HYPERBOLIC CONSERVATIVE MODEL FOR COMPRESSIBLE TWO–PHASE FLOW

E. Romenski[1], D. Zeidan[2], A. Slaouti[3] and E.F. Toro[4]

[1] Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia; e-mail: evrom@math.nsc.ru
[2] Applied Mathematics & Operational Research, Engineering Systems Department, Cranfield University, Royal Military College of Science, Shrivenham, Swindon SN6 8LA, UK; e-mail: d.zeidan@rmcs.cranfield.ac.uk
[3] Department of Engineering and Technology, Manchester Metropolitan University, Manchester M1 5GD, UK; e-mail: a.slaouti@mmu.ac.uk
[4] Laboratory of Applied Mathematics, Faculty of Engineering, University of Trento, 38050 Mesiano di Povo, Trento, Italy; e-mail: toro@ing.unitn.it

Abstract. A mathematical model for compressible two–phase flow governed by a non-linear system that is hyperbolic and conservative is presented. Extended thermodynamics principles are applied in the derivation of the three–dimensional governing equations and of their corresponding closure relations. The derivation of the equation of state for the mixture from the known individual phase equations of state is proposed. The hyperbolicity analysis is carried out for the one–dimensional two–phase flow case. This analysis shows that the system has real eigenvalues and a set of linearly independent eigenvectors for physically acceptable states. The sound wave propagation in static two–fluid media is also derived and its dependence on the volume concentration relaxation is discussed.

Key words. extended thermodynamics, hyperbolic conservative model, compressible two–phase flow

1. Introduction

Two–phase flow modeling has undergone intensive development in recent years. However, there are still no clear theoretical bases for deriving the final mathematical expressions when modeling two–phase flow. There are several approaches to model two-pase flow processes. In one of these approaches (structural continuum fluid model) the governing equations are directly formulated according to conservation principles and by treating a two–phase mixture as a set of interacting subregions of individual phases. But as a rule the mixture consists of a great number of subregions with complex changing shape and it is difficult to realize this approach in practice. In another approach (averaged continuum fluid model), the equations are derived from structural continuum fluid models and the mathematical model is expressed in terms of partial differential balance equations by treating a two–phase mixture as one or two averaged continua [7]. The governing equations represent the physical balance of mass, momentum and energy taking into account interphase exchange at phase interfaces. Widely used two–fluid models are typical examples of this latter approach. This modelling approach assumes each phase as a separate continuum which is interacting with the other by interfacial transfer. Note that in this approach there are no physical postulates fully determining the form of governing equations and only empirical reasons can be used in their derivation. The variational viewpoint [3,5] offers a more rigorous approach, from the mathematical and physical points of view, to the derivation of multi-phase flow models. Here the governing equations are derived by the minimization of an appropriate Lagrangian. This approach also leads to the systems of balance laws for each phase, but their interaction is governed by the terms prescribed by the Lagrangian substance.

Early two–phase models did not result in a system of hyperbolic equations (the eigenvalues can be complex-valued) [16]. Such models, although used in practice with a degree of success, lead to ill–

posed initial–value problem and hence are physically unacceptable. Different hyperbolic models have been proposed in the past [1–3,11,15,16,19]. These models are not in conservative form, which means that the definition of discontinuous solutions (contacts and shocks) is not a straigtforward procedure. Nevertheless certain approach in defining a weak (discontinuous) solution for two–phase model in variational formulation has been developed in [5,14].

There are two main issues that any of the approaches followed for the construction of mathematical models must address, namely the equations must be hyperbolic and must be expressed in conservation-law form. These and related difficulties can be addressed successfully by formulating two–phase problem in terms of extended thermodynamics. The model proposed in this paper is developed by using such principles [4,10,13], which provide reliable results for modeling complicated media with rapidly varying and strongly inhomogeneous processes. The field equations based on the laws of extended thermodynamics form a symmetric hyperbolic system of balance equations in conservative form. It guarantees the well–posedeness of the initial–value problem (and solvability, at least locally in time). The present model is a new model, at least in understanding the mathematical character of the governing equations as well as in studying the physical character of two–phase flow models. From the numerical point of view, the hyperbolic character of the equations and their conservation-law form constitute two major advantages. Firstly, one can develop, at least in principle, upwind-based numerical methods that have proved so successful in the area of single-phase compressible fluid dynamics [17]. Secondly, one can use the finite volume conservative variety of these upwind methods. In the past there have been some reasonably successful attempts at using Godunov-type methods, even for mixed-elliptic hyperbolic and non-conservative systems [18]. This has been possible by (i) assuming smooth solutions and transfering non-conservative terms to the right-hand side and treating these as source terms, and (ii) by splitting the phases at each time step, which has allowed the construction of the Riemann problem solution for each phase separately. The model presented in this paper opens the way to the development of proper upwind schemes for multiphase flows in the near future.

In this paper, we initially focus (section 2) on the assumptions made which lead to a conservative two–phase flow model and we discuss the choice of the equation of state and dissipative interfacial interaction. In section 3 we carry out the hyperbolicity analysis of the one–dimensional equations. Finally, in section 4, the acoustic velocity is derived and used to study the wave propagation for a static water–air mixture.

2. General conservative equations for two–phase flow

In describing two–phase compressible flows, we treat the two–fluid media as a mixture of constituents in which the stress tensor reduces to the pressure. The system of governing equations generalises the model developed by Romensky [13]. In this paper the conservative system of balance laws has been proposed consisting of the well-known conservation laws of density, momentum and energy for the two-phase mixture and completed by the equation for the relative velosity in a conservative form. All the equations can be derived from the general system of thermodynamically compatible system of balance laws, which is formulated by the principles of extended thermodynamics. Such general system includes as an examples various equations from different fields of continuum mechanics, such as gas dynamics, magnetohydrodynamics, hydrodynamics of superfluid helium, nonlinear elasticity, electrodynamics of moving media, etc. This system is formulated in terms of generalized thermodynamic potential depending on generalized parameters of state, which are connected with physical variables. Each partial system describing one or another physical model is determined by the definition of generalized potential and physical meaning of generalized variables. Note that the general thermodinamically compatible system is overdetermined, that is the number of conservation laws is greater then the number of variables. The remarkable property of thermodynamically compatible system is a possibility to reduce it to the complete symmetric hyperbolic system (hyperbolicity is provided by the convexity of generalized thermodynamic potential). A complete description of the thermodynamically compatible conservation laws in continuum mechanics can be found in [4,12,13].

