Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors

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Abstract

This paper is devoted to stady the existence and the time-asymptotic of multi-dimensional quantum hydrodynamic equations for the electron particle density, the current density and the electrostatic potential in spatial periodic domain. The equations are formally analogous of the classical hydrodynamics but differ in the momentum equation, which is forced by an additional nonlinear dispersion term, (due to the quantum Bohm potential,) and are used in the modelling of quantum effects on semiconductors devices.

We prove the local-in-time existence of the solutions, in the case of *general, nonconvex* pressure-density relation and *large and regular* initial data. Furthermore we propose a "subsonic" type stability condition related to that one of the classical hydrodynamical equations. When this condition is satisfied, the local-in-time solutions exist globally in-time and converge time exponentially toward the corresponding steady-state. Since for this problem many classical methods, like for instance the Friedrichs theory for symmetric hyperbolic systems, cannot be used then we investigate, via an iterative procedure, a suitable extended system which incorporates the quantum hydrodynamics as a special case. In particular, by using this new approach, the nonlinear dispersive terms are tranformed into a fourth-order wave type operator, with linear principal part.

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1 Introduction and main results

Quantum hydrodynamic models become important and necessary to model and simulate electron transport, affected by extremely high electric fields, in ultra-small sub-micron semiconductor devices, such as resonant tunnelling diodes, where quantum effects (like particle tunnelling through potential barriers and built-up in quantum wells [10, 21]) take place and dominate the process. Such kinds of quantum mechanical phenomena cannot be simulated by classical hydrodynamical models. The advantage of the macroscopic quantum hydrodynamical models relies in the facts that they are not only able to describe directly the dynamics of physical observable and simulate the main characters of quantum effects but also numerically less expensive than those microscopic models like Schrödinger and Wigner-Boltzmann equations. Moreover, even in the process of semi-classical (or zero dispersion) limit, the macroscopic quantum quantities like density, momentum, and temperature converge in some sense to these of Newtonian fluid-dynamical quantities [13]. Similar macroscopic quantum models are also used in other physical area such as superfluid [26] and superconductivity [5]

The idea to derive quantum fluid-type equations goes back to Madelung's in 1927 [27, 24], where the relation between (linear) Schrödinger equation and quantum fluid equation was described in view of nonlinear geometric optic (WKB)–ansatz of the wave function for irrotational flow away from vacuum. This in fact gives a way to derive quantum fluid type equations, i.e., to make use of WKB–expansion and derive the equations for (macroscopic) density and momentum from the single-state Schrödinger equation, or these with temperature involved from the mixed-state Schrödinger equation [14, 18, 13]. Another practicable way to derive quantum hydrodynamic equations is to take advantage of the kinetic structure behind the Schrödinger Hamiltonian through Wigner transformation [36]. In fact, the action of Wigner transformation on the wave function describes the equivalence between (linear) Schrödinger equation and Wigner-Boltzmann equation [31], the quantum kinetic transport equation. The application of the moment method to the Wigner-Boltzmann (or Wigner-Poisson) equation, yields to the macroscopic quantities density, momentum and temperature, whose time-evolutions obey to the quantum hydrodynamic equations [10, 11]. This is done in analogy with derivation the first three moments equations, in the moment expansion for the Wigner (distribution) function of Wigner-Boltzmann equation, under appropriate closure conditions [15] near "quantum Maxwellian". For further references on the quantum modelling of semiconductor devices, we refer to [32, 10, 14, 18, 11] and the references quoted therein.

We are interested in the mathematical analysis of quantum hydrodynamic model for semiconductors. In the present paper we consider the initial value problem (IVP) of the quantum hydrodynamic model for semiconductors where an additional relaxation term is involved in the linear momentum equation to model the interaction between electron and crystal lattice. The re-scaled multi-dimensional quantum hydrodynamic models for semiconductors (QHD) then is given by

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \rho\nabla V + \frac{1}{4}\varepsilon^2\nabla\cdot\left(\rho\nabla^2\log\rho\right) - \frac{\rho\mathbf{u}}{\tau}, \tag{1.2}$$

$$-\lambda^2\Delta V = \rho - \mathcal{C}(x), \tag{1.3}$$

$$\rho(x,0) = \rho_1(x),\ \mathbf{u}(x,0) = \mathbf{u}_1(x), \tag{1.4}$$

where $\rho > 0$, $\mathbf{u}$, $J = \rho\mathbf{u}$ denote the density, velocity and momentum respectively. $\varepsilon > 0$ the scaled Planck constant, $\tau > 0$ is the (scaled) momentum relaxation time, $\lambda > 0$ the (re-scaled) Debye length, and $\mathcal{C}(x)$ the doping profile simulating the semiconductor device under consideration [18, 32]. The pressure $P = P(\rho)$, like in classical fluid dynamics, often satisfies the γ-law expression

$$P(\rho) = \frac{T}{\gamma}\rho^\gamma, \quad \rho \geq 0, \quad \gamma \geq 1$$

with the temperature $T > 0$ [10, 17]. Notice that the particle temperature is $T(\rho) = T\rho^{\gamma-1}$. Moreover, the nonlinear dispersive term

$$\frac{1}{4}\varepsilon^2\nabla\cdot\left(\rho\nabla^2\log\rho\right) = \frac{1}{2}\varepsilon^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$$

is produced by the gradient of quantum Bohm potential

$$Q(\rho) = \frac{1}{2}\varepsilon^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}},$$

which requires the strict positivity of density for classical solution.

Recently, many efforts have been made on the existence of (steady-state or time-dependent) solutions of QHD (1.1)–(1.3). The existence and uniqueness of (classical) steady-state solutions to the QHD (1.1)–(1.3) for current density $J = 0$ (thermal equilibrium) has been studied in one dimensional and multi-dimensional bounded domain for density and electrostatic potential boundary conditions [1, 12]. The stationary QHD (1.1)–(1.3) for $J > 0$ (non-thermal equilibrium) has been considered in [9, 17, 37] for general monotone pressure functions, but, with different boundary conditions, i.e., Dirichlet data for the velocity potential S [17] or by using nonlinear boundary conditions [9, 37]. The existence of the one-dimensional steady-state solutions to (1.1)–(1.3) subject to boundary conditions on the density and the electrostatic potential has been proved in

[16], for the case of a linear pressure function $P(\rho)=\rho$, and in [19] for general pressure functions $P(\rho)$. The local in-time existence of classical solution was obtained in one-dimensional bounded domain [20] (subject to boundary conditions on the density and the electrostatic potential). In this case additional boundedness restriction on initial velocity were required to keep the strict positivity of density. The case of large initial data and strictly convex pressure function in $\mathbb{R}^n$ has been investigated by [25] . In both of these cases, the classical solutions exist globally in time for initial data which are small perturbation of stationary states (which are time exponentially stable). up [20, 25].

In the present paper we consider the initial value problem (1.1)–(1.4) for *general, nonconvex* pressure function in multi-dimension, and we focus on the *local* existence of the classical solutions $(\rho, \mathbf{u}, V)$ of IVP (1.1)–(1.4) for regular *large* initial data, and their time-asymptotic convergence to asymptotic state under small perturbation. We give a general framework to show the local in-time existence of classical solutions for general (nonconvex) pressure density function and for regular large initial data. Then, we propose a (generic) "subsonic" condition to prove the global existence of the classical solutions in "subsonic" region and investigate their large time behavior.

It is convenient to make use of the variable transformation $\rho=\psi^2$ in (1.1)–(1.4). Then, we derive the corresponding IVP for $(\psi, \mathbf{u}, V)$:

$$2\psi\cdot\partial_t\psi+\nabla\cdot(\psi^2\mathbf{u})=0, \tag{1.5}$$

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla h(\psi^2)+\frac{\mathbf{u}}{\tau}=\nabla V+\frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi}{\psi}\right), \tag{1.6}$$

$$\Delta V=\psi^2-\mathcal{C}, \tag{1.7}$$

$$\psi(x,0)=\psi_1(x):=\sqrt{\rho_1(x)},\ \mathbf{u}(x,0)=\mathbf{u}_1(x), \tag{1.8}$$

with $\rho h'(\rho)=P'(\rho)$. Note here the two problems (1.1)–(1.4) and (1.5)–(1.8) are equivalent for classical solutions. For simplicity in this paper we consider the initial value problem (1.5)–(1.8) on the multi-dimensional torus $\mathbb{T}^n$, with $\mathbb{T}=[0,L]$ and $L>0$ representing the period length. Because of the periodicity in the space variables, the solution of Poisson equation is not unique since each combination of one solution and a constant is another solution. It is natural to consider the Poisson equation (1.7) in homogeneous Sobolev space. Note, however, the physical meaning of electrostatic potential V, we can consider the Poisson equation (1.7) for V satisfying

$$\int_{\mathbb{T}^n}(V+V_{ext}-h(\mathcal{C}))(x,t)dx=0, \qquad t\geq 0,$$

where V_{ext} is the given external potential used to model the (exterior) quantum well in semiconductor devices. After an appropriate choice, we can let $V_{ext}=h(\mathcal{C})$, then from the mathematical point of view, we reduce to consider only potentials satisfying the condition

$$\int_{\mathbb{T}^n}V(x,t)dx=0, \quad t\geq 0.$$

In analogy, the right hand side term of the Eq. (1.7) is required to belong to the homogeneous Sobolev space, i.e.,

$$\int_{\mathbb{T}^n} (\psi^2 - \mathcal{C})(x,t)dx = 0, \quad t \geq 0.$$

This can be guaranteed due to the conservation (neutrality) of density (1.5) and neutrality assumption on the initial datum

$$\int_{\mathbb{T}^n} (\psi_1^2 - \mathcal{C})(x)dx = 0. \tag{1.9}$$

In the present paper we consider the problem (1.5)–(1.8) for ir-rotational (quantum) flow. We describe some ideas to prove both the local and the global existence and we investigate the large time behavior in the "subsonic" regime. The general situation for rotational flow is quite more complicated and it is expected to be investigated in a forthcoming paper.

The first result is the following local existence theorem:

Theorem 1.1 *Suppose $P(\rho) \in C^5(0,+\infty)$. Assume $(\psi_1, \mathbf{u}_1) \in H^6(\mathbb{T}^n) \times H^5(\mathbb{T}^n)$ ($n = 2,3$) satisfying* (1.9)*, $\nabla \times \mathbf{u}_1 = 0$, and $\min_{x\in[0,1]} \psi_1(x) > 0$. Then, there exists $T_{**} > 0$, such that there exists a unique solution $(\psi, \mathbf{u}, V)$ to the IVP* (1.5)–(1.8)*, with $\psi > 0$, which satisfies*

$$\psi \in C^i([0,T_{**}]; H^{6-2i}(\mathbb{T}^n)) \bigcap C^3([0,T_{**}]; L^2(\mathbb{T}^n)), \quad i = 0,1,2;$$
$$\mathbf{u} \in C^i([0,T_{**}]; H^{5-2i}(\mathbb{T}^n)),\ i = 0,2; \quad V \in C^1([0,T_{**}]; \dot{H}^4(\mathbb{T}^n)).$$

Remark 1.2 The irrotationality assumption on the velocity vector fields $\mathbf{u}$ is consistent with the equation (1.6), namely it keeps this property as long as it is true initially. This can be justified via standard arguments as used in the case of ideal fluids in classical hydrodynamics based on the Kelvin's theorem and the Stokes's theorem, see for instance [23] for details.

The proof of above local-in-time existence is based on the construction of approximate solutions and the application of compactness arguments. The main difficulties are given by the following facts. The former arises since the general pressure $P(\rho)$ can be non-convex (even zero), then the left part of (1.5)–(1.7) (or (1.1)–(1.3) resp.) may not be hyperbolic anymore and we cannot apply the theory of quasilinear symmetric hyperbolic systems like [25] to obtain the local existence. The latter is given by the nonlinear dispersion term in (1.6), which requires the density ψ (or ρ resp.) to be strictly positive, for regular solutions. Hence we have to establish the local-in-time existence of solutions in a less traditional way.

Indeed we are going to construct approximate solutions and to prove the local in-time existence of classical solutions $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$ for an extended system, which incorporates our problem, constructed in a suitable way based on (1.5)–(1.8). Note that in this new system, there are two additional equations for the variable $\mathbf{v}$, the artificial "velocity" (a sort of Lagrangian type velocity), and the artificial "density" $\varphi > 0$. The key point is that the local in-time existence of classical solutions for this extended system for the unknown $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$ will be equivalent to the original one given by (1.5)–(1.8), when $\mathbf{v} = \mathbf{u}$ and $\psi = \varphi$ (see section 3 for proof in details).

In order to extend the local-in-time solution globally in time, we will need uniform a-priori estimates, that can be proved by assuming the initial data close to the time-asymptotic (stationary) state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V})$. Actually it will be possible to extend globally, the local-in-time solutions, in the "subsonic" region (in the sense defined by (1.10) or (1.12) below); namely we will prove the global existence of the local-in-time solution when it starts near a stationary state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V})$ located in the so called "subsonic" region (being this notion to be provided later in a more precise fashion).

The well-posedness of stationary state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V})$ of the boundary value problem (1.5)–(1.7) subject to density and electrostatic potential boundary conditions was established in one dimension [19] for general (nonconvex) pressure function $P(\rho)$, and was obtained for multi-dimensional irrotational flow [17] for monotone enthalpy function where additional boundary condition was imposed for the Fermi potential. The argument [17, 19] could be applied also here to obtain the existence of stationary solution with periodic boundary conditions. However, since here we are focusing our attention only on the global existence, for simplicity we will bound ourselves to consider only the particular stationary state $(\bar{\psi}, \bar{\mathbf{u}}, \bar{V}) = (\mathcal{C}, 0, h(\mathcal{C}))$. By noting that here we can replace $h(\mathcal{C})$ by 0 just by changing the value of the external potential, we will consider here only the situation where the initial data are assumed in a small neighborhood of the stationary solution $(\mathcal{C}, 0, 0)$ to (1.5)–(1.7).

Theorem 1.3 *Let $P(\rho) \in C^5(0, +\infty)$ satisfying*

$$A_0 =: \frac{\pi^2}{L^2}\varepsilon^2 + P'(\mathcal{C}) > 0, \tag{1.10}$$

where $L > 0$ is the space period length. Let us assume $(\psi_1 - \sqrt{\mathcal{C}}, \mathbf{u}_1) \in H^6(\mathbb{T}^n) \times H^5(\mathbb{T}^n)$ $(n = 2, 3)$, the condition (1.9) and moreover $\nabla \times \mathbf{u}_1 = 0$. There exists $\eta > 0$ such that, if $\|\psi_1 - \sqrt{\mathcal{C}}\|_{H^6(\mathbb{T}^n)} + \|\mathbf{u}_1\|_{H^5(\mathbb{T}^n)} \le \eta$, the solution $(\psi, \mathbf{u}, V)$ of the IVP (1.5)–(1.8) exists globally in time and moreover one has

$$\|(\psi - \sqrt{\mathcal{C}})(t)\|^2_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|^2_{H^5(\mathbb{T}^n)} + \|V(t)\|^2_{H^4(\mathbb{T}^n)} \le C\delta_0 e^{-\Lambda_0 t},$$

for all $t \ge 0$, where $C > 0$, $\Lambda_0 > 0$ are suitable constants, and

$$\delta_0 = \|\psi_1 - \sqrt{\mathcal{C}}\|^2_{H^6(\mathbb{T}^n)} + \|\mathbf{u}_1\|^2_{H^5(\mathbb{T}^n)}. \tag{1.11}$$

Remark 1.4 (1) Although in the Theorem 1.3 we choose the special stationary state $(\sqrt{\mathcal{C}},0,0)$, we claim that the method used here can be applied to prove the time-asymptotic convergence toward any stationary state of (1.5)–(1.7), say $(\bar{\psi},\bar{\mathbf{u}},\bar{V})$, with $\nabla\times\bar{\mathbf{u}}=0$. The well-posedness of them can be obtained by applying the arguments of [17], with suitable modifications. In this case, the corresponding "subsonic" condition have to be changed in the following way

$$\frac{\pi^2}{L^2}\varepsilon^2+P'(\bar{\psi}^2)>|\bar{\mathbf{u}}|^2. \tag{1.12}$$

(2) It is known that classical solutions of hydrodynamical model for semiconductors (without dispersion term) for large initial data may blow up in finite time to form singularities [3]. Analogous results on the existence of L^∞ solution and one or two dimensional transonic solutions for the hydrodynamical model for semiconductors was proven [6, 7]. However when dispersive regularity is involved in (1.10) or (1.12), it may prevent the formation of singularities and classical solutions exist globally in time even in the transonic or supersonic region, in the classical sense [2, 4].

