SHOCK WAVES IN COMPRESSIBLE TWO-PHASE FLOWS

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Abstract

Hyperbolic models for compressible two-phase flows including a conservative symmetric hyperbolic model are reviewed. The basis for a theory of shock waves is developed within the framework of this conservative model. The analysis of small amplitude discontinuities allows us to conclude that in general there are two types of shocks corresponding to two sound waves. The problem of transition between a pure phase and a mixture (the *phase vacuum* problem) is analysed. It is proved that for some models the smooth centred wave solution can not provide such a transition. Within the framework of our conservative model there is the possibility of constructing discontinuous solutions which can resolve the phase vacuum problem.

Key words shock wave, hyperbolic conservative model, compressible two–phase flow

1 Introduction

A very large variety of scientific and technological problems are intrinsically of a multi-phase flow nature. Examples arise in the petrochemical industry, nuclear processes, environmental flows and propulsion technology, to name but a few. In this paper we are concerned with mathematical models for processes in which at least of the phases is compressible. In some cases, and depending on the particular application in mind, simplifying assumptions may lead to useful limiting cases. One such case is the dusty gas approximation in which more than one velocity vector are admitted but the volume occupied by one of the phases is neglected. Another simplification is the multi-component flow model, in which a single velocity vector is assumed, namely that of the carrier fluid. Of course the simplest case is that of a single phase, in which the multi-phase nature of the process is incorporated via an appropriate equation of state. In any event, the physical nature of multi-phase flows is by now only partially understood and the construction of mathematical models contains uncertainties. Incorporating other phenomena, such as combustion, makes the problem of modelling even more challenging. For simple two-phase flow situations, such as in stratified and annular flow, in which the two phases are separated by a single interface, one may write down balance equations for the dynamics of each phase together with interfacial conditions. For more complex situations, such as those involving a mixture of water droplets and a gas, any attempts at following the evolution of all interfaces involved would lead to an intractable mathematical model.

Focusing only on the dynamical description of multi-phase flow processes leads to still largely unresolved physical and mathematical issues. For instance, the initial-value problem for some of mathematical models for compressible flow in current use is ill-posed. This is a consequence of the equations being mixed elliptic-hyperbolic. See Stewart and Wendroff (1984) for a comprehensive review of the subject. A practical consequence on this is the situation that numerical computations, based on these mixed models, on course meshes and or diffusive numerical methods, provide reasonably-looking solutions that are comparable to experimental measurements. But when the mesh is sufficiently refined or and the numerical method used is sufficiently accurate the solution *blows up*. In addition to hyperbolicity there is another difficulty: most current mathematical models cannot be expressed in conservation-law form. This may appear paradoxical, as the equations are in all cases derived from the application of physical conservation principles. This difficulty

means that the definition of discontinuous solutions, such as shock waves, is not a straightforward procedure (see for example Serre(1993), Gouin & Gavrilyuk(1999)). It also means that modern, conservative shock-capturing methods (Toro 1999) cannot be directly applied to solve multi-phase flow problems.

The area of multiphase flow model is currently a very active area and the following are are partial list of relevant references: Baer & Nunziato (1986); Bdzil *et al.* (1999); Bestion (1999); Coquel *et al.* (1997); Drew *et al.* (1979); Drew & Passman (1998); Fitt (1993); Gavrilyk & Saurel (2001); Ishii (1975); Nigmatulin (1991); Ransom & Hicks (1984); Resnyansky *et al.* (1997); Saurel & Abgrall (1999); Saurel & LeMetayer (2001); Städtke *et al.* (2001); Stewart & Wendroff (1984); Toro (1989); Toumi & Kumbaro (1996); Toumi *et al.* (1999); Wallis (1982).

