THE RIEMANN PROBLEM FOR FLUID FLOWS
IN A NOZZLE WITH DISCONTINUOUS CROSS-SECTION

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ABSTRACT. The system of balance laws describing a compressible fluid flow in a nozzle forms a non-strictly hyperbolic systems of partial differential equations which, also, is not fully conservative due to the effect of the geometry. First, we investigate the general properties of the system and determine all possible wave combinations. Second, we construct analytically the solutions of the Riemann problem for any values of the left- and right-hand states. For certain values we obtain up to three solutions whose structure is carefully described here. In some range of Riemann data, no solution exists. When three solutions are available, then exactly one of them contains two stationary waves which are super-imposed in the physical space. We include also numerical plots of these solutions.

1. INTRODUCTION

We consider the Riemann problem for the following system describing the evolution of an isothermal fluid in a nozzle with discontinuous cross-sectional area $a(x) > 0$:

$$
\begin{aligned}
&\partial_t(a\rho) + \partial_x(a\rho u) = 0, \\
&\partial_t(a\rho u) + \partial_x(a(\rho u^2 + p(\rho))) = p(\rho)\partial_x a, \\
&\partial_t a = 0, \qquad x \in \mathbf{R},\ t > 0.
\end{aligned}
\tag{1.1}
$$

Here, ρ and u stand for the density and the particle velocity of the fluid under consideration, respectively, and the pressure function $p = p(\rho)$ is given by

$$
p(\rho) = \kappa\rho^\gamma, \quad 1 < \gamma < 5/3.
\tag{1.2}
$$

Observe that the third equation in (1.1) is trivial since the function a depends only on x. Since we are interested in discontinuous functions a it is convenient (following [13]) to consider the full set of three equations. The third equation is associated with a linearly degenerate field with constant characteristic speed.

First, observe that the system (1.1) does not take the usual form of a system of conservation laws and is not fully conservative. This is due to the effect of the geometry modeled by the function a. When a admits discontinuities, Dirac masses appear on the right-hand side of the second equation of (1.1). Therefore, the usual notion of weak solutions for systems of conservation laws does not apply. The product still makes sense as a measure within the framework introduced by Dal Maso, LeFloch, and Murat [5]. Throughout this paper, the function a will be assumed to be piecewise constant, so that the application of the theory in [5] is particularly immediate, as we see below.

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For smooth solutions $(x, t) \mapsto (\rho, u, a)$, the system (1.1) is equivalent to the following three conservation laws:

(1.3)
$$\partial_t(a\rho) + \partial_x(a\rho u) = 0,$$
$$\partial_t u + \partial_x(\frac{u^2}{2} + h(\rho)) = 0,$$
$$\partial_t a = 0,$$

where the function h is defined by $h'(\rho) = p'(\rho)/\rho$, thus

$$h(\rho) := \frac{\kappa\gamma}{\gamma - 1}\rho^{\gamma-1}.$$

On the other hand, the Rankine-Hugoniot relation associated with the third equation in (1.1) takes the form

(1.4)
$$-\lambda[a] = 0,$$

where λ denotes the speed of the discontinuity, $[a] := a_+ - a_-$ is the jump of the quantity a, and $a_\pm$ denotes its left- and right-hand traces. The relation (1.4) implies that :

(i) the component a remains constant across the shock, or

(ii) a is discontinuous but the shock velocity vanishes.

The following discussion provides us the list of admissible waves for solving the Riemann problem, and is central to this paper.

Let us assume first that the component a remains constant across some shock wave. Since a is piecewise constant, it should be constant in a neighborhood of the shock. Eliminating a from (1.1), we obtain the following system of two conservation laws

(1.5)
$$\partial_t(\rho) + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x((\rho u^2 + p(\rho))) = 0.$$

Thus, the left- and right-hand states are related by the Rankine Hugoniot relations corresponding to (1.5)

(1.6)
$$-\lambda[\rho] + [\rho u] = 0,$$
$$-\lambda[\rho u] + [\rho u^2 + p(\rho)] = 0,$$

where $[\rho] := \rho_+ - \rho_+$, etc.

Suppose next that the component a is discontinuous and that, therefore, the shock speed vanishes. The solution is independent of time, ad it is natural to search for a solution as the limit of a sequence of time-independent smooth solutions of (1.1). Following Marchesin-Paes-Leme [18] and LeFloch [13], this motivates us to consider the system of ordinary differential equations

(1.7)
$$(a\rho u)' = 0,$$
$$(\frac{u^2}{2} + h(\rho))' = 0.$$

The integral curve of (1.7) passing through each point (ρ_0, u_0, a_0) can be parameterized by u, say

$$u \mapsto (\rho(u), u, a(u)),$$

and satisfie

$$\frac{u^2}{2} + \frac{\kappa\gamma}{\gamma - 1}\rho^{\gamma-1}(u) = \frac{u_0^2}{2} + \frac{\kappa\gamma}{\gamma - 1}\rho_0^{\gamma-1},$$
$$a(u) = \frac{a_0\rho_0 u_0}{\rho(u)u}.$$

Letting $u \to u_\pm$ and setting $\rho_\pm = \rho(u_\pm)$, $a_\pm = a(u_\pm)$, we see that the states $(\rho_\pm, u_\pm, a_\pm)$ satisfy the Rankine-Hugoniot relations associated with (1.3), but with zero shock speed:

$$(1.8) \qquad \begin{aligned} &[a\rho u] = 0, \\ &[\frac{u^2}{2} + h(\rho)] = 0. \end{aligned}$$

It is easy to check (LeFloch [13]) that discontinuities satisfying (1.8) are solutions of the nonconservative system (1.1) in the sense of DalMaso-LeFloch-Murat for the family of paths based on the ODE trajectories (1.7).

In this paper, we will construct solutions of the Riemann problem associated with (1.1), made of elementary waves defined as follows.

Definition 1.1. The admissible waves for the system (1.1) are the following ones:

- the **rarefaction waves**, which are smooth solutions of (1.1) with constant component a depending only on the self-similarity variable x/t;
- the **shock waves** which satisfy (1.6) and Lax shock inequalities (see [12]) and have constant component a;
- and the **stationary waves** which have zero propagation speed and are given by (1.8).

As will become clear in Section 2, the system under consideration in this paper is *not strictly hyperbolic*. This makes the analysis of the Riemann problem particularly challenging. For previous works in this direction we refer to [3,6,9,10,11,16,17,18]. For numerical work on hyperbolic systems with source-term we refer to [1, 2, 7, 8]. An outline of this paper is as follows: Section 2 presents basic material on the system (1.1). Then, in Section 3, for each wave family we construct one-parameter sets of states which can be connected to a given state by using shock and rarefaction waves in one family plus a stationary wave. In Sections 4 and 5, we exhibit the solution (or solutions) of the Riemann problem: first using a single stationary wave (Section 4) and, second, using two stationary waves (Section 5). The Riemann solutions are finally plotted numerically in Section 6.

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2. Preliminaries

It will be convenient to write $U = (\rho, u, a)$. For smooth solutions the system (1.3) can be written in the standard form

$$(2.1) \qquad \partial_t U \,+\, A(U)\,\partial_x U \,=\, 0,$$

where

$$A = \begin{bmatrix} u & \rho & \rho u/a \\ h'(\rho) & u & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix A is not the Jacobian of a function. It admits the following three eigenvalues and right-eigenvectors:

(2.2)
$$\lambda_1 := u - \sqrt{p'(\rho)}, \quad \lambda_2 := 0, \quad \lambda_3 := u + \sqrt{p'(\rho)},$$
$$r_1 := (\rho, -\sqrt{p'(\rho)}, 0)^t, \quad r_2 := (\rho u, -p'(\rho), a(u - \frac{p'(\rho)}{u}))^t, \quad r_3 := (\rho, \sqrt{p'(\rho)}, 0)^t.$$

The first and the second characteristic speeds may coincide, and so do the second and the third characteristic speeds. More precisely, setting

(2.3)
$$\mathcal{C}^{\pm} : u = \pm\sqrt{\kappa\gamma}\,\rho^{\frac{\gamma-1}{2}},$$

we see that
$$\lambda_2 = \lambda_1 \quad \text{on} \quad \mathcal{C}^+,$$
$$\lambda_2 = \lambda_3 \quad \text{on} \quad \mathcal{C}^-$$

and (1.1) is *not strictly hyperbolic*.

In (ρ, u)-plan, the *boundaries of strict hyperbolicity* $C_{\pm}$ separates the half-plane $\rho > 0$ into three (open) domains. For convenience, we will refer to them as the "lower region" $\mathcal{G}_1$, the "middle region" $\mathcal{G}_2$, and the "upper region" $\mathcal{G}_3$:
$$\mathcal{G}_1 = \{(\rho, u) : u < -\sqrt{p'(\rho)}\},$$
$$\mathcal{G}_2 = \{(\rho, u) : |u| < \sqrt{p'(\rho)}\},$$
$$\mathcal{G}_3 = \{(\rho, u) : u > \sqrt{p'(\rho)}\}.$$

In each of these regions the system (2.1) is strictly hyperbolic and we have
$$\lambda_1 < \lambda_3 < \lambda_2, \quad \text{in} \quad \mathcal{G}_1,$$
$$\lambda_1 < \lambda_2 < \lambda_3, \quad \text{in} \quad \mathcal{G}_2,$$
$$\lambda_2 < \lambda_1 < \lambda_3, \quad \text{in} \quad \mathcal{G}_3.$$

It will be also convenient to use the notation
$$\mathcal{G}_2^+ = \{(\rho, u) \in \mathcal{G}_2 : u > 0\},$$
$$\mathcal{G}_2^- = \{(\rho, u) \in \mathcal{G}_2 : u < 0\}.$$

The second characteristic family is *linearly degenerate*, while the 1- and the 3-characteristic families are *genuinely nonlinear*:
$$-\nabla\lambda_1 \cdot r_1 = \nabla\lambda_3 \cdot r_3 = \frac{1}{2\sqrt{p'(\rho)}}(\rho p''(\rho) + 2p'(\rho)) > 0.$$

A shock wave in the first or third family connecting a given left-hand state U_0 to a right-hand state U must satisfy the **Lax shock inequalities** (see [12])

(2.4)
$$\lambda_i(U) < \bar{\lambda}_i(U, U_0) < \lambda_i(U_0), \quad i = 1, 3,$$

where $\bar{\lambda}_i(U, U_0)$ denotes the shock speed.

Shock waves from 1- and 3-families are constrained by the jump conditions (1.6). The so-called Hugoniot set is determined
$$(u - u_0)^2 = \kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^{\gamma} - \rho_0^{\gamma}).$$

We can distinguish between two families. Using the condition that the 1-Hugoniot curve issued from U_0 is tangent to the 1-eigenvector vector $r_1(U)$, we find

$$\mathcal{H}_1(U_0) : u = \begin{cases} u_0 - \left(\kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho > \rho_0, \\ u_0 + \left(\kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, & \rho < \rho_0. \end{cases}$$

Similar arguments lead to $\mathcal{H}_3(U_0)$. Using Lax shock inequalities (2.4), we can define the 1-wave and a 3-wave (left-to-right) shock curve $\mathcal{S}_1(U_0)$ and $\overrightarrow{\mathcal{S}}_3(U_0)$ consisting of all right-hand states U that can be connected to U_0 by a shock by:

(2.5)
$$\mathcal{S}_1(U_0) : \omega_1(\rho; U_0) := u_0 - \left(\kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, \quad \rho > \rho_0,$$

$$\overrightarrow{\mathcal{S}}_3(U_0) : \omega_3(\rho) := u_0 - \left(\kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, \quad \rho < \rho_0.$$

The condition $\rho > \rho_0$ for $\mathcal{S}_1(U_0)$ (respectively $\rho < \rho_0$ for $\overrightarrow{\mathcal{S}}_3(U_0)$) is derived from the Lax shock inequalities (2.4).

Similarly, the 1-wave and 3-wave (right-to-left) shock curve $\overleftarrow{\mathcal{S}}_1(U_0), \mathcal{S}_3(U_0)$ consisting of all left-hand states U that can be connected to U_0 by a Lax shock are :

(2.6)
$$\omega_1(\rho) := u_0 + \left(\kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, \quad \rho < \rho_0,$$

$$\omega_3(\rho) := u_0 + \left(\kappa(\frac{1}{\rho_0} - \frac{1}{\rho})(\rho^\gamma - \rho_0^\gamma)\right)^{1/2}, \quad \rho > \rho_0.$$

Next, we search for rarefaction waves, i.e., smooth self-similar solutions to the system (1.3). As usual, we consider for ordinary differential equations

(2.7)
$$\frac{dU}{d\xi} = \frac{r_i(U)}{\nabla \lambda_i \cdot r_i(U)}, \quad i = 1, 3,$$
$$U(\xi_0) = U_0.$$

Therefore, we can determine the 1-wave (left-to-right) rarefaction curve $\mathcal{R}_1(U_0)$ by

(2.8)
$$\omega_1(\rho) = u_0 - \int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho$$
$$= u_0 - \frac{2\sqrt{\kappa\gamma}}{\gamma - 1}(\rho^{(\gamma-1)/2} - \rho_0^{(\gamma-1)/2}), \quad \rho \leq \rho_0,$$
$$\rho^{\frac{\gamma-1}{2}}(\xi) = \rho_0^{\frac{\gamma-1}{2}} - \frac{\gamma-1}{\sqrt{\kappa\gamma}(\gamma+1)}(\xi - \xi_0), \quad \xi \geq \xi_0.$$

and the 3-wave (right-to-left) rarefaction curve $\mathcal{R}_3(U_0)$ by

(2.9)
$$\omega_3(\rho) = u_0 + \int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho, \quad \rho \leq \rho_0,$$
$$\rho^{\frac{\gamma-1}{2}}(\xi) = \rho_0^{\frac{\gamma-1}{2}} + \frac{\gamma-1}{\sqrt{\kappa\gamma}(\gamma+1)}(\xi - \xi_0), \quad \xi \leq \xi_0.$$

If we take $\xi \le \xi_0$ and $\rho \ge \rho_0$ in the formulas (2.8), (2.9) instead, we get the 1-wave (right-to-left) and 3-wave (left-to-right) rarefaction curve $\overleftarrow{\mathcal{R}}_1(U_0), \overrightarrow{\mathcal{R}}_3(U_0)$.