The closed system of balance equations for two–phase media with different phase velocities comprises the following conservation laws of density, momentum and energy for the mixture and additional

closure governing equations:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_k}(\rho u_k) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho u_l) + \frac{\partial}{\partial x_k}(\rho u_l u_k + \mathcal{P}\delta_{kl} + \rho w_l \mathcal{E}_{w_k}) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}\Big(\rho(\mathcal{E} + \frac{u_l u_l}{2})\Big) + \frac{\partial}{\partial x_k}\Big(\rho u_k(\mathcal{E} + \frac{u_l u_l}{2}) + \Pi_k\Big) = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho\alpha) + \frac{\partial}{\partial x_k}(\rho u_k \alpha) = -\phi, \tag{2.4}$$

$$\frac{\partial}{\partial t}(\rho c) + \frac{\partial}{\partial x_k}(\rho u_k c + \rho \mathcal{E}_{w_k}) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial t} w_k + \frac{\partial}{\partial x_k}(u_l w_l + \mathcal{E}_c) = -(e_{klj} u_l \omega_j + \pi_k). \tag{2.6}$$

Here equations (2.1)–(2.3) correspond to the mass, momentum and energy conservation laws for the mixture (although they could also be applied to each one of the two phases). Equation (2.4) is the balance law for the volume concentration of the second phase, equation (2.5) is the conservation law for the mass concentration of the second phase and the last equation (2.6) is the balance law for the relative velocity. Subscripts k, l and j denote one of the three–dimensional components of the physical variables. Other subscripts denote differentiation with respect to these subscripts.

In system (2.1)–(2.6), t is the time, x_k are the Cartesian coordinates, $\alpha = \alpha_2$ is volume concentrations for the second phase which is related to the first phase volume concentrations α_1 by the relation

$$\alpha_1 + \alpha_2 = (1 - \alpha) + \alpha = 1.$$

The mixture density ρ is given by

$$\rho = \alpha_1\rho_1 + \alpha_2\rho_2, \tag{2.7}$$

where ρ_1, ρ_2 are the mass densities of phases.
The mass concentrations for the phases are

$$c_2 = \frac{\alpha\rho_2}{\rho} = c, \tag{2.8}$$

$$c_1 = \frac{(1-\alpha)\rho_1}{\rho} = (1 - c). \tag{2.9}$$

The mixture momentum is given by

$$\rho u_l = (\alpha_1\rho_1 u_{1l} + \alpha_2\rho_2 u_{2l}),$$

where u_{1l} and u_{2l} are the velocities of the phases.
The relative velocity w_l is such that

$$w_l = u_{2l} - u_{1l}.$$

We also define the mixture internal energy as

$$\mathcal{E}(\rho, \alpha, c, w_l, \mathcal{S}) = \mathfrak{e}(\rho, \alpha, c, \mathcal{S}) + c(1-c)\frac{w_l w_l}{2}, \tag{2.10}$$

where $\mathcal{S}$ is the entropy of the mixture and $\mathfrak{e}(\rho, \alpha, c, \mathcal{S})$ is the specific potential energy of the mixture.
The pressure of the mixture is defined by

$$\mathcal{P} = \rho^2 \mathcal{E}_\rho = \rho^2 \mathfrak{e}_\rho.$$

and ω_j is the vorticity vector. The term Π_k in equation (2.3) is given by

$$\Pi_k = \mathcal{P}u_k + \rho u_k w_l \mathcal{E}_{w_l} + \rho \mathcal{E}_c \mathcal{E}_{w_k}.$$

The source term in the volume concentration equation (2.4) describes the relaxation of volume concentration to equilibrium state with relaxation time τ which can be a function of parameters of state, that is,

$$\phi = \frac{\rho}{\tau}\mathcal{E}_\alpha.$$

The source term in the relative velocity equation (2.6) dominate the interfacial friction

$$\pi_k = \kappa \mathcal{E}_{w_k} = \kappa c(1-c)w_k,$$

(κ can depend on the parameters of state) and the relative velocity vorticity vector ω_j (e_{jkl} is the unit pseudoscalar).

The system (2.1)–(2.6) is compatible with two additional conservation laws [13]

$$\frac{\partial w_k}{\partial x_l} - \frac{\partial w_l}{\partial x_k} = e_{klj}\omega_j, \tag{2.11}$$

$$\frac{\partial \omega_k}{\partial t} + \frac{\partial (u_l\omega_k - u_k\omega_l + e_{klj}\pi_j)}{\partial x_l} = 0. \tag{2.12}$$

Equation (2.11) is the definition of the relative velocity vorticity vector, while equation (2.12) is the compatibility condition between vorticity and interfacial friction. Here, the vorticity ω_j, which is not a parameter of state, allows us to write the conservative form for the relative velocity equation. More precisely, from equation (2.11), equation (2.6) can be written as

$$\frac{\partial w_k}{\partial t} + u_l\frac{\partial w_k}{\partial x_l} + \frac{\partial \mathcal{E}_c}{\partial x_k} + w_l\frac{\partial u_l}{\partial x_k} = -\pi_k. \tag{2.13}$$

The equations (2.1)–(2.5) and (2.13) form the closed system for $\rho, u_l, \mathcal{S}, \alpha, c, w_k$ and after solving these equations, the vorticity ω_l can be found using (2.11).

An additional entropy balance law is valid for the solution to the system (2.1)–(2.6)

$$\frac{\partial \rho \mathcal{S}}{\partial t} + \frac{\partial \rho u_k \mathcal{S}}{\partial x_k} = \frac{\rho}{\mathcal{E}_{\mathcal{S}}}\left(\mathcal{E}_\alpha \phi + \mathcal{E}_{w_k}\pi_k\right).$$

The right hand side in the above equation is the entropy production and it is a non–negative quantity due to appropriate definitions of ϕ and π_k.

The important fact is the possibility to reduce the system (2.1)–(2.6) to symmetric hyperbolic form with the use of stationary law for the vorticity (2.11). The proof can be done by introducing a special generating potential and the hyperbolicity is provided by the convexity of this potential [4,13]. The symmetric hyperbolicity was proved for a simpler system in which a balance law for the volume concentration was not included in the complete system of balance laws.

