^a(\boldsymbol{\omega})|$. To do this we embed the time-discrete convolution $\sum_{m=0}^n q_m^a \varepsilon_{n-m}^a = \mathcal{E}_n$ into a time-continuous problem

$$\sum_{m=0}^{\infty} q_m^a(\boldsymbol{\omega})\varepsilon^a(\boldsymbol{\omega}, t - t_m) = \mathcal{E}(\boldsymbol{\omega}, t) \tag{5.10}$$

where $\mathcal{E}(\boldsymbol{\omega}, t)$ and $\varepsilon^a(\boldsymbol{\omega}, t)$ interpolate $\mathcal{E}_n$ and ε_n^a at time levels $t = t_n$. The aim is to obtain a bound on $\|\varepsilon^a(\boldsymbol{\omega}, \cdot)\|_{H^1}$ and hence on the point values $|\varepsilon_n^a(\boldsymbol{\omega})| = |\varepsilon^a(\boldsymbol{\omega}, t_n)|$ via (2.19). We generalise the formula (5.2) for $\mathcal{E}_n$ to obtain the interpolant

$$\mathcal{E}(\boldsymbol{\omega}, t) = 2\pi\int_0^\infty J_0(\omega R)\,\widehat{u}(\boldsymbol{\omega}, t - R)dR - \sum_{m=0}^{\infty} q_m(\boldsymbol{\omega})\,\widehat{u}(\boldsymbol{\omega}, t - t^{m+(1-\beta)/2})\,, \tag{5.11}$$

and note that it follows from causality of u that $\mathcal{E}(\boldsymbol{\omega}, t_n) = \mathcal{E}_n(\boldsymbol{\omega})$.

We bound $\|\varepsilon^a(\boldsymbol{\omega}, \cdot)\|_{H^1(\mathbb{R}^+)}$ via the Laplace transform of the time-continuous problem (5.10):

$$\bar{\varepsilon^a}(\boldsymbol{\omega}, s)\, Zq^a(\boldsymbol{\omega}, s) = \bar{\mathcal{E}}(\boldsymbol{\omega}, s)\,.$$

This implies

$$|\bar{\varepsilon^a}(\boldsymbol{\omega}, s)| \le |Zq^a(\boldsymbol{\omega}, s)|^{-1}\, |\bar{\mathcal{E}}(\boldsymbol{\omega}, s)|\,,$$

with the upper bound on $|Zq^a|^{-1}$ given in Lemma 5.5. Using this bound, multiplying by $1 + |s|$ and applying the equivalence inequality (2.9), then gives

$$\|e^{-\sigma t}\varepsilon^a(\boldsymbol{\omega}, t)\|_{H^1(\mathbb{R}^+)} \le \begin{cases} C(1+\omega)\|e^{-\sigma t}\mathcal{E}(\boldsymbol{\omega}, t)\|_{H^2(\mathbb{R}^+)}\,, & \beta = 0 \\ C(1+\omega)^2\|e^{-\sigma t}\mathcal{E}(\boldsymbol{\omega}, t)\|_{H^3(\mathbb{R}^+)}\,, & \beta = 1\,. \end{cases} \tag{5.12}$$

The pointwise result

$$|\varepsilon^a(\boldsymbol{\omega}, t)| \le \frac{e^{\sigma T}}{\sqrt{2\pi}}\,\|e^{-\sigma t}\varepsilon^a(\boldsymbol{\omega}, t)\|_{H^1(\mathbb{R}^+)} \tag{5.13}$$

for $t \in (0, T)$, then follows from (2.19), and the next lemma provides the crucial $O(h^2)$ term that leads to the second order convergence result.

LEMMA 5.6. *The error term $\mathcal{E}(\boldsymbol{\omega}, t)$ defined by (5.11) satisfies*

$$\|e^{-\sigma t}\mathcal{E}(\boldsymbol{\omega}, t)\|_{H^m(0,T)} \le Ch^2(1+\omega)^3\|\widehat{a}(\boldsymbol{\omega}, \cdot)\|_{H^{m+3}(0,T)}$$

for $1 \le m \le 2 + \beta$.