In conclusion, we can define the wave curves

$$
\begin{aligned}
\mathcal{W}_1(U_0) &:= \mathcal{S}_1(U_0) \cup \mathcal{R}_1(U_0), \\
\overleftarrow{\mathcal{W}}_1(U_0) &:= \overleftarrow{\mathcal{S}}_1(U_0) \cup \overleftarrow{\mathcal{R}}_1(U_0), \\
\mathcal{W}_3(U_0) &:= \mathcal{S}_3(U_0) \cup \mathcal{R}_3(U_0), \\
\overrightarrow{\mathcal{W}}_3(U_0) &:= \overrightarrow{\mathcal{S}}_3(U_0) \cup \overrightarrow{\mathcal{R}}_3(U_0).
\end{aligned}
$$
(2.10)

Let us investigate properties of the wave curves. A state U_0 being given we consider the wave curves in the (ρ, u)-plan issuing from U_0. It follows from (2.6) that

$$
(2.11a) \qquad \frac{d\omega_1(U_0, \rho)}{d\rho} = -\frac{\kappa\left[\frac{1}{\rho^2}(\rho^\gamma - \rho_0^\gamma) + (\frac{1}{\rho_0} - \frac{1}{\rho})\gamma\rho^{\gamma-1}\right]}{2|\omega_1(U_0, \rho) - u_0|} < 0, \quad \rho > \rho_0,
$$

and, clearly,

$$
(2.11b) \qquad \frac{d\omega_1(U_0, \rho)}{d\rho} = -\sqrt{\kappa\gamma}\rho^{\frac{\gamma-3}{2}} < 0, \quad \rho \le \rho_0.
$$

Similar calculations show that

$$
(2.12) \qquad \frac{d\omega_3(U_0, \rho)}{d\rho} > 0, \quad \rho > 0.
$$

From (2.11)-(2.12), we have the well-known property:

Lemma 2.1. (1- and 3-wave curves)

The wave curve $\mathcal{W}_1(U_0) : \rho \mapsto \omega_1(U_0, \rho), \rho > 0$ is strictly decreasing. The wave curve $\overrightarrow{\mathcal{W}}_3(U_0) : \rho \mapsto \omega_3(U_0, \rho), \rho > 0$ is strictly increasing.

As observed in Section 1, a stationary shock from a given state U_0 to some state U must satisfy

$$
\begin{aligned}
(2.13) \qquad & \mathcal{W}_2(U_0) : u = \omega_2(U_0, \rho) := \mathrm{sgn}(u_0)\left(u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1})\right)^{1/2}, \quad u_0 \ne 0, \\
& \Sigma(U_0, \rho; a) := \omega_2(U_0, \rho)\rho - \frac{a_0 u_0 \rho_0}{a} = 0.
\end{aligned}
$$

If $u_0 = 0$, then the equations (2.13) determine three points $(\rho_0, 0), (0, \pm\sqrt{(2\kappa\gamma)/(\gamma-1)}\rho_0^{(\gamma-1)/2})$. Assume $u_0 \ne 0$. The function $\rho \mapsto \omega_2(U_0, \rho)$ is defined provided the expression under the square root is non-negative:

$$
u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) \ge 0,
$$

which requires that

$$
(2.14) \qquad \rho \le \bar{\rho}(U_0) := \left(\frac{\gamma-1}{2\kappa\gamma}u_0^2 + \rho_0^{\gamma-1}\right)^{\frac{1}{\gamma-1}}.
$$

In the interval $[0, \bar{\rho}(U_0)]$, the function $\rho \mapsto \omega_2(U_0, \rho)$ is decreasing for $u_0 > 0$ and increasing for $u_0 < 0$. On the other hand,

$$
(2.15) \qquad \frac{\partial\Sigma(U_0, \rho; a)}{\partial\rho} = \frac{u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}}{\left(u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1})\right)^{1/2}}.
$$

Assume, for simplicity, that $u_0 > 0$. The last expression means that

$$
(2.16) \qquad
\begin{aligned}
\frac{\partial \Sigma(U_0, \rho; a)}{\partial \rho} &> 0, \quad \rho < \rho_{\max}(\rho_0, u_0), \\
\frac{\partial \Sigma(U_0, \rho; a)}{\partial \rho} &< 0, \quad \rho > \rho_{\max}(\rho_0, u_0),
\end{aligned}
$$

where

$$
(2.17) \qquad \rho_{\max}(\rho_0, u_0) := \Big(\frac{\gamma - 1}{\kappa \gamma (\gamma + 1)} u_0^2 + \frac{2}{\gamma + 1} \rho_0^{\gamma - 1} \Big)^{\frac{1}{\gamma - 1}}.
$$

The function $\rho \mapsto \Sigma(U_0, \rho; a)$ takes negative values at the endpoints. Thus, it admits some root if and only if the maximum value is non-negative. This is equivalent to saying that

$$
(2.18) \qquad a \geq a_{\min}(U_0) := \frac{a_0 \rho_0 |u_0|}{\sqrt{\kappa \gamma} \rho_{\max}^{\frac{\gamma + 1}{2}}(\rho_0, u_0)}.
$$

For $u_0 < 0$, the same properties hold provided we reverse the inequalities (2.16) and replace "negative" by "positive". Based on these observations, we have

Lemma 2.2. (Waves associated with the linearly degenerate field.)

Given U_0, a stationary shock issuing from U_0 and connecting to some state $U = (\rho, u, a)$ exists if and only if $a \geq a_{\min}(U_0)$. More precisely, we have :

(i) *If $a < a_{\min}(U_0)$, there are no stationary shocks.*

(ii) *If $a > a_{\min}(U_0)$, then there are exactly two values $\varphi_1(U_0, a) < \rho_{\max}(U_0) < \varphi_2(U_0, a)$ such that*

$$
(2.19) \qquad \Sigma(U_0, \varphi_1(U_0, a), a) = \Sigma(U_0, \varphi_2(U_0, a), a) = 0.
$$

Accordingly, along the curve $\mathcal{W}_2(U_0)$, there are two distinct points that can be attained from U_0 using a stationary shock.

(iii) *If $a = a_{\min}(U_0)$, then on the curve $\mathcal{W}_2(U_0)$ there is a unique point that can be attained from U_0 using a stationary shock.*

The following lemma compare together terms arising in the previous lemma.

Lemma 2.3.

i) We have

$$
(2.20) \qquad
\begin{aligned}
\rho_{\max}(\rho_0, u_0) &< \rho_0, \quad (\rho_0, u_0) \in \mathcal{G}_2, \\
\rho_{\max}(\rho_0, u_0) &> \rho_0, \quad (\rho_0, u_0) \in \mathcal{G}_3 \cup \mathcal{G}_1, \\
\rho_{\max}(\rho_0, u_0) &= \rho_0, \quad (\rho_0, u_0) \in \mathcal{C}_{\pm}.
\end{aligned}
$$

ii) After a stationary jump, the state $(\varphi_1(U_0, a), \omega_2(U_0, \varphi_1(U_0, a)))$ belongs to $\mathcal{G}_1$ if $u_0 < 0$, and belongs to $\mathcal{G}_3$ if $u_0 > 0$, while the state $(\varphi_2(U_0, a), \omega_2(U_0, \varphi_2(U_0, a)))$ always belongs to $\mathcal{G}_2$. Moreover, for $u_0 \neq 0$, we have

$$
(2.21) \qquad
\begin{aligned}
(\rho_{\max}(U_0, a), \omega_2(U_0, \rho_{\max}(U_0, a))) &\in \mathcal{C}^+ \quad if \quad u_0 > 0, \\
(\rho_{\max}(U_0, a), \omega_2(U_0, \rho_{\max}(U_0, a))) &\in \mathcal{C}^- \quad if \quad u_0 < 0.
\end{aligned}
$$

In addition, we have

  – *If $a > a_0$, then*

$$
(2.22a) \qquad \varphi_1(U_0, a) < \rho_0 < \varphi_2(U_0, a).
$$

  – *If $a < a_0$, then*

$$
(2.22b) \qquad
\begin{aligned}
\rho_0 &< \varphi_1(U_0, a) \quad for \quad U_0 \in \mathcal{G}_1 \cup \mathcal{G}_3, \\
\rho_0 &> \varphi_2(U_0, a) \quad for \quad U_0 \in \mathcal{G}_2.
\end{aligned}
$$

iii)

$$
\begin{aligned}
a_{\min}(U, a) &< a, \quad (\rho, u) \in \mathcal{G}_i, \ i = 1, 2, 3, \\
a_{\min}(U, a) &= a, \quad (\rho, u) \in C_{\pm}, \\
a_{\min}(U, a) &= 0, \quad \rho = 0 \quad or \quad u = 0.
\end{aligned}
$$

(2.23)

Proof. The inequalities (2.20) are straightforward. To demonstrate ii) and iii), we can assume for simplicity that $u_0 > 0$. Let us define the function

$$
\begin{aligned}
g(U_0, \rho) &= \omega_2^2(U_0, \rho) - \kappa\gamma\rho^{\gamma-1} \\
&= u_0^2 - \frac{2\kappa\gamma}{\gamma - 1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}.
\end{aligned}
$$

Then, a straightforward calculation gives

$$
g(U_0, \rho_{\max}(U_0)) = 0,
$$

which proves (2.21). On the other hand, since

$$
\frac{dg(U_0, \rho)}{d\rho} = -(\gamma + 1)\kappa\gamma\rho^{\gamma-2} < 0,
$$

and that $\varphi_1(U_0, a) < \rho_{\max}(U_0, a) < \varphi_2(U_0, a)$ it holds that

$$
g(U_0, \varphi_1(U_0, a)) > g(U_0, \rho_{\max}(U_0)) = 0 > g(U_0, \varphi_1(U_0, a)).
$$

The last two inequalities justify the statement in ii). Moreover,

$$
\Sigma(U_0, \rho_0; a) = \rho_0 u_0(1 - a_0/a) > 0 \quad \text{iff} \quad a > a_0,
$$

which proves (2.22a), and shows that ρ_0 is located outside of the interval $[\varphi_1(U_0, a), \varphi_2(U_0, a)]$ in the opposite case. It is then derived from (2.15) that

$$
\frac{\partial\Sigma(U_0, \rho_0; a)}{\partial\rho} = \frac{u_0^2 - \kappa\gamma\rho_0^{\gamma-1}}{u_0} < 0 \quad \text{iff} \quad U_0 \in \mathcal{G}_2,
$$

which, together with the earlier observation, implies (2.22b).

We next check (2.23) for $a = a_0$. It comes from the definition of $a_{\min}(U_0)$ that $a_{\min}(U_0) < a_0$ if and only if

$$
\sqrt{\kappa\gamma}\rho^{*\frac{\gamma+1}{2}} > \rho_0|u_0|,
$$

that can be equivalently written as

$$
l(m) := \frac{2}{\gamma + 1}m - (\kappa\gamma)^{\frac{1-\gamma}{\gamma+1}}m^{\frac{2}{\gamma+1}} + \frac{\gamma - 1}{\kappa\gamma(\gamma + 1)} > 0,
$$

where $m := \rho_0^{\gamma-1}/u_0^2$. Then, we can see that

(2.24) $l(1/\kappa\gamma) = 0,$

which, in particular shows that the second equation in (2.23) holds, since $(\rho_0, u_0) \in \mathcal{C}_{\pm}$ for $m = 1/\kappa\gamma$. Moreover,

$$
\frac{dl(m)}{dm} = \frac{2}{\gamma + 1}(1 - (\kappa\gamma m)^{\frac{1-\gamma}{\gamma+1}}),
$$

which is positive for $m > 1/\kappa\gamma$ and negative for $m < \kappa\gamma$. This together with (2.24) establish the first statement in (2.23). The third statement in (2.23) is straightforward. This completes the proof of Lemma 2.3. $\square$

Proposition 2.4. *i) The 1-shock speed $\bar{\lambda}_1(U_0, U)$, (for $\rho > \rho_0$) may change sign along the 1-shock curve $\mathcal{S}_1(U_0)$. More precisely, if $U_0 \in \mathcal{G}_1 \cup \mathcal{G}_2$ then $\bar{\lambda}_1(U_0, U)$ remains negative:*

$$\text{(2.25a)} \qquad\qquad \bar{\lambda}_1(U_0, U) < 0, \quad U \in \mathcal{S}_1(U_0).$$

If $U_0 \in \mathcal{G}_3$, then $\bar{\lambda}_1(U_0, U)$ vanishes once at some point $\tilde{U}_0$ corresponding to a value $\rho = \psi_1(U_0) > \rho_0$ on the 1-shock curve $\mathcal{S}_1(U_0) : \rho \mapsto U = (\rho, \omega_1(\rho; U_0))$, such that

$$\text{(2.25b)} \qquad \begin{aligned} &\bar{\lambda}_1(U_0, \tilde{U}_0) = 0, \\ &\bar{\lambda}_1(U_0, U) > 0, \quad \rho \in (\rho_0, \tilde{\rho}_0), \\ &\bar{\lambda}_1(U_0, U) < 0, \quad \rho \in (\tilde{\rho}_0, +\infty), \\ &\tilde{U}_0 \in \mathcal{G}_2^+. \end{aligned}$$

ii) The 3-shock speed $\bar{\lambda}_3(U_0, U)$, (for $\rho > \rho_0$) may change sign along the 3-shock backward curve $\mathcal{S}_3(U_0)$. More precisely, if $U_0 \in \mathcal{G}_2 \cup \mathcal{G}_3$ then $\bar{\lambda}_3(U_0, U)$ remains positive:

$$\text{(2.26a)} \qquad\qquad \bar{\lambda}_3(U_0, U) < 0, \quad U \in \mathcal{S}_3(U_0).$$