To close the system of governing equations it is necessary to define the equation of state (2.10), namely the specific potential energy $\mathfrak{e}$. Assuming that the equations of state for each constituent are given functions of their own parameters of state we develop the derivation of a common equation of state for the mixture. From the definition of the mixture parameters of state follows that the mass densities of each phase are

$$\rho_1 = \frac{(1-c)\rho}{(1-\alpha)},$$

and

$$\rho_2 = \frac{c\rho}{\alpha}.$$

Let us suppose that $\mathfrak{e}_1(\rho_1, \mathcal{S}_1)$ and $\mathfrak{e}_2(\rho_2, \mathcal{S}_2)$ are the equations of state for the two phases and define the specific potential energy for the mixture as

$$\tilde{\mathfrak{e}}(\rho, \alpha, c, \mathcal{S}_1, \mathcal{S}_2) = c_1\mathfrak{e}_1(\rho_1, \mathcal{S}_1) + c_2\mathfrak{e}_2(\rho_2, \mathcal{S}_2) + \mathfrak{e}_0(c), \tag{2.14}$$

where $\mathfrak{e}_0(c)$ is the excess energy providing in particular the thermodynamic equilibrium condition $\mathfrak{e}_c = 0$ if $c = 0$ or $c = 1$ at normal conditions (by convention "normal conditions" can be regarded as the state under atmospheric pressure and temperature 293K and its parameters can be defined as $\rho_1 = \rho_1^0$, $\rho_2 = \rho_2^0$, $\mathcal{S} = 0$). Further we choose

$$\mathfrak{e}_0(c) = \frac{1}{2}B_1c_1^2 + \frac{1}{2}B_2c_2^2 = \frac{1}{2}B_1(1-c)^2 + \frac{1}{2}B_2c^2.$$

The definition (2.14) of the equation of state of the mixture allows us to derive the governing equations in terms of the individual parameters of state of the phases. Moreover, using the relations (2.7),(2.8),(2.9) we derive a thermodynamic identity as

$$\begin{aligned} d\tilde{\mathfrak{e}}(\rho,\alpha,c,\mathcal{S}_1,\mathcal{S}_2) &= \tilde{\mathfrak{e}}_\rho d\rho + \tilde{\mathfrak{e}}_c dc + \tilde{\mathfrak{e}}_\alpha d\alpha + \tilde{\mathfrak{e}}_{\mathcal{S}_1} d\mathcal{S}_1 + \tilde{\mathfrak{e}}_{\mathcal{S}_2} d\mathcal{S}_2 \\ &= \frac{1}{\rho^2}(\alpha_1\mathcal{P}_1 + \alpha_2\mathcal{P}_2)d\rho + \frac{1}{\rho}(\mathcal{P}_1 - \mathcal{P}_2)d\alpha \\ &+ (\mathfrak{e}_2 - \mathfrak{e}_1 + \frac{\mathcal{P}_2}{\rho_2} - \frac{\mathcal{P}_1}{\rho_1} + \mathfrak{e}_{0_c})dc \\ &+ c_1\mathcal{T}_1 d\mathcal{S}_1 + c_2\mathcal{T}_2 d\mathcal{S}_2, \end{aligned} \tag{2.15}$$

where the pressures $\mathcal{P}_j$ $(j = 1, 2)$ and temperatures $\mathcal{T}_j$ $(j = 1, 2)$ of the two phases are

$$\begin{aligned} \mathcal{P}_j &= \rho_j^2(\mathfrak{e}_j)_{\rho_j}, \\ \mathcal{T}_j &= (\mathfrak{e}_j)_{\mathcal{S}_j}. \end{aligned}$$

Now we pass to the new thermodynamic variables connected with the phase entropies and temperatures. We can define

$$\begin{aligned} \mathcal{S}_1 &= \mathcal{S} - \Delta, \\ \mathcal{S}_2 &= \mathcal{S} + \Delta, \end{aligned}$$

where $\mathcal{S}$ is the mean entropy of the mixture and Δ is the parameter characterizing thermal nonequilibrity. Using the above definition, equality (2.15) can be rewritten in the form

$$\begin{aligned} d\tilde{\mathfrak{e}}(\rho,\alpha,c,\mathcal{S}_1,\mathcal{S}_2) = d\hat{\mathfrak{e}}(\rho,\alpha,c,\mathcal{S},\Delta) &= \hat{\mathfrak{e}}_\rho d\rho + \hat{\mathfrak{e}}_c dc + \hat{\mathfrak{e}}_\alpha d\alpha + \hat{\mathfrak{e}}_{\mathcal{S}} d\mathcal{S} + \hat{\mathfrak{e}}_\Delta d\Delta \\ &= \frac{1}{\rho^2}(\alpha_1\mathcal{P}_1 + \alpha_2\mathcal{P}_2)d\rho + \frac{1}{\rho}(\mathcal{P}_1 - \mathcal{P}_2)d\alpha \\ &+ (\mathfrak{e}_2 - \mathfrak{e}_1 + \frac{\mathcal{P}_2}{\rho_2} - \frac{\mathcal{P}_1}{\rho_1} + \mathfrak{e}_{0_c})dc \\ &+ (c_1\mathcal{T}_1 + c_2\mathcal{T}_2)d\mathcal{S} + (c_2\mathcal{T}_2 - c_1\mathcal{T}_1)d\Delta \end{aligned}$$

The final step in the definition of the equation of state for our model is the assumption that the potential energy for the mixture does not depend on Δ, that is

$$\mathfrak{e}(\rho,\alpha,c,\mathcal{S}) = \hat{\mathfrak{e}}(\rho,\alpha,c,\mathcal{S},\Delta)|_{\Delta=0}.$$

This assumption is correct for the processes in which the variation of entropy is not too large. A more complicated case could be considered in a more sophisticated two–temperature model of two–phase flow with thermal nonequilibrity.

Thus we have the thermodynamic identity defining the equation of state for the system (2.1)–(2.6)

$$\begin{aligned} d\mathfrak{e}(\rho,\alpha,c,\mathcal{S}) &= \mathfrak{e}_\rho d\rho + \mathfrak{e}_c dc + \mathfrak{e}_\alpha d\alpha + \mathfrak{e}_{\mathcal{S}} d\mathcal{S} \\ &= \frac{1}{\rho^2}(\alpha_1\mathcal{P}_1 + \alpha_2\mathcal{P}_2)d\rho + \frac{1}{\rho}(\mathcal{P}_1 - \mathcal{P}_2)d\alpha \\ &+ (\mathfrak{e}_2 - \mathfrak{e}_1 + \frac{\mathcal{P}_2}{\rho_2} - \frac{\mathcal{P}_1}{\rho_1} + \mathfrak{e}_{0_c})dc \\ &+ (c_1\mathcal{T}_1 + c_2\mathcal{T}_2)d\mathcal{S}, \end{aligned} \tag{2.16}$$

from which we conclude that

$$\begin{aligned} \mathcal{P} &= \rho^2\mathfrak{e}_\rho = \alpha_1\mathcal{P}_1 + \alpha_2\mathcal{P}_2, \\ \mathfrak{e}_\alpha &= \frac{1}{\rho}(\mathcal{P}_1 - \mathcal{P}_2), \\ \mathfrak{e}_c &= (\mathfrak{e}_2 - \mathfrak{e}_1 + \frac{\mathcal{P}_2}{\rho_2} - \frac{\mathcal{P}_1}{\rho_1} + \mathfrak{e}_{0_c}), \\ \mathcal{T} &= \mathfrak{e}_{\mathcal{S}} = c_1\mathfrak{e}_{1_{\mathcal{S}}} + c_2\mathfrak{e}_{2_{\mathcal{S}}}. \end{aligned}$$

Here $\mathfrak{e}_c$ is the difference of the enthalpies [10] for the two phases supplemented by the normalizing term $\mathfrak{e}_{0_c}$ arising from excess energy and $\mathcal{T}$ is the temperature of the mixture.