(3) Note here that the conditions (1.10) and (1.12) are exactly the subsonic conditions in the classical sense [2], when the re-scaled Planck constant ε goes to zero. If $\varepsilon>0$ and $P'(\rho)>0$, the "sound" speed $\tilde{c}(\bar{\rho})=\sqrt{\pi^2\varepsilon^2/L^2+P'(\bar{\rho})}$ is bigger than the sound speed $c(\rho)=\sqrt{P'(\bar{\rho})}$ for the classical hydrodynamic equations. □

The theorems 1.1–1.3 can be extended to the multi-dimensional torus $\mathbb{T}^n$, $n\geq 2$, for the IVP (1.5)–(1.8) with smooth initial data. Indeed, we have

Theorem 1.5 *Let $P\in C^m(0,\infty)$, with $m\geq s-1$ and $s>\left[\frac{n}{2}\right]+5$. Let us assume that $(\psi_1,\mathbf{u}_1)\in H^s(\mathbb{T}^n)\times H^{s-1}(\mathbb{T}^n)$, $\nabla\times\mathbf{u}_1=0$, and $\min_{x\in[0,1]}\psi_1(x)>0$, then, there exists $T'>0$ such that a solution $(\psi,\mathbf{u},V)(t)\in H^s(\mathbb{T}^n)\times H^{s-1}(\mathbb{T}^n)\times H^{s-2}(\mathbb{T}^n)$ of the IVP (1.5)–(1.8), with $\psi>0$, exists on $[0,T']$.*

Moreover, assume that $(\bar{\psi},\bar{\mathbf{u}},\bar{V})$, with $\nabla\times\bar{\mathbf{u}}=0$ and $\bar{\psi}>0$, is a classical stationary state of (1.5)–(1.7) with small oscillation and satisfies (1.12). Then, if $\|\psi_1-\bar{\psi}\|_{H^s(\mathbb{T}^n)}+\|\mathbf{u}_1-\bar{\mathbf{u}}\|_{H^{s-1}(\mathbb{T}^n)}$ is sufficiently small,then the solution $(\psi,\mathbf{u},V)(t)$ of IVP (1.5)–(1.8) exists globally in time and satisfies

$$\|(\psi-\bar{\psi})(t)\|^2_{H^s(\mathbb{T}^n)}+\|(\mathbf{u}-\bar{\mathbf{u}})(t)\|^2_{H^{s-1}(\mathbb{T}^n)}+\|(V-\bar{V})(t)\|^2_{H^{s-2}(\mathbb{T}^n)}\leq C\delta_1 e^{-\Lambda_2 t},$$

with $\Lambda_2>0$ and

$$\delta_1=\|(\psi_1-\bar{\psi})\|^2_{H^s(\mathbb{T}^n)}+\|(\mathbf{u}_1-\bar{\mathbf{u}})\|^2_{H^{s-1}(\mathbb{T}^n)}.$$

Remark 1.6 Once we prove the local existence (resp. global existence) of solutions $(\psi,\mathbf{u},V)$ of IVP (1.5)–(1.8), we can obtain the local existence (resp. global existence) of solutions $(\rho,\mathbf{u},V)$ of IVP (1.1)–(1.4) by setting $\rho=\psi^2$. □

This paper is organized in the following way. In the section 2, we present preliminary results on divergence equation, Poisson equation, and a fourth order semilinear wave type equation on $\mathbb{T}^n$, then we list some known calculus inequalities. We prove the Theorem 1.1 in the section 3. After the construction of our extended system in the section 3.1, we show the construction of the approximate solutions,we derive the uniform estimates, and we prove the Theorem 1.1 in the section 3.2. The section 4 is concerned with the proof of Theorem 1.3. After the reformulation of original problem in the section 4.1, we establish the a-priori estimates on the local solutions in the section 4.2, and prove the global existence and the large time behavior in the remaining part.

Notation. C always denotes generic positive constant. $L^2(\mathbb{T}^n)$ is the space of square integral functions on $\mathbb{T}^n$ with the norm $\|\cdot\|$. $H^k(\mathbb{T}^n)$ with integer $k \geq 1$ denotes the usual Sobolev space of function f, satisfying $\partial_x^i f \in L^2$ $(0 \leq i \leq k)$, with norm

$$\|f\|_k = \sqrt{\sum_{0\leq|l|\leq m} \|D^l f\|^2},$$

here and after $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ for $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$ and $\partial_j = \partial_{x_j}$, $j = 1, 2, ..., n$, for abbreviation. In particular, $\|\cdot\|_0 = \|\cdot\|$. $\dot{H}^k(\mathbb{T}^n)$ denotes the subspace of function in $H^k(\Omega)$ satisfying

$$\int_\Omega u(x)\,dx = 0.$$

Let $T > 0$ and let $\mathcal{B}$ be a Banach space. $C^k(0, T; \mathcal{B})$ ($C^k([0, T]; \mathcal{B})$ resp.) denotes the space of $\mathcal{B}$-valued k-times continuously differentiable functions on $(0, T)$ (or $[0, T]$ resp.), $L^2([0, T];\ \mathcal{B})$ the space of $\mathcal{B}$-valued L^2-functions on $[0, T]$, and $H^k([0, T]; \mathcal{B})$ the spaces of functions f, such that $\partial_t^i f \in L^2([0, T]; \mathcal{B})$, $1 \leq i \leq k$, $1 \leq p \leq \infty$.

2 Preliminaries

In this section, we prove the existence and uniqueness of solutions of the divergence equation on $\mathbb{T}^n$ and we recall a known result on multi-dimensional Poisson equation with periodic boundary conditions. Then, we turn to prove the well-posedness for an abstract second order semi-linear evolution equation. Finally, some calculus inequalities are listed without proofs.

First, we have the following theorem on the divergence operator and Laplace operator on $\mathbb{T}^n$:

Theorem 2.1 *Let $f \in \dot{H}^s(\mathbb{T}^n)$, $s \geq 0$. There exists a unique solution $u \in (H^{s+1}(\mathbb{T}^n))^n$ satisfying*

$$\nabla\cdot \mathbf{u} = f, \quad \nabla\times\mathbf{u} = 0, \quad \int_{\mathbb{T}^n} (\mathbf{u} - \hat{u})\,dx = 0, \tag{2.1}$$

and

$$\|(\mathbf{u} - \hat{u})\|_{H^{s+1}(\mathbb{T}^n)} \le c_1 \|f\|_{\dot{H}^s(\mathbb{T}^n)}, \tag{2.2}$$

where $c_1 > 0$ is a suitable constant and $\hat{u}$ a vector in $\mathbb{R}^n$.

Theorem 2.2 *Let $f \in \dot{H}^s(\mathbb{T}^n)$, $s \ge 0$. There exists a unique solution $u \in \dot{H}^{s+2}(\mathbb{T}^n)$ to the Poisson equation*

$$\Delta u = f$$

satisfying

$$\|u\|_{\dot{H}^{s+2}(\mathbb{T}^n)} \le c_2 \|f\|_{\dot{H}^s(\mathbb{T}^n)} \tag{2.3}$$

with $c_2 > 0$.

The proofs of Theorems 2.1–2.2 can be completed with the help of Fourier series expansion of the functions $\mathbf{u}$, u and f. Here we omit the details . □

Based on Theorem 2.2, we obtain the initial potential V_1 through (1.7) in view of initial density:

$$\Delta V_1 = \psi_1^2 - \mathcal{C}, \quad \int_{\mathbb{T}^n} V_1(x)dx = 0. \tag{2.4}$$

By (1.9) and $\psi_1 - \sqrt{\mathcal{C}} \in H^3$, we obtain that $V_1 \in \dot{H}^5$ and satisfies

$$\|V_1\|_{\dot{H}^5(\mathbb{T}^n)} \le c_3 \|\psi_1^2 - \mathcal{C}\|_{\dot{H}^3(\mathbb{T}^n)} \le c_4 \|\psi_1 - \sqrt{\mathcal{C}}\|_{H^3(\mathbb{T}^n)}, \tag{2.5}$$

with $c_3, c_4 > 0$ constants.

Finally, let us consider the abstract initial value problem in the periodic Hilbert space $L^2(\mathbb{T}^n)$:

$$u'' + \frac{1}{\tau}u' + Au + \mathcal{L}u' = F(t), \tag{2.6}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.7}$$

Hereafter u' denotes $\frac{du}{dt}$. The operator A is defines by

$$Au = \nu_0 \Delta^2 u + \nu_1 u \tag{2.8}$$

where Δ is the Laplacian operator on $\mathbb{R}^n$, and $\nu_0, \nu_1 > 0$ are given constants. The domain of the linear operator A is $D(A) = H^4(\mathbb{T}^n)$. Related to the operator A, we define a continuous and symmetric bilinear form $a(u, v)$ on $H^2(\mathbb{T}^n)$

$$a(u, v) = \int_{\mathbb{T}^n} (\nu_0 \Delta u \Delta v + \nu_1 uv)dx, \quad \forall\, u, v \in H^2(\mathbb{T}^n), \tag{2.9}$$

which is coercive, i.e.,

$$\exists\, \nu > 0, \quad a(u,u) \geq \nu \|u\|_{H^2(\mathbb{T}^n)}, \quad \forall\, u \in H^2(\mathbb{T}^n). \tag{2.10}$$

This means that there are a complete orthogonal family $\{r_l\}_{l\in N}$ of $L^2(\mathbb{T}^n)$ and a family $\{\mu_l\}_{l\in N}$ consisting of the eigenvectors and eigenvalues of operator A

$$\begin{aligned} &Ar_l = \mu_l r_l, \quad l = 1, 2, \cdots, \\ &0 < \mu_1 \leq \mu_2, \cdots, \quad \mu_l \to \infty \text{ as } l \to \infty. \end{aligned} \tag{2.11}$$

The family $\{r_l\}_{l\in N}$ is also orthogonal for $a(u,v)$ on $H^2(\mathbb{T}^n)$, i.e.,

$$a(r_l, r_j) =< Ar_l, r_j >= \mu_l(r_l, r_j) = \mu_l \delta_{lj}, \quad \forall\, l, j,$$

where δ_{lj} denotes the Kronecker symbol.

Related to $\mathcal{L}u$ and $F(t)$, we have

$$< \mathcal{L}u, v >= \int_{\mathbb{T}^n} (b(x,t) \cdot \nabla u) v dx, \quad u,\ v \in H^2(\mathbb{T}^n), \tag{2.12}$$

$$< F(t), v >= \int_{\mathbb{T}^n} f(x,t) v dx, \quad v \in H^2(\mathbb{T}^n), \tag{2.13}$$

where $b : \mathbb{T} \times [0,T] \to \mathbb{R}^n$ and $f : \mathbb{T} \times [0,T] \to \mathbb{R}$ are measurable functions.

By applying the Faedo-Galerkin method [35, 38], we can obtain the existence of solutions to (2.6)-(2.7) in a standard way.

Theorem 2.3 *Let $T > 0$, $n = 2, 3$, and assume that*

$$F \in H^1([0,T]; L^2(\mathbb{T}^n)), \quad b \in L^2([0,T]; H^3(\mathbb{T}^n)) \bigcap H^1([0,T]; H^2(\mathbb{T}^n)). \tag{2.14}$$

Then, if $u_0 \in H^4(\mathbb{T}^n)$ and $u_1 \in H^2(\mathbb{T}^n)$, the solution to (2.6)*–*(2.7) *exists and satisfies*

$$u \in C^i([0,T]; H^{4-2j}(\mathbb{T}^n)),\ j = 0, 1, \quad u'' \in L^\infty([0,T]; L^2(\mathbb{T}^n)). \tag{2.15}$$

Moreover, assume that

$$F',\ F \in L^2([0,T]; H^2(\mathbb{T}^n)), \tag{2.16}$$

then, if $u_0 \in H^6(\mathbb{T}^n)$ and $u_1 \in H^4(\mathbb{T}^n)$, it follows

$$u \in C^i([0,T]; H^{6-2j}(\mathbb{T}^n)),\ j = 0, 1, 2, \quad u''' \in L^\infty([0,T]; L^2(\mathbb{T}^n)). \tag{2.17}$$

Proof: The statement (2.17) follows from (2.15), if we consider the same type of problem for new unknown $v = \mathrm{D}^2 u$. The statement (2.15) can be proved by applying the Faedo-Galerkin method. We omit the details here since everything is quite standard. For general stability theory of abstract second order equations, the reader can refer to [29, 30].

□

Remark 2.4 Note that if (2.14) is replaced by

$$F \in C^1([0,T];L^2(\mathbb{T}^n)), \quad b \in C^i[0,T];H^{3-i}(\mathbb{T}^n)),\ i=0,1, \tag{2.18}$$

then in (2.15) it follows

$$u'' \in C([0,T];L^2(\mathbb{T}^n)).$$

Furthermore, when (2.16) is replaced by

$$F \in C^1([0,T];H^2(\mathbb{T}^n)), \tag{2.19}$$

it also holds in (2.17) that

$$u''' \in C([0,T];L^2(\mathbb{T}^n)).$$

Finally, we list below the Moser-type calculus inequalities [22, 28, 34]:

Lemma 2.5 *Let $f,g \in L^\infty(\mathbb{T}^n)\bigcap H^s(\mathbb{T}^n)$. Then, it follows*

$$\|D^\alpha(fg)\| \le C\|g\|_{L^\infty}\|D^\alpha f\| + C\|f\|_{L^\infty}\|D^\alpha g\|, \tag{2.20}$$

$$\|D^\alpha(fg) - fD^\alpha g\| \le C\|g\|_{L^\infty}\|D^\alpha f\| + C\|f\|_{L^\infty}\|D^{\alpha-1} g\|, \tag{2.21}$$

for $1 \le |\alpha| \le s$.

3 Local existence

This section is concerned in the proof of Theorem 1.1. We construct the new extended system based on (1.5)–(1.8) in the section 3.1, then we build up the approximate solutions, derive the uniform estimates, and prove the Theorem 1.1 in the section 3.2. For simplicity, we set $\tau = 1$.