If $U_0 \in \mathcal{G}_1$, then $\bar{\lambda}_3(U_0, U)$ vanishes once at some point $\tilde{U}_0$ corresponding to a value $\rho = \psi_2(U_0) > \rho_0$ on the 3-shock backward curve $\mathcal{S}_3(U_0) : \rho \mapsto U = (\rho, \omega_3(\rho; U_0))$, such that

$$\text{(2.26b)} \qquad \begin{aligned} &\bar{\lambda}_3(U_0, \tilde{U}_0) = 0, \\ &\bar{\lambda}_3(U_0, U) < 0, \quad \rho \in (\rho_0, \tilde{\rho}_0), \\ &\bar{\lambda}_3(U_0, U) > 0, \quad \rho \in (\tilde{\rho}_0, +\infty), \\ &\tilde{U}_0 \in \mathcal{G}_2^-. \end{aligned}$$

Proof. We only prove i), since similar arguments can be used for ii). From (1.6) and (2.6) we deduced that

$$\text{(2.27)} \qquad \bar{\lambda}_1(U_0, U) = \frac{\rho \omega_1(U_0, \rho) - \rho_0 u_0}{\rho - \rho_0} = u_0 - \left(\kappa \frac{\rho}{\rho_0} \frac{\rho^\gamma - \rho_0^\gamma}{\rho - \rho_0} \right)^{1/2}.$$

Thus, if $U_0 \in \mathcal{G}_1 \cup \mathcal{C}^-$, then $u_0 < 0$ and $\bar{\lambda}_1(U_0, U) < 0$. If $U_0 \in \mathcal{G}_2$, then, since the function $\rho \mapsto \rho^\gamma$ is convex and $\rho > \rho_0$, we deduce from (2.21) that

$$\bar{\lambda}_1(U_0, U) < u_0 - \sqrt{\kappa \gamma} \rho_0^{\frac{\gamma-1}{2}} < 0,$$

where the last inequality holds by $U_0 \in \mathcal{G}_2$. Letting $U_0 \to \mathcal{C}^+$ in the last inequality, we obtain

$$\bar{\lambda}_1(U_0, U) < 0, \quad U \neq U_0, U_0 \in \mathcal{C}^+.$$

Thus, (2.25a) is established.

Assume now that $U_0 \in \mathcal{G}_3$. Set

$$\text{(2.28)} \qquad l(\rho) := \kappa \frac{\rho}{\rho_0} \frac{\rho^\gamma - \rho_0^\gamma}{\rho - \rho_0} - u_0^2 = 0, \rho \geq \rho_0.$$

Observe that the function $l(\rho)$ and $\bar{\lambda}_1(U_0, U)$ have the same roots and take values of opposite signs. Then, a straightforward calculation shows that

$$\begin{aligned} &l(\rho_0) = \kappa \gamma \rho_0^{\gamma-1} - u_0^2 < 0, \quad l(+\infty) = +\infty, \\ &l'(\rho) = \kappa \frac{(\gamma-1)\rho^\gamma + \rho_0^\gamma}{\rho_0(\rho - \rho_0)} > 0, \end{aligned}$$

which implies the existence of the value $\psi_1(U_0)$ as indicated in (2.25b). Moreover, the three statements in the beginning of (2.25b) are satisfied. In view of the jump relation

$$[\rho u] = 0,$$

for a Lax shock with zero speed, the value $\omega_1(U_0, \psi_1(U_0))$ must be positive. The fact that the function $\rho \mapsto \rho^\gamma$ is convex and that the function ω_1 is decreasing in ρ yields

$$
\begin{aligned}
0 = \bar{\lambda}_1(U_0, \tilde{U}_0) &= u_0 - \left(\kappa \frac{\rho}{\rho_0} \frac{\rho^\gamma - \rho_0^\gamma}{\rho - \rho_0}\right)^{1/2} \\
&> u_0 - \sqrt{\kappa\gamma}\psi_1(U_0)^{\frac{\gamma-1}{2}} \\
&> \omega_1(U_0, \psi_1(U_0)) - \sqrt{\kappa\gamma}\psi_1(U_0)^{\frac{\gamma-1}{2}} \\
&= \bar{\lambda}_1(\psi_1(U_0), \omega_1(U_0, \psi_1(U_0))),
\end{aligned}
$$

which, since $\omega_1(U_0, \psi_1(U_0)) > 0$, proves the last statement of (2.25b). This completes the proof of Proposition 2.4. $\square$

Let us introduce two other curves in the (ρ, u)-plane:

$$
\text{(2.29)} \qquad
\begin{aligned}
\mathcal{C}_* &: u_*(\rho) = -\frac{2\sqrt{\kappa\gamma}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}, \\
\mathcal{C}^* &: u^*(\rho) = \frac{2\sqrt{\kappa\gamma}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}},
\end{aligned}
$$

Formally, the curve $\mathcal{C}_*$ is given by $\mathcal{R}_1(\rho_0 = 0, u_0 = 0)$, and the curve $\mathcal{C}^*$ is given by $\overrightarrow{\mathcal{R}}_3(\rho_0 = 0, u_0 = 0)$. At this stage, we have:

Lemma 2.5. *i) If U_0 is located above the curve $\mathcal{C}_*$, then the curve $\mathcal{W}_1(U_0)$ has a unique intersection with each curve $\mathcal{C}_\pm$, $\mathcal{C}_*$ and $\mathcal{C}^*$. If U_0 is located below $\mathcal{C}_*$, no such an intersection is available.*

ii) If U_0 is located below or on the curve $\mathcal{C}^*$, then the curve $\mathcal{W}_3(U_0)$ has a unique intersection with each curve $\mathcal{C}_\pm$, $\mathcal{C}_*$ and $\mathcal{C}^*$. If U_0 is located above $\mathcal{C}^*$, no such an intersection is available.

3. Two-parameter sets of composite waves

Solution curves of the Riemann problem for (1.1) are understood to be either wave curves that were already defined in the previous section or sets of composite waves $\mathcal{W}_{1,2}(U_0, a), \overleftarrow{\mathcal{W}}_{1,2}(U_0, a)$ (for 1- and 2-wave families) or $\mathcal{W}_{2,3}(U_0, a), \overrightarrow{\mathcal{W}}_{2,3}(U_0, a)$ (for 2- and 3-wave families). For simplicity, we restrict our attention in this section to the situation that there is only one stationary shock in each composite wave. Two stationary shocks in a composite wave is also possible and this will be discussed later in Section 5.

Definition 3.1. The composite curve $\mathcal{W}_{1\rightarrow 2}^i(U_0, a)$ is the set of all states $U = (\rho, u, a)$ such that there exists a state $U' \in \mathcal{W}_1(U_0)$ and the 1-wave from U_0 to U' can be followed by the stationary shock from U' to U by using $\varphi_i(U', a)$, $i = 1, 2$. The composite curve $\mathcal{W}_{2\rightarrow 1}^i(U_0, a)$ is the set of all the states $U = (\rho, u, a)$ such that the (fixed) stationary jump from U_0 to some U' (using $\varphi_i(U_0, a)$) can be followed by the 1-waves from U' to U, $i = 1, 2$.

Under the transformation

$$\text{(3.1)} \qquad x \mapsto -x, \quad u \mapsto -u,$$

a right-hand state $U = (\rho, u, a)$ becomes a left-hand state of the form $(\rho, -u, a)$. Therefore, without loss of generality, we can always assume from now on that

$$\text{(3.2)} \qquad a_0 < a.$$
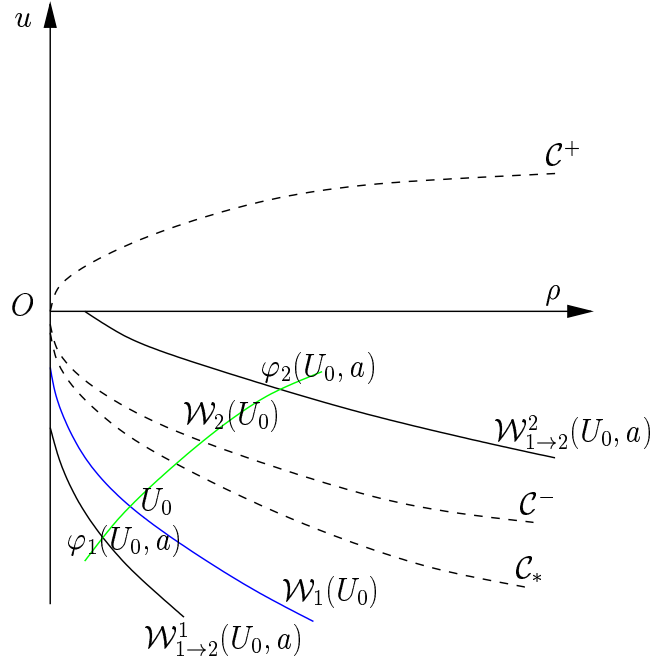
Figure 3.1: U_0 below C_*

We need only construct $\mathcal{W}_{1,2}(U_0, a)$, as similar arguments can be used for other cases. Set

$$(3.3) \qquad U_{\pm} := \mathcal{W}_1(U_0) \cap \mathcal{C}_{\pm}, \quad \mathcal{Z}(U_0) = \mathcal{W}_1(U_0) \cap \{u = 0\}.$$

First, assume that U_0 is below or on the curve $\mathcal{C}_*$ (Figure 3.1). The curve $\mathcal{W}_1(U_0)$ always remains in $\mathcal{G}_1$ and does not cross the strict hyperbolicity boundary $\mathcal{C}_{\pm}$. Therefore, all the 1-waves have negative speeds. Consequently, the 2-waves can not be followed by 1-waves. The only way is that the 1-waves are followed by 2-waves. Thus, we have in this case

$$(3.4) \qquad \mathcal{W}_{1,2}(U_0, a) \supset \mathcal{W}_{1\to2}^1(U_0, a) \cup W_{1\to2}^2(U_0, a).$$

Second, U_0 is between C_* and $\mathcal{C}^+$ or on the curve $\mathcal{C}^+$ (Figure 3.2).

The construction can be a 1-wave from U_0 to $U = (\rho, u) \in \mathcal{W}_1(U_0)$ as long as U do not belong to $\mathcal{G}_3$, followed by a stationary jump by either using $\varphi_1(U_0, a)$ to a state U_1 with

$$(3.5) \qquad \begin{aligned} U_1 &\in \mathcal{G}_1 \quad \text{if} \quad u \leq 0, \\ U_1 &\in \mathcal{G}_3 \quad \text{if} \quad u \geq 0, \end{aligned}$$

or using $\varphi_2(U_0, a)$ to a state $U_2 \in \mathcal{G}_2$. Such states U_2 form $\mathcal{W}_{1\to2}^2(U_0, a)$. The states $U_1 \in \mathcal{G}_1$ form the set $\mathcal{W}_{1\to2}^1(U_0, a)$. In the case $U_1 \in \mathcal{G}_3$ reached by a stationary jump from $U \in \bar{\mathcal{G}}_2^+$, we have

$$(3.6) \qquad \lambda_1(U_1) > 0,$$
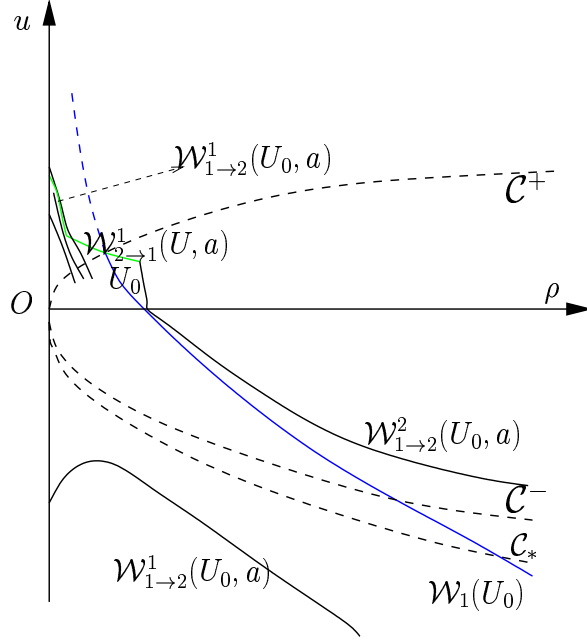
Figure 3.2: U_0 between C_+ and C_*

the construction can therefore be continued with $\mathcal{W}_1(U_1)$ as long as the 1-shock speed from U_1 is non-negative. This is $\mathcal{W}^1_{2\to1}(U, a)$ with any U belongs to $\mathcal{W}_1(U_0)$ between U_+ and $Z(U_0)$. Hence, we have in this case two curves and a one-parameter family of solutions described by

$$(3.7) \qquad \begin{aligned} \mathcal{W}_{1,2}(U_0, a) &\supset \mathcal{W}^1_{1\to2}(U_0, a) \cup W^2_{1\to2}(U_0, a) \cup \mathcal{W}^1_{2\to1}(U, a) \\ U &\in \mathcal{W}_1(U_0), \quad U \quad \text{between}\, U_+ \,\text{and}\, \mathcal{Z}(U_0). \end{aligned}$$

Finally, assume that U_0 is above $\mathcal{C}^+$, i.e, $U_0 \in \mathcal{G}_3$ (Figure 3.3).

In a neighborhood of U_0, the shock speed and the characteristic speed are positive, so that stationary shocks can be followed by 1-waves, only: using $\varphi_1(U_0, a)$, the solution can begin with a stationary shock to a state $U_1 \in \mathcal{G}_3$, followed by using $\mathcal{W}_1(U_1)$ as long as the shock speed is non-negative, i.e.,

$$(3.8) \qquad U \in \mathcal{W}_1(U_1) \,:\, \rho \le \psi_1(U_1).$$

This is the curve $\mathcal{W}^1_{2\to1}(U_0, a)$. Clearly, this curve crosses the curve $\mathcal{C}^+$. If the solution jumps by a stationary shock using $\varphi_2(U_0, a)$ to a state U_2, then we know by Lemma 2.2 that $U_2 \in \mathcal{G}_2$ in which 1-wave speeds are always negative so that 1-waves can not follow. In other words, $\mathcal{W}^2_{2\to1}(U_0, a)$ is empty.