Using the above relationships between the parameters of state for the mixture and individual phases the closed system of balance laws selected from the above equations can be written in terms of parameters $\rho_1, \rho_2, \alpha, u_{1l}, u_{2l}, \mathcal{S}$ as follows:

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha_1\rho_1 + \alpha_2\rho_2) + \frac{\partial}{\partial x_k}(\alpha_1\rho_1 u_{1k} + \alpha_2\rho_2 u_{2k}) &= 0,\\
\frac{\partial}{\partial t}(\alpha_1\rho_1 u_{1l} + \alpha_2\rho_2 u_{2l}) + \frac{\partial}{\partial x_k}(\alpha_1\rho_1 u_{1l}u_{1k} + \alpha_2\rho_2 u_{2l}u_{2k} + \mathcal{P}\delta_{lk}) &= 0,\\
\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)\alpha) + \frac{\partial}{\partial x_k}((\alpha_1\rho_1 u_{1k} + \alpha_2\rho_2 u_{2k})\alpha) &= -\phi,\\
\frac{\partial}{\partial t}(\alpha_2\rho_2) + \frac{\partial}{\partial x_k}(\alpha_2\rho_2 u_{2k}) &= 0,\\
\frac{\partial}{\partial t}(u_{2k} - u_{1k}) + \frac{\partial}{\partial x_k}(\frac{u_{2l}u_{2l}}{2} - \frac{u_{1l}u_{1l}}{2} + \mathfrak{e}_2 - \mathfrak{e}_1 + \frac{\mathcal{P}_2}{\rho_2} - \frac{\mathcal{P}_1}{\rho_1} + \mathfrak{e}_{0c}) &= -\mathit{\Gamma}_k,\\
\frac{\partial}{\partial t}((\alpha_1\rho_1 + \alpha_2\rho_2)\mathcal{S}) + \frac{\partial}{\partial x_k}((\alpha_1\rho_1 u_{1k} + \alpha_2\rho_2 u_{2k})\mathcal{S}) &= \mathit{\Lambda},\\
\frac{\partial w_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} &= \mathit{\Theta},
\end{aligned}$$

where the source terms are

$$\begin{aligned}
\phi &= \frac{1}{\tau}(\mathcal{P}_1 - \mathcal{P}_2),\\
\mathit{\Gamma}_k &= e_{klj}u_l\omega_j + \pi_k, \quad \pi_k = \kappa c(1-c)w_k,\\
\mathit{\Lambda} &= \frac{1}{\rho\tau}(\mathcal{P}_1 - \mathcal{P}_2)^2 + \kappa c(1-c)w_k w_k \geq 0,\\
\mathit{\Theta} &= -e_{klj}\omega_j.
\end{aligned}$$

We can see that the two–phase flow conservation laws are supplemented here by the balance law for the relative velocity. This law postulates that the relative velocity arises if there exists a gradient of the difference of enthalpies and (or) kinetic energies of phases.

It is significant to note that if the relaxation process leading to the equilibrium of volume concentration is sufficiently fast ($\tau \to 0$), then the above system gives the single–pressure approximation within the framework of formulated model, because in this case the approximation

$$\mathcal{E}_\alpha = \frac{1}{\rho}(\mathcal{P}_1 - \mathcal{P}_2) = 0,$$

can be used instead of the equation for α.

3. Hyperbolicity analysis of the one–dimensional equations

The analysis is presented in a one–dimensional setting. At the outset, the equations for conservative, one–dimensional two–phase flow can be written in the form of conservation laws:

$$\partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = \mathbb{S}(\mathbb{U}), \quad -\infty < x < \infty, \quad t > 0, \tag{3.1}$$

where $\mathbb{U}$, $\mathbb{F}(\mathbb{U})$, $\mathbb{S}(\mathbb{U})$ are the set of conservative variables, flux vector, and source term vector respectively. Here $\mathbb{U}(x,t)$ is a vector–valued function. $\mathbb{R}\times[0,\infty) \to \mathit{\Omega}$, $\mathit{\Omega}$ is the set of (physically acceptable) states and we assume that each component of $\mathbb{U}$ is a piecewise C^1 function in the sense that there is only a finite set in $\mathbb{R}$ which is not in C^1 and where the shock conditions hold. The components of $\mathbb{U}$

are the conserved quantities

$$\mathbb{U} = \begin{pmatrix} \rho \\ \rho\alpha \\ \rho u \\ \rho c \\ w \\ \rho(\mathcal{E} + \frac{u^2}{2}) \end{pmatrix},$$

the flux function is

$$\mathbb{F}(\mathbb{U}) = \begin{pmatrix} \rho u \\ \rho u\alpha \\ \rho u^2 + \mathcal{P} + \rho w \mathcal{E}_w \\ \rho uc + \rho \mathcal{E}_w \\ uw + \mathcal{E}_c \\ \rho u(\mathcal{E} + \frac{u^2}{2}) + \mathcal{P}u + \rho uw\mathcal{E}_w + \rho \mathcal{E}_c \mathcal{E}_w \end{pmatrix},$$

and the source term vector is

$$\mathbb{S}(\mathbb{U}) = \begin{pmatrix} 0 \\ -\frac{\rho}{\tau}\mathcal{E}_\alpha \\ 0 \\ 0 \\ -\pi \\ 0 \end{pmatrix}.$$

The system is closed by equations of state for the mixture

$$\mathcal{E}(\rho, \alpha, c, w, \mathcal{S}) = \mathfrak{e}(\rho, \alpha, c, \mathcal{S}) + c(1-c)\frac{w^2}{2},$$

where $\mathfrak{e}(\rho, \alpha, c, \mathcal{S})$ is defined by (2.16). The pressure and derivatives of $\mathcal{E}$ with respect to w, c are

$$\mathcal{P} = \rho^2 \mathcal{E}_\rho = \rho^2 \mathfrak{e}_\rho, \quad \mathcal{E}_w = c(1-c)w, \quad \mathcal{E}_c = \mathfrak{e}_c + (1-2c)\frac{w^2}{2}.$$