3.1 Construction of the extended system

We construct the extended system in this subsection. For ir-rotational flow, the velocity field can be represented as the gradient field of a phase function S:

$$\mathbf{u} = \nabla S. \tag{3.1}$$

In analogy, the continuous equation (1.6) for the ir-rotational velocity vector field $\mathbf{u}$ is changed into

$$\partial_t \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{u}|^2) + \nabla h(\psi^2) + \mathbf{u} = \nabla V + \frac{\varepsilon^2}{2}\nabla\Big(\frac{\Delta\psi}{\psi}\Big), \tag{3.2}$$

which, together with the initial data $\mathbf{u}(x,0) = \mathbf{u}_1(x)$, provides the time-decay of mean velocity on $\mathbb{T}^n$:

$$\int_{\mathbb{T}^n} \mathbf{u}(x,t)dx = \bar{\mathbf{u}}(t) =: e^{-t} \int_{\mathbb{T}^n} \mathbf{u}_1(x)dx, \quad t \geq 0. \tag{3.3}$$

For $\psi > 0$ the equation (1.5) becomes

$$2\partial_t\psi + 2\mathbf{u}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{u} = 0. \tag{3.4}$$

We want to explain the main steps that we will use in the next subsection to implement an iterative procedure. Once we know $\mathbf{u}$ and ψ based on the Eq. (3.4) and the previous observation, we introduce two new equations for the artificial "velocity" $\mathbf{v}$ and artificial "density" $\varphi > 0$

$$\nabla\cdot\mathbf{v} = -\frac{2(\partial_t\psi + \mathbf{u}\cdot\nabla\psi)}{\varphi}, \quad \nabla\times\mathbf{v} = 0, \quad \int_{\mathbb{T}^n} \mathbf{v}(x,t)dx = \bar{\mathbf{u}}(t). \tag{3.5}$$

$$\partial_t\varphi + \frac{1}{2}\varphi\nabla\cdot\mathbf{v} + \mathbf{u}\cdot\nabla\psi = 0, \quad \varphi(x,0) = \psi_1(x) > 0. \tag{3.6}$$

Clearly to re-initialize the procedure, we have to determine ψ and $\mathbf{u}$ as long as we know φ and $\mathbf{v}$ (we will propose the corresponding equations, used in the next subsection, for ψ and $\mathbf{u}$ below). By a simple combination of the equations (3.5)–(3.6), we obtain

$$\partial_t[\varphi - \psi](x,t) = 0, \quad \forall\, x \in \mathbb{T}^n,$$

which implies

$$[\varphi - \psi](x,t) = 0 \text{ for } (x,t) \in \mathbb{T}^n \times (0,\infty), \quad \text{ if } [\varphi - \psi](x,0) = 0. \tag{3.7}$$

By applying to (3.6) a standard argument in the theory of O.D.E. namely by multiplying Eq. (3.6) by the function $\exp\{\frac{1}{2}\int_0^t \nabla\cdot\mathbf{v}(x,s)ds\}$ and by integrating the resulting equation with respect to time, we can represent φ for $(x,t) \in \mathbb{T}^n \times [0,+\infty)$ by the identity

$$\varphi(x,t) = \psi_1(x)e^{-\frac{1}{2}\int_0^t \nabla\cdot\mathbf{v}(x,s)ds} - \int_0^t \mathbf{u}\cdot\nabla\psi(x,s)e^{-\frac{1}{2}\int_s^t \nabla\cdot\mathbf{v}(x,\xi)d\xi}ds. \tag{3.8}$$

This means that for short time (smooth) solutions (if they exist) satisfy

$$\varphi(x,t) > 0, \quad \text{if} \quad \psi_1(x) > 0, \quad x \in \mathbb{T}^n.$$

Based on the Eq. (3.4) and Eq. (3.2), we show how to reconstruct the density ψ. Here we use the following second order evolutional problem

$$\psi_{tt} + \psi_t + \frac{1}{4}\varepsilon^2\Delta^2\psi - \frac{1}{4}\varepsilon^2\frac{|\Delta\psi|^2}{\varphi} - \frac{1}{2\varphi}\Delta P(\psi^2) + \frac{1}{2}\psi\Delta V + \nabla\psi\cdot\nabla V$$

$$+ (\mathbf{u}+\mathbf{v})\cdot\nabla\psi_t - \frac{1}{2}\nabla\psi\cdot\nabla(|\mathbf{v}|^2) - \frac{1}{2}\psi\nabla\mathbf{v}:\nabla\mathbf{v} + \mathbf{v}\cdot\nabla(\mathbf{u}\cdot\nabla\psi)$$
$$- \frac{1}{\varphi}(\psi_t+\mathbf{u}\cdot\nabla\psi)(\mathbf{v}\cdot\nabla\psi) - \frac{\psi_t}{\varphi}(\psi_t+\mathbf{u}\cdot\nabla\psi) = 0, \tag{3.9}$$

with initial data

$$\psi(x,0)=\psi_1,\ \psi_t(x,0)=\psi_0 =: -\frac{1}{2}\psi_1\nabla\cdot\mathbf{u}_1 - \mathbf{u}\cdot\nabla\psi_1, \tag{3.10}$$

where $\mathbf{v}=(v^1,v^2,...,v^n)$ and

$$\nabla\mathbf{v}:\nabla\mathbf{v} = \sum_{i,j}|\partial_j v^i|^2.$$

Indeed, let us multiply (3.2) by ψ^2, take divergence of the resulting equation, then use (3.4), the irrotationality assumption of velocity vector fields plus the relation

$$\nabla\cdot\left[\psi^2\nabla\Big(\frac{\Delta\psi}{\psi}\Big)\right] = \psi\left[\Delta^2\psi - \frac{|\Delta\psi|^2}{\psi}\right],$$

replace the nonlinear term $\frac{1}{4\psi}\nabla\cdot(\psi^2\nabla(|\mathbf{u}|^2))$ by

$$\frac{1}{2}\nabla\psi\cdot\nabla(|\mathbf{v}|^2) + \frac{1}{2}\psi\nabla\mathbf{v}:\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\psi_t - (\mathbf{v}\cdot\nabla)(\mathbf{u}\cdot\nabla\psi) + \frac{1}{\psi}(\psi_t+\mathbf{u}\cdot\nabla\psi)(\mathbf{v}\cdot\nabla\psi),$$

and finally replace $\frac{1}{\psi}$ in the resulting equation by $\frac{1}{\varphi}$, we get the equation (3.9).

Similarly we can construct from (3.2) the equation for reconstructing the velocity $\mathbf{u}$

$$\partial_t\mathbf{u} + \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{v}|^2) + \nabla h(\psi^2) = \nabla V + \frac{\varepsilon^2}{2}\Big(\frac{\nabla\Delta\psi}{\varphi} - \frac{\Delta\psi\nabla\psi}{\varphi^2}\Big), \quad \mathbf{u}(x,0)=\mathbf{u}_1(x). \tag{3.11}$$

Here we have used the identity

$$\nabla\Big(\frac{\Delta\psi}{\psi}\Big) = \Big(\frac{\nabla\Delta\psi}{\psi} - \frac{\Delta\psi\nabla\psi}{\psi^2}\Big), \tag{3.12}$$

and we replaced $\frac{1}{\psi}$ and $|\mathbf{u}|^2$ by $\frac{1}{\varphi}$ and $|\mathbf{v}|^2$ respectively in (3.2).

Finally, from (1.7) the reconstruction of V is done directly by using the Poisson equation on $\mathbb{T}^n$ and involves only ψ:

$$\Delta V = \psi^2 - \mathcal{C} - \frac{1}{L^n}\int_{\mathbb{T}^n}(\psi^2-\mathcal{C})(x,t)dx, \quad \int_{\mathbb{T}^n}V(x,t)dx = 0. \tag{3.13}$$

So far, we have construct an extended coupled and closed system for the new unknown $U=(\mathbf{v},\varphi,\psi,\mathbf{u},V)$, which consists of two O.D.E.s (3.6) for φ and (3.11) for $\mathbf{u}$, a second order evolutional equation (3.9) for ψ, a divergence equation (3.5) for $\mathbf{v}$, and an elliptic equation (3.13) for V. The most important fact (which we will be able to show later on) is to note that this extended system for $U=(\mathbf{v},\varphi,\psi,\mathbf{u},V)$ is equivalent to the original equations (1.5)–(1.7) of $(\psi,\mathbf{u},V)$, as far as we look for classical solutions, when $\mathbf{u}=\mathbf{v}$ and $\psi=\varphi>0$.

3.2 Iteration scheme and local existence

Now, we consider the corresponding problem for an approximate solution $\{U^i\}_{i=1}^{\infty}$ with $U^p = (\mathbf{v}_p, \varphi_p, \psi_p, \mathbf{u}_p, V_p)$ based on the extended system constructed in the subsection (3.1). The iteration scheme for the approximate solution $U^{p+1} = (\mathbf{v}_{p+1}, \varphi_{p+1}, \psi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$, $p \geq 1$, is defined by solving the following problems on $\mathbb{T}^n$:

$$\nabla\cdot \mathbf{v}_{p+1} = r_p(t), \quad \nabla\times\mathbf{v}_{p+1} = 0, \quad \int_{\mathbb{T}^n} \mathbf{v}_{p+1}(x,t)\,dx = \bar{\mathbf{u}}(t), \tag{3.14}$$

$$\begin{cases} \varphi'_{p+1} + \frac{1}{2}(\nabla\cdot \mathbf{v}_p)\varphi_{p+1} + \mathbf{u}_p\cdot\nabla\psi_p = 0, \quad t>0, \\ \varphi_{p+1}(x,0) = \psi_1(x), \end{cases} \tag{3.15}$$

$$\begin{cases} \psi''_{p+1} + \psi'_{p+1} + \nu\Delta^2\psi_{p+1} + \nu\psi_{p+1} + k_p(t)\cdot\nabla\psi'_{p+1} = h_p(t), \quad t>0, \\ \psi_{p+1}(x,0) = \psi_1(x),\ \psi'_{p+1}(x,0) = \psi_0 =: -\frac{1}{2}\psi_1\nabla\cdot\mathbf{u}_1 - \mathbf{u}\cdot\nabla\psi_1, \end{cases} \tag{3.16}$$

$$\begin{cases} \mathbf{u}'_{p+1} + \mathbf{u}_{p+1} = g_p(t), \quad t>0, \\ \mathbf{u}_{p+1}(0) = \mathbf{u}_1,\ \nabla\times\mathbf{u}_1 = 0, \end{cases} \tag{3.17}$$

$$\Delta V_{p+1} = q_p(t), \quad \int_{\mathbb{T}^n} V_{p+1}(x,t)dx = 0, \tag{3.18}$$

where $\nu = \frac{1}{4}\varepsilon^2$, and

$$r_p(t) = r_p(x,t) = -\frac{2(\psi'_p + \mathbf{u}_p\cdot\nabla\psi_p)}{\varphi_p} + \frac{1}{L^n}\int_{\mathbb{T}^n}\frac{2(\psi'_p + \mathbf{u}_p\cdot\nabla\psi_p)}{\varphi_p}(x,t)dx, \tag{3.19}$$

$$k_p(t) = k_p(x,t) = \mathbf{u}_p(x,t) + \mathbf{v}_p(x,t), \tag{3.20}$$

$$\begin{aligned} h_p(t) = h_p(x,t) =& \frac{|\psi'_p|^2}{\varphi_p} + \frac{\psi'_p}{\varphi_p}\mathbf{u}_p\cdot\nabla\psi_p + \frac{\varepsilon^2}{4}\frac{|\Delta\psi_p|^2}{\varphi_p} - \frac{1}{2}\psi_p\Delta V_p - \nabla\psi_p\cdot\nabla V_p \\ &+ \frac{1}{2}\frac{\Delta P(\psi_p^2)}{\varphi_p} + \nu\psi_p + \frac{1}{2}\nabla\psi_p\cdot\nabla(|\mathbf{v}_p|^2) + \frac{1}{2}\psi_p\sum_{j,i}|\partial_j v_p^i|^2 \\ &- \mathbf{v}_p\cdot\nabla(\mathbf{u}_p\cdot\nabla\psi_p) + \frac{1}{\varphi_p}(\psi'_p + \mathbf{u}_p\cdot\nabla\psi_p)\mathbf{v}_p\cdot\nabla\psi_p, \end{aligned} \tag{3.21}$$

$$g_p(t) = g_p(x,t) = \nabla V_p - \frac{1}{2}\nabla\big(|\mathbf{v}_p|^2\big) - \nabla h(\psi_p^2) + \frac{1}{2}\varepsilon^2\left(\frac{\nabla\Delta\psi_p}{\varphi_p} - \frac{(\Delta\psi_p)\nabla\psi_p}{\varphi_p^2}\right), \tag{3.22}$$

$$q_p(t) = q_p(x,t) = \psi_p^2 - \mathcal{C} - \frac{1}{L^n}\int_{\mathbb{T}^n}(\psi_p^2 - \mathcal{C})(x,t)dx, \tag{3.23}$$

where $\mathbf{u}_p = (u_p^1, u_p^2, \cdots, u_p^n)$ and $\mathbf{v}_p = (v_p^1, v_p^2, \cdots, v_p^n)$.

Let us emphasize that here the functions $r_p(0), k_p(0), h_p(0), g_p(0), q_p(0)$ depend only upon the initial data $(\psi_1, \mathbf{u}_1)$ and moreover they are periodic in the space variables.

The main result in this section is the following concerning “a-priori estimates”.

Lemma 3.1 *Let us assume that* $P \in C^5(0,\infty)$ *and* $(\psi_1, \mathbf{u}_1) \in H^6 \times H^5$, $\nabla\times\mathbf{u}_1 = 0$, *such that*

$$\psi^* = \max_{x\in\mathbb{T}^n} \psi_1(x), \quad \psi_* =: \min_{x\in\mathbb{T}^n} \psi_1(x) > 0. \tag{3.24}$$

Then, there exist a positive time T_* *and a sequence* $\{U^p\}_{p=1}^{\infty}$ *of approximate solutions, which solve the system* (3.14)–(3.18) *for* $t \in [0, T_*]$ *and satisfy*

$$\begin{cases} \mathbf{v}_p \in C^j([0,T_*]; H^{4-j}(\mathbb{T}^n)) \bigcap C^2([0,T_*]; H^1(\mathbb{T}^n)), \quad j = 0,1, \\ \varphi_p \in C^1([0,T_*]; H^3(\mathbb{T}^n)) \bigcap C^2([0,T_*]; H^2(\mathbb{T}^n)) \bigcap C^3([0,T_*]; L^2(\mathbb{T}^n)), \\ \psi_p \in C^l([0,T_*]; H^{6-2l}(\mathbb{T}^n)) \bigcap C^3([0,T_*]; L^2(\mathbb{T}^n)), \quad l = 0,1,2, \\ \mathbf{u}_p \in C^1([0,T_*]; H^3(\mathbb{T}^n)) \bigcap C^2([0,T_*]; H^1(\mathbb{T}^n)), \\ V_p \in C([0,T_*]; \dot{H}^4(\mathbb{T}^n)) \bigcap C^1([0,T_*]; \dot{H}^4(\mathbb{T}^n)). \end{cases} \tag{3.25}$$

Moreover, there is a positive constant M_* *so that for all* $t \in [0, T_*]$, *we have*

$$\begin{cases} \|(\mathbf{u}_p, \mathbf{u}_p')(t)\|_3^2 + \|(\mathbf{u}_p'', \mathbf{v}_p'')(t)\|_1^2 + \|\mathbf{v}_p(t)\|_4^2 + \|\mathbf{v}_p'(t)\|_3^2 + \|(V_p, V_p')(t)\|_4^2 \le M_*, \\ \|(\psi_p, \psi_p', \psi_p'', \psi_p''')(t)\|^2_{H^6\times H^4\times H^2\times L^2} + \|(\varphi_p, \varphi_p', \varphi_p'', \varphi_p''')(t)\|^2_{H^3\times H^3\times H^2\times L^2} \le M_*, \end{cases} \tag{3.26}$$

uniformly with respect to $p \ge 1$.

Proof: Step 1: estimates for $\mathbf{p} = 1$**.** Obviously, $U^1 = (\mathbf{u}_1(x), \psi_1(x), \psi_1(x), \mathbf{u}_1(x), V_1(x))$ satisfies (3.25)–(3.26) for the time interval $[0,1]$ with M_* replaced by some constant $B_1 > 0$ and V_1 is determined by (2.4).