On the other hand, from U_0, the construction can begin with a non-positive shock from U_0 to some state U with $\rho \ge \psi_1(U_0)$, followed by a stationary shock using either $\varphi_1(U, a)$ to a state U_1 with

$$\begin{aligned} U_1 &\in \mathcal{G}_1 \quad \text{if} \quad u \le 0, \\ U_1 &\in \mathcal{G}_3 \quad \text{if} \quad u \ge 0, \end{aligned}$$
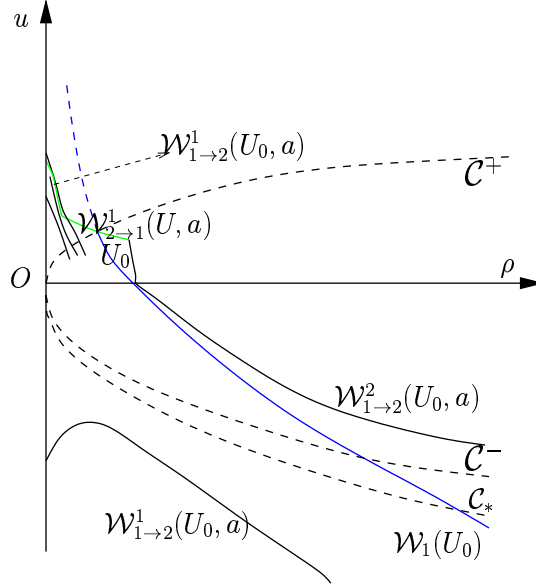
Figure 3.3: U_0 above $\mathcal{C}^+$

or using $\varphi_2(U,a)$ to a state $U_2 \in \mathcal{G}_2$, These two ways determine two composite curves $\mathcal{W}^1_{1\to2}(U_0,a), W^2_{1\to2}(U_0,a)$. When using $\varphi_1(U,a)$ to attain a point $U_1 \in \mathcal{G}_3$, similarly, by virtue of (3.6) the construction can be continued by using $\mathcal{W}_1(U_1)$ as much as 1-shock speeds from U_1 are non-negative. In this case, the composite wave set consists of three curves and a one-parameter family of solutions:

(3.9)
$$\mathcal{W}_{1,2}(U_0,a) \supset \mathcal{W}^1_{1\to2}(U_0,a) \cup \mathcal{W}^2_{1\to2}(U_0,a) \cup \mathcal{W}^1_{2\to1}(U_0,a) \cup \mathcal{W}^1_{2\to1}(U,a)$$
$$U \in \mathcal{W}_1(U_0), \quad U \quad \text{between } \tilde{U}_0 \text{ and } \mathcal{Z}(U_0),$$

where $\tilde{U}_0 := (\psi_1(U_0), \omega_1(U_0, \psi_1(U_0)))$ the point at which the 1-shock from U_0 has zero speed. The construction is complete.

Obviously, the curve $\mathcal{W}^1_{2\to1}(U_0,a)$ is monotone, as a part of $\mathcal{W}_1(U_1)$ for some U_1. Moreover, it is derived from the above construction that the two curves $\mathcal{W}^1_{1\to2}(U_0,a), \mathcal{W}^2_{1\to2}(U_0,a)$ can be parameterized by ρ:

(3.10)
$$\rho \mapsto (\varphi_i(\rho, \omega_1(U_0; \rho); a), \omega_2((\rho, \omega_1(U_0; \rho)); \varphi_i(\rho, \omega_1(U_0; \rho), a))), \quad i = 1, 2,$$

where $\varphi_i(\rho, \omega_1(U_0; \rho))$ is defined by the implicit equation

(3.11)
$$\Sigma((\rho, \omega_1(U_0; \rho)), \varphi_i(\rho, \omega_1(U_0; \rho)); a) = 0.$$

We will show that parts of these two curves are monotone. These parts are sufficient to the construction of Riemann solutions in the next section.

Lemma 3.1. *The u-component of the curve $W^2_{1,2}(U_0,a)$ is a decreasing function of ρ. The u-component of the curve $\mathcal{W}^1_{1\to2}(U_0,a)$, as a function of ρ, is increasing for $\rho < \rho^*$ and decreasing for $\rho > \rho^*$, where the point $(\rho^*, \omega_1(U_0, \rho^*)) \in \mathcal{G}_2$.*

Proof. Without loss of generality we can assume that $u_0 > 0$. For simplicity we may omit U_0 in (3.10). Obviously, for both components u of the two curves we have

$$(3.12) \qquad u^2(\rho) = \omega_1^2(\rho) - \frac{2\kappa\gamma}{\gamma - 1}\Big[\Big(\frac{a_0\omega_1\rho}{au}\Big)^{\gamma-1} - \rho^{\gamma-1}\Big].$$

Taking the derivative with respect to ρ on both sides we find

$$uu' = \omega_1\omega_1' - \kappa\gamma\Big[\Big(\frac{a_0\omega_1\rho}{a\omega_2}\Big)^{\gamma-2}\frac{a_0}{a}\frac{(\omega_1 + \omega_1'\rho)\omega_2 - \omega_2'\omega_1\rho}{\omega_2^2} - \rho^{\gamma-2}\Big]$$

$$= \omega_1\omega_1' - \kappa\gamma\Big(\varphi_i^{\gamma-1}\frac{(\omega_1 + \omega_1'\rho)\omega_2 - \omega_2'\omega_1\rho}{\omega_1\omega_2\rho} - \rho^{\gamma-2}\Big).$$

Combining together terms having ω_2' of the last equality, we obtain

$$(3.13) \qquad (u^2 - \kappa\gamma\varphi_i^{\gamma-1})\frac{u'}{u} = \omega_1\omega_1' + \kappa\gamma\rho^{\gamma-2} - \frac{\kappa\gamma\varphi_i^{\gamma-1}(\omega_1 + \omega_1'\rho)}{\omega_1\rho}.$$

Observe that the first factor on the left-hand side of (3.13)

$$(3.14) \qquad \begin{aligned} u^2(\rho) - \kappa\gamma\varphi_i(\rho)^{\gamma-1} &> 0 \quad \text{for } i = 1, \\ u^2(\rho) - \kappa\gamma\varphi_i(\rho)^{\gamma-1} &< 0 \quad \text{for } i = 2. \end{aligned}$$

For the right-hand side we proceed as follows. For U on the curve $\mathcal{W}_1(U_0)$ we have

$$(3.15) \qquad \omega_1\omega_1' + \kappa\gamma\rho^{\gamma-2} - \frac{\kappa\gamma\varphi_i^{\gamma-1}(\omega_1 + \omega_1'\rho)}{\omega_1\rho} = \frac{\omega_1'}{\omega_1}(\omega_1^2 - \kappa\gamma\varphi_i^{\gamma-1}) + \frac{\kappa\gamma}{\rho}(\rho^{\gamma-1} - \varphi_i^{\gamma-1}) < 0,$$

which is

- positive from U_- down to infinity for $i = 1$,
- but negative between U_0 and U_- for $i = 2$.

From now on we need only consider the case $i = 2$ as the other case is similar. Therefore we just consider the part $\rho > \rho_-$ in which

$$(3.16) \qquad (\rho, \omega_1(\rho)) \in \mathcal{G}_1.$$

The right-hand side of (3.13) is estimated as

$$(3.17) \qquad \begin{aligned} \omega_1\omega_1' + \kappa\gamma\rho^{\gamma-2} - \frac{\kappa\gamma\varphi_i^{\gamma-1}(\omega_1 + \omega_1'\rho)}{\omega_1\rho} &< \omega_1\omega_1' + \omega_1^2/\rho - \frac{\kappa\gamma\varphi_i^{\gamma-1}(\omega_1 + \omega_1'\rho)}{\omega_1\rho} \\ &\le \frac{\omega_1 + \omega_1'\rho}{\omega_1\rho}(\omega_1^2 - \kappa\gamma\varphi_i^{\gamma-1}). \end{aligned}$$

The first factor on the right-hand side of (3.17) is positive. We will show that the second factor is negative. Actually,

$$(3.18) \qquad \omega_1^2 - \kappa\gamma\varphi_i^{\gamma-1} = \frac{\gamma - 1}{2}(u^2 - \omega_1^2) + \omega_1^2 - \kappa\gamma\rho^{\gamma-1} := l(\rho).$$

Taking the derivative of the function $l(\rho)$ we obtain

$$l'(\rho) = (\gamma - 1)(uu' - \omega_1\omega_1') + 2\omega_1\omega_1' - \kappa\gamma(\gamma - 1)\rho^{\gamma-2},$$

or

$$(3.19) \quad \begin{aligned} \frac{2}{\gamma-1} l'(\rho) &= \frac{\kappa\omega_1}{u_0 - \omega_1}\Big[\frac{\rho^\gamma - \rho_0^\gamma}{\rho^2} + \gamma\rho^{\gamma-1}\big(\frac{1}{\rho_0} - \frac{1}{\rho}\big)\Big] - \kappa\gamma\rho^{\gamma-2} \\ &= \frac{\kappa\omega_1(\rho - \rho_0)}{(u_0 - \omega_1)\rho^2}\Big(\frac{\rho^\gamma - \rho_0^\gamma}{\rho - \rho_0} + \frac{\gamma\rho^\gamma}{\rho_0}\Big) - \kappa\gamma\rho^{\gamma-2} \\ &\leq \frac{2\kappa\gamma\rho_0^{\gamma-3}\omega_1(\rho - \rho_0)}{u_0 - \omega_1} - \kappa\gamma\rho^{\gamma-2} < 0, \quad \rho > \rho_0, \end{aligned}$$

and tends to $-\infty$ as $\rho \to +\infty$. The fact that the right-hand side of (3.13) takes a negative value at U_- and the inequality (3.19) implies that the right-hand side of (3.13) is always negative. It is then derived from (3.14) that the function defined in (3.11) is decreasing for $u < 0$. It is similar to prove the case $u > 0$. The proof of Lemma 3.1 is complete. $\square$

Lemma 3.2. *The ρ-component of the curve $\mathcal{W}_{1\to2}^2(U_0, a)$ is increasing in ρ at least when $(\rho, \omega_1(\rho))$ remains in $\mathcal{G}_2$.*

Proof. Let the function u be defined as in (3.12). Taking the derivative in ρ of the identity

$$u(\rho)\varphi_2(\rho) = a_0\omega_1(\rho)\rho/a,$$

we obtain

$$\varphi_2' u + u'\varphi_2 = a_0(\omega_1 + \omega_1'\rho)/a,$$

or

$$\varphi_2'(\rho) = -\varphi_2 u'/u + a_0(\omega_1 + \omega_1'\rho)/a.$$

Replacing u'/u from (3.13) in the last equality and after rearranging the terms, we obtain

$$(3.20) \qquad\qquad \varphi_2'(\rho) = \varphi_2(\rho)\Big(\frac{\omega_1 + \omega_1'\rho}{\omega_1\rho} - \frac{u'}{u}\Big).$$

As long as $(\rho, \omega_1(\rho))$ belongs to $\mathcal{G}_2$, it follows from (3.13) that

$$(u^2 - \kappa\gamma\varphi_i^{\gamma-1})\frac{u'}{u} = \omega_1\omega_1' + \kappa\gamma\rho^{\gamma-2} - \frac{\kappa\gamma\varphi_i^{\gamma-1}(\omega_1 + \omega_1'\rho)}{\omega_1\rho}$$

$$\omega_1\omega_1' + \omega_1^2/\rho - \frac{\kappa\gamma\varphi_i^{\gamma-1}(\omega_1 + \omega_1'\rho)}{\omega_1\rho}$$

$$= \frac{\omega_1 + \omega_1'\rho}{\omega_1\rho}(\omega_1^2 - \kappa\gamma\varphi_i^{\gamma-1}).$$

Thus,

$$\frac{\omega_1 + \omega_1'\rho}{\omega_1\rho} \geq \frac{u'}{u}\frac{u^2 - \kappa\gamma\varphi_i^{\gamma-1}}{\omega_1^2 - \kappa\gamma\varphi_i^{\gamma-1}} > \frac{u'}{u},$$

so that (3.20) yields

$$\varphi_2'(\rho) > 0.$$

This completes the proof of Lemma 3.2.

$\square$

Corollary 3.3. *The curve $\mathcal{W}^2_{1\to2}(U_0, a)$ can be parameterized by expressing the ρ-component as a function of the u-component, i.e.,*

$$\rho = \rho(u).$$

Moreover, this function is monotone decreasing, at least for $(\rho, \omega_1(\rho)) \in \mathcal{G}_2$.

Proof. As seen by Lemma 3.1, the component $u = u(\rho)$ of $\mathcal{W}^2_{1\to2}(U_0, a)$ is a decreasing function of ρ. So, it admits in inverse

$$\rho = u^{-1}(u).$$

Using this identity into the expression of the component ρ in (3.10), we get the component

$$\rho = \rho(u).$$

Lemmas 3.1 and 3.2 yield the desired monotonicity property.

□

4. The monotonicity criterion : Solutions with one stationary wave

As seen in the previous section that the Riemann problem for (1.1) may admit up to a one-parameter family of solutions. This phenomenon can be avoided by requiring Riemann solutions to satisfy a monotone condition on the component a.

Monotonicity Criterion.

   (a) Along any stationary curve $\mathcal{W}_2(U_0)$, the cross-section area a is monotone as a function of ρ.
   (b) The total variation of the cross-section component of any Riemann solution must not exceed (and, therefore, is equal to) $|a_l - a_r|$, where a_l, a_r are left-hand and right-hand cross-section levels.

A similar criterion was used by Isaacson and Temple [9, 10] and by Goatin and LeFloch [6].

In this section we will consider Riemann solutions containing *one stationary wave only*. Consequently, only the requirement (a) of the a-monotone criterion affects on the construction of solutions. The requirement (b) will be taken into account for solutions containing two stationary waves, which will be discussed in the next section. Under our monotonicity criterion, we will show that there exist *two solutions containing one stationary wave at most*.