Proof. The Laplace transform of (5.11) is

$$\bar{\mathcal{E}}(\boldsymbol{\omega}, s) = E(\boldsymbol{\omega}, s)\,\bar{\widehat{u}}(\boldsymbol{\omega}, s) \tag{5.14}$$

where

$$E(\boldsymbol{\omega}, s) \stackrel{\text{def}}{=} \left(\frac{2\pi}{\sqrt{\omega^2 + s^2}} - Zq(\boldsymbol{\omega}, s)e^{s\Delta t(\beta-1)/2}\right)\,.$$

Multiplying by $(1 + |s|)^m$, square integrating over $\mathbb{R}$ and using (2.9) gives

$$\|e^{-\sigma t}\,\mathcal{E}(\boldsymbol{\omega}, t)\|^2_{H^m(\mathbb{R}^+)} \le C\int_{-\infty}^{\infty}(1 + |s|)^{2m}|E(\boldsymbol{\omega}, s)|^2|\bar{\widehat{u}}(\boldsymbol{\omega}, s)|^2 d\eta$$

where $s = \sigma + i\eta$. We obtain two different bounds for $|E(\boldsymbol{\omega}, s)|$, valid for "high" and "low" values of $|\eta|$.

When $|\eta\Delta t| > \pi$ the triangle inequality implies that

$$|E(\boldsymbol{\omega}, s)| \leq \frac{2\pi}{|\sqrt{\omega^2 + s^2}|} + e^{(\beta-1)\sigma\Delta t/2} \, |Zq(\boldsymbol{\omega}, s)|$$

and we consider each term separately. If $|\eta\Delta t| > \pi$ then

$$\frac{1}{|\omega^2 + s^2|^2} \leq \frac{\Delta t^2}{2\sigma^2\pi^2} \leq C$$

when Δt is small. The second term can also be bounded by a constant: by definition

$$|Zq(\boldsymbol{\omega}, s)| \leq \sum_{n=0}^{\infty} |q_n \, e^{-sn\Delta t}| \leq C\Delta t \sum_{n=0}^{\infty} e^{-\sigma n\Delta t}$$

since (4.10) and (4.11) imply that each $|q_n| < C\Delta t$. Hence

$$|Zq(\boldsymbol{\omega}, s)| \leq \frac{C\Delta t}{1 - e^{-\sigma\Delta t}} \leq C$$

if Δt is sufficiently small.

Thus we have shown that if $|\eta\Delta t| > \pi$ then $|E(\boldsymbol{\omega}, s)| \leq C$. In this region $|s| > |\eta| > \pi/\Delta t$, and so $|s|\Delta t/\pi > 1$, which means that

$$|E(\boldsymbol{\omega}, s)| \leq C < C\,(|s|\Delta t/\pi)^2 = C\,|s|^2\,h^2$$

since $\Delta t/h$ is fixed.

When $|\eta\Delta t| \leq \pi$ we use Lemma 5.4 and consider the cases $\beta = 0$ and $\beta = 1$ separately. Define

$$E_0 = \frac{1}{s} - \frac{\Delta t}{e^{s\Delta t/2} - e^{-s\Delta t/2}}\,, \quad E_1 = \frac{1}{s} - \frac{\Delta t}{2}\left(\frac{e^{s\Delta t} + 1}{e^{s\Delta t} - 1}\right)$$

so that $E = E_\beta - Zq^b\, e^{(\beta-1)s\Delta t/2}$. Lemma 5.4 implies that $|q_n^b| \leq C\Delta t(h\omega)^2$ in either case, and so it follows from an identical argument to that used above to bound $|Zq|$ that $|Zq^b\, e^{(\beta-1)s\Delta t/2}| \leq e^{(\beta-1)\sigma\Delta t/2}\, C\,(h\omega)^2 \leq C\,(h\omega)^2$ if Δt is sufficiently small. It can be shown (again by considerable algebraic manipulation) that $|E_\beta| \leq \Delta t^2\,|s|$ for $\beta = 0,\,1$ when $|\eta\Delta t| \leq \pi$. Hence if $|\eta\Delta t| \leq \pi$ and Δt is sufficiently small we get $|E| \leq C\,\left(\Delta t^2\,|s| + h^2\omega^2\right)$.