We start the iterative process with $U^1 = (\mathbf{u}_1, \psi_1, \psi_1, \mathbf{u}_1, V_1)$, then by solving the problems (3.14)–(3.18) for $p = 1$, we can prove the (local in time) existence of a solution $U^2 = (\mathbf{v}_2, \psi_2, \varphi_2, \mathbf{u}_2, V_2)$ which also satisfies (3.25)–(3.26) for a time interval (which without loss of generality is chosen to be $[0,1]$ since we focus on local in-time existence of solutions) and with M_* replaced by another constant $B_2 > 0$. In fact, for $U^1 = (\mathbf{u}_1, \psi_1, \psi_1, \mathbf{u}_1, V_1)$ the functions $r_1,\ k_1,\ h_1,\ g_1,\ q_1$ depend only on the initial data $(\psi_1, \mathbf{u}_1)$, i.e.,

$$r_1(x,t) = \tilde{r}_1(x),\ k_1(x,t) = \tilde{k}_1(x),\ h_1(x,t) = \tilde{h}_1(x),$$
$$g_1(x,t) = \tilde{g}_1(x),\ q_1(x,t) = \tilde{q}_1(x),$$

and

$$\|\tilde{r}_1\|_2^2 + \|\tilde{k}_1\|_3^2 + \|\tilde{h}_1\|_3^2 + \|\tilde{g}_1\|_3^2 + \|\tilde{q}_1\|_2^2 \le N a_0 I_0^4 e^{N\|\mathbf{u}_1\|_3}. \tag{3.27}$$

¿From now on, $N > 0$ denotes a generic constant independent of $U^p, p \ge 1$,

$$a_0 = \frac{(1+\psi^*)^m}{\psi_*^m}, \qquad \text{for a integer } m \ge 10, \tag{3.28}$$

and

$$I_0 = \|(\psi_1 - \sqrt{\mathcal{C}})\|^2 + \|\nabla\psi_1\|_5^2 + \|\mathbf{u}_1\|_5^2. \tag{3.29}$$

The system (3.14)–(3.18) with $p = 1$ is linear on the unknown $U^2 = (\mathbf{v}_2, \psi_2, \varphi_2, \mathbf{u}_2, V_2)$, therefore it can be solved based on the estimates (3.27) for the corresponding right hand side terms as follows. Namely, by the Theorem 2.1, we obtain the existence of solution $\mathbf{v}_2$ to the divergence equation (3.14), with $r_1(x,t)$ replaced by $\tilde{r}_1(x)$, satisfying

$$\mathbf{v}_2 \in C^j([0,1]; H^{4-j}(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)), \quad j = 0, 1.$$

Then by making use of the theory of linear O.D.E. system, we prove the existence of solution $\mathbf{u}_2$ of (3.17) for $g_1(x,t) = \tilde{g}_1(x)$ and then φ_2 of (3.15):

$$\mathbf{u}_2 \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)),$$
$$\varphi_2 \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^2(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)).$$

By applying the Theorem 2.3 to (3.16), with $b(x,t) = 2\mathbf{u}_1(x)$ in (2.12) and $f(x,t) = \tilde{h}_1(x)$ in (2.13), we obtain the existence of a solution ψ_2 satisfying

$$\psi_2 \in C^j([0,1]; H^{6-2j}(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \ j = 0, 1, 2.$$

Finally, the existence of a solution V_2 satisfying

$$V_2 \in C^1([0,1]; \dot{H}^4(\mathbb{T}^n))$$

follows from the application of the Theorem 2.2 to Eq. (3.18) on $\mathbb{T}^n$, with $q_1(x,t)$ replaced by $\tilde{q}_1(x)$.

Moreover, based on the estimates (3.27), we conclude there is a constant $B_2 > 0$, such that U^2 satisfies

$$\begin{cases} \|(\mathbf{u}_2, \mathbf{u}_2')(t)\|_3^2 + \|(\mathbf{u}_2'', \mathbf{v}_2'')(t)\|_1^2 + \|\mathbf{v}_2(t)\|_4^2 + \|\mathbf{v}_2'(t)\|_3^2 + \|(V_2, V_2')(t)\|_4^2 \le B_2, \\ \|(\varphi_2, \varphi_2', \varphi_2'', \varphi_2''')(t)\|_{H^6\times H^3\times H^2\times L^2}^2 + \|(\psi_2, \psi_2', \psi_2'', \psi_2''')(t)\|_{H^6\times H^4\times H^2\times L^2}^2 \le B_2, \end{cases}$$

for all $t \in [0,1]$.

Step 2: estimates for $\mathbf{p} \ge 2$. Now, assume that $\{U^i\}_{i=1}^p$ ($p \ge 2$) exist in the time interval $[0,1]$, solve the system (3.14)–(3.18), and satisfy (3.25)–(3.26), with M_* replaced by the max B_p ($\ge \max_{1\le j\le p-1}\{B_j\}$). For given U^p, the system (3.17)–(3.18) is linear in $U^{p+1} = (\mathbf{v}_{p+1}, \varphi_{p+1}, \psi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$. As before, we apply the Theorem 2.1 to the Eq. (3.14) for $\mathbf{v}_{p+1}$, the theory of linear O.D.E. systems to the Eq. (3.15) for φ_{p+1} and the Eq. (3.17) for $\mathbf{u}_{p+1}$, the Theorem 2.3 to wave type equation (3.16) for ψ_{p+1} with $f(x,t) = h_p(t)$ and $b(x,t) = k_p(t)$, and the Theorem 2.2 to the Poisson equation (3.18) for V_{p+1}. Therefore we obtain the existence of $U^{p+1} = (\mathbf{v}_{p+1}, \psi_{p+1}, \varphi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$ on the time interval $[0,1]$ and moreover it follows

$$\begin{cases} \mathbf{v}_{p+1} \in C^j([0,1]; H^{4-j}(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)), \ j=0,1, \\ \varphi_{p+1} \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^2(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \\ \psi_{p+1} \in C^j([0,1]; H^{6-2j}(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \ j=0,1,2, \\ \mathbf{u}_{p+1} \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^1(\mathbb{T}^n)), \\ V_{p+1} \in C([0,1]; \dot{H}^4(\mathbb{T}^n)) \bigcap C^1([0,1]; \dot{H}^4(\mathbb{T}^n)). \end{cases}$$

Now our goal is to deduce uniform bounds for U^{j+1}, $1 \le j \le p$, for some time interval. Let us first estimate the L^2 norms of the initial value of $\psi_{p+1}, \psi'_{p+1}, \psi''_{p+1}$, where the initial value Ψ_0 of ψ''_{p+1} is obtained through $(3.16)_1$ at $t=0$, where ψ_{p+1} and ψ'_{p+1} are replaced by the initial data ψ_1, ψ_0:

$$\Psi_0 = -\psi_0 - \nu\Delta^2\psi_0 - \nu\psi_1 - 2\mathbf{u}_1 \cdot \nabla\psi_0 + \tilde{h}(0), \tag{3.30}$$

and $\tilde{h}(0) = h_p(0)$ depending only on $(\psi_1, \mathbf{u}_1)$. Hence these initial values will depend only on $(\psi_1, \mathbf{u}_1)$ and are periodic functions of the space variables. Obviously, there is a constant $M_2 > 0$, such that the initial values of $\psi_{p+1}, \psi'_{p+1}, \psi''_{p+1}$ for $p \ge 1$ are bounded by

$$M_2 I_0 \ge \max\left\{ \|\psi_1\|_2^2, \ \|\psi_0\|_2^2, \ \|\Psi_0\|_2^2, \ \|\mathbf{u}_1\|_3^2 \right\}. \tag{3.31}$$

Here, we recall that I_0 is defined by means of (3.29).

Denote by

$$M_0 = 40 M_2 I_0 \cdot \max\{1, \nu^{-1}\}, \tag{3.32}$$
$$M_1 = 3 N a_0^2 (I_0 + 1 + M_0)^7 \cdot \max\{1, \nu^{-2}\}, \tag{3.33}$$

and choose

$$T_* = \min\left\{ 1, \ \frac{\psi_*}{4M_0}, \ \frac{M_2 I_0}{N M_3}, \ \frac{\ln 2}{N M_4}, \ \frac{2 M_2 I_0}{N M_5}, \ \frac{2 M_2 I_0}{N M_6} \right\}, \tag{3.34}$$

where

$$\begin{aligned} &M_3 = 5a_0^2(I_0+1+M_0+M_1)^6, \ M_4 = 2a_0^3(I_0+1+M_0+M_1)^8, \\ &M_5 = a_0^2(I_0+1+M_0+M_1)^7, \quad M_6 = a_0^5(I_0+1+M_0+M_1)^{14}. \end{aligned} \tag{3.35}$$

As before $N \ge M_2$ denotes a generic constant independent of $U^p, p \ge 1$, and a_0 is defined by (3.28).

Step 2.1: we claim that

If the solution $\{U^j\}_{j=1}^p$, $(p \geq 2)$, *to the problems* (3.14)–(3.18) *satisfies*

$$\begin{cases} \|\mathbf{u}_j(t)\|_3^2 + \|(\psi_j, \psi_j')(t)\|_4^2 + \|\psi_j''(t)\|_2^2 \leq M_0, \\ \|\mathbf{v}_j(t)\|_4^2 + \|D\Delta\psi_j(t)\|_1^2 \leq a_0 M_1, \end{cases} \tag{3.36}$$

for all $1 \leq j \leq p$ *and* $t \in [0, T_*]$, *then this is also true for* U^{p+1}, *namely*

$$\begin{cases} \|\mathbf{u}_{p+1}(t)\|_3^2 + \|(\psi_{p+1}, \psi_{p+1}')(t)\|_4^2 + \|\psi_{p+1}''(t)\|_2^2 \leq M_0, \\ \|\mathbf{v}_{p+1}(t)\|_4^2 + \|D\Delta\psi_{p+1}(t)\|_1^2 \leq a_0 M_1, \end{cases} \tag{3.37}$$

for all $t \in [0, T_*]$. *Here* M_0 *and* M_1 *are given by* (3.32) *and* (3.33).

We prove (3.37) in following steps 2.2–2.4, namely, we first obtain the uniform bounds for V_{j+1} $(1 \leq j \leq p)$ based on (3.36), then we estimate uniform bounds of φ_{j+1}, $\mathbf{v}_{j+1}$, $\mathbf{u}_{j+1}(1 \leq j \leq p)$ and their time derivatives in Sobolev space and prove that $\mathbf{v}_{p+1}, \mathbf{u}_{p+1}$ satisfy (3.37), and finally we estimate ψ_{j+1} $(1 \leq j \leq p)$. Meanwhile, related we can get uniform estimates on the time derivatives of $\mathbf{u}_{p+1}, \mathbf{v}_{p+1}$ and on ψ_{p+1}'''.

Step 2.2: estimate on V_{j+1}**.** Based on (3.36) we derive the estimates on V_{j+1} $(1 \leq j \leq p)$ by solving the Poisson equation (3.18) on $\mathbb{T}^n$ for V_{j+1}, $1 \leq j \leq p$. Since it always holds

$$\int_{\mathbb{T}^n} q_j(x,t)dx = 0, \qquad 1 \leq j \leq p, \quad t \in [0, T_*],$$

by using the Theorem 2.2 there exists a unique solution V_{j+1} of Eq. (3.18) satisfying

$$\|V_{j+1}(t)\|_4^2 \leq N\|q_j(t)\|_2^2 \leq N\|\psi_j(t)\|_2^4 \leq NM_0^2, \qquad t \in [0, T_*],\ 1 \leq j \leq p, \tag{3.38}$$

$$\|V_{j+1}'(t)\|_4^2 \leq N\|q_j'(t)\|_2^2 \leq N\|(\psi_j', \psi_j)(t)\|_2^4 \leq NM_0^2, \quad t \in [0, T_*],\ 1 \leq j \leq p. \tag{3.39}$$

Thus, we conclude that $V_{p+1} \in C^1([0, T_*]; \dot{H}^4(\mathbb{T}^n))$ is uniformly bounded so long as (3.36) is true.

Step 2.3: estimates on $\varphi_j, \mathbf{v}_j, \mathbf{u}_j$**.** We estimate $\varphi_j, \mathbf{v}_j, \mathbf{u}_j$, $1 \leq j \leq p$ for $(x,t) \in \mathbb{T}^n \times [0, T_*]$ based on (3.36). For $(x,t) \in \mathbb{T}^n \times [0, T_*]$ by using the same ideas as in deriving (3.8) it follows for φ_{j+1} from (3.15) that

$$\begin{cases} \varphi_{j+1}(x,t) = \left(\psi_1(x) - \int_0^t e^{\frac{1}{2}\int_0^s \nabla\cdot\mathbf{v}_j(x,\xi)d\xi}\mathbf{u}_j \cdot \nabla\psi_j(x,s)ds\right)e^{-\frac{1}{2}\int_0^t \nabla\cdot\mathbf{v}_j(x,s)ds}, \\ \varphi_{j+1} \in C^1([0,1]; H^3(\mathbb{T}^n)) \bigcap C^2([0,1]; H^2(\mathbb{T}^n)) \bigcap C^3([0,1]; L^2(\mathbb{T}^n)), \end{cases} \tag{3.40}$$

which satisfies for all $(x,t) \in \mathbb{T}^n \times [0, T_*]$

$$\frac{1}{4}\psi_* \leq \frac{1}{2}\psi_* e^{-N(1+M_1)T_*} \leq \varphi_{j+1}(x,t) \leq (\psi^* + \psi_*)e^{N(1+M_1)T_*} \leq 2(\psi^* + \psi_*). \tag{3.41}$$

Moreover the L^2 norm of φ_{j+1}, with $1 \le j \le p$, and its derivatives are bounded for all $t \in [0, T_*]$, through those one of $\mathbf{v}_j, \mathbf{u}_j$ and through the initial data by

$$\|\varphi_{j+1}(t)\|_3^2 \le Ne^{N(1+M_1)T_*}(\|\psi_1\|_3^2 + T_*(\|\mathbf{u}_j(t)\|_3^2 \cdot \|\psi_j(t)\|_4^2)) \le NI_0, \tag{3.42}$$

and

$$\begin{aligned} \big\|\varphi'_{j+1}(t)\big\|_3^2 \le & N\big(I_0 + \|\mathbf{v}_j\|_4^2 + \|\mathbf{u}_j(t)\|_3^2 + \|\psi_j(t)\|_4^2\big)^2 \\ \le & Na_0(I_0 + 1 + M_0 + M_1)^2, \end{aligned} \tag{3.43}$$

$$\begin{aligned} \big\|\varphi''_{j+1}(t)\big\|_2^2 \le & NM_0\big(\|\varphi'_{j+1}(t)\|_2^2 + \|\mathbf{u}'_j(t)\|_2^2 + M_0\big) + N\|\varphi_{j+1}(t)\|_3^2 \cdot \|\mathbf{v}'_j(t)\|_3^2 \\ \le & Na_0^2(I_0 + 1 + M_0 + M_1)^3 + N(I_0 + M_0)\|(\mathbf{u}'_j, \mathbf{v}'_j)(t)\|_3^2, \end{aligned} \tag{3.44}$$

$$\begin{aligned} \big\|\varphi'''_{j+1}(t)\big\|^2 \le & N(I_0 + M_0)\big(\|\varphi''_{j+1}(t)\|^2 + \|(\mathbf{u}'_j, \mathbf{u}''_j)(t)\|^2 + \|\mathbf{v}''_j(t)\|_1^2 + M_0\big) \\ & + N\big(\|\varphi'_{j+1}(t)\|_3^4 + \|\mathbf{v}'_j(t)\|_3^4\big) \\ \le & N\big(\|\mathbf{v}'_j(t)\|_3^4 + (I_0 + M_0)^2\|(\mathbf{u}'_j, \mathbf{v}'_j)(t)\|_3^2\big) \\ & + N(I_0 + M_0)\big(\|\mathbf{u}''_j(t)\|^2 + \|\mathbf{v}''_j(t)\|_1^2\big) \\ & + Na_0^2(I_0 + 1 + M_0 + M_1)^4. \end{aligned} \tag{3.45}$$