As before, for definiteness we assume in this section that

$$(4.1) \qquad\qquad\qquad\qquad\qquad a_l < a_r.$$

Moreover, the notation

$$W_i(U_1, U_2) \oplus W_j(U_2, U_3),$$

will be used when the i-wave connecting some state U_1 to some state U_2 is followed by a j-wave connecting U_2 to U_3 (here $i, j = 1, 2, 3$).

Lemma 4.1. *The Monononicity Criterion (a) is equivalent to saying that any stationary shock does not cross the boundary of strict hyperbolicity. In other words:*

   (i) *If $U_0 \in \mathcal{G}_1 \cup \mathcal{G}_3$, then only the stationary shock based on the value $\varphi_1(U_0, a)$ is allowed.*
   (ii) *If $U_0 \in \mathcal{G}_2$, then only the stationary shock using $\varphi_2(U_0, a)$ is allowed.*

Proof. Taking the derivative with respect to ρ in the identity

$$a^2(\omega_2\rho)^2 = (a_0 u_0 \rho_0)^2,$$

we get

$$(4.2) \qquad\qquad a(\rho)a'(\rho)(\omega_2\rho)^2 + 2a^2(\omega_2\rho)(\omega_2'\rho + \omega_2) = 0.$$

To demonstrate the lemma, it is sufficient to show that the last factor of the second term in (4.2) remains of a constant sign whenever the curve $\mathcal{W}_2(U_0)$ does not cross the boundary of strict hyperbolicity $\mathcal{C}_\pm$. Indeed, assume for simplicity that $u_0 > 0$, then

$$\omega_2'(\rho)\rho + \omega_2 = \frac{-\kappa\gamma\rho^{\gamma-1}}{\omega_2} + \omega_2$$
$$= \frac{\omega_2^2 - \kappa\gamma\rho^{\gamma-1}}{\omega_2},$$

which remains of a constant sign if and only if $\mathcal{W}_2(U_0)$ does not cross $\mathcal{C}_\pm$. This completes the proof of Lemma 4.1.

$\square$
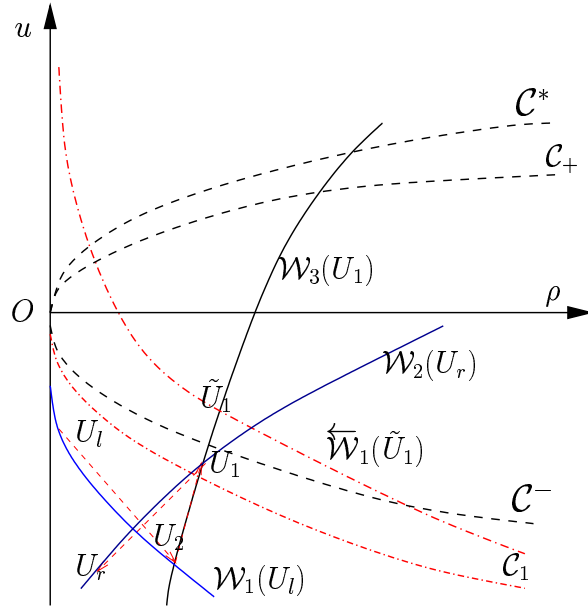
Figure 4.1: U_r below or on $\mathcal{C}_1$

Let us denote by $\mathcal{W}_{1\to2}^{i,a}(U_0)$ the part of $\mathcal{W}_{1\to2}^i(U_0, a)$, $i = 1, 2$, which is admissible for Monotonicity Criterion. In view of Lemma 4.1 we have

(4.3)
$$\mathcal{W}_{1\to2}^{1,a}(U_0) = \{U \in \mathcal{W}_{1\to2}^1(U_0, a), \quad \rho \geq \rho_-\},$$
$$\mathcal{W}_{1\to2}^{2,a}(U_0) = \{U \in \mathcal{W}_{1\to2}^2(U_0, a), \quad \rho \leq \rho_-\},$$

where ρ_- is the value at which the curve $\mathcal{W}_1(U_0)$ intersects the boundary $\mathcal{C}^-$.

Lemma 4.2. *When U describes the curve $\mathcal{W}_3(U_0), U_0 \in \mathcal{G}_1$ from the boundary $\mathcal{C}^-$ to the horizontal axis $\{u = 0\}$, the critical value $a_{\min}(U)$ decreases to zero.*

Given left-hand and right-hand states $U_l = (\rho_l, u_l, a_l), U_r = (\rho_r, u_r, a_r)$, we now discuss the construction algorithm for the corresponding Riemann solution. We need some further notation. First, define the two curves $\mathcal{C}_1, \mathcal{C}_2$ made of states reached by stationary jumps from points on $\mathcal{C}^-$ when the variable a jumps from a_l to a_r. Here we use $\varphi_1(U(a = a_l), a_r)$ and $\varphi_2(U(a = a_l), a_r)$, $U \in \mathcal{C}^-$, respectively. Obviously, we have

$$\mathcal{C}_1 \in \mathcal{G}_1, \quad \mathcal{C}_2 \in \mathcal{G}_2.$$

Lemma 4.3. *The curves $\mathcal{C}_1, \mathcal{C}_2$ can be parameterized so that the u-component is decreasing as a function of the ρ-component.*

Second, it is convenient to introduce the two points at which the curve $\mathcal{W}_1(U_l)$ crosses the boundary of strict hyperbolicity:

(4.4) $\{U_\pm\} := \mathcal{W}_1(U_l) \cap \mathcal{C}_\pm.$

In view of Proposition 2.4, for each $U \in \mathcal{C}_2$, there exists a point $\tilde{U} \in \mathcal{G}_1$ such that the 3-shock speed vanishes:

$$\bar{\lambda}_3(U, \tilde{U}) = 0.$$

Such states $\tilde{U}$ where $U \in \mathcal{C}_2$ form a curve denoted by $\tilde{\mathcal{C}}_2$.
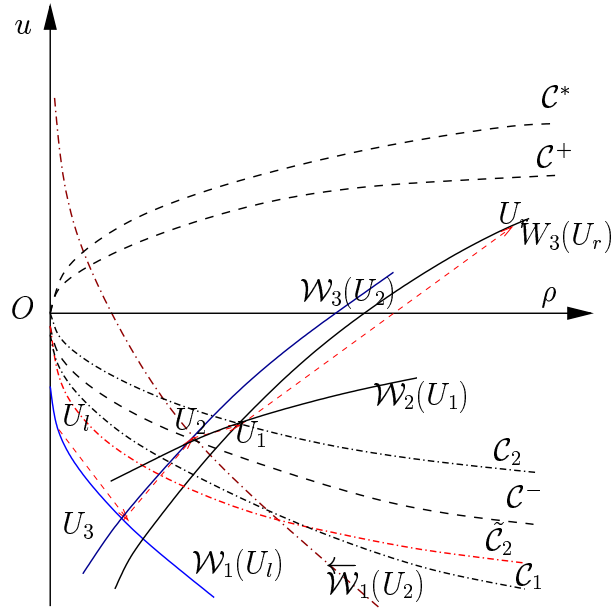
Figure 4.2: U_r between or on $\tilde{\mathcal{C}}_2, \mathcal{C}^*$

Construction 1. U_r *is below or on the curve* $\mathcal{C}_1$ *(Figure 4.1).*

The solution may have a (right-to-left) jump from U_r to $U_1 \in \mathcal{G}_1 \cup \mathcal{C}^-$ by $\varphi_1(U_r, a_l)$. In view of Proposition 2.4, there exists a point $\tilde{U}_1 \in \mathcal{W}_3(U_1)$ at which the 3-shock speed vanishes. Then, define U_2 by

$$\{U_2\} = \mathcal{W}_1(U_l) \cap \mathcal{W}_3(U_1).$$

The solution can be

(4.5)
$$W_1(U_l, U_2) \oplus W_3(U_2, U_1) \oplus W_2(U_1, U_r),$$

which makes sense if and only if the 3-shock speed $\bar{\lambda}_3(U_2, U_1)$ is non-positive. In other words, U_l is below or on the curve $\overleftarrow{\mathcal{W}}_1(\tilde{U}_1)$.
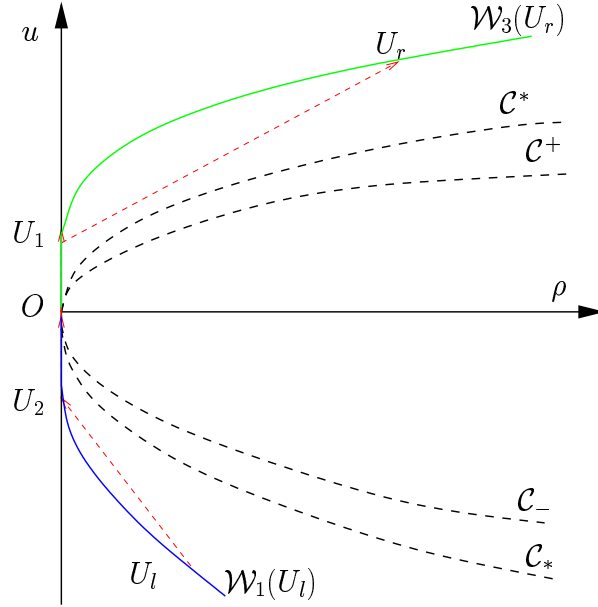
Figure 4.3: U_l below $\mathcal{C}_*$, U_r above $\mathcal{C}^*$

Construction 2. U_r *is between or on the two curves* $\tilde{\mathcal{C}}_2, \mathcal{C}^*$ *(Figure 4.2).*

Let
$$\{U_1\} = \mathcal{W}_3(U_r) \cap \mathcal{C}_2.$$

Since $U_1 \in \mathcal{C}_2$, by definition, there exists a point $U_2 \in \mathcal{C}^-$ which can be attained from U_1 by using a stationary shock. Let
$$\{U_3\} = \mathcal{W}_1(U_l) \cap \mathcal{W}_3(U_2).$$

The solution is then

(4.6) $W_1(U_l, U_3) \oplus W_3(U_3, U_2) \oplus W_2(U_2, U_1) \oplus W_3(U_1, U_r),$
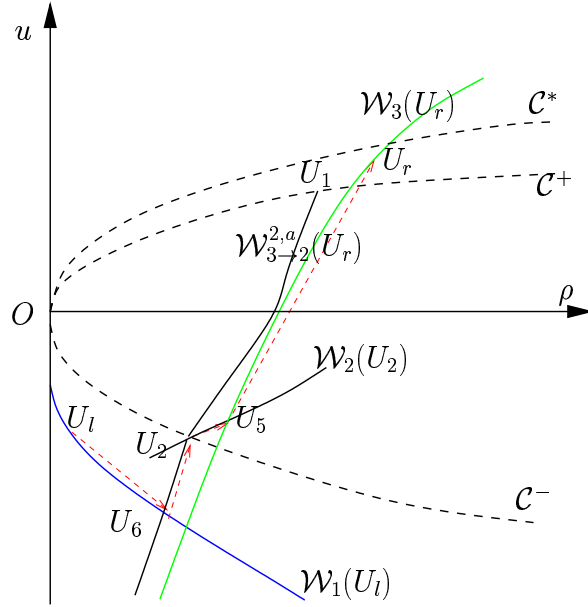
provided U_2 is above or on the curve $\mathcal{W}_1(U_l)$. In other words, U_l is below or on the curve $\overleftarrow{\mathcal{W}}_1(U_2)$.

Construction 3. U_l is below or on the curve $\mathcal{C}_*$, and U_r is above or on the curve $\mathcal{C}^*$ (Figure 4.3).

Set
$$\{U_1\} = \mathcal{W}_3(U_r) \cap \{\rho = 0\},$$
$$\{U_2\} = \overleftarrow{\mathcal{W}}_1(U_l) \cap \{\rho = 0\}.$$

The solution is then

(4.7) $R_1(U_l, U_2) \oplus R_1(U_2, (O, a_l)) \oplus W_2((O, a_l), (O, a_r)) \oplus R_3((O, a_r), U_1) \oplus R_3(U_1, U_r).$
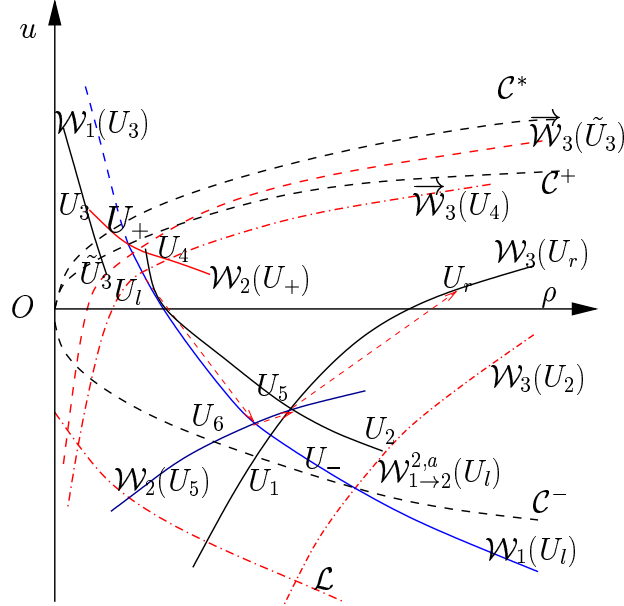
Figure 4.4: U_r between $W_3(U_2), W_3(U_4)$

Construction 4. U_l is below or on $\mathcal{C}_+$.

At U_-, the solution can jump by a stationary wave using either $\varphi_1(U_-, a_r)$ to $U_1 \in \mathcal{G}_1$, or $\varphi_2(U_-, a_r)$ to a state $U_2 \in \mathcal{G}_2$. At U_+, the solution can jump by a stationary wave using either $\varphi_1(U_+, a_r)$ to $U_3 \in \mathcal{G}_3$, or $\varphi_2(U_+, a_r)$ to a state $U_4 \in \mathcal{G}_2$. It is easy to see that U_i is an endpoint of $\mathcal{W}_{1 \to 2}^{i,a}(U_l, a_r)$, $i = 1, 2$.