The mathematical structure of the system is more clearly revealed and the characteristic analysis given below is simplified in terms of primitive variables [20]. For the conservative two–fluid model case one possibility is to choose a vector $\mathbb{W}$ of primitive variables defined as

$$\mathbb{W} = \Big(\rho, \alpha, u, c, w, \mathcal{S}\Big)^T.$$

Thus, in quasilinear form we have

$$\partial_t \mathbb{W} + \mathbb{A}(\mathbb{W})\partial_x \mathbb{W} = \mathbb{Q}(\mathbb{U}), \tag{3.2}$$

where the Jacobian matrix $\mathbb{A}(\mathbb{W})$ is

$$\begin{pmatrix} u & 0 & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 & 0 \\ \frac{1}{\rho}(\mathcal{P}_\rho + c(1-c)w^2) & \frac{\mathcal{P}_\alpha}{\rho} & u & \frac{1}{\rho}(\mathcal{P}_c + \rho(1-2c)w^2) & 2c(1-c)w & \frac{\mathcal{P}_\mathcal{S}}{\rho} \\ c(1-c)\frac{w}{\rho} & 0 & 0 & u + (1-2c)w & c(1-c) & 0 \\ \mathfrak{e}_{c\rho} & \mathfrak{e}_{\alpha c} & w & \mathfrak{e}_{cc} - w^2 & u + (1-2c)w & \mathfrak{e}_{c\mathcal{S}} \\ 0 & 0 & 0 & 0 & 0 & u \end{pmatrix},$$

and $\mathbb{Q}(\mathbb{U}) = (\mathbb{U}_\mathbb{W})^{-1}\, \mathbb{S}(\mathbb{U})$.

Let us introduce the set Ω of physically admissible states, $\Omega = \big\{\mathbb{U} \in \mathbb{R}^6; \rho, \mathcal{S} > 0, \alpha, c \in]0,1[, u, w \in \mathbb{R}\big\}$ Further we will see that there exists a positive number $w^\star$ such that for any $\mathbb{U}$ that lies in the set $\Omega^\star$ defined by $\Omega^\star = \big\{\mathbb{U} \in \Omega; |w| \leq w^\star\big\}$ the eigenvalues $\lambda_\mathfrak{j}$, $1 \leq \mathfrak{j} \leq 6$ are real for w small enough. Thus the eigenvalues of the matrix $\mathbb{A}(\mathbb{W})$ are the solutions of the characteristic polynomial

$$\det\Big[\mathbb{A}(\mathbb{W}) - \lambda I\Big] = 0,$$

which becomes

$$\mathcal{Y}^{\mathbf{2}}(\mathcal{Y}^{\mathbf{4}} + \varsigma_{\mathbf{3}}\mathcal{Y}^{\mathbf{3}} + \varsigma_{\mathbf{2}}\mathcal{Y}^{\mathbf{2}} + \varsigma_{\mathbf{1}}\mathcal{Y} + \varsigma_{\mathbf{0}}) = \mathbf{0}, \tag{3.3}$$

where

$$\mathcal{Y} \equiv u - \lambda,$$

and

$$\begin{aligned}
\varsigma_3 &= 2(1-2c)w, \\
\varsigma_2 &= -(\mathcal{P}_\rho + c(1-c)\mathfrak{e}_{cc}) + b_0 w^2, \\
\varsigma_1 &= 2(2\rho c(1-c)\mathfrak{e}_{c\rho} - (1-2c)\mathcal{P}_\rho)w + b_1 w^2 + b_2 w^3 + b_3 w^4, \\
\varsigma_0 &= c(1-c)(\mathcal{P}_\rho \mathfrak{e}_{cc} - \mathcal{P}_c \mathfrak{e}_{c\rho}) + b_4 w^2 + b_5 w^4,
\end{aligned} \tag{3.4}$$

and $b_0, b_1, b_2, b_3, b_4, b_5$ are coefficients depending on the state parameters [20]. Solving equation (3.3) for the non-trivial roots is diffcult. Note that the coefficients (3.4) of equation (3.3) have the form

$$\varsigma_\imath = \varsigma_\imath^0 + \varsigma_\imath^1 w + \mathcal{O}(w^2), \quad \imath = 0, 1, 2, 3. \tag{3.5}$$

Assuming that the relative velocity w is sufficiently small we can use a perturbation method around $w = 0$ and write the solution in the following form

$$\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}_1 w + \mathcal{O}(w^2),$$

At zero order in w we derive from (3.3)

$$\mathcal{Y}_0^4 + \varsigma_2^0 \mathcal{Y}_0^2 + \varsigma_0^0 = 0, \tag{3.6}$$

and the roots of this equation are

$$\mathcal{Y}_0 = \pm\left[\frac{-\varsigma_2^0 \pm \sqrt{(\varsigma_2^0)^2 - 4\varsigma_0^0}}{2}\right]^{\frac{1}{2}}. \tag{3.7}$$

We mentioned that the system under consideration can be transformed to a symmetric hyperbolic system and its hyperbolicity is guaranteed by the convexity of the equation of state. Now we prove that the convexity of the specific potential energy $\mathfrak{e}$ guarantees that all roots $\mathcal{Y}_0$ of the equation (3.6) are real. To do this it is convenient to use the variables $V = (\frac{1}{\rho}, c, \mathcal{S})$ where V is the specific volume. In terms of these variables we have $\mathcal{P} = -\mathfrak{e}_V$ and the following formulae:

$$\begin{aligned}
\mathcal{P}_\rho &= V^2 \mathfrak{e}_{VV}, \\
\mathcal{P}_c &= -\mathfrak{e}_{Vc}, \\
\mathfrak{e}_{\rho c} &= -V^2 \mathfrak{e}_{Vc}, \\
\mathcal{P}_\rho + c(1-c)\mathfrak{e}_{cc} &= \left[\mathfrak{e}_{VV} + \frac{c(1-c)}{V^2}\mathfrak{e}_{cc}\right]^2 V^2.
\end{aligned}$$