Let us consider the divergence equation (3.14) for $\mathbf{v}_{j+1}$, with $1 \le j \le p$. Since one has

$$\int_{\mathbb{T}^n} r_j(x,t)\,dx = 0, \quad 1 \le j \le p, \quad t \in [0, T_*],$$

the application of the Theorem 2.1 yields to the existence of a unique solution $\mathbf{v}_{j+1}$ of Eq. (3.14) for $t \in [0, T_*]$, which, in view of (3.40)–(3.44) and (3.36), satisfies the following bounds

$$\begin{aligned} \|\mathbf{v}_{j+1}(t)\|_4^2 \le N\|r_j(t)\|_3^2 \le & Na_0\|\varphi_j(t)\|_3^2\big(\|\psi'_j(t)\|_3^2 + \|\psi_j(t)\|_4^2 + \|\mathbf{u}_j(t)\|_3^2\big) \\ \le & Na_0(I_0 + 1 + M_0)^3 && (3.46) \\ \le & \frac{1}{3}M_1, \qquad t \in [0, T_*],\ 1 \le j \le p, && (3.47) \end{aligned}$$

$$\begin{aligned} \|\mathbf{v}'_{j+1}(t)\|_3^2 \le & N\|r'_j(t)\|_2^2 \le Na_0\|\varphi_j\|_2^2\big(\|\psi''_j(t)\|_2^2 + M_0\|(\psi'_j, \mathbf{u}'_j)(t)\|_2^2\big) \\ & \qquad + Na_0I_0\|\varphi'_j(t)\|_3^2\big(M_0\|\mathbf{u}_j\|_2^2 + \|\psi'_j(t)\|_2^2\big) \\ \le & Na_0^2(I_0 + 1 + M_0 + M_1)^5 \\ & + Na_0(I_0 + M_0)^2\|\mathbf{u}'_j(t)\|_2^2, \qquad t \in [0, T_*], 1 \le j \le p, \end{aligned} \tag{3.48}$$

and

$$\|\mathbf{v}''_{j+1}(t)\|_1^2 \le N\|r''_j(t)\|^2$$

$$
\begin{aligned}
&\le N a_0\big(\|\psi_j'''(t)\|^2 + M_0\|(\mathbf{u}_j'',\mathbf{u}_j')(t)\|^2 + M_0\|\psi_j''(t)\|^2\big)\\
&\quad + N a_0\|\varphi_j''(t)\|^2\big(\|\psi_j'(t)\|_2^2 + M_0\|\mathbf{u}_j(t)\|_2^2\big)\\
&\quad + N a_0\|\varphi_j'(t)\|_3^2\big(\|\varphi_j''(t)\|^2 + \|\varphi_j'(t)\|_2^2 + \|\varphi_j(t)\|_4^2 + \|(\mathbf{u}_j,\mathbf{u}_j')(t)\|^2\big)\\
&\le N a_0\big(\|\psi_j'''(t)\|_3^2 + M_0\|\mathbf{u}_j''(t)\|^2\big) + N a_0^3(I_0+1+M_0+M_1)^5\\
&\quad + N a_0(I_0+1+M_0+M_1)^3\big(\|\mathbf{u}_j'(t)\|_3^2 + \|\mathbf{v}_j'(t)\|_3^2\big)\\
&\le N a_0\big(\|\psi_j'''(t)\|_3^2 + M_0\|\mathbf{u}_j''(t)\|^2\big) + N a_0^3(I_0+1+M_0+M_1)^8\\
&\quad + N a_0^2(I_0+1+M_0+M_1)^3\|\mathbf{u}_j'(t)\|_3^2\\
&\quad + N a_0^2(I_0+1+M_0)^5\|\mathbf{u}_{j-1}'(t)\|_2^2, \qquad t\in[0,T_*],\ 2\le j\le p, \qquad (3.49)
\end{aligned}
$$

where we have already used (3.48) for $\mathbf{v}_j'$.

For the functions U^i $(1\le j\le p)$ satisfying (3.36), it is easy to verify that g_j, g_j' $(1\le j\le p)$ belong to $H^3(\mathbb{T}^n)$ and $H^1(\mathbb{T}^n)$. By (3.36), (3.38)–(3.43) and (3.47)–(3.48), we can obtain the L^2 norm of $g_j, \mathbf{u}_j', \mathbf{v}_{j+1}'$ $(1\le j\le p)$ and those one of their derivatives as follows. We observe that

$$
\begin{aligned}
\|g_j(t)\|_3^2 &\le N a_0\big(\|\psi_j(t)\|_6^2 + \|\varphi_j(t)\|_3^2\big)^5 + N\big(\|\nabla V_j(t)\|_3^2 + \|\mathbf{v}_j(t)\|_4^4\big)\\
&\le N a_0^2(I_0+1+M_0+M_1)^6, \qquad t\in[0,T_*],\ 1\le j\le p. \qquad (3.50)
\end{aligned}
$$

Then from (3.17) and (3.36) one has

$$
\begin{aligned}
\|\mathbf{u}_j'(t)\|_3^2 &\le N\big(\|\mathbf{u}_j\|_3^2 + \|g_{j-1}(t)\|_3^2\big)\\
&\le N a_0^2(I_0+1+M_0+M_1)^6, \quad t\in[0,T_*],\ 1\le j\le p. \qquad (3.51)
\end{aligned}
$$

And we can estimate $\mathbf{v}_{j+1}'$ in view of (3.48) as follows

$$
\begin{aligned}
\|\mathbf{v}_{j+1}'(t)\|_3^2 &\le N a_0^2(I_0+1+M_0+M_1)^5 + N a_0(I_0+M_0)^2\|\mathbf{u}_j'(t)\|_2^2\\
&\le N a_0^3(I_0+1+M_0+M_1)^8, \qquad t\in[0,T_*],\ 1\le j\le p. \qquad (3.52)
\end{aligned}
$$

By differentiating (3.22) with respect to t, and using (3.36), (3.39), (3.42)–(3.43), (3.47) and (3.52), we obtain

$$
\begin{aligned}
\|g_j'(t)\|_1^2 &\le N a_0\big(\|(\psi_j',\psi_j)(t)\|_4^2 + \|\varphi_j(t)\|_3^2\big)^3\\
&\quad + N a_0\|\varphi_j'(t)\|_3^2\big(\|\psi_j(t)\|_4^2 + \|\varphi_j(t)\|_3^2\big)^4\\
&\quad + N\big(\|\nabla V_j'(t)\|_1^2 + \|\mathbf{v}_j(t)\|_3^2\cdot\|\mathbf{v}_j'(t)\|_3^2\big)\\
&\le N a_0^4(I_0+1+M_0+M_1)^{11}, \qquad t\in[0,T_*],\ 1\le j\le p. \qquad (3.53)
\end{aligned}
$$

Hence, we obtain, after differentiating (3.17) with respect to time, that

$$
\begin{aligned}
\|\mathbf{u}_j''(t)\|^2 &\le N(\|\mathbf{u}_j'\|_1^2 + \|g_{j-1}'(t)\|^2)\\
&\le N a_0^4(I_0+1+M_0+M_1)^{11}, \quad t\in[0,T_*],\ 2\le j\le p, \qquad (3.54)
\end{aligned}
$$

and from (3.49) that

$$\begin{aligned}\|\mathbf{v}''_{j+1}(t)\|_1^2 \le& N a_0\big(\|\psi'''_j(t)\|_3^2 + M_0\|\mathbf{u}''_j(t)\|^2\big) + N a_0^3(I_0+1+M_0+M_1)^8\\ &+ N a_0^2(I_0+1+M_0+M_1)^3\|\mathbf{u}'_j(t)\|_3^2\\ &+ N a_0^2(I_0+1+M_0)^5\|\mathbf{u}'_{j-1}(t)\|_2^2\\ \le& N a_0^5(I_0+1+M_0+M_1)^{12}\\ &+ N a_0\|\psi'''_j(t)\|_3^2, \qquad t\in[0,T_*],\ 1\le j\le p. \end{aligned}\tag{3.55}$$

By the previous estimates, it is easy to obtain the estimates for $\mathbf{u}_{p+1}$. In fact, by taking the inner product between $D^\alpha(3.17)_1$ $(0\le|\alpha|\le 3)$ and $2D^\alpha\mathbf{u}_{p+1}$ over $\mathbb{T}^n$, we obtain

$$\frac{d}{dt}\|D^\alpha\mathbf{u}_{p+1}\|^2 + \|D^\alpha\mathbf{u}_{p+1}\|^2 \le \|D^\alpha g_p(t)\|^2. \tag{3.56}$$

Hence by summing (3.56) with respect to $|\alpha| = 0,1,2,3$, and integrating it over $[0,t]$, and by the Gronwall lemma, we have

$$\begin{aligned}\|\mathbf{u}_{p+1}(t)\|_3^2 \le& \|\mathbf{u}_1\|_3^2 + \int_0^t \|g_p(s)\|_3^2 e^{-(t-s)}ds\\ \le& M_2 I_0 + T_* N a_0^2(I_0+1+M_0+M_1)^6 \le \frac{2}{5}M_0, \quad t\in[0,T_*],\end{aligned}\tag{3.57}$$

with T_* defined by (3.34). With the help of (3.50), (3.53) and (3.57), the corresponding H^3 and H^1 norms of $\mathbf{u}'_{p+1}$ and $\mathbf{u}''_{p+1}$ are bounded, similarly to (3.51) and (3.54), by

$$\|\mathbf{u}'_{p+1}(t)\|_3^2 \le N\big(\|\mathbf{u}_p(t)\|_3^2 + \|g_p(t)\|_3^2\big) \le N a_0^2(I_0+1+M_0+M_1)^6, \tag{3.58}$$

$$\|\mathbf{u}''_{p+1}(t)\|^2 \le N(\|\mathbf{u}'_p(t)\|^2 + \|g'_p(t)\|^2) \le N a_0^4(I_0+1+M_0+M_1)^{11}, \tag{3.59}$$

for $t\in[0,T_*]$.

In addition, with the help of previous estimates on $\mathbf{v},\mathbf{u}$ (i.e., (3.51), (3.52), (3.54), and (3.55)), we obtain from (3.44)–(3.45) that

$$\begin{aligned}\big\|\varphi''_{j+1}(t)\big\|_2^2 \le& N a_0^2(I_0+1+M_0+M_1)^3 + N(I_0+M_0)\|(\mathbf{u}'_j,\mathbf{v}'_j)(t)\|_3^2\\ \le& N a_0^3(I_0+1+M_0+M_1)^9,\end{aligned}\tag{3.60}$$

and

$$\begin{aligned}\big\|\varphi'''_{j+1}(t)\big\|^2 \le& N\big(\|\mathbf{v}'_j(t)\|_3^4 + (I_0+M_0)^2\|(\mathbf{u}'_j,\mathbf{v}'_j)(t)\|_3^2\big)\\ &+ N(I_0+M_0)\big(\|\mathbf{u}''_j(t)\|^2 + \|\mathbf{v}''_j(t)\|_1^2\big)\\ &+ N a_0^2(I_0+1+M_0+M_1)^4\\ \le& N a_0^6(I_0+1+M_0+M_1)^{16} + N a_0(I_0+M_0)\|\psi'''_j(t)\|_3^2.\end{aligned}\tag{3.61}$$

So far, we have proved that $\mathbf{v}_{p+1}$ and $\mathbf{u}_{p+1}$ satisfy (3.37) (i.e., (3.47) and (3.57)) as long as (3.36) holds, and the time derivatives of them (i.e., (3.52), (3.58), and (3.59)) are also bounded uniformly in Sobolev space, with the exception (3.55) for $\mathbf{v}''_{j+1}$ relative to ψ'''_{j+1} $(1 \le j \le p)$. Furthermore, from (3.42)–(3.43) and (3.60)–(3.61) we conclude that φ_{p+1} and its time derivatives are uniformly bounded in Sobolev space, with the exception of φ'''_{j+1}, i.e., (3.61), relative to ψ'''_{j+1} $(1 \le j \le p)$.

Step 2.4: estimates on ψ_{j+1}, $\mathbf{v}''_{j+1}$, φ'''_{j+1}. We estimate ψ_{j+1} and then $\mathbf{v}''_{j+1}$ and φ'''_{j+1}, $1 \le j \le p$, for $(x,t) \in \mathbb{T}^n \times [0,T_*]$. By (3.36), (3.46), (3.51), and (3.52), it is easy to verify the upper bounds of k_j, k'_j in $H^3(\mathbb{T}^n)$, for all $t \in [0,T_*]$, namely

$$\|k_j(t)\|_3^2 \le N(\|\mathbf{u}_j(t)\|_3^2 + \|\mathbf{v}_j(t)\|_3^2) \le N a_0 (I_0 + 1 + M_0)^3, \quad 1 \le j \le p, \tag{3.62}$$

and

$$\|k'_j(t)\|_3^2 \le N(\|\mathbf{u}'_j(t)\|_3^2 + \|\mathbf{v}'_j(t)\|_3^2) \le N a_0^3 (I_0 + 1 + M_0 + M_1)^8, \ 1 \le j \le p. \tag{3.63}$$

With the help of (3.36), (3.38)–(3.39), (3.42), (3.43), (3.46), (3.51) and (3.52), we obtain, from (3.21), the following bounds on $h_p(t), h'_p(t)$

$$\begin{aligned}
\|h_j(t)\|_2^2 \le & N a_0 \big(\|\varphi_j(t)\|_3^2 + \|\psi_j(t)\|_4^2 + \|\psi'_j(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\big)^4 \\
& + N a_0 \|\mathbf{v}_j(t)\|_4^2 \big(\|\psi_j(t)\|_4^2 + \|(\psi'_j, \varphi_j)(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\big)^3 \\
& + N\|\psi_j(t)\|_4^2 \big(\|V_j(t)\|_4^2 + \|\mathbf{v}_j(t)\|_4^4\big) \\
\le & N a_0^2 (I_0 + 1 + M_0)^7, \qquad 1 \le j \le p, \ t \in [0,T_*],
\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
\|h'_j(t)\|_2^2 \le & N a_0 \big(\|(\varphi'_j, \varphi_j)(t)\|_3^2 + \|(\psi'_j, \psi_j)(t)\|_4^2 + \|\psi''_j(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\big)^5 \\
& + N a_0 \|\mathbf{u}'_j(t)\|_3^2 \big(1 + \|\mathbf{v}_j(t)\|_4^2\big)\big(\|\varphi_j(t)\|_2^2 + \|(\psi'_j, \psi_j)(t)\|_4^2\big)^4 \\
& + N a_0 \|\mathbf{v}_j(t)\|_4^2 \big(\|(\psi'_j, \psi_j)(t)\|_4^2 + \|(\psi''_j, \varphi'_j, \varphi_j)(t)\|_2^2 + \|\mathbf{u}_j(t)\|_3^2\big)^4 \\
& + N a_0 \|\mathbf{v}'_j(t)\|_4^2 \big(\|\psi_j(t)\|_4^2 + \|(\psi'_j, \varphi_j)(t)\|_2^2 + \|\mathbf{u}_j(t)\|_4^2\big)^5 \\
& + N\|\psi_j(t)\|_4^2 \big(\|V'_j(t)\|_4^2 + \|\mathbf{v}'_j(t)\|_3^2 \cdot \|\mathbf{v}_j(t)\|_4^2\big) \\
& + N\|\psi'_j(t)\|_4^2 \big(\|V_j(t)\|_4^2 + \|\mathbf{v}_j(t)\|_4^4\big) \\
\le & N a_0^5 (I_0 + 1 + M_0 + M_1)^{14}, \qquad 2 \le j \le p, \ t \in [0,T_*].
\end{aligned} \tag{3.65}$$