A common theory of two-phase compressible flow is based on the consideration and averaging of local balance laws for each phase. In such a theory the two-phase medium is supposed to be an averaged continuum in which the interfacial interaction is taken into account (see for example Drew & Passman 1998; Ishii 1975). There are different approaches leading to hyperbolic-two phase models Baer & Nunziato (1986); Bdzil *et al.* (1999); Gavrilyk & Saurel (2001). In this paper we are concerned with a single pressure two-fluid model Ishii (1975). But we note that the basic single pressure model is not hyperbolic (Drew & Passman 1998). Further development of two-phase single pressure models is based on more detailed consideration of interfacial interaction and consists of including extra differential terms to governing balance equations, such as virtual mass and interfacial pressure forces (see for example Drew & Passman 1998; Städtke *et al.* 2001). The system of governing equations modified by these additional differential terms have real eigenvalues and a complete set of linearly independent eigenvectors and hence is hyperbolic. Note that there exists a comprehensive theory of solvability of the initial value problem for multidimensional quasilinear symmetric hyperbolic systems introduced by Friedrichs (1954,1958). That is why the hyperbolic two-phase flow models for which the governing equations can be written in symmetric form are preferable, because the symmetric form guarantees the existence and uniqueness of solution for some kinds of initial and boundary value problem (Godunov 1987; Okazava&Unai 1993).

In this paper (Section 2) we review two two-phase hyperbolic models including a *conservative symmetric hyperbolic* model. That follows the phenomenological approach is based on the principle of extended thermodynamics and was proposed in Romensky (1998, 2001). In Section 3 we consider the one-dimensional conservative equations, study their characters, and formulate shock conditions. Characteristic analysis shows that there are two types of sound waves and hence two types of shock waves exist. The analysis of shock waves of small amplitude confirms this conclusion. In Section 4 we study the problem of construction of solutions providing the transition between pure phase and mixture. In practice this problem is very important to understand whether the numerical simulations of processes in which there are regions of pure phase are correct. It is shown that some of reviewed models have no centred wave solutions providing the transition between pure phase and mixture. But there is the possibility of constructing a discontinuous solution to resolve this problem.

2 A hyperbolic model of two-phase media

2.1 Single pressure models

In general the modelling of multiphase flows can be based on two different approaches. One of them uses separate balance equations for each of the phases and coupled by terms describing momentum and energy interaction. Another approach is the phenomenological one, supposing the multiphase medium as homogeneous. In such a methodology the system of governing equations is derived for the parameters of state which are assumed to take into account the multiphase character of the flow. In this paper we only consider two-phase two-fluid models. If dissipative processes, such as viscous friction or thermoconductivity are neglected, then the typical form of the governing equations in the first approach consist of balance laws for mass, momentum and energy for each of the phases (with numbers $i = 1, 2$) (see for example Ishii 1975; Drew & Passman 1998):

$$\begin{aligned}
&\frac{\partial \alpha^i \rho^i}{\partial t} + \frac{\partial \alpha^i \rho^i v_k^i}{\partial x_k} = 0, \\
&\frac{\partial \alpha^i \rho^i v_k^i}{\partial t} + \frac{\partial \alpha^i \rho^i v_k^i v_j^i}{\partial x_j} + \alpha^i \frac{\partial p^i}{\partial x_k} = F_k^i, \\
&\frac{\partial \alpha^i \rho^i (e^i + \frac{|v^i|^2}{2})}{\partial t} + \frac{\partial (\alpha^i \rho^i v_k^i (e^i + \frac{|v^i|^2}{2}) + v_k^i p^i)}{\partial x_k} + p^i \frac{\partial \alpha^i}{\partial t} = v_\alpha^i F_\alpha^i.
\end{aligned} \tag{1}$$

Here α^i are the volume concentrations of phases ($\alpha^1 + \alpha^2 = 1$), ρ^i are mass densities, v_k^i are velocities, p^i are pressures, $e^i = e^i(\rho^i, S^i)$ are equations of state (specific internal energy), S^i are specific entropies of phases and F_k^i are body forces.

It is supposed that the first law of thermodynamics holds for each of the phases separately:

$$de^i(\rho^i, S^i) = -p^i d\left(\frac{1}{\rho^i}\right) + T^i dS^i,$$

where T^i is the temperature of i-th phase. It is easy to see that the number of unknown variables is greater than the number of differential equations. To close the system it is necessary to make an additional assumption. The simplest way leads to the widely known single pressure model, which rests on the assumption that the pressure of the two phases are equal, that is $p^1 = p^2 = p$. Hence, the closing relationships can be chosen in the form:

$$\rho^1 = \rho(p, S^1), \quad \rho^2 = \rho(p, S^2). \tag{2}$$

The body forces F_k^i in the momentum equations must satisfy the requirement $F_k^1 + F_k^2 = 0$, which follows from the total momentum conservation law. These forces take into account the different processes of momentum exchange. The simplest example of such a process is the interfacial friction. It means that F_k^i is proportional to the relative velocity $v_k^2 - v_k^1$ and is an algebraic (not differential) term of the equation. In this case the study of hyperbolicity of system (1) can be done under assumption $F_k^i = 0$.