We thus have the bound

$$|E(\boldsymbol{\omega}, s)| \leq Ch^2(1+\omega)^2(1+|s|)^2 \quad \forall \eta \in \mathbb{R}\,.$$

Inserting this into the integral in (5.14) gives

$$\|e^{-\sigma t}\mathcal{E}(\boldsymbol{\omega}, t)\|_{H^m(\mathbb{R}^+)} \leq Ch^2(1+\omega)^2\|e^{-\sigma t}\widehat{u}(\boldsymbol{\omega}, t)\|_{H^{m+2}(\mathbb{R}^+)}$$

and the result follows from Lemma 2.4. □

We now use this result to bound $|\varepsilon_n|$: (5.9) implies that

$$|\varepsilon_n(\boldsymbol{\omega})| \leq C(1+\omega)^2 \max_{n \leq T/\Delta t} |\varepsilon_n^a(\boldsymbol{\omega})|$$

for $n\Delta t \leq T$, and using bounds (5.12) and (5.13) gives

$$|\varepsilon_n(\boldsymbol{\omega})| \leq Ch^2(1+\omega)^{6+\beta}\|\widehat{a}(\boldsymbol{\omega}, \cdot)\|_{H^{5+\beta}(0,T)} \equiv h^2\zeta_\beta(\boldsymbol{\omega}) \tag{5.15}$$

for $\beta = 0, 1$. Hypothesis (H1) guarantees that $\zeta_\beta(\boldsymbol{\omega}) \in L_2(\mathbb{R}^2)$, which completes the small $h\omega$ bound calculation.

5.4. A bound for $\|\varepsilon_n\|_{\mathcal{F}_h}$. We split the range of integration of the Fourier norm $\|\varepsilon_n\|_{\mathcal{F}_h}$ into a "low" frequency section $\boldsymbol{\omega} \in L_h \equiv \{\boldsymbol{\omega} : h\omega < \gamma\}$ where inequality (5.15) is used (where the constant γ is chosen to be less than $1/(\rho\sqrt{2})$ so that all the small $h\omega$ bounds hold), and a "high" frequency section $\boldsymbol{\omega} \in S_h \backslash L_h$ where inequality (5.5) is used. The result is

$$\|\varepsilon_n\|_{\mathcal{F}_h} \leq Ch^2 \left(\int_{L_h} |\zeta_\beta(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} \right)^{1/2} + C \left(\int_{S_h \backslash L_h} |\zeta(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} \right)^{1/2}$$

where $\zeta \in L^2(\mathbb{R}^2)$ was introduced in §5.2. The integral over low frequencies satisfies

$$\int_{L_h} |\zeta_\beta(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} \leq \int_{\mathbb{R}^2} |\zeta_\beta(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} \leq \|a\|^2_{H_*^{5+\beta}(0,T;H^{6+\beta}(\mathbb{R}^2))} .$$

Following the arguments used by Thomeé [33], the high frequency integral satisfies

$$\int_{S_h \backslash L_h} |\zeta(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} \leq \int_{S_h \backslash L_h} \left| \left(\frac{\omega h}{\gamma} \right)^2 \zeta(\boldsymbol{\omega}) \right|^2 \, d\boldsymbol{\omega} \leq Ch^4 ,$$

since $\omega^2 \zeta(\boldsymbol{\omega}) \in L^2(\mathbb{R}^2)$ by Lemma 5.3.

Combining the low and high frequency bounds above and using Lemma 5.2 with $r = 2$ yields the final result.

THEOREM 5.7. *Under hypotheses (H1) and (H2) for $\beta = 0, 1$, the global error for schemes S1Tβ satisfies the bound*

$$\|u(\cdot, t^{n-(1-\beta)/2}) - U(\cdot, t^{n-(1-\beta)/2})\|_h \leq Ch^2$$

as $h \to 0$ whenever $t^n \leq T$, where C is a constant.

6. Conclusions. We have presented two new schemes for the RPIE (2.1) that appear stable over a wide range of mesh ratio values, and are hence likely to be useful and reliable in practice. We have also given what we believe is the first rigorous convergence proof with reasonable, checkable hypotheses that RPIE collocation schemes converge at the optimal $O(h^2)$ rate one would expect from the underlying approximation methods. This is great improvement on our earlier work [11] where we obtained proof of covergence at the rate $O(1/|\ln h|)$ for all but extremely smooth incident fields, whose spatial Fourier transforms decay faster than $e^{-\gamma_0 \omega}$ for a constant γ_0.

This improved result is mostly due to a change in approach to the error analysis for low spatial frequencies (§5.3) from a Volterra integral equation analysis in the style of [17, 23, 24], to an approach using Z and Laplace transforms in the style of Lubich [27]. We believe that our new smoothness requirements may be relaxed further by more refined or alternative methods of proof, and we conjecture that this convergence rate will be achieved for a wider class of excitations.

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