To obtain the bounds on the L^2 norm of ψ_{p+1} and its derivatives, we first take the inner product between the Eq. $(3.16)_1$ and $2\psi'_{p+1}$ and then we integrate by parts. By using Lemma 2.5, we have

$$\begin{aligned}
& \frac{d}{dt}\big(\|\psi'_{p+1}(t)\|^2 + \nu\|\psi_{p+1}(t)\|^2 + \nu\|\Delta\psi_{p+1}(t)\|^2\big) \\
\le & |\nabla \cdot k_p(t)|_{L^\infty}\|\psi'_{p+1}\|^2 + \|h_p(t)\|^2
\end{aligned}$$

$$\leq N(1 + \|k_p(t)\|_3^2)\|\psi'_{p+1}(t)\|^2 + \|h_p(t)\|^2. \tag{3.66}$$

Take the inner product between Eq. $D^\alpha(3.16)_1$ and $2D^\alpha\psi'_{p+1}$ with $1 \leq |\alpha| \leq 2$ and integrate it by parts over $\mathbb{T}^n$. It follows

$$\begin{aligned}
&\frac{d}{dt}(\|D^\alpha\psi'_{p+1}(t)\|^2 + \nu\|D^\alpha\psi_{p+1}(t)\|^2 + \nu\|\Delta D^\alpha\psi_{p+1}(t)\|^2)\\
\leq&|\nabla\cdot k_p(t)|_{L^\infty}\|D^\alpha\psi'_{p+1}(t)\|^2 + \|D^\alpha h_p(t)\|^2 + N\int_{\mathbb{T}^n}|H_\alpha(\psi'_{p+1},k_p)|^2dx\\
\leq&N(1+\|k_p(t)\|_3^2)\|D\psi'_{p+1}(t)\|^2 + \|D^\alpha h_p(t)\|^2 + N\int_{\mathbb{T}^n}|H_\alpha(\psi'_{p+1},k_p)|^2dx,
\end{aligned} \tag{3.67}$$

where

$$H_\alpha(\psi,k) = D^\alpha(k\cdot\nabla\psi) - k\cdot\nabla(D^\alpha\psi).$$

By the Lemma 2.5, (3.62), we get

$$\int_{\mathbb{T}^n}|H_\alpha(\psi,k)|^2dx \leq \begin{cases} N(1+\|k(t)\|_3^2)\|D\psi\|^2, & |\alpha| = 1,\\ N(1+\|k(t)\|_3^2)(\|D\psi\|^2 + \|D^\alpha\psi\|^2), & |\alpha| = 2. \end{cases} \tag{3.68}$$

By substituting (3.68) into (3.67) and taking summation of these differential inequalities with respect to $|\alpha| = 0, 1, 2$, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\psi'_{p+1}(t)\|_2^2 + \nu\|\psi_{p+1}(t)\|_2^2 + \nu\|\Delta\psi_{p+1}(t)\|_2^2)\\
\leq&N(1+\|k_p(t)\|_3^2)\big(\|\psi'_{p+1}(t)\|_2^2 + \nu\|\psi_{p+1}(t)\|_2^2 + \nu\|\Delta\psi_{p+1}(t)\|_2^2\big)\\
&+\|h_p(t)\|_2^2.
\end{aligned} \tag{3.69}$$

By applying the Gronwall inequality and by using (3.62), (3.64), we obtain

$$\begin{aligned}
&\|\psi'_{p+1}(t)\|_2^2 + \|\psi_{p+1}(t)\|_2^2 + \|\Delta\psi_{p+1}(t)\|_2^2\\
\leq&\max\{1,\ \nu^{-1}\}\cdot(\|\psi_0\|_2^2 + \|\psi_1\|_4^2 + T_*NM_5)e^{T_*Na_0(1+M_0+M_1)^3}\\
\leq&2(2M_2I_0 + T_*M_5)\cdot\max\{1,\ \nu^{-1}\}\\
\leq&8M_2I_0 = \frac{1}{5}M_0, \quad t\in[0,T_*],\ p\geq 1,
\end{aligned} \tag{3.70}$$

where we recall that M_0, T_* and M_5 are defined by (3.32), (3.34), and (3.35) respectively.

Let us take the inner product between the Eq. $D^\alpha\partial_t(3.16)_1$ and $2D^\alpha\psi''_{p+1}$, with $0 \leq |\alpha| \leq 2$, and integrate by parts over $\mathbb{T}^n$, then by summing the resulting differential inequality with respect to α, by (3.68) and by following estimates

$$\int_{\mathbb{T}^n}|D^\alpha(k'_p\cdot\nabla\psi'_{p+1})|^2 \leq \begin{cases} N\nu\|k'_p(t)\|_2^2\big(\|\psi'_{p+1}(t)\|^2 + \|\Delta\psi'_{p+1}(t)\|^2\big), & \alpha = 0,\\ N\nu\|k'_p(t)\|_2^2\big(\|D\psi'_{p+1}\|_1^2 + \|\Delta\psi'_{p+1}(t)\|^2\big), & |\alpha| = 1,\\ N\nu\|k'_p(t)\|_2^2\big(\|D\psi'_{p+1}\|^2 + \|\Delta\psi'_{p+1}\|_1^2\big), & |\alpha| = 2, \end{cases}$$

we obtain, in analogy to (3.66), (3.69), that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\psi''_{p+1}(t)\|_2^2+\nu\|\psi'_{p+1}(t)\|_2^2+\nu\|\Delta\psi'_{p+1}(t)\|_2^2\big)\\
\le& N(1+\|k_p(t)\|_3^2)\big(\|D\psi''_{p+1}(t)\|_1^2+\nu\|D\psi'_{p+1}(t)\|_1^2+\nu\|\Delta D\psi'_{p+1}(t)\|_1^2\big)\\
&+\|h'_p(t)\|_2^2+N\sum_{0\le|\alpha|\le 2}\int_{\mathbb{T}^n}\big(|D^\alpha\big(k'_p\cdot\nabla\psi'_{p+1}\big)|^2+|H_\alpha(\psi''_{p+1},k_p)|^2\big)\,dx\\
\le& NB_1\big(\|\psi''_{p+1}\|_2^2+\nu\|\psi'_{p+1}(t)\|_2^2+\nu\|\Delta\psi'_{p+1}(t)\|_2^2\big)+\|h'_p(t)\|_2^2,
\end{aligned}\tag{3.71}$$

where

$$B_1=a_0^3(I_0+1+M_0+M_1)^8.$$

By applying the Gronwall inequality to (3.71), it follows

$$\begin{aligned}
&\|\psi''_{p+1}(t)\|_2^2+\|\psi'_{p+1}(t)\|_2^2+\|\Delta\psi'_{p+1}(t)\|_2^2\\
\le& \max\{1,\nu^{-1}\}\cdot(\|\Psi_0\|_2^2+\|\psi_0\|_4^2+T_*NM_6)e^{T_*Na_0^3(I_0+1+M_0+M_1)^8}\\
\le& 2(2M_2I_0+T_*NM_6)\cdot\max\{1,\nu^{-1}\}\\
\le& 8M_2I_0=\frac{1}{5}M_0,\qquad t\in[0,T_*],\ p\ge 1,
\end{aligned}\tag{3.72}$$

where we recall that M_0, T_* and M_6 are defined by (3.32), (3.34), and (3.35) respectively.

To estimate the L^2 bounds of $D^5\psi_{p+1}$ and $D^6\psi_{p+1}$, it is sufficient to estimate these one of $\Delta^2D\psi_{p+1}$ and $\Delta^2D^2\psi_{p+1}$. By differentiating the Eq. $(3.16)_1$ twice with respect to x and by taking the inner product with $\Delta^2D\psi_{p+1}$ and $\Delta^2D^2\psi_{p+1}$ over $\mathbb{T}^n$, and using the estimates (3.62), (3.64), (3.70), and (3.72), one has

$$\begin{aligned}
\|\Delta^2D\psi_{p+1}(t)\|^2\le&\frac{N}{\nu^2}\big(\|\psi''_{p+1}(t)\|_1^2+\|\psi'_{p+1}(t)\|_1^2+\|\psi_{p+1}(t)\|_1^2\big)\\
&+\frac{N}{\nu^2}\|D(k_p\cdot\nabla\psi'_{p+1})(t)\|^2+\frac{N}{\nu^2}\|h_p(t)\|_1^2\\
\le&\frac{N}{\nu^2}a_0^2(I_0+1+M_0)^7\le\frac{1}{3}M_1,\qquad t\in[0,T_*],\ p\ge 1,
\end{aligned}\tag{3.73}$$

$$\begin{aligned}
\|\Delta^2D^2\psi_{p+1}(t)\|^2\le&\frac{N}{\nu^2}\big(\|\psi''_{p+1}(t)\|_2^2+\|\psi'_{p+1}(t)\|_2^2+\|\psi_{p+1}(t)\|_2^2\big)\\
&+\frac{N}{\nu^2}\|D^2(k_p\cdot\nabla\psi'_{p+1})(t)\|^2+\frac{N}{\nu^2}\|h_p(t)\|_2^2\\
\le&\frac{N}{\nu^2}a_0^2(I_0+1+M_0)^7\le\frac{1}{3}M_1,\qquad t\in[0,T_*],\ p\ge 1,
\end{aligned}\tag{3.74}$$

where we recall M_1 and T_* are defined by (3.33) and (3.34) respectively.

We have now to show the L^2 norm of ψ'''_{j+1} and $\mathbf{v}''_{j+1}$ for $1\le j\le p$. By taking the inner product between $\partial_t(3.16)_1$ and ψ'''_{p+1} and using above estimates, we obtain

$$\|\psi'''_{p+1}(t)\|^2\le N\big(\|\psi''_{p+1}(t)\|^2[1+\|k_p(t)\|_2^2]+\|h'_p(t)\|^2\big)$$

$$
\begin{aligned}
&+ N\|\psi'_{p+1}(t)\|_4^2\big(1+\|k'_p(t)\|_2^2\big)\\
\le& N a_0^5 (I_0+1+M_0+M_1)^{14}, \quad t\in[0,T_*],\ 1\le j\le p,
\end{aligned} \tag{3.75}
$$

which gives from (3.55) that

$$
\begin{aligned}
\|\mathbf{v}''_{j+1}(t)\|_1^2 \le& N a_0^5 (I_0+1+M_0+M_1)^{12} + N a_0\|\psi'''_j(t)\|_3^2\\
\le& N a_0^6 (I_0+1+M_0+M_1)^{14}, \quad t\in[0,T_*],\ 1\le j\le p,
\end{aligned} \tag{3.76}
$$

and from (3.61) that

$$
\begin{aligned}
\left\|\varphi'''_{j+1}(t)\right\|^2 \le& N a_0^6 (I_0+1+M_0+M_1)^{16} + N a_0 (I_0+M_0)\|\psi'''_j(t)\|_3^2\\
\le& N a_0^6 (I_0+1+M_0+M_1)^{16}, \quad t\in[0,T_*],\ 1\le j\le p.
\end{aligned} \tag{3.77}
$$

Step 3: end of proof. By the previous estimates (3.38)–(3.39) on V_{p+1}, (3.47), (3.52), and (3.76) on $\mathbf{v}_{p+1}$, (3.42)–(3.43), (3.60), and (3.77) on φ_{p+1}, (3.57)–(3.59) on $\mathbf{u}_{p+1}$, and (3.70) and (3.72)–(3.75) on ψ_{p+1}, we conclude that the approximate solution $U^{p+1} = (\mathbf{v}_{p+1}, \varphi_{p+1}, \psi_{p+1}, \mathbf{u}_{p+1}, V_{p+1})$ is uniformly bounded in the time interval $[0,T_*]$ and it satisfies (3.37) for each $p\ge 1$ as long as U^p satisfies (3.36) with M_0, M_1, and T_* defined by (3.32), (3.33), and (3.34) respectively, which are independent of U^{p+1}, $p\ge 1$. By repeating the procedure used above, we can construct the approximate solution $\{U^i\}_{i=1}^{\infty}$, which solves (3.25)–(3.26) on $[0,T_*]$, with T_* defined by (3.34) and the constant $M_*>0$ chosen by

$$
M_* = \max\left\{M_0,\ M_1,\ N a_0^6 (I_0+1+M_0+M_1)^{16}\right\}. \tag{3.78}
$$

Let us recall here that M_0, M_1 and a_0 are defined by (3.32), (3.33) and (3.28) respectively and $N>0$ is a generic constant independent of U^{p+1}, $p\ge 1$. Therefore, the proof of Lemma 3.1 is completed. □

Proof of Theorem 1.1. By means of the Lemma 3.1, we obtain an approximate solution sequence $\{U^i\}_{i1}^{\infty}$ satisfying (3.25)–(3.26). Therefore, the proof of Theorem 1.1 is completed if we show that the whole sequence converges. Indeed, based on Lemma 3.1, we can obtain the estimates of the difference $Y^{p+1} =: U^{p+1} - U^p$, $p\ge 1$, of the approximate solution sequence $\{U^p\}_{p=1}^{\infty}$. Let us denote $Y^{p+1} = (\bar{\mathbf{v}}_{p+1}, \bar\varphi_{p+1}, \bar\psi_{p+1}, \bar{\mathbf{u}}_{p+1}, \bar V_{p+1})$ by

$$
\begin{aligned}
\bar{\mathbf{v}}_{p+1} = \mathbf{v}_{p+1} - \mathbf{v}_p, \quad \bar\varphi_{p+1} = \varphi_{p+1} - \varphi_p,\\
\bar\psi_{p+1} = \psi_{p+1} - \psi_p, \quad \bar{\mathbf{u}}_{p+1} = \mathbf{u}_{p+1} - \mathbf{u}_p, \quad \bar V_{p+1} = V_{p+1} - V_p.
\end{aligned}
$$

We can obtain for $p\ge 4$

$$
\|\bar{\mathbf{v}}_{p+1}(t)\|_4^2 + \|(\bar V_{p+1}, \bar V'_{p+1})(t)\|_3^2 \le N_*\big(\|(\bar\psi_p, \bar\psi'_p)(t)\|_4^2 + \|(\bar\varphi_p, \bar{\mathbf{u}}_p)(t)\|_3^2\big),
$$

$$\|(\bar{\mathbf{v}}'_{p+1}, \bar{\mathbf{u}}'_{p+1}, \bar{\varphi}'_{p+1})(t)\|_3^2 \leq N_* \sum_{j=0}^{2} \|(\bar{\psi}_{p-j}, \bar{\psi}'_{p-j})(t)\|_4^2 + N_* \sum_{j=0}^{2} \big(\|\bar{\psi}''_{p-j}(t)\|_2^2 + \|(\bar{\varphi}_{p-j}, \bar{\mathbf{u}}_{p-j})(t)\|_3^2\big),$$

$$\sum_{5\leq|\alpha|\leq 6} \|D^\alpha \bar{\psi}_{p+1}(t)\|^2 \leq N_* \|\bar{\psi}''_{p-j}(t)\|_2^2 + N_* \sum_{j=0}^{2} \|(\bar{\psi}_{p+1-j}, \bar{\psi}'_{p+1-j})(t)\|_4^2 + N_* \sum_{j=0}^{1} \big(\|\bar{\psi}''_{p-j}(t)\|_2^2 + \|(\bar{\varphi}_{p-j}, \bar{\mathbf{u}}_{p-j})(t)\|_3^2\big).$$