It follows from (3.4), (3.5) that

$$\begin{aligned}
\frac{1}{V^4}((\varsigma_2^0)^2 - 4\varsigma_0^0) &= \left[\mathfrak{e}_{VV} + \frac{c(1-c)}{V^2}\mathfrak{e}_{cc}\right]^2 - \frac{4c(1-c)}{V^2}(\mathfrak{e}_{VV}\mathfrak{e}_{cc} - \mathfrak{e}_{Vc}^2) \\
&= \left[\mathfrak{e}_{VV} - \frac{c(1-c)}{V^2}\mathfrak{e}_{cc}\right]^2 + \frac{4c(1-c)}{V^2}\mathfrak{e}_{Vc}^2 > 0,
\end{aligned}$$

which means that the values

$$\mathcal{Y}_0^2 = \frac{1}{2}(-\varsigma_2^0 \pm \sqrt{(\varsigma_2^0)^2 - 4\varsigma_0^0}),$$

defined by (3.7) are real. Furthermore the positiveness of $\mathcal{Y}_0^2$ (and hence the reality of $\mathcal{Y}_0$) is guaranteed by the inequality

$$\mathfrak{e}_{VV}\mathfrak{e}_{cc} - \mathfrak{e}_{Vc}^2 > 0. \tag{3.8}$$

The above inequality (3.8) is one of the convexity conditions which are the positive definiteness conditions of the matrix

$$\begin{pmatrix} \mathfrak{e}_{VV} & \mathfrak{e}_{Vc} & \mathfrak{e}_{V\mathcal{S}} \\ \mathfrak{e}_{cV} & \mathfrak{e}_{cc} & \mathfrak{e}_{c\mathcal{S}} \\ \mathfrak{e}_{\mathcal{S}V} & \mathfrak{e}_{\mathcal{S}c} & \mathfrak{e}_{\mathcal{S}\mathcal{S}} \end{pmatrix}.$$

From (3.3) the first-order terms in w lead to

$$2(\varsigma_2^0 + 2\mathcal{Y}_0^2)\mathcal{Y}_1 + \varsigma_3^1\mathcal{Y}_0^2 + \varsigma_1^1 = 0,$$

and the solution is

$$\mathcal{Y}_1 = -\frac{\varsigma_1^1 + \varsigma_3^1\mathcal{Y}_0^2}{2(\varsigma_2^0 + 2\mathcal{Y}_0^2)}.$$

Thus, the system (3.1) admits the following six eigenvalues up to first order in w

$$\begin{aligned} \lambda_1 &= u - a_1 + \mathcal{Y}_1(a_1)w + \mathcal{O}(w^2), \\ \lambda_2 &= u - a_2 + \mathcal{Y}_1(a_2)w + \mathcal{O}(w^2), \\ \lambda_3 &= \lambda_4 = u, \\ \lambda_5 &= u + a_2 + \mathcal{Y}_1(a_2)w + \mathcal{O}(w^2), \\ \lambda_6 &= u + a_1 + \mathcal{Y}_1(a_1)w + \mathcal{O}(w^2), \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{2}}\Big[-\varsigma_2^0 - \sqrt{(\varsigma_2^0)^2 - 4\varsigma_0^0}\Big]^{1/2}, \\ a_2 &= \frac{1}{\sqrt{2}}\Big[-\varsigma_2^0 + \sqrt{(\varsigma_2^0)^2 - 4\varsigma_0^0}\Big]^{1/2}, \end{aligned}$$

and

$$\mathcal{Y}_1(a_j) = -\frac{\varsigma_1^1 + \varsigma_3^1 a_j^2}{2(\varsigma_2^0 + 2a_j^2)}.$$

The eigenvalues $\lambda_{3,4}$ are clearly the mixture velocity u, whereas the other eigenvalues correspond to the acoustic wave propagation for the mixture. The associated linearly independent set of eigenvectors

$$\mathcal{K}_i = (k_1, k_2, k_3, k_4, k_5, k_6)_i^T, \quad i = 1, ..., 6$$

can be found from the equation

$$\Big(\mathbb{A}(\mathbb{W}) - \lambda I\Big)\mathcal{K} = 0,$$

and determined as

$$\mathcal{K}_3 = (0, 0, 1, 0, 0, 0)^T, \quad \mathcal{K}_4 = (0, 0, 0, 1, 0, 0)^T,$$

$$\mathcal{K}_i = (k, k_2(a_i, k), 0, 0, k_5(a_i, k), k_6(a_i, k))^T, \quad i = 1, 2,$$

$$\mathcal{K}_{4+i} = (k, k_2(-a_i, k), 0, 0, k_5(-a_i, k), k_6(-a_i, k))^T, \quad i = 1, 2,$$

where k is an arbitrary parameter, $k_2(a_i, k) = -\frac{a_i}{\rho}k$, and $k_5(a_i, k), k_6(a_i, k)$ are the solutions to the linear system of algebraic equations:

$$\begin{aligned} (a_i + c(1-c)w)k_3 + c(1-c)k_4 &= \frac{1}{\rho}c(1-c)wk \\ \mathfrak{e}_{cc}k_3 + (a_i + c(1-c)w)k_4 &= (\mathfrak{e}_{c\rho} - \frac{1}{\rho}a_i w)k \end{aligned}$$

Further, the characteristic fields corresponding to the acoustic waves were shown to be genuinely non–linear while the last two are linearly degenerate [20]. Thus the considered one–dimensional conservative two–fluid model is hyperbolic.

4. Study of the sound wave propagation

The analysis of the eigenstructure given in the previous section has shown that there are four eigenvalues corresponding to two sound velocities. Now we clarify the values of the velocity of sound in static two–fluid media and its dependence on the volume concentration relaxation illustrated by the example of a water–air mixture. Consider the partial case of the system (3.2) ignoring a source term in the equation for the relative velocity w and leaving fixed the source term in the equation for the volume concentration as given in equation (2.4)

$$\partial_t \mathbb{W} + \mathbb{A}(\mathbb{W})\partial_x \mathbb{W} = \mathbb{Q}(\mathbb{W}), \tag{4.1}$$

where

$$\mathbb{Q}(\mathbb{W}) = \Big(0, -\frac{1}{\tau}\mathcal{E}_\alpha, 0, 0, 0, \frac{\mathcal{E}_\alpha^2}{\tau \mathcal{E}_{\mathbb{S}}}\Big)^T.$$

We study small–amplitude wave propagation in the static two–fluid media which is also in the thermodynamic equilibrium (in particular $\mathcal{E}_\alpha = 0$). A solution of (4.1) can be represented in the form

$$\begin{aligned}\mathbb{W} &= \mathbb{W}^0 + \mathbb{W}^1 \\ &= (\rho^0 + \rho^1, \alpha^0 + \alpha^1, u^1, c^0 + c^1, w^1, \mathbb{S}^1),\end{aligned}$$

where $\mathbb{W}^1$ is the vector of small perturbation of the primitive variables. The smallness of $\mathbb{W}^1$ allows us to derive a linear system of partial differential equations

$$\partial_t \mathbb{W}^1 + \mathbb{A}(\mathbb{W}^0)\partial_x \mathbb{W}^1 = \mathbb{Q}_{\mathbb{W}}(\mathbb{W}^0)\mathbb{W}^1, \tag{4.2}$$