The analysis of the characteristics of system (1) with closing relationships (2) and $F_k^i = 0$ shows that there exists a large region for the relative velocity values for which the system is not hyperbolic (Stewart & Wendroff 1984). In order to resolve difficulties associated with non-hyperbolic models, additional non-dissipative momentum exchange forces are usually introduced into the momentum balance equations (see for example Drew & Passman 1998). The general form of such forces is given by the functions

$$F_k^1 = -F_k^2 = F_k\left(\alpha^1, p, v_k^i, S^i, \frac{\partial \alpha^1}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial v_k^i}{\partial t}, \frac{\partial S^i}{\partial t}, \frac{\partial \alpha^1}{\partial x_k}, \frac{\partial p}{\partial x_k}, \frac{\partial v_k^i}{\partial x_k}, \frac{\partial S^i}{\partial x_k}\right).$$

Naturally, the vector function F_k must be Galilean invariant, that is rotation and translation invariant. This restriction is caused by the requirement of invariance of the governing equations.

Concerning the physical meaning of the various new terms, it is argued that these correspond to interfacial forces of different nature (virtual mass, interfacial pressure and others) (see Bestion 1990; Drew & Passman 1998; Drew *et al.* 1979; Ransom & Hicks 1984; Städtke *et al.* 2001):

$$F_j^1 = -k\rho\alpha^1\alpha^2\left(\frac{d^1v_j^2}{dt} - \frac{d^2v_j^1}{dt}\right) + \alpha^1\alpha^2(\alpha^1\rho^2 - \alpha^2\rho^1)(v_i^1 - v_i^2)\frac{\partial(v_i^1 - v_i^2)}{\partial x_j} -$$

$$-\alpha^1\alpha^2(\rho^1 + \rho^2)(v_j^1 - v_j^2)(v_i^1 - v_i^2)\frac{\partial\alpha_1}{\partial x_j} - \alpha^1\alpha^2(\rho^1 + \rho^2)(v_j^1 - v_j^2)\left(\frac{\alpha^1}{\rho^1}\frac{d^1\rho^1}{dt} + \frac{\alpha^2}{\rho^2}\frac{d^2\rho^2}{dt}\right),$$

where $\frac{d^i}{dt} = \frac{\partial}{\partial t} + v_k^i\frac{\partial}{\partial x_k}$, $k \in [0, \infty)$ is a parameter open to choice. The new differential terms, introduced in the manner just described, allow the modified system to be hyperbolic, and even explicit formulae for characteristics are in some cases derived (Städtke *et al.* 2001).

Unfortunately, it appears as if the resulting system cannot be reduced to a symmetric form. In addition, the modified system cannot be expressed in conservative form, and this is why the definition of discontinuous solutions, such as shock waves and contacts, do not admit a correct mathematical formulation. Thus the properties of the governing equations in the approach just reviewed are not suitable for a complete mathematical analysis. Needless to say, the generalization of the above approach to modelling multiphase flow for the case when the number of phase is greater than two is unclear.

There are other ways of constructing models of two-phase flow, see for example the Baer-Nunziato model (Baer & Nunziato 1986) or variational approach leading to a closed system of governing equations presented in Gavriljuk & Saurel (2002). We remark however that, in common with the approach described above, the equations derived by these methods are not in conservative form.

2.2 Phenomenological conservative model

We now describe an approach which is based on the *phenomenological theory* of continuum mechanics and the principles of extended thermodynamics, which has been developed in the last decades on the basis of the analysis of various well-posed equations of mathematical physics. It is applicable to creating a new well-posed model of complicated media (Godunov & Romenski 1995; Godunov & Romenski 1998; Müller & Ruggeri 1998; Romensky 1998; Romensky 2001).

We emphasize that this theory allows the derivation of symmetric hyperbolic systems of conservation laws for multiphase fluid flow in which the number of phases can be greater than two (Romenski 2001). The resulting models are then well suited for the study of shock waves in multiphase media and the application of modern conservative shock-capturing numerical methods (Toro 1999) to solve technological and scientific problems of current interest.