Here N_* denotes a constant dependent of M_*. By using the previous estimates, the Lemma 3.1, and an arguments similar to the one used to get (3.42), (3.57), (3.70), and (3.72), we show, after a tedious computation, that there exists $0 < T_{**} \leq T_*$, such that the difference $Y^{p+1} = U^{p+1} - U^p$, $p \geq 1$, of the approximate solution sequence satisfies the following estimates

$$\sum_{p=1}^{\infty} \Big(\|(\bar{\mathbf{u}}_{p+1}, \bar{\varphi}_{p+1})\|^2_{C^1([0,T_{**}];H^3)} + \|\bar{V}_{p+1}\|^2_{C^1([0,T_{**}];\dot{H}^4)}\Big) \leq C_*, \tag{3.79}$$

$$\sum_{p=1}^{\infty} \Big(\|\bar{\psi}_{p+1}\|^2_{C^i([0,T_{**}];H^{6-2i})} + \|\bar{\mathbf{v}}_{p+1}\|^2_{C([0,T_{**}];H^4)} + \|\bar{\mathbf{v}}'_{p+1}\|^2_{C([0,T_{**}];H^3)}\Big) \leq C_*, \tag{3.80}$$

where $i = 0, 1, 2$, and $C_* = C_*(N, M_*)$ denotes a positive constant depending on N and M_*. Then by applying the Ascoli-Arzela Theorem (to the time variable) and the Rellich-Kondrachev theorem (to the spatial variables) [33], we prove, in a standard way (see for instance [28]), that there exists a (unique) $U = (\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$, such that as $p \to \infty$ it holds

$$\begin{cases} \mathbf{v}_p \to \mathbf{v} & \text{strongly in } C^i([0,T_{**}]; H^{4-i-\sigma}(\mathbb{T}^n)), \\ \varphi_p \to \varphi & \text{strongly in } C^1([0,T_{**}]; H^{3-\sigma}(\mathbb{T}^n)) \bigcap C^2([0,T_{**}]; H^{2-\sigma}(\mathbb{T}^n)), \\ \psi_p \to \psi & \text{strongly in } C^i([0,T_{**}]; H^{6-2i-\sigma}(\mathbb{T}^n)) \bigcap C^2([0,T_{**}]; H^{2-\sigma}(\mathbb{T}^n)), \\ \mathbf{u}_p \to \mathbf{u} & \text{strongly in } C^i([0,T_{**}]; H^{3-\sigma}(\mathbb{T}^n)), \\ V_p \to V & \text{strongly in } C^i([0,T_{**}]; \dot{H}^{4-\sigma}(\mathbb{T}^n)), \end{cases} \tag{3.81}$$

with $i = 0, 1$, and $\sigma > 0$. Moreover, by (3.41) one has

$$\varphi(x,t) \geq \frac{1}{4}\psi_* > 0, \quad (x,t) \in \mathbb{T}^n \times [0, T_{**}]. \tag{3.82}$$

If we take $\sigma \ll 1$ in (3.81) and we pass into the limit as $p \to \infty$ in (3.14)–(3.18), we obtain the (short time) existence and uniqueness of classical solution of the system (3.5), (3.6), (3.9), (3.11), and (3.13) constructed in the section 3.1.

Next, we claim the local in-time classical solution $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)$, with initial data $(\mathbf{v}, \varphi, \psi, \mathbf{u}, V)(x, 0) = (\mathbf{u}_1, \psi_1, \psi_1, \mathbf{u}_1, V_1)(x)$ also satisfies

$$\psi = \varphi, \quad \mathbf{u} = \mathbf{v}, \tag{3.83}$$

and then solves the IVP (1.5)–(1.8). Indeed by passing into the limit in $(3.18)_1$, we have

$$\varphi_t + \mathbf{u}\cdot\nabla\psi + \frac{1}{2}\varphi\nabla\cdot\mathbf{v} = 0, \tag{3.84}$$

which yields

$$\frac{2(\varphi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi} = -\nabla\cdot\mathbf{v}, \tag{3.85}$$

$$\int_{\mathbb{T}^n} \frac{2(\varphi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi}(x,t)dx = -\int_{\mathbb{T}^n} \nabla\cdot\mathbf{v}(x,t)dx = 0. \tag{3.86}$$

Let us note here that $\varphi > 0$. Then by taking the limiting equation of $\mathbf{v}$ (passing into the limit in (3.14) and (3.19))

$$\nabla\cdot\mathbf{v} = -\frac{2(\psi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi} + \frac{1}{L^n}\int_{\mathbb{T}^n} \frac{2(\psi_t + \mathbf{u}\cdot\nabla\psi)}{\varphi}(x,t)dx \tag{3.87}$$

and using (3.85), (3.86), one has

$$\frac{(\varphi - \psi)_t(x,t)}{\varphi} - \frac{1}{L^n}\int_{\mathbb{T}^n} \frac{(\varphi-\psi)_t}{\varphi}(x,t)dx = 0, \quad \forall\, x \in \mathbb{T}^n,\ t \geq 0. \tag{3.88}$$

Since by a straightforward computation we obtain $(\varphi - \psi)_t(x, 0) = 0$ from (3.85) and (3.87) with $t = 0$, then, from (3.88), we conclude that

$$(\varphi - \psi)_t(x,t) = \varphi(x,t)f(t), \quad t \geq 0,$$

for any $f \in C^2([0, T_{**}])$, with $f(0) = 0$. In particular we can choose

$$f(t) = 0, \quad t \geq 0,$$

hence by (3.82) and the fact

$$\varphi(x,0) = \psi(x,0) = \psi_1(x) \quad \Rightarrow \quad (\varphi - \psi)(x,0) = 0,$$

we obtain

$$\psi(x,t) = \varphi(x,t) \geq \frac{1}{4}\psi_* > 0, \qquad t \in [0, T_{**}],\ x \in \mathbb{T}^n, \tag{3.89}$$

$$\psi_t + \mathbf{u}\cdot\nabla\psi + \frac{1}{2}\psi\nabla\cdot\mathbf{v} = 0, \qquad t \in [0, T_{**}],\ x \in \mathbb{T}^n. \tag{3.90}$$

By passing into the limit $p \to \infty$ in (3.17) we recover the equation for $\mathbf{u}$, i.e., (3.11). By using (3.89) and (3.12), from (3.11), one has

$$\partial_t \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{v}|^2) + \nabla h(\psi^2) + \mathbf{u} = \nabla V + \frac{\varepsilon^2}{2}\nabla\Big(\frac{\Delta\psi}{\psi}\Big). \tag{3.91}$$

This equation, together with the fact $\nabla\times\mathbf{u}_1(x) = 0$, implies

$$\nabla\times\mathbf{u} = 0, \qquad \forall\, x \in \mathbb{T}^n,\ t \geq 0. \tag{3.92}$$

Similarly, by passing into limit in (3.16) we recover the equation (3.9) for ψ, hence recombining the various terms, with the help of (3.89) and (3.90), we get

$$\psi_{tt} + \psi_t + \mathbf{u}\cdot\nabla\psi_t + \frac{1}{2}\psi_t(\nabla\cdot\mathbf{v}) - \frac{1}{4\psi}\nabla\cdot(\psi^2\nabla(|\mathbf{v}|^2)) - \frac{1}{2\psi}\Delta P(\psi^2)$$
$$+\frac{1}{2\psi}\nabla\cdot(\psi^2\nabla V) + \frac{1}{4\psi}\varepsilon^2\nabla\cdot\Big(\psi^2\nabla\Big(\frac{\Delta\psi}{\psi}\Big)\Big) = 0. \tag{3.93}$$

¿From (3.90) we have $\psi_t = -\mathbf{u}\cdot\nabla\psi - \frac{1}{2}\psi\nabla\cdot\mathbf{v}$, then by substituting it into (3.93) and by representing $\mathbf{u}_t$ by (3.91), it follows

$$\nabla\cdot(\mathbf{u}-\mathbf{v})_t + \nabla\cdot(\mathbf{u}-\mathbf{v}) = 0.$$

By integrating previous above equation with respect to time on $[0, T_{**}]$, since $\nabla\cdot(\mathbf{u}-\mathbf{v})(x,0) = 0$ and

$$\int_{\mathbb{T}^n}\mathbf{u}(x,t)dx = \int_{\mathbb{T}^n}\mathbf{v}(x,t)dx = \bar{\mathbf{u}}(t),$$

we get the conclusion, by applying the Theorem 2.1 where we choose $f = 0$, namely we have $\hat{u} = 0$, that

$$\mathbf{u}(x,t) = \mathbf{v}(x,t), \quad t \in [0, T_{**}],\ x \in \mathbb{T}^n, \tag{3.94}$$

for irrotational flow. Thus, by (3.91) and (3.94), we recover the equation for $\mathbf{u}$ which is exactly Eq. (3.2) (and then Eq. (1.6) for ir-rotational flow). Multiplying (3.90) by ψ and by using (3.94) we recover the equation for ψ (which is exactly the Eq. (1.5))

$$\partial_t(\psi^2) + \nabla\cdot(\psi^2\mathbf{u}) = 0. \tag{3.95}$$

¿From (3.95) it follows the conservation (neutrality) of the density

$$\int_{\mathbb{T}^n}(\psi^2 - \mathcal{C})(x,t)dx = \int_{\mathbb{T}^n}(\psi_1^2 - \mathcal{C})(x)dx = 0, \quad t > 0. \tag{3.96}$$

Therefore passing into limit as $p \to \infty$, by (3.18) and by the Theorem 2.2 one has that $V \in C^1([0, T_{**}]; \dot{H}^4)$ is the unique solution of the periodic boundary problem of Poisson equation:

$$\Delta V = \psi^2 - \mathcal{C}, \quad \int_{\mathbb{T}^n} V dx = 0.$$

Therefore $(\psi, \mathbf{u}, V)$ with $\psi \geq \frac{1}{2}\psi_* > 0$ is the unique local (in time) solution of IVP (1.5)–(1.8). By a straightforward computation once more, we get

$$\psi \in C^i([0,T_{**}]; H^{6-2i}(\mathbb{T}^n)) \bigcap C^3([0,T_{**}]; L^2(\mathbb{T}^n)), \quad i = 0,1,2;$$
$$\mathbf{u} \in C^i([0,T_{**}]; H^{5-2i}(\mathbb{T}^n)),\ i = 0,1,2; \quad V \in C^1([0,T_{**}]; \dot{H}^4(\mathbb{T}^n)).$$

The proof of Theorem 1.1 is completed. □

4 Global existence and large time behavior

We prove here uniform a-priori estimates for the local classical solutions $(\psi, \mathbf{u}, V)$ of IVP (1.5)–(1.8) for any fixed $T > 0$, when $(\psi, \mathbf{u}, V)$ is close to the steady state $(\sqrt{\mathcal{C}}, 0, 0)$.

4.1 Reformulation of original problem

In this subsection, we reformulate the original problem (1.5)–(1.8) into an equivalent one for classical solutions. For simplicity, we still set $\tau = 1$.

Set

$$w = \psi - \sqrt{\mathcal{C}}.$$

By using (1.5), (1.7) and (3.9), we have the following systems for $(w, \mathbf{u}, V)$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u} = f_1(x,t), \tag{4.1}$$
$$w_{tt} + w_t + \frac{1}{4}\varepsilon^2\Delta^2 w + \mathcal{C}w = f_2(x,t) + f_3(x,t), \tag{4.2}$$
$$\Delta V = (2\sqrt{\mathcal{C}} + w)w. \tag{4.3}$$

and the corresponding initial values are

$$w(x,0) = w_1(x),\ w_t(x,0) = w_2(x),\ \mathbf{u}(x,0) = \mathbf{u}_1(x), \tag{4.4}$$

with

$$w_1(x) =: \psi_1 - \sqrt{\mathcal{C}},\ w_2(x) =: \mathbf{u}_1\cdot\nabla(\sqrt{\mathcal{C}} + w_1) - \frac{1}{2}(\sqrt{\mathcal{C}} + w_1)\nabla\cdot\mathbf{u}_1. \tag{4.5}$$

Here

$$f_1(x,t) = \nabla V - \nabla(h((\sqrt{\mathcal{C}} + w)^2) - h(\mathcal{C})) + \frac{1}{2}\varepsilon^2\nabla\left(\frac{\Delta w}{w + \sqrt{\mathcal{C}}}\right), \tag{4.6}$$
$$f_2(x,t) = -2\mathbf{u}\cdot\nabla w_t + P'(\mathcal{C})\Delta w, \tag{4.7}$$

$$f_3(x,t) = -\frac{w_t^2}{w+\sqrt{\mathcal{C}}} - \frac{1}{2}w^2(3\sqrt{\mathcal{C}}+w) - \nabla w\cdot\nabla V + \frac{\varepsilon^2}{4}\frac{|\Delta w|^2}{(\sqrt{\mathcal{C}}+w)}$$
$$+ (P'((\sqrt{\mathcal{C}}+w)^2) - P'(\mathcal{C}))\Delta w + \frac{(P'((\sqrt{\mathcal{C}}+w)^2)(\sqrt{\mathcal{C}}+w))'}{\sqrt{\mathcal{C}}+w}|\nabla w|^2$$
$$+ \frac{1}{2(\sqrt{\mathcal{C}}+w)}\nabla^2\cdot\Big([\sqrt{\mathcal{C}}+w]^2\mathbf{u}\otimes\mathbf{u}\Big) + 2\mathbf{u}\cdot\nabla w_t. \tag{4.8}$$

The derivatives of w and $\mathbf{u}$ satisfies:

$$2w_t + 2\mathbf{u}\cdot\nabla(\sqrt{\mathcal{C}}+w) + (\sqrt{\mathcal{C}}+w)\nabla\cdot\mathbf{u} = 0. \tag{4.9}$$

4.2 The a-priori estimates

For all $T>0$, define a suitable function space for the unknown $(w,\mathbf{u},V)$ of the IVP (4.2)–(4.4) in the following way

$$X(T) = \{(w,\mathbf{u},V)\in H^6(\mathbb{T}^n)\times H^5(\mathbb{T}^n)\times \dot{H}^4(\mathbb{T}^n),\quad 0\le t\le T\}$$

with norm

$$M(0,T) = \max_{0\le t\le T}\{\|w(t)\|_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|_{H^5(\mathbb{T}^n)} + \|V(t)\|_{\dot{H}^4(\mathbb{T}^n)}\},$$

and assume that

$$\delta_T = \max_{0\le t\le T}(\|w(t)\|_{H^6(\mathbb{T}^n)} + \|\mathbf{u}(t)\|_{H^5(\mathbb{T}^n)}) \ll 1. \tag{4.10}$$

Under the assumption (4.10), it follows immediately

$$-\frac{1}{2}\sqrt{\mathcal{C}} \le w \le \frac{1}{2}\sqrt{\mathcal{C}}. \tag{4.11}$$

Lemma 4.1 *Let $(w,\mathbf{u},V)\in X(T)$, let the multi-index α satisfies $0\le|\alpha|\le 4$, then following inequality holds*

$$|\nabla V|^2 + \|V\|_5^2 \le C\|w\|_3^2,\quad |V_t|^2 + |\nabla V_t|^2 + \|V_t\|_4^2 \le C\|D^\alpha w_t\|_2^2, \tag{4.12}$$
$$\|D^\alpha\mathbf{u}\|^2 \le C\|D^\alpha\mathbf{u}_1\|^2e^{-t} + C\|D^\alpha(\nabla V, w_t, w, \nabla w, \Delta w)\|^2, \tag{4.13}$$
$$\|D^\alpha f_3\|^2 \le C\|D^\alpha\mathbf{u}_1\|^2e^{-t} + C\delta_T\|D^\alpha(\nabla V, w_t, w, \nabla w, \Delta w)\|^2, \tag{4.14}$$

provided that $\delta_T \ll 1$.

Proof: The estimates (4.12) follows from the Theorem 2.3, since the integral of right hand side term of (4.3) equals to zero due to the conservation of density and (1.9). By (4.12) and by (4.10), we have

$$|\nabla V| + |V_t| + |\nabla V_t| + ||(\nabla V,\nabla V_t)|| \le C\delta_T. \tag{4.15}$$

In order to estimate (4.13) we take the inner product between (4.1) and $\mathbf{u}$ on $\mathbb{T}^n$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 =& -\frac{1}{2}\int_{\mathbb{T}^n} \mathbf{u}\cdot\nabla(|\mathbf{u}|^2)dx + \int_{\mathbb{T}^n} f_1\cdot \mathbf{u}\, dx \\ &\le \left(\frac{1}{4} + C\delta_T\right)\|\mathbf{u}\|^2 + C\|\nabla\cdot\mathbf{u}\|^2 + C\|(w, \nabla V, \Delta w)\|^2. \end{aligned} \tag{4.16}$$

By replacing $\nabla\cdot\mathbf{u}$ in (4.16) by (4.9) and by (4.12), one has

$$\frac{d}{dt}\|\mathbf{u}\|^2 + 2(1 - C\delta_T)\|\mathbf{u}\|^2 \le C\|(\nabla w, w_t, \Delta w)\|^2 dx. \tag{4.17}$$

By applying the Gronwall Lemma, by taking δ_T small enough such that $1 - C\delta_T \le 1/2$, we get (4.13) for $\alpha = 0$.