where

$$\mathbb{A}(\mathbb{W}^0) = \begin{pmatrix} 0 & 0 & \rho^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mathcal{P}_\rho}{\rho^0} & \frac{\mathcal{P}_\alpha}{\rho^0} & 0 & \frac{\mathcal{P}_c}{\rho^0} & 0 & \frac{\mathcal{P}_{\mathbb{S}}}{\rho^0} \\ 0 & 0 & 0 & 0 & c^0(1-c^0) & 0 \\ \mathfrak{e}_{c\rho} & \mathfrak{e}_{\alpha c} & 0 & \mathfrak{e}_{cc} & 0 & \mathfrak{e}_{c\mathbb{S}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the the Jacobian matrix $\mathbb{Q}_{\mathbb{W}}$ is

$$\mathbb{Q}_{\mathbb{W}}(\mathbb{W}^0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\tau}\mathfrak{e}_{\alpha\rho} & -\frac{1}{\tau}\mathfrak{e}_{\alpha\alpha} & 0 & -\frac{1}{\tau}\mathfrak{e}_{\alpha c} & 0 & -\frac{1}{\tau}\mathfrak{e}_{\alpha\mathbb{S}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here all derivatives of thermodynamic parameters are computed at the values $\rho = \rho^0, \alpha = \alpha^0, c = c^0, \mathbb{S} = 0$. Note that the right hand side in the equation for entropy vanishes due to quadratic degree of smallness.

The sound waves are harmonic solutions of the form

$$\mathbb{W}^1 = \mathbb{W}_0^1 \exp(i(\nu t - kx)).$$

If we substitute this expression into (4.2) we observe that only the solution with $\alpha^1 = 0$, $\mathbb{S}^1 = 0$ exists. The equation connecting the frequency ν and wavelength k can then be derived:

$$\nu^4 - \sigma_2(i\nu)\nu^2 k^2 + \sigma_0(i\nu)k^4 = 0. \tag{4.3}$$

Here

$$\begin{aligned}\sigma_2(i\nu) &= \mathcal{P}_\rho + c^0(1-c^0)\mathfrak{e}_{cc} - \frac{\rho^2\mathfrak{e}_{\alpha\rho}^2 + \mathfrak{e}_{\alpha c}^2}{i\nu\tau + \mathfrak{e}_{\alpha\alpha}}, \\ \sigma_0(i\nu) &= c^0(1-c^0)\left(\mathcal{P}_\rho - \frac{\rho^2\mathfrak{e}_{\alpha\rho}^2}{i\nu\tau + \mathfrak{e}_{\alpha\alpha}}\right)\left(\mathfrak{e}_{cc} - \frac{\mathfrak{e}_{\alpha c}^2}{i\nu\tau + \mathfrak{e}_{\alpha\alpha}}\right) \\ &\quad - c^0(1-c^0)\left(\mathcal{P}_c - \frac{\rho^2\mathfrak{e}_{\alpha\rho}\mathfrak{e}_{\alpha c}}{i\nu\tau + \mathfrak{e}_{\alpha\alpha}}\right)\left(\mathfrak{e}_{c\rho} - \frac{\mathfrak{e}_{\alpha\rho}\mathfrak{e}_{\alpha c}}{i\nu\tau + \mathfrak{e}_{\alpha\alpha}}\right).\end{aligned}$$

It is obvious that if the wavelength is prescribed then the sound wave behaviour depends on the value of relaxation time τ (note that τ can depend on the thermodynamic parameters of state). In particular, we may consider two interesting limiting cases, $\tau = \infty$ and $\tau = 0$, which correspond to the cases when the characteristic time of a process is respectively less or greater than the relaxation time.

For both cases we have the following formula for sound wave velocities obtained by solving equation (4.3) (we choose only positive roots):

$$a_{1,2} = \frac{\nu}{k} = \sqrt{\frac{1}{2}(\sigma_2 \pm \sqrt{\sigma_2^2 - 4\sigma_0})}. \tag{4.4}$$

Let us consider now an example of a water-air mixture and find out the dependence of the velocity of sound on the volume concentration of the gas (void fraction) for the two cases described above supposing that the mixture is at normal conditions. An equation of state for such a mixture is given by (2.16) and determined by equations of state for water and air. Equation of state for air is the perfect gas EOS [6,8] given by

$$\mathfrak{e}_2 = \frac{A_2}{\gamma_2 - 1}\left(\frac{\rho_2}{\rho_2^0}\right)^{\gamma_2 - 1} \exp\left(\frac{\mathcal{S}}{c_V^2}\right).$$

The equation of state for water is defined as

$$\mathfrak{e}_1 = \frac{A_1}{\gamma_1 - 1}\left(\frac{\rho_1}{\rho_1^0}\right)^{\gamma_1 - 1} \exp\left(\frac{\mathcal{S}}{c_V^1}\right) + A_0\frac{\rho_1^0}{\rho_1},$$

which is the stiffened gas equation of state written in terms of density and entropy [6,9]. All the constants are defined in table 4.1.

Table 4.1. Constants for the study of the sound wave propagation.

Constants in the equations of state for liquid and gas	
$\gamma_1 = 2.8$	$\gamma_2 = 1.4$
$\rho_1^0 = 10^3 \quad kg/m^3$	$\rho_2^0 = 1 \quad kg/m^3$
$c_v^1 = 1495 \quad J/kgK$	$c_v^2 = 720 \; J/kgK$
$A_0 = 8.4999\ 10^5 \; m^2/s^2$	$A_2 = 10^5 \; m^2/s^2$
$A_1 = 8.5\ 10^5 \quad m^2/s^2$	

Direct calculation of the speeds of sound at normal conditions ($\rho_1 = \rho_1^0, \rho_2 = \rho_2^0, \mathcal{S} = 0$) gives the following results for the case $\tau = \infty$:

$$a_1 = \sqrt{\gamma_1 A_1}, \quad a_2 = \sqrt{\gamma_2 A_2}.$$

We can see that both speeds of sound do not depend on the void fraction and coincide with the speeds of sound in the pure water and air respectively.

The dependence of the speeds of sound on the void fraction in the case when $\tau = 0$ (equal pressure approximation) was calculated using the formula (4.4). It is found that one of the speeds is equal to 0. The nonzero speed of sound is presented in figure 4.1.

5. Conclusions

Based on an extended thermodynamics approach, a new model for two–phase flow has been developed. We have examined the mathematical character of a two–phase mixture description used to model fully–compressible non–equilibrium two-phase flows. Additionally, the model is conservative and hyperbolic. Due to the nice mathematical properties the model in the case of dissipation–free flow can be used to formulation of the simple wave solutions (shock, centred wave, contact) and further to solve the Riemann problem. It is also expected that the conservative form could be useful to develop an accurate numerical methods for the proposed model. It is hoped that this paper will motivate other researchers to use, and further develop, extended thermodynamics methodology for the analysis of two–phase flow in systems and processes of practical concern.