Note that the rather complicated model taking into account process of relaxation of the difference between phases pressure is considered in Romenski *et al.* (2003). The fundamental assumption in this approach is the existence of a common equation of state (internal energy) for two-phase medium, $e = e(\alpha^1\rho^1, \alpha^2\rho^2, S)$, where S is the entropy of the medium. Note that the momentum balance laws in this phenomenological model can be written in the usually employed manner

$$\frac{\partial\alpha^i\rho^iv_k^i}{\partial t} + \frac{\partial\alpha^i\rho^iv_k^iv_j^i}{\partial x_j} + \alpha^i\frac{\partial p}{\partial x_k} = F_k^i\,.$$

But the conservation requirement of the governing equations leads to the following definition of momentum interfacial force:

$$F_k^1 = -F_k^2 = -\frac{\alpha^1\alpha^2(\rho^1-\rho^2)}{\rho}\frac{\partial p}{\partial x_k} + \frac{\alpha^1\alpha^2}{\rho}\frac{\partial n}{\partial x_k} -$$

$$-\frac{\alpha^1\alpha^2}{\rho}\left[v_k^1\left(\frac{\partial v_k^1}{\partial x_i}-\frac{\partial v_i^1}{\partial x_k}\right) - v_k^2\left(\frac{\partial v_k^2}{\partial x_i}-\frac{\partial v_i^2}{\partial x_k}\right)\right],$$

where $\rho = \alpha^1\rho^1 + \alpha^1\rho^1$ is the mass density of the medium.

Here an additional thermodynamic parameter n is introduced into the momentum exchange force. Further we determine its connection with the parameters of state.

For processes without dissipation the two-phase flow is governed by the system of conservation laws (Romenski 1998; Romenski 2001)

$$\begin{aligned}
&\frac{\partial\alpha^1\rho^1}{\partial t} + \frac{\partial\alpha^1\rho^1 v_k^1}{\partial x_k} = 0,\\
&\frac{\partial\alpha^2\rho^2}{\partial t} + \frac{\partial\alpha^2\rho^2 v_k^2}{\partial x_k} = 0,\\
&\frac{\partial(\alpha^1\rho^1 v_k^1 + \alpha^2\rho^2 v_k^2)}{\partial t} + \frac{\partial(\alpha^1\rho^1 v_k^1 v_j^1 + \alpha^2\rho^2 v_k^2 v_j^2 + p\delta_{kj})}{\partial x_j} = 0,\\
&\frac{\partial(v_i^2 - v_i^1)}{\partial t} + \frac{\partial(v_j^2 v_j^2/2 - v_j^1 v_j^1/2 + n)}{\partial x_i} = 0,\\
&\frac{\partial(\alpha^1\rho^1 + \alpha^2\rho^2)S}{\partial t} + \frac{\partial(\alpha^1\rho^1 v_k^1 + \alpha^2\rho^2 v_k^2)S}{\partial x_k} = 0.
\end{aligned} \tag{3}$$

Here the first two equations are the mass conservation laws for the phases, the third equation is the total momentum conservation law, the fourth equation is the conservation law for the relative velocity, and the fifth one is the entropy conservation law. It is more convenient to introduce new parameters of state: $\rho = \alpha^1\rho^1 + \alpha^2\rho^2$ is the mass density of the medium, $c = (\alpha^2\rho^2)/\rho$ is the mass concentration of the 2-nd phase, $v_k = \frac{\alpha^1\rho^1}{\rho}v_k^1 + \frac{\alpha^2\rho^2}{\rho}v_k^2$ – the average velocity, $w_k = v_k^2 - v_k^1$ – the relative velocity, and S – the specific entropy of the medium.