In order to get higher order estimates, we set $\hat{u} = D^\alpha \mathbf{u}$. It satisfies the equation¶

$$\hat{u}_t + (\mathbf{u}\cdot\nabla)\hat{u} + \hat{u} = f_5 + \nabla f_6, \tag{4.18}$$

where

$$f_5(x,t) = \nabla(D^\alpha V) - D^\alpha \nabla h(\sqrt{\mathcal{C}} + w) - [D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)D^\alpha\mathbf{u}], \tag{4.19}$$

$$f_6(x,t) = \frac{1}{2}\varepsilon^2 D^\alpha\left(\frac{\Delta w}{\sqrt{\mathcal{C}} + w}\right). \tag{4.20}$$

Let us take the inner product between (4.18) and $\hat{u}$ and integrate by parts over $\mathbb{T}^n$. Then, it follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2 + \left(\frac{3}{4} - \frac{1}{2}\nabla\cdot\mathbf{u}\right)\|\hat{u}\|^2 \\ \le& - C\|f_5\|^2 + \frac{1}{2}\varepsilon^2\int_{\mathbb{T}^n} |D^\alpha\nabla\cdot\mathbf{u}|\left|D^\alpha((\sqrt{\mathcal{C}} + w)^{-1}\Delta w)\right| dx \\ \le& C\|D^\alpha(\nabla V, w, \nabla w, \Delta w)\|^2 + C\delta_T\|\hat{u}\|^2 + \frac{1}{4}\|\nabla\cdot(D^\alpha\mathbf{u})\|^2. \end{aligned} \tag{4.21}$$

By Lemma 2.5 and by (4.9), one has

$$\begin{aligned}\|\nabla\cdot(D^\alpha\mathbf{u})\|^2 \le& C\|D^\alpha((\sqrt{\mathcal{C}} + w)^{-1}w_t)\|^2 + C\|D^\alpha((\sqrt{\mathcal{C}} + w)^{-1}(\mathbf{u}\cdot\nabla)w)\|^2 \\ \le& C\|D^\alpha(w_t, w, \nabla w)\|^2 + C\delta_T\|D^\alpha\mathbf{u}\|^2. \end{aligned} \tag{4.22}$$

By substituting (4.22) into (4.21) and by using the Gronwall inequality, one obtains (4.13) for $1 \le |\alpha| \le 4$, provided that δ_T is small enough.

¶For the proof of the case $|\alpha| = 4$, we can assume that the solutions $(w, \mathbf{u}, V)$ have high order regularity to have enough smooth derivatives, since the a-priori estimates (4.24) and (4.31) below are still valid for these solutions when smoothed by the Friedrich's mollifier under assumptions similar to (4.10). We omit all the detail here.

Finally, we estimate (4.14), with the help of Lemma 2.5, (4.10)–(4.13), (4.9), as

$$\begin{aligned}\|D^\alpha f_3\|^2 \leq & C\delta_T\|D^\alpha(\nabla V, w, w_t, \nabla w, \Delta w, \mathbf{u}, D^\alpha\nabla^2 w)\|^2 + C\delta_T\sum_{l,j}\|D^\alpha\partial_l u_j\|^2 \\ \leq & C\delta_T\|D^\alpha(\nabla V, w, w_t, \nabla w, \Delta w, \mathbf{u}, \nabla\cdot\mathbf{u})\|^2 \\ \leq & C\|D^\alpha\mathbf{u}_1\|^2 e^{-t} + C\delta_T\|D^\alpha(\nabla V, w, w_t, \nabla w, \Delta w)\|^2. \end{aligned} \tag{4.23}$$

Thus, the proof of Lemma 4.1 is complete. □

We have the following basic estimates:

Lemma 4.2 *Let $(w, \mathbf{u}, V) \in X(T)$, then there exists $\beta_1 > 0$, such that*

$$\|(w, \nabla w, \Delta w, w_t)(t)\|^2 + \|\mathbf{u}(t)\|_1^2 + \|V(t)\|_2^2 \leq C(||w_1||_2^2 + ||\mathbf{u}_1||_1^2)e^{-\beta_1 t}, \tag{4.24}$$

provided that δ_T is small enough.

Proof: Take the inner product between (4.2) and $w + 2w_t$ and integrate by parts over $\mathbb{T}^n$. Therefore one has

$$\begin{aligned} & \frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{1}{2}w^2 + ww_t + w_t^2 + \mathcal{C}w^2 + \frac{1}{4}\varepsilon^2|\Delta w|^2\right)dx \\ & + \frac{1}{4}\varepsilon^2\|\Delta w\|^2 + \mathcal{C}\|w\|^2 + \|w_t\|^2 \\ = & \int_{\mathbb{T}^n}(f_2 + f_3)(w + 2w_t)dx \\ \leq & C\delta_T\|(w_t, w, \nabla w, \Delta w)\|^2 + C\|\mathbf{u}_1\|^2 e^{-t} \\ & + \frac{1}{4}\mathcal{C}\|w\|^2 + \frac{1}{4}\|w_t\|^2 + \int_{\mathbb{T}^n} f_2(w + 2w_t)dx. \end{aligned} \tag{4.25}$$

By integration by parts and (4.9), the last term on the right hand side of (4.25) can be estimated by

$$\begin{aligned}\int_{\mathbb{T}^n} f_2(w + 2w_t)dx = & \int_{\mathbb{T}^n}(2ww_t\nabla\cdot\mathbf{u} + 2w_t\mathbf{u}\cdot\nabla w + w_t^2\nabla\cdot\mathbf{u})dx \\ & - P'(\mathcal{C})\frac{d}{dt}\|\nabla w\|^2 - P'(\mathcal{C})\|\nabla w\|^2 \\ \leq & C\delta_T\|(w, w_t, \nabla w)\|^2 - P'(\mathcal{C})\frac{d}{dt}\|\nabla w\|^2 - P'(\mathcal{C})\|\nabla w\|^2. \end{aligned} \tag{4.26}$$

Since

$$\|\nabla w\|^2 \leq \frac{L^2}{4\pi^2}\|\Delta w\|^2, \tag{4.27}$$

it follows

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{1}{2}w^2+ww_t+w_t^2+\mathcal{C}w^2+\frac{1}{4}\varepsilon^2|\Delta w|^2+P'(\mathcal{C})|\nabla w|^2\right)dx\\&+\left(\frac{1}{4}A_0-C\delta_T\right)\|\Delta w\|^2+\left(\frac{3}{4}\mathcal{C}-C\delta_T\right)\|w\|^2+\left(\frac{3}{4}-C\delta_T\right)\|w_t\|^2\\\leq&C\|\mathbf{u}_1\|^2e^{-t},\end{aligned}\tag{4.28}$$

where A_0 is defined by the “subsonic” condition (1.10)

$$A_0=\frac{\pi^2}{L^2}\varepsilon^2+P'(\mathcal{C})>0.$$

Note that there are positive constants $\kappa_1,\ \beta_0$ such that

$$\begin{aligned}&||(w,w_t,\nabla w,\Delta w)||^2\\\leq&\kappa_1\int_{\mathbb{T}^n}\left(\frac{1}{2}w^2+ww_t+w_t^2+\mathcal{C}w^2+\frac{1}{4}\varepsilon^2|\Delta w|^2+P'(\mathcal{C})|\nabla w|^2\right)dx\\\leq&\kappa_1\beta_0^{-1}\|(w_t,w,\Delta w)\|^2.\end{aligned}\tag{4.29}$$

Hence, by applying the Gronwall lemma to (4.28) and using (4.29), we get

$$||(w,w_t,\nabla w,\Delta w)||^2\leq C(||w_1||_2^2+||\mathbf{u}_1||_1^2)e^{-\beta_1 t}\tag{4.30}$$

with $0<\beta_1<\min\{1,\ \kappa_2\beta_0\}$, provided that δ_T is sufficiently small to have

$$\min\left\{\frac{1}{4}A_0-C\delta_T,\ \frac{3}{4}\mathcal{C}-C\delta_T,\ \frac{3}{4}-C\delta_T\right\}=:\kappa_2>0.$$

The combination of (4.30) and (4.12)–(4.13) with $\alpha=0$ yields to (4.24). □

In order to obtain higher order estimates, we differentiate (4.1)–(4.2) with respect to x, therefore by repeating the previous steps and by using the Lemmas 4.1–4.2, we have

Lemma 4.3 *Let$(w,\mathbf{u},V)\in X(T)$, then there exists $\beta_4>0$, such that the following inequality holds*

$$\|(w,\Delta w,w_t)(t)\|_{|\alpha|}^2+\|\mathbf{u}(t)\|_{1+|\alpha|}^2+\|V(t)\|_4^2\leq C(||w_1||_{2+|\alpha|}^2+||\mathbf{u}_1||_{1+|\alpha|}^2)e^{-\beta_4 t}\tag{4.31}$$

for $1\leq|\alpha|\leq 4$, provided that $\delta_T\ll 1$.

Proof: Let $\tilde{w}=D^\alpha w$, with $1\leq|\alpha|\leq 4$. Then $\tilde{w}$ satisfies the equation

$$\tilde{w}_{tt}+\tilde{w}_t+\frac{1}{4}\varepsilon^2\Delta^2\tilde{w}+\mathcal{C}\tilde{w}=D^\alpha f_2(x,t)+D^\alpha f_3(x,t).\tag{4.32}$$

Let us take inner product between (4.32) by $\tilde{w}+2\tilde{w}_t$ and integrate it by parts over $\mathbb{T}^n$. By using (4.10), (4.11), and (4.14), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^n}\Big(\frac{1}{2}\tilde{w}^2+\tilde{w}\tilde{w}_t+\tilde{w}_t^2+\mathcal{C}\tilde{w}^2+\frac{1}{4}\varepsilon^2|\Delta\tilde{w}|^2\Big)dx\\
&+\frac{1}{4}\varepsilon^2\|\Delta\tilde{w}\|^2+\mathcal{C}\|\tilde{w}\|^2+\|\tilde{w}_t\|^2\\
\le& C\delta_T\|(\tilde{w}_t,\tilde{w},\nabla\tilde{w},\Delta\tilde{w},\nabla V)\|^2+\frac{1}{8}\mathcal{C}\|\tilde{w}\|^2+\frac{1}{8}\|\tilde{w}_t\|^2\\
&+C\|D^\alpha\mathbf{u}_1\|^2\exp\{-t\}+\int_{\mathbb{T}^n}D^\alpha f_2(\tilde{w}+2\tilde{w}_t)dx. \qquad (4.33)
\end{aligned}$$

By integrating by parts and by using (4.9), (4.13), the last term on the right hand side of (4.33) can be estimated as follows

$$\begin{aligned}
\int_{\mathbb{T}^n}D^\alpha f_2(\tilde{w}+2\tilde{w}_t)dx =& -2\int_{\mathbb{T}^n}[D^\alpha(\mathbf{u}\cdot\nabla w_t)-\mathbf{u}\cdot\nabla\tilde{w}_t](\tilde{w}+2\tilde{w}_t)dx\\
&+\int_{\mathbb{T}^n}(2\tilde{w}\tilde{w}_t\nabla\cdot\mathbf{u}+2\tilde{w}_t\mathbf{u}\cdot\nabla\tilde{w}+\tilde{w}_t^2\nabla\cdot\mathbf{u})dx\\
&-P'(\mathcal{C})\frac{d}{dt}\|\nabla\tilde{w}\|^2-P'(\mathcal{C})\|\nabla\tilde{w}\|^2\\
\le& C\delta_T\|(D^\alpha\mathbf{u},\nabla w_t,\tilde{w},\nabla\tilde{w})\|^2+\frac{1}{8}\mathcal{C}\|\tilde{w}\|^2+\frac{1}{8}\|\tilde{w}_t\|^2\\
&-P'(\mathcal{C})\frac{d}{dt}\|\nabla\tilde{w}\|^2-P'(\mathcal{C})\|\nabla\tilde{w}\|^2\\
\le& C\delta_T\|(w_t,w,\Delta w,D^\alpha\nabla V)\|^2+C\|D^\alpha\mathbf{u}_1\|^2e^{-t}\\
&+\frac{1}{8}\mathcal{C}\|\tilde{w}\|^2+\frac{1}{8}\|\tilde{w}_t\|^2\\
&-P'(\mathcal{C})\frac{d}{dt}\|\nabla\tilde{w}\|^2-P'(\mathcal{C})\|\nabla\tilde{w}\|^2, \qquad (4.34)
\end{aligned}$$

where we used the Nirenberg type inequality

$$\|\nabla\tilde{w}\|\le C(\|\tilde{w}\|^2+\|\Delta\tilde{w}\|^2). \qquad (4.35)$$

By substituting (4.34) into (4.33), by using the Gronwall inequality, (4.24), (4.35), and an argument similar to the one for (4.29), we have, for $1\le|\alpha|\le 4$, that

$$\|(\tilde{w},\tilde{w}_t,\tilde{w},\Delta\tilde{w})\|^2\le C(\|w_1\|^2_{2+|\alpha|}+\|\mathbf{u}_1\|^2_{1+|\alpha|})e^{-\beta_2 t}. \qquad (4.36)$$

where β_2 is a suitable positive constant.

Finally we have

$$\begin{aligned}
\|D^{\alpha+1}\mathbf{u}\|^2\le& C\|\nabla\cdot(D^\alpha\mathbf{u})\|^2\le C\|D^\alpha(w_t,w,\nabla w,\mathbf{u})\|^2\\
\le& C\|D^\alpha(w_t,w,\nabla w,\Delta w)\|^2+C\|D^\alpha\mathbf{u}_1\|^2e^{-t}
\end{aligned}$$

$$\leq C(\|w_1\|_{2+|\alpha|}^2 + \|\mathbf{u}_1\|_{1+|\alpha|}^2)e^{-\beta_3 t} \tag{4.37}$$

with $\beta_3 = \min\{\beta_2,\ 1\}$.

The estimates (4.31) follows from (4.36)–(4.37) and the Lemma 4.1. □

Hence by the Lemmas 4.1–4.3, (4.35) and by the Sobolev embedding theorem, we get the following result.

Theorem 4.4 *Let $(w, \mathbf{u}, V) \in X(T)$, then the following inequality holds*

$$\|w(t)\|_{H^6(\mathbb{T}^n)}^2 + \|\mathbf{u}(t)\|_{H^5(\mathbb{T}^n)}^2 + \|V(t)\|_{\dot{H}^4(\mathbb{T}^n)}^2 \leq C\delta_0 e^{-\beta_5 t}, \tag{4.38}$$

provided that $\delta_T \ll 1$. Here $\beta_5 = \min\{\beta_4,\ \beta_1\}$ and δ_0 is given by (1.11).

Proof of the Theorem 1.3. Based on Theorem 4.4, we can prove that (4.10) is true for the classical solution existing locally in time, as long as $\delta_0 = ||\psi_1 - \sqrt{\mathcal{C}}||_6^2 + ||\mathbf{u}_1||_5^2$ is small enough (e.g. $C\delta_0 \ll 1$). Then via the classical continuity argument and the uniform a-priori bounds (4.38) we have the global existence, and the time-asymptotic behavior of our solutions. □

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