On the other hand, by virtue of Proposition 2.4, from any point $U \in \mathcal{W}_{1 \to 2}^{2,a}(U_l, a_r)$, the exists a point $\tilde{U} \in \vec{\mathcal{W}}_3(U) \cap \mathcal{G}_1$ such that the 3-shock speed from U to $\tilde{U}$ vanishes:

$$\bar{\lambda}_3(U, \tilde{U}) = 0.$$

Such states $\tilde{U}$ form a curve in $\mathcal{G}_1$, denoted by L.

On the curve $\mathcal{W}_1(U_3)$ there exists a point $\tilde{U}_3 \in \mathcal{G}_2$ at which the 1-shock speed vanishes:

$$\bar{\lambda}_1(U_3, \tilde{U}_3) = 0.$$

This construction gives a (unique) solution in the following two cases:
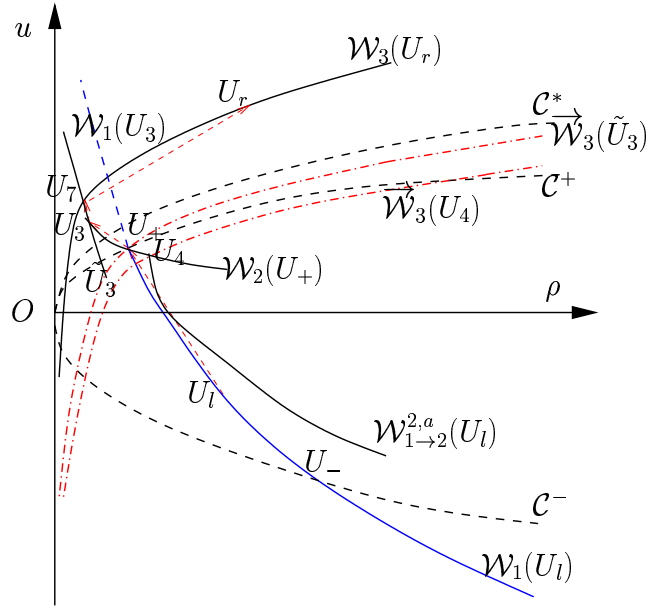
Figure 4.5: U_r above $W_3(\tilde{U}_3)$

(i) Assume that U_r is above or on the curves L and $W_3(U_2)$, and is below or on the curve $W_3(U_4)$ (Figure 4.4).
Set
$$U_5 \in \mathcal{W}_3(U_r) \cap \mathcal{W}_{1 \to 2}^{i,a}(U_l, a_r),$$
$$U_6 \in \mathcal{W}_2(U_5) \cap \mathcal{W}_1(U_l).$$

The solution is then

(4.10) $$W_1(U_l, U_6) \oplus W_2(U_6, U_5) \oplus W_3(U_5, U_r).$$

(ii) Assume that U_r is above or on the curve $W_3(\tilde{U}_3)$ (Figure 4.5).
  If
$$\mathcal{W}_3(U_r) \cap \mathcal{W}_1(U_3) = \emptyset,$$

then the solution contains a component of empty density. Otherwise, set

$$\{U_7\} = \mathcal{W}_3(U_r) \cap \mathcal{W}_1(U_3).$$

The solution is then

(4.11) $R_1(U_l, U_+) \oplus W_2(U_+, U_3) \oplus W_1(U_3, U_7) \oplus W_3(U_7, U_r).$
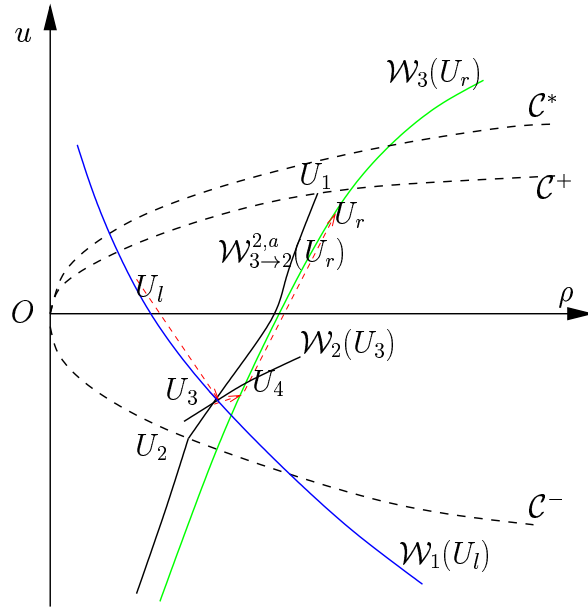
Figure 4.6: U_2 above $\mathcal{W}_1(U_l)$

Construction 5. *U_r is above $\mathcal{C}^-$ and is below $\mathcal{C}^*$. This construction gives the same solutions as Constructions 2, 4.*

Let us denote by $\mathcal{W}_{3\to2}^{2;a}(U_r)$ the composite (backward) curve consisting of all the states U such that there exists some state $U_1 \in \overrightarrow{\mathcal{W}}_3(U_r)$ and the stationary shock from U using $\varphi_2(U, a_r)$ admissible to the a-monotone criterion to U_1 can be followed by the 3-wave from U_1 to U_r. We then define

$$\{U_1\} = \mathcal{W}_{3\to2}^{2;a}(U_r) \cap \mathcal{C}^+,$$
$$\{U_2\} = \mathcal{W}_{3\to2}^{2;a}(U_r) \cap \mathcal{C}^-.$$

Figure 4.7: U_2 below $\mathcal{W}_1(U_l)$

Then, if U_l is below or on the curve $\overleftarrow{\mathcal{W}}_1(U_1)$, the following construction gives a unique solution:

(i) If U_2 is above or on the curve $\mathcal{W}_1(U_l)$, then

$$\mathcal{W}^{2,a}_{3\to2}(U_r) \cap \mathcal{W}_1(U_l) = \{U_3\}.$$

See Figure 4.6.

By definition, the point U_3 determines a point $U_4 \in \mathcal{W}_3(U_r)$ to which the solution jumps by a stationary shock. The solution is then

(4.12) $$W_1(U_l, U_3) \oplus W_2(U_3, U_4) \oplus W_3(U_4, U_r).$$

This solution however coincides with the one given by Construction 4.

(ii) If U_2 is below the curve $\mathcal{W}_1(U_l)$, then the point U_2 determines a point $U_5 \in \mathcal{W}_3(U_r)$ to which the solution can use a stationary jump, and

$$\mathcal{W}_1(U_l) \cap \mathcal{W}_3(U_2) = \{U_6\}.$$

See Figure 4.7.

The solution is then

(4.13) $$W_1(U_l, U_6) \oplus R_3(U_6, U_2) \oplus W_2(U_2, U_5) \oplus W_3(U_5, U_r),$$

which coincides with the one given by Construction 2.
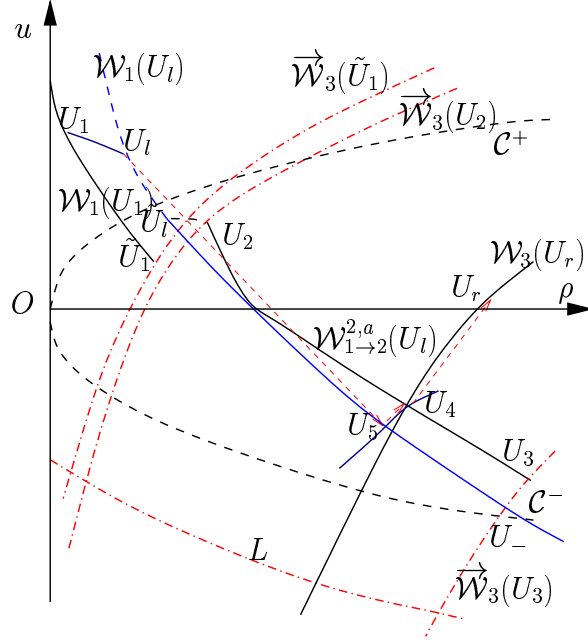
Figure 4.8: U_r between $\overrightarrow{\mathcal{W}}_3(U_2)$ and $\overrightarrow{\mathcal{W}}_3(U_3)$

Construction 6. $U_l \in \mathcal{G}_3$.

From U_l, a stationary jump using $\varphi_1(U_l, a_r)$ attains a state U_1, the solution can be continued by using $\mathcal{W}_1(U_1)$ as long as the 1-shock speed from U_1 is non-negative, i.e., until a state $\tilde{U}_1$ at which the 1-shock speed from U_1 vanishes. On the other hand, the solution can jump by a non-positive shock from U_l to any state U below or at a state $\tilde{U}_l$ at which the 1-shock speed from U_l vanishes. The solution then can be continued by a stationary jump using $\varphi_2(U, a_r)$ where $U \in \mathcal{W}_1(U_l)$. Let us denote U_2, U_3 the states reached by a stationary jump from $\tilde{U}_l, U_-$, respectively, using $\varphi_2(\tilde{U}_l, a_r), \varphi_2(U_-, a_r)$, respectively.

(i) Assume that U_r is above or on the curve $\overrightarrow{\mathcal{W}}_3(U_3)$, and is below or on the curve $\overrightarrow{\mathcal{W}}_3(U_2)$ (Figure 4.8).
   Then
$$\mathcal{W}_{1\to2}^{2,a}(U_l) \cap \mathcal{W}_3(U_r) = \{U_4\}.$$

By definition, the point U_4 determines a point U_5 belonging to $\mathcal{W}_1(U_l)$ corresponding to a stationary jump. The solution is then

(4.14) $$S_1(U_l, U_5) \oplus W_2(U_5, U_4) \oplus W_3(U_4, U_r),$$

provided the 3-shock speed $\bar{\lambda}_3(U_4, U_r)$ is non-negative, i.e., U_r is above or on a curve $\mathcal{L}$ - the set of all states at which the 3-shock speed from any state U on the part $\{u \leq 0\}$ of the curve $\mathcal{W}_{1\to2}^{2,a}(U_l)$ vanishes.
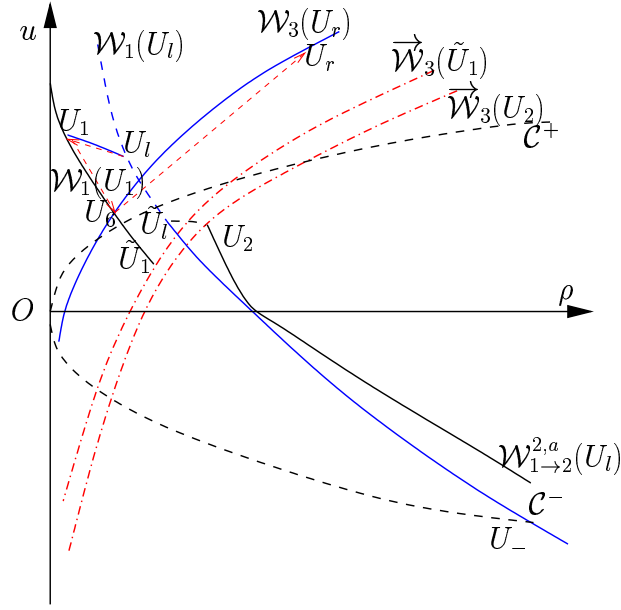
Figure 4.9: U_r above $\vec{\mathcal{W}}_3(\tilde{U}_1)$

(ii) Now, let U_r be above or on the curve $\vec{\mathcal{W}}_3(\tilde{U}_1)$ (Figure 4.9).
If

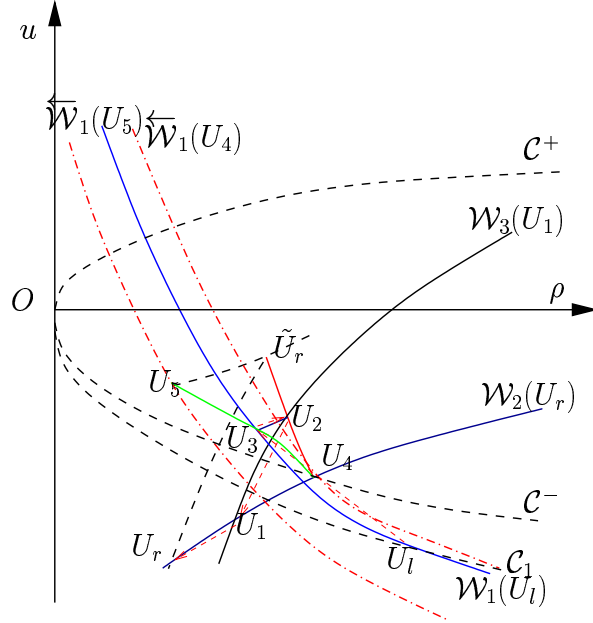$$(4.15) \qquad\qquad \mathcal{W}_3(U_r) \cap \mathcal{W}_1(U_1) = \{U_6\}.$$

The solution is then

$$(4.16) \qquad\qquad W_2(U_l, U_1) \oplus W_1(U_1, U_6) \oplus W_3(U_6, U_r).$$

If (4.15) is not satisfied, i.e.,

$$\mathcal{W}_1(U_1) \cap \{\rho = 0\} = \{U_7\},$$
$$\mathcal{W}_3(U_r) \cap \{\rho = 0\} = \{U_8\},$$

so that U_8 is above U_7, the solution then contains a component of empty density.

5. Solutions with two stationary waves

Figure 5.1: $U_r \in \mathcal{G}_1$

In this section we will complete the description of Riemann solutions by considering solutions that contain two stationary waves. Under the Monotonicity Criterion, this happens when a stationary shock jumps from the level $a = a_l$ to an intermediate level a between a_l and a_r, and then we use a k-shock with zero speed which keeps constant the level a, $k = 1, 3$, then we finally use another stationary shock jumping from the level a to a_r. Hence, there are only two possibilities:

   (i) U_r belongs to $\mathcal{G}_1$ and the shock with zero speed is a 3-shock,
   (ii) U_l belongs to $\mathcal{G}_3$ and the shock with zero speed is a 1-shock.

   We are going to discuss these two situations.