We introduce now the new generalized thermodynamic potential

$$E = e(\rho, c, S) + c(1-c)\frac{w_k w_k}{2} = e(\rho, c, S) + \frac{\rho^1}{\rho}\frac{v_k^1 v_k^1}{2} + \frac{\rho^2}{\rho}\frac{v_k^2 v_k^2}{2} - \frac{v_k v_k}{2}.$$

Then system (3) can be written in the form

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \frac{\partial\rho v_k}{\partial x_k} = 0,\\
&\frac{\partial\rho c}{\partial t} + \frac{\partial(\rho c v_k + \rho E_{w_k})}{\partial x_k} = 0,\\
&\frac{\partial\rho v_i}{\partial t} + \frac{\partial(\rho v_i v_k + p\delta_{ik} + \rho w_i E_{w_k})}{\partial x_k} = 0,\\
&\frac{\partial w_k}{\partial t} + \frac{\partial(v_j w_j + E_c)}{\partial x_k} = 0,\\
&\frac{\partial\rho S}{\partial t} + \frac{\partial\rho v_k S}{\partial x_k} = 0.
\end{aligned} \tag{4}$$

Here $p = \rho^2 E_\rho = \rho^2 e_\rho$ is the pressure, $E_c = n + (1-2c)\frac{w_k w_k}{2}$, $n = e_c$. Note that the system (4) admits an additional steady conservation law caused by the structure of the equation for the relative velocity:

$$\frac{\partial w_k}{\partial x_i} - \frac{\partial w_i}{\partial x_k} = \frac{\partial (v_k^2 - v_k^1)}{\partial x_i} - \frac{\partial (v_i^2 - v_i^1)}{\partial x_k} = 0. \tag{5}$$

This conservation law follows directly from the fourth equation of system (4). Using this equation for w_k it is easy to see that

$$\frac{\partial}{\partial t}\left(\frac{\partial w_k}{\partial x_i} - \frac{\partial w_i}{\partial x_k}\right) = 0.$$

Hence, if the equation (5) holds for $t = 0$, then it holds for $t \geq 0$. The conservation law (5) postulates that the relative velocity vorticity for the non-dissipative processes is equal to zero. Further, we shall see that dissipation leads to the generation of relative velocity vorticity.

The proof of hyperbolicity of system (4) (and hence the system (3)), and the method of reducing this system to a symmetric form, can be found in Romenski (2001). The hyperbolicity presupposes the potential energy $e(\rho, c, S)$ to be a convex function of variables $V = 1/\rho, c, S$, that is the matrix

$$\begin{pmatrix} e_{VV} & e_{Vc} & e_{VS} \\ e_{cV} & e_{cc} & e_{cS} \\ e_{SV} & e_{Sc} & e_{SS} \end{pmatrix}$$

is positive definite.

The proof of hyperbolicity is based on the methods of extended thermodynamics (Godunov & Romenski 1995; Romenski 2001). It is necessary to introduce the generated thermodynamic potential

$$L(q_\omega, q_0, v_1, v_2, v_3, n, j_1, j_2, j_3) = -E_V + \rho w_k E_{w_k}$$

where

$$q_\omega = T = E_S, \quad q_0 = E - SE_S - VE_V - cE_c - \frac{v_i v_i}{2}, \quad n = E_c, \quad j_k = \rho E_{j_k}.$$

In terms of these new thermodynamic variables the system (4) and additional steady conservation laws (5) can be written in the form

$$\begin{aligned}
&\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (v_k L)_{q_0}}{\partial x_k} = 0, \\
&\frac{\partial L_n}{\partial t} + \frac{\partial ((v_k L)_n + j_k)}{\partial x_k} = 0, \\
&\frac{\partial L_{v_i}}{\partial t} + \frac{\partial ((v_k L)_{v_i} + j_k L_{j_i} - \delta_{ik} j_\alpha L_{j_\alpha})}{\partial x_k} = 0, \\
&\frac{\partial L_{j_k}}{\partial t} + \frac{\partial (v_\alpha L_{j_\alpha} + n)}{\partial x_k} = 0, \\
&\frac{\partial L_{q_\omega}}{\partial t} + \frac{\partial (u_k L)_{q_\omega}}{\partial x_k} = 0, \\
&\frac{\partial L_{j_k}}{\partial x_\alpha} - \frac{\partial L_{j_\alpha}}{\partial x_k} = 0.
\end{aligned}$$