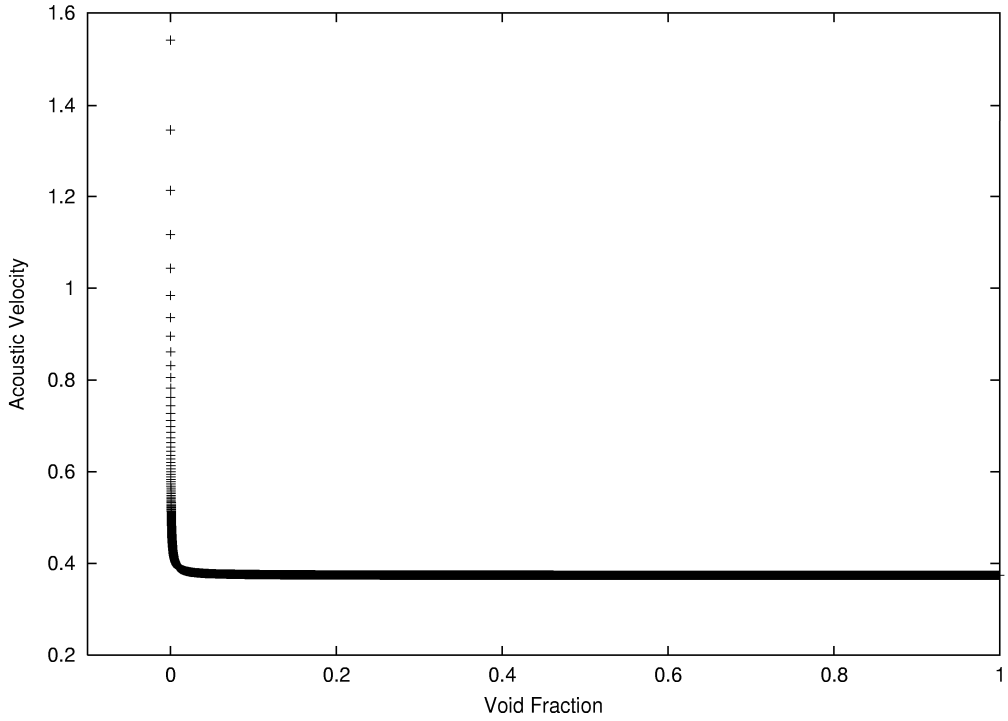

Fig. 4.1. The acoustic velocity for a water/air mixture for the case of single pressure approximation as a function of void fraction α. The water and air are at rest under atmospheric conditions.

Acknowledgements

This work has been partially supported by the EU Comission under contract FIKS-CT-2000-00050, ASTAR project. The paper was finalized during the stay of the first author at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge and participation in the programme *Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics.* The fourth author acknowledges the support provided by the Isaac Newton Institute for Mathematical Sciences, University of Cambridge UK, as organizer of the six-month programme on *Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics*, January to July 2003, and the associated EPSRC senior visiting fellowship, grant No GR N09276.

References

1. Baer M. and Nunziato J. A Two–Phase Mixture Theory for the Deflagration–to–Detonation Transition (DDT) in Reactive Granular Materials. Int. J. Multiphase Flow **12**, (1986) 861–889.
2. Bdzil J., Menikoff R., Son S., Kapila A. and Stewart D. Two–Phase Modeling of Deflagration–To–Detonation Transition in Granular Materials: a Critical Examination of Modeling Issues. Physics of Fluids **11**, (1999) 378–402.
3. Gavrilyuk S. and Saurel R. Mathematical and Numerical Modeling of Two–Phase Compressible Flows with Micro–Inertia. J. of Comp. Phys. **175**, (2002) 326–360.
4. Godunov S. and Romensky E. Thermodynamics, Conservation Laws and Symmetric Forms of Differential Equations in Mechanics of Continuous Media. In *Comput. Fluid Dynamic Review* (John Wiley & Sons, New York 1995) pp.19–31.
5. Gouin H. and Gavrilyuk S. Hamilton's Principle and Rankine–Hugoniot Conditions for General Motions of Mixtures. *Meccanica.* (1999) **34**: 39–47.
6. Harlow F. and Amsden A. *Fluid Dynamics.* Monograph LA–4700, Los Alamos National Laboratory, Los Alamos, NM, 1971.
7. Ishii M. *Thermo–Fluid Dynamic Theory of Two–Phase Flow.* (Eyrolles, Paris 1975).
8. Mader C. *Numerical Modeling of Detonations.* University of California Press Ltd. London, England, 1979.
9. Menikoff R, Plohr B. The Riemann Problem for Fluid Flow of Real Materials. *Rev. of Mod. Phys.* (1989) **61**: 75–130.
10. Müller I. and Ruggeri T. *Rational Extended Thermodynamics* (Springer–Verlag, New York 1998).

11. Ransom V. and Hicks D. Hyperbolic Two–Pressure Models for Two–Phase Flow. J. of Comp. Phys. **53**, (1984) 124–151.
12. Romensky E. Hyperbolic Systems of Thermodynamically Compatible Conservation Laws in Continuum Mechanics. Math. Comput. Modelling **28**, (1998) 115–130.
13. Romensky E. Thermodynamics and Hyperbolic Systems of Balances Laws in Continuum Mechanics. In *Godunov Methods: theory and applications*, E. F. Toro (ed.), (Kluwer Academic/Plenum publishers 2001) pp. 745–761.
14. Serre D. Sur le principe variationnel des équations de la mécanique des fluides parfaits, Modél. Math. Anal. Numér. **27**, (1993) 739–758.
15. Städtke H., Worth B. & Franchello G. On the hyperbolic nature of two-phase flow equations: characteristic analysis and related numerical methods. In *Godunov Methods: theory and applications*, E. F. Toro (ed.), (Kluwer Academic/Plenum publishers 2001) pp. 841–862.
16. Stewart H. and Wendroff B. Two–phase flow: models and methods, J. of Comp. Phys. **56**, (1984) 363–409.
17. Toro, E. F. *Riemann Solvers and Numerical Methods for Fluid Dynamics. Second Edition* (Springer-Verlag, Berlin Heidelberg 1999)
18. Toro E. Riemann–problem Based Techniques for Computing Reactive Two-Phase Flows. In *Numerical Combustion*, A. Dervieux & B. Larrouturou (eds.), (Springer Lecture Notes in Physics, 1989) Vol 351, pp. 427–481.
19. Wallis G. Review–theoretical models of gas–liquid flows. J. of Fluids Engineering **104**, (1982) 279–283.
20. Zeidan D. Mathematical and numerical study of two–phase flow models. *Ph.D. Thesis* (Manchester Metropolitan University, Manchester UK 2003).