Construction 7. $U_r \in \mathcal{G}_1$ *(Figure 5.1).*

Any $a \in (a_l, a_r)$ determines a state $U_1 \in \mathcal{W}_2(U_r)$ which can be attained by a stationary jump using $\varphi_1(U_r, a)$. In view of Proposition 2.4, there exists a point $U_2 \in \mathcal{W}_3(U_1)$ such that the 3-shock speed $\bar{\lambda}_3(U_2, U_1) = 0$. The state U_2 determines a state $U_3 \in \mathcal{W}_2(U_2)$ attainable by a stationary jump using $\varphi_2(U_2(a), a_l)$. The curve $\mathcal{W}_1(U_l)$ passing through U_3 will determine the Riemann solution:

$$(5.1) \qquad W_1(U_l, U_3) \oplus W_2(U_3, U_2) \oplus S_3(U_2, U_1) \oplus W_2(U_1, U_r).$$

In order for this construction to make sense, the following requirements must be imposed

$$(5.2) \qquad \begin{cases} a \geq a_{\min}(U_r), \\ a_l \geq a_{\min}(U_2). \end{cases}$$
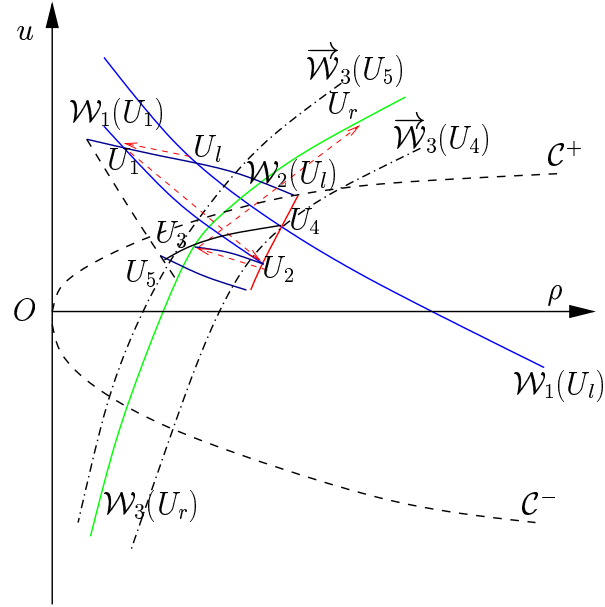
Figure 5.2: $U_l \in \mathcal{G}_3$

Observe that $a_{\min}(U_r) < a_l \ (< a)$ if and only if U_r is below the curve $\mathcal{C}_1$ so that the first line in (5.2) is obvious. The condition (5.2) can be seen as a constraint on the contracting duct.

Construction 8. $U_l \in \mathcal{G}_3$ *(Figure 5.2).*

Any $a \in (a_l, a_r)$ determines a state $U_1 \in \mathcal{W}_2(U_l)$. It is derived from Proposition 2.4 that there exists a point $U_2 \in \mathcal{W}_1(U_1)$ such that the 1-shock speed $\bar{\lambda}_1(U_1, U_2) = 0$. The state U_2 determines a state $U_3 \in \mathcal{W}_2(U_2)$ attainable by a stationary jump using $\varphi_2(U_2(a), a_r)$. The curve $\overrightarrow{\mathcal{W}}_3(U_r)$ passing through U_3 will determine the Riemann solution:

(5.3) $$W_2(U_l, U_1) \oplus S_1(U_1, U_2) \oplus W_2(U_2, U_3) \oplus W_3(U_3, U_r).$$

There is no constraint on expanding duct.

6. Numerical results

In this section we plot the solution of the Riemann solution for various range of values of the data.

Construction 1. (See Figure 4.1, Section 4)
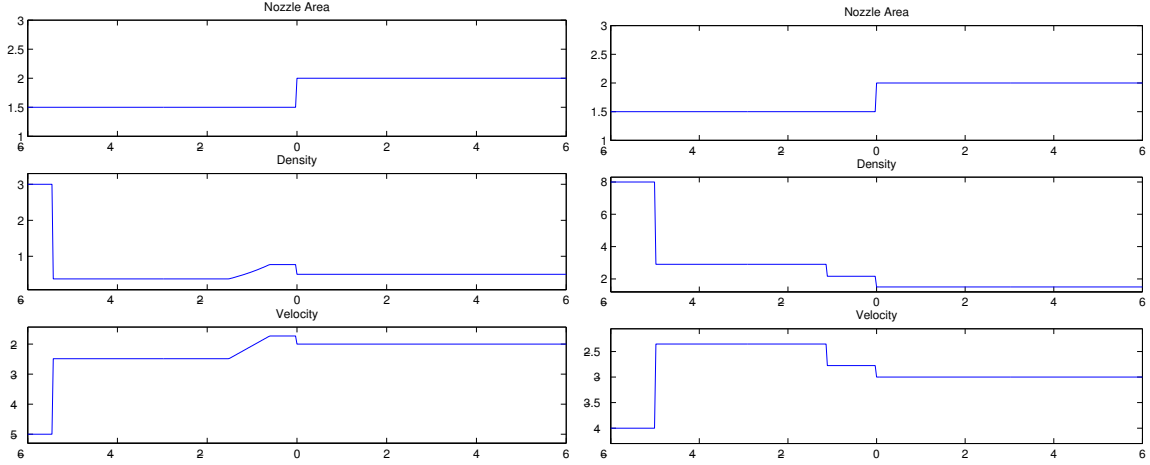
Figure C1.1: $S_1 \oplus R_3 \oplus Z$ Figure C1.2: $S_1 \oplus S_3 \oplus Z$

Figure C1.1: The solution is a 1-shock from $U_l = (\rho_l, u_l, a_l)$ to U_2 followed by a 3-rarefaction wave from U_2 to U_1, followed by a stationary wave from U_1 to $U_r = (\rho_r, u_r, a_r)$.

Figure C1.2: The solution is a 1-shock from $U_l = (\rho_l, u_l, a_l)$ to U_2 followed by a 3-shock from U_2 to U_1, followed by a stationary wave from U_1 to $U_r = (\rho_r, u_r, a_r)$.

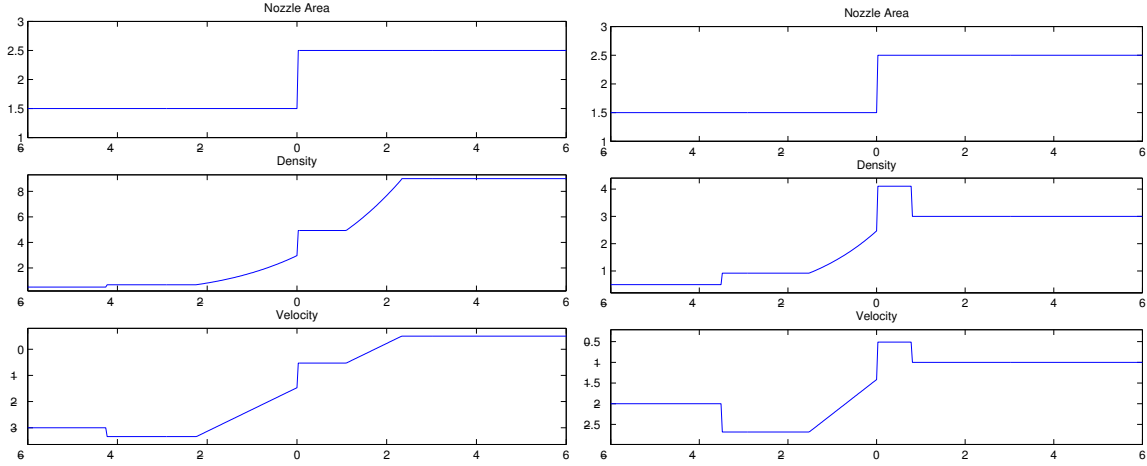**Construction 2.** (See Figure 4.2, Section 4)
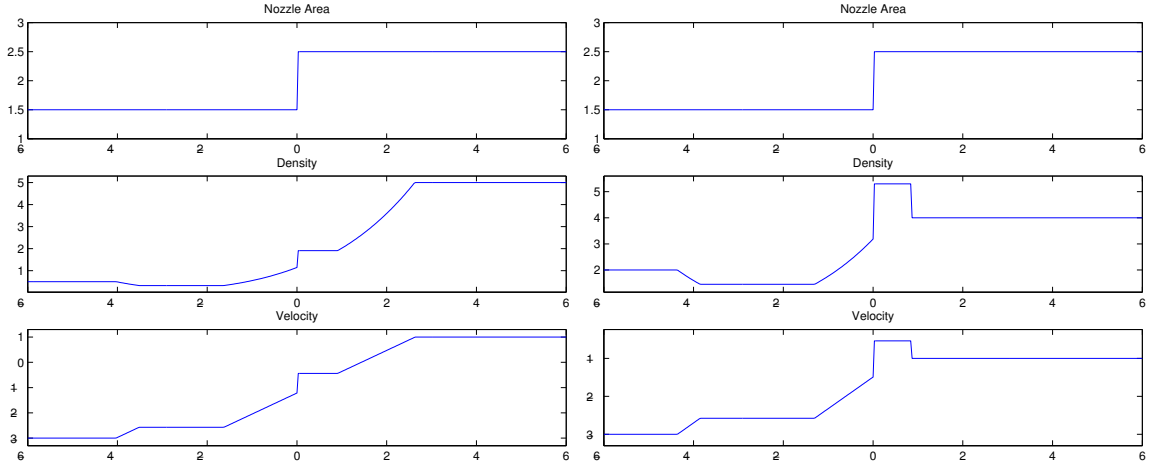
Figure C2.1: $R_1 \oplus R_3 \oplus Z \oplus R_3$ Figure C2.2: $R_1 \oplus R_3 \oplus Z \oplus S_3$
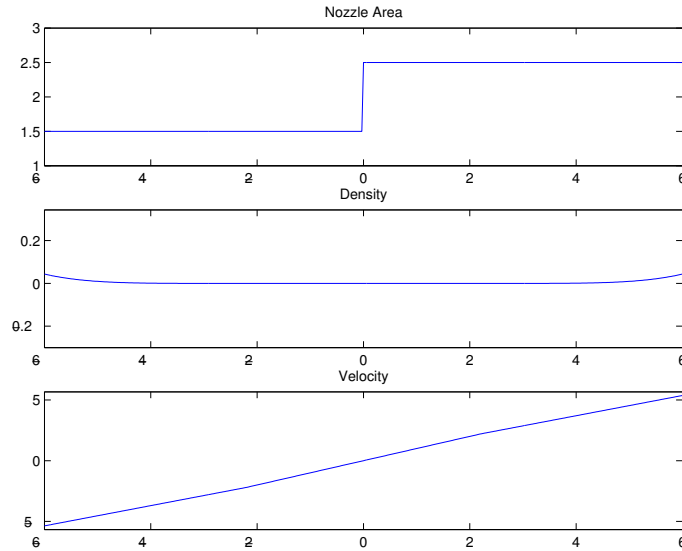
Figure C2.1: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_3 followed by a 3-rarefaction wave from U_3 to U_2, followed by a stationary wave from U_1 to U_1, followed by a 3-rarefaction wave from U_1 to $U_r = (\rho_r, u_r, a_r)$.

Figure C2.3: $S_1 \oplus R_3 \oplus Z \oplus R_3$ Figure C2.4: $S_1 \oplus R_3 \oplus Z \oplus S_3$

Figure C2.2: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_3 followed by a 3-rarefaction wave from U_3 to U_2, followed by a stationary wave from U_1 to U_1, followed by a 3-shock from U_1 to $U_r = (\rho_r, u_r, a_r)$.

Figure C2.3: The solution is a 1-shock wave from $U_l = (\rho_l, u_l, a_l)$ to U_3 followed by a 3-rarefaction wave from U_3 to U_2, followed by a stationary wave from U_1 to U_1, followed by a 3-rarefaction wave from U_1 to $U_r = (\rho_r, u_r, a_r)$.

Figure C2.4: The solution is a 1-shock wave from $U_l = (\rho_l, u_l, a_l)$ to U_3 followed by a 3-rarefaction wave from U_3 to U_2, followed by a stationary wave from U_1 to U_1, followed by a 3-shock from U_1 to $U_r = (\rho_r, u_r, a_r)$.

Construction 3. (See Figure 4.3, Section 4)

Figure C3.1: $R_1 \oplus R_1 \oplus Z \oplus R_3 \oplus R_3$

Figure C3.1: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_2 followed by a rarefaction wave with zero density, i.e. in empty, from U_2 to O at which it suffers a shift on the component a,

followed by a rarefaction wave with zero density from O to U_1, followed by a 3-rarefaction wave from U_1 to $U_r = (\rho_r, u_r, a_r)$.

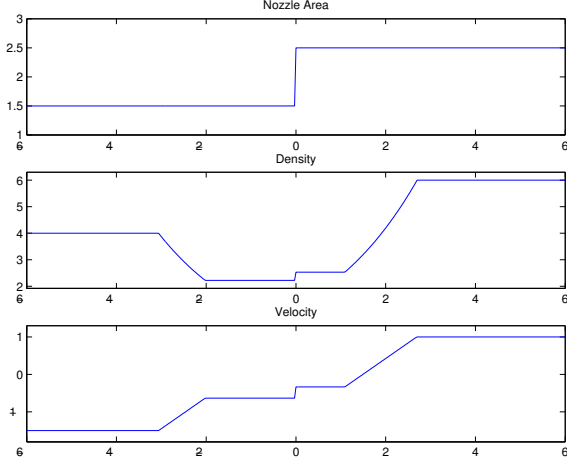**Construction 4.** (See Figure 4.4, Section 4)

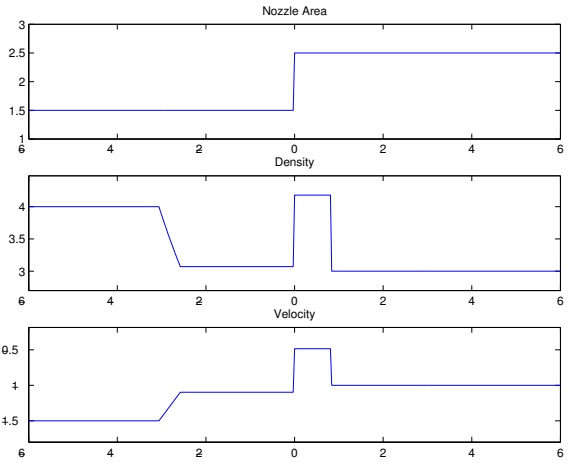Figure C4.1: $R_1 \oplus Z \oplus R_3$ Figure C4.2: $R_1 \oplus Z \oplus S_3$

Figure C4.1: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_6 followed by a stationary wave from U_6 to U_5, followed by a 3-rarefaction wave from U_5 to $U_r = (\rho_r, u_r, a_r)$.