One can proved that the five first equations of this system can be transformed, with the help of the last steady equation, to the following system:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (v_k L)_{q_0}}{\partial x_k} = 0,$$

$$
\begin{aligned}
&\frac{\partial L_n}{\partial t} + \frac{\partial (v_k L)_n}{\partial x_k} + \frac{\partial j_k}{\partial x_k} = 0,\\
&\frac{\partial L_{v_i}}{\partial t} + \frac{\partial (v_k L)_{v_i}}{\partial x_k} + L_{j_i}\frac{\partial j_k}{\partial x_k} - L_{j_\alpha}\frac{\partial j_\alpha}{\partial x_i} = 0,\\
&\frac{\partial L_{j_i}}{\partial t} + \frac{\partial (v_k L)_{j_i}}{\partial x_k} + L_{j_\alpha}\frac{\partial v_\alpha}{\partial x_i} - L_{j_i}\frac{\partial v_k}{\partial x_k} + \frac{\partial n}{\partial x_i} = 0,\\
&\frac{\partial L_{q_\omega}}{\partial t} + \frac{\partial (u_k L)_{q_\omega}}{\partial x_k} = 0.
\end{aligned}
$$

This system is symmetric because the first two terms of each equation can be written by the matrices of the second derivatives of functions L and $v_k L$. The symmetry of other terms is quite clear. The hyperbolicity condition is the convexity of the generating potential L and is equivalent to the convexity of the equation of state on its arguments.

Note that smooth solutions of system (4) satisfy the additional energy conservation law:

$$
\frac{\partial}{\partial t}\rho\left(E + \frac{v_i v_i}{2}\right) + \frac{\partial}{\partial x_k}\left[\rho v_k\left(E + \frac{v_i v_i}{2}\right) + p v_k + \rho v_i w_i E_{w_k} + \rho n E_{w_k}\right] = 0, \tag{6}
$$

which is used for the definition of the generated thermodynamic potential L and and variables q_ω, q_0, n, j_k (Romenski 1998; Romenski 2001). This energy conservation law will be used later in this paper to study discontinuous solutions.

Up to now we have discussed the governing equations for two-phase media without any dissipation. Certainly, models of real processes should take into account various dissipative processes. For the case of two-phase flow, when the dissipation consists of the phase interfacial friction and diffusion, such a model can be derived on the basis of system (4) and takes the form:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0,\\
&\frac{\partial \rho c}{\partial t} + \frac{\partial (\rho v_k c + \rho E_{w_k})}{\partial x_k} = \frac{\partial}{\partial x_k}\varepsilon\frac{\partial E_c}{\partial x_k},\\
&\frac{\partial \rho v_i}{\partial t} + \frac{\partial (\rho v_i v_k + p\delta_{ik} + \rho w_i E_{w_k})}{\partial x_k} = 0,\\
&\frac{\partial w_k}{\partial t} + \frac{\partial (v_\alpha w_\alpha + E_c)}{\partial x_k} = -(e_{k\alpha\beta} v_\alpha \omega_\beta + \pi_k),\\
&\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v_k S}{\partial x_k} = \frac{E_{w_k}\pi_k}{E_S} + \frac{\varepsilon}{E_S}\frac{\partial E_c}{\partial x_k}\frac{\partial E_c}{\partial x_k}.
\end{aligned} \tag{7}
$$

These unsteady equations must be supplemented by two equations providing the compatibility of system (7):

$$
\frac{\partial w_k}{\partial x^\alpha} - \frac{\partial w_\alpha}{\partial x^k} = e_{k\alpha\beta}\omega_\beta, \quad \frac{\partial \omega_k}{\partial t} + \frac{\partial (v_i\omega_k - v_k\omega_i + e_{ki\mu}\pi_\mu)}{\partial x_i} = 0. \tag{8}
$$

Here $e_{k\alpha\beta}$ is the unit pseudoscalar, ω_β is the vorticity vector and π_k is the interfacial friction force.

To prove the compatibility of equations (7) and (8) it is necessary to differentiate the equation for w_k of system (7) with respect to x_α and subtract the equation for w_α differentiated with respect to x_k. Using the second equation from (8) we derive

$$
\frac{\partial}{\partial t}\left(\frac{\partial w_k}{\partial x_\alpha} - \frac{\partial w_\alpha}{\partial x_k} - e_{k\alpha\beta}\omega_\beta\right) = 0.
$$