Figure C4.2: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_6 followed by a stationary wave from U_6 to U_5, followed by a 3-shock from U_5 to $U_r = (\rho_r, u_r, a_r)$.
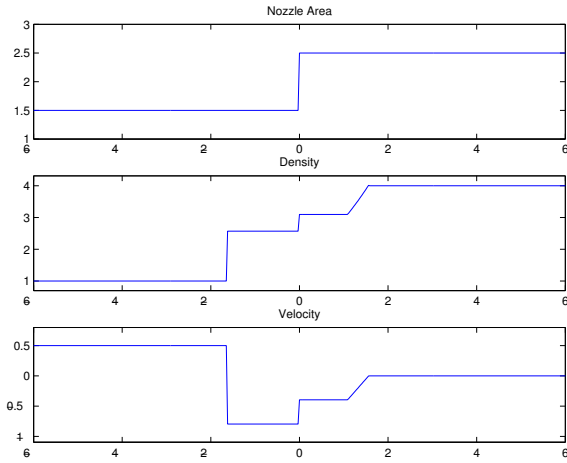

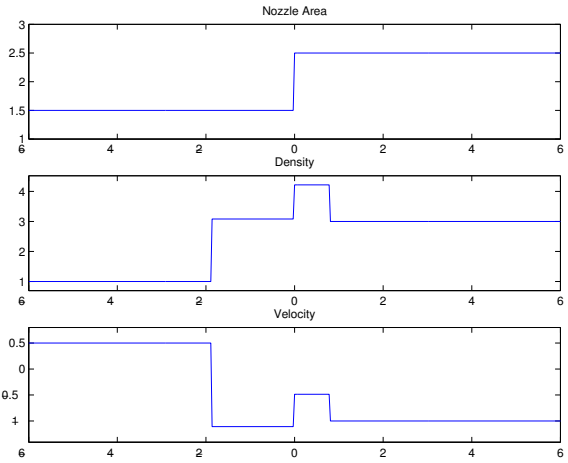Figure C4.3: $S_1 \oplus Z \oplus R_3$ Figure C4.4: $S_1 \oplus Z \oplus S_3$

Figure C4.3: The solution is a 1-shock wave from $U_l = (\rho_l, u_l, a_l)$ to U_6 followed by a stationary wave from U_6 to U_5, followed by a 3-rarefaction wave from U_5 to $U_r = (\rho_r, u_r, a_r)$.

Figure C4.4: The solution is a 1-shock wave from $U_l = (\rho_l, u_l, a_l)$ to U_6 followed by a stationary wave from U_6 to U_5, followed by a 3-shock from U_5 to $U_r = (\rho_r, u_r, a_r)$.

Construction 5. (See Figure 4.5, Section 4)

Figure C5.1: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_+ followed by a stationary wave from U_+ to U_3, followed by a 1-rarefaction wave from U_3 to U_7, followed by a 3-rarefaction wave from U_7 to $U_r = (\rho_r, u_r, a_r)$.
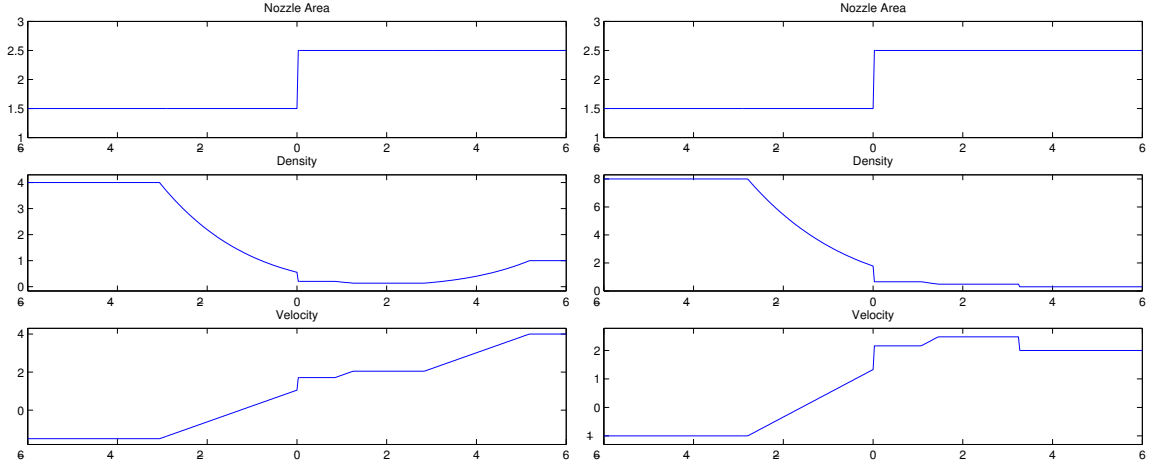
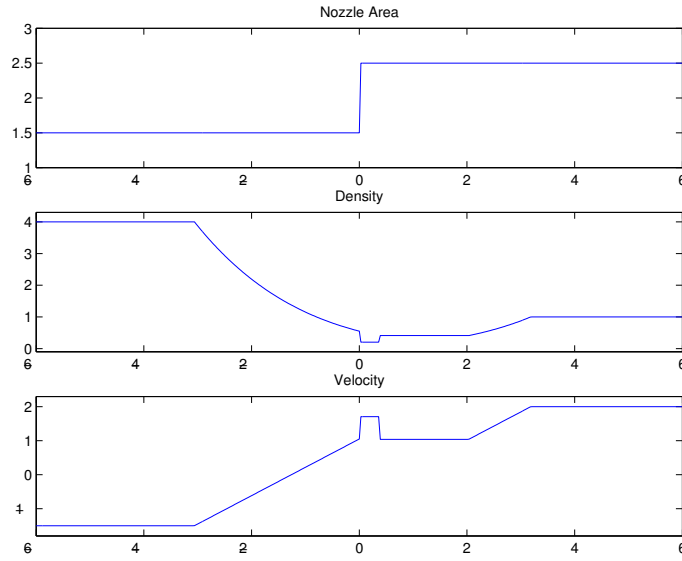Figure C5.1: $R_1 \oplus Z \oplus R_1 \oplus R_3$ Figure C5.2: $R_1 \oplus Z \oplus R_1 \oplus S_3$

Figure C5.2: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_+ followed by a stationary wave from U_+ to U_3, followed by a 1-rarefaction wave from U_3 to U_7, followed by a 3-shock wave from U_7 to $U_r = (\rho_r, u_r, a_r)$.

Figure C5.3: $R_1 \oplus Z \oplus S_1 \oplus R_3$

Figure C5.3: The solution is a 1-rarefaction wave from $U_l = (\rho_l, u_l, a_l)$ to U_+ followed by a stationary wave from U_+ to U_3, followed by a 1-shock wave from U_3 to U_7, followed by a 3-rarefaction wave from U_7 to $U_r = (\rho_r, u_r, a_r)$.

Construction 6. (See Figure 4.8, Section 4)

Figure C6.1: The solution is a 1-shock wave from $U_l = (\rho_l, u_l, a_l)$ to U_5 followed by a stationary wave from U_5 to U_4, followed by a 3-rarefaction wave from U_4 to $U_r = (\rho_r, u_r, a_r)$.

Figure C6.2: The solution is a 1-shock wave from $U_l = (\rho_l, u_l, a_l)$ to U_5 followed by a stationary wave from U_5 to U_4, followed by a 3-shock from U_4 to $U_r = (\rho_r, u_r, a_r)$.

Construction 7. (See Figure 4.9, Section 4)
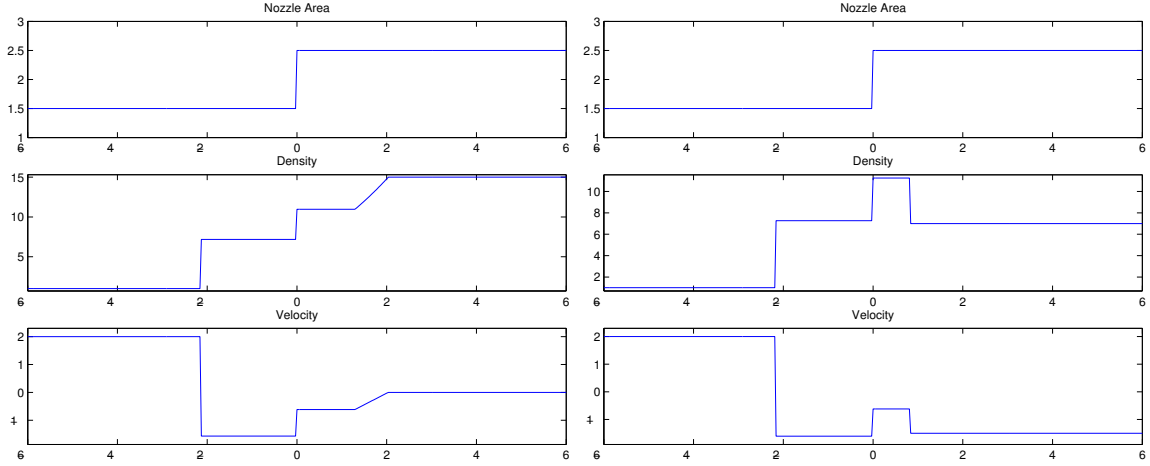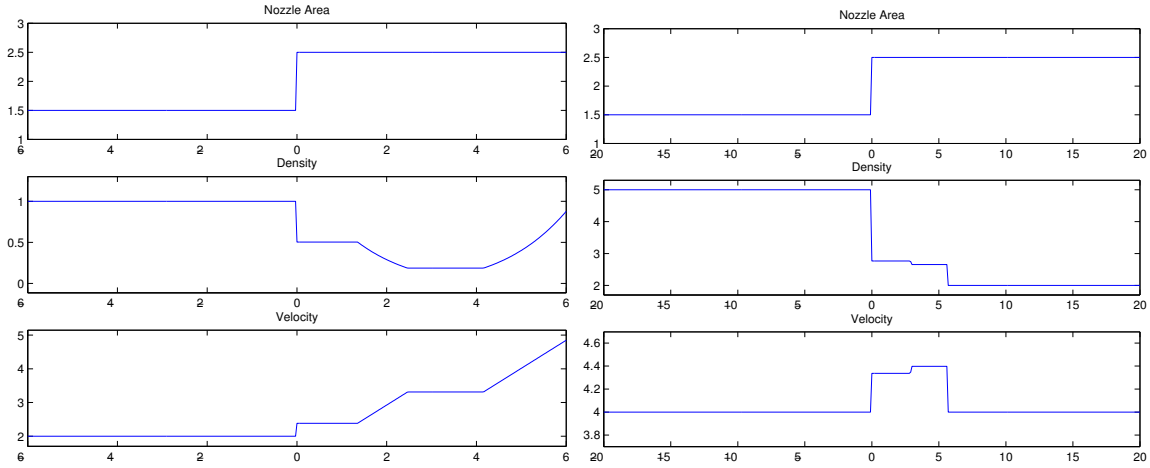
Figure C6.1: $S_1 \oplus Z \oplus R_3$ Figure C6.2: $S_1 \oplus Z \oplus S_3$

Figure C7.1: $Z \oplus R_1 \oplus R_3$ Figure C7.2: $Z \oplus R_1 \oplus S_3$

Figure C7.1: The solution is a stationary shock from $U_l = (\rho_l, u_l, a_l)$ to U_1 followed by a 1-rarefaction wave stationary wave from U_1 to U_6, followed by a 3-rarefaction wave from U_6 to $U_r = (\rho_r, u_r, a_r)$.

Figure C7.2: The solution is a stationary shock from $U_l = (\rho_l, u_l, a_l)$ to U_1 followed by a 1-rarefaction wave stationary wave from U_1 to U_6, followed by a 3-shock wave from U_6 to $U_r = (\rho_r, u_r, a_r)$.

Figure C7.3: The solution is a stationary wave $U_l = (\rho_l, u_l, a_l)$ to U_1, followed by a 1-shock wave from U_1 to U_6, followed by a 3-rarefaction wave from U_6 to $U_r = (\rho_r, u_r, a_r)$.

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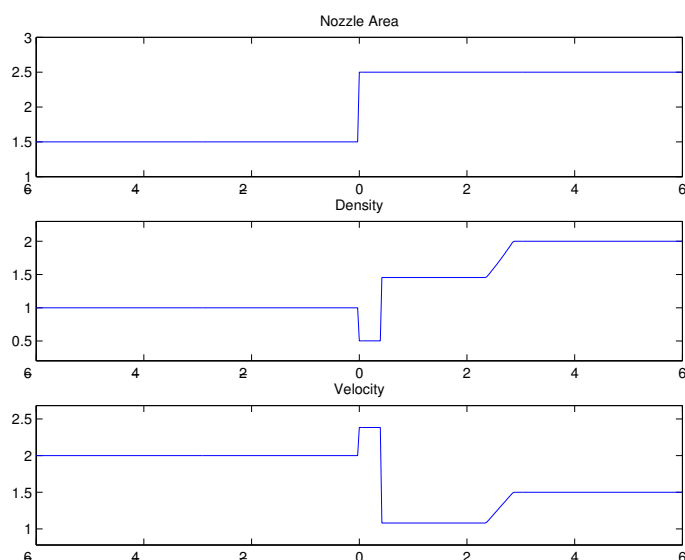
Figure C7.3: $Z \oplus S_1 \oplus R_3$

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