In fact the vorticity vector ω_i is the additional variable and is not a parameter of state. To close the system (7),(8) it is enough to define the coefficient of phase diffusion ε and the interfacial friction force $\pi_k(\rho, c, S, w_i)$ as a function of the state parameters. After that it is possible to derive a closed system of governing equations:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0,\\
&\frac{\partial \rho c}{\partial t} + \frac{\partial (\rho v_k c + \rho E_{w_k})}{\partial x_k} = \frac{\partial}{\partial x_k} \varepsilon \frac{\partial n}{\partial x_k},\\
&\frac{\partial \rho v_i}{\partial t} + \frac{\partial (\rho v_i v_k + p\delta_{ik} + \rho w_i E_{w_k})}{\partial x_k} = 0,\\
&\frac{\partial w_k}{\partial t} + v_\alpha \frac{\partial w_k}{\partial x_\alpha} + \frac{\partial n}{\partial x_k} + w_\alpha \frac{\partial v_\alpha}{\partial x_k} = -\pi^k,\\
&\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v_k S}{\partial x_k} = \frac{\rho E_{w_k} \pi_k}{E_S} + \frac{\varepsilon}{E_S} \frac{\partial n}{\partial x_k} \frac{\partial n}{\partial x_k}.
\end{aligned} \tag{9}$$

Note that in this system the fourth equation is not in a conservative form.

Systems (7),(8) or (9),(8) admit an additional energy conservation law in the form:

$$\frac{\partial \rho \left(E + \frac{v_\alpha v_\alpha}{2}\right)}{\partial t} + \frac{\partial \left(\rho v_k \left(E + \frac{p}{\rho} + w_\alpha E_{w_\alpha} + \frac{v_\alpha v_\alpha}{2}\right) + \rho E_c E_{w_k} + \varepsilon E_c \frac{\partial E_c}{\partial x_k}\right)}{\partial x_k} = 0. \tag{10}$$

To study discontinuous solutions it is preferable to use system (7),(8) and (10), where all equations are in conservative form. The interfacial friction force can be chosen in the form

$$\pi_k = \chi E_{w_k} = \chi c(1-c) w_k, \tag{11}$$

where the coefficient of friction χ can be a function of the parameters of state. The diffusion coefficient ε also can be a function of the state.

Note that the friction force (11) provides the non-negativeness of the entropy production automatically:

$$Q = \frac{\rho E_{w_k} \pi_k}{E_S} + \frac{\varepsilon}{E_S} \frac{\partial E_c}{\partial x_k} \frac{\partial E_c}{\partial x_k} \geq 0.$$

Finally, we emphasize that the described phenomenological approach can be similarly expanded for multiphase flow models when the number of phases is greater than two (Romenski 2001).

3 Shock waves

3.1 One-dimensional equations, shock conditions, characteristics

As mentioned earlier, conservative models admit correctly defined discontinuous solutions. That is why we now deal with the phenomenological conservative model described in the Subsection 2.2. Consider one-dimensional flow of a two-phase medium. Suppose that the medium moves along the $x_1 = x$ axis and that there is only one component of the velocity vector $v_1 = v$ and one component of the relative velocity $w = w_1$ vector. Under such assumptions we shall study the closed system of balance laws that consists of the first four equations of system (7) and the energy conservation law (10), written for the one-dimensional case. That is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0,$$

$$
\begin{aligned}
&\frac{\partial \rho c}{\partial t} + \frac{\partial(\rho v c + \rho E_w)}{\partial x} = \frac{\partial}{\partial x}\varepsilon\frac{\partial E_c}{\partial x},\\
&\frac{\partial \rho v}{\partial t} + \frac{\partial(\rho v^2 + p + \rho w E_w)}{\partial x} = 0,\\
&\frac{\partial w}{\partial t} + \frac{\partial(v w + E_c)}{\partial x} = -\pi,\\
&\frac{\partial \rho\left(E + \frac{v^2}{2}\right)}{\partial t} + \frac{\partial\left(\rho v\left(E + \frac{v^2}{2} + \frac{p}{\rho} + \rho v w E_w\right) + \rho E_c E_w + \varepsilon E_c \frac{\partial E_c}{\partial x}\right)}{\partial x} = 0.
\end{aligned}
\tag{12}
$$

To study discontinuous solutions here, we include in the closed system the energy conservation law instead of the entropy balance law. The entropy balance law is a consequence of system (11) and has the form:

$$
\frac{\partial \rho S}{\partial t} + \frac{\partial \rho v S}{\partial x} = \frac{\rho E_w \pi}{E_S} + \frac{\varepsilon}{E_S}\left(\frac{\partial E_c}{\partial x}\right)^2. \tag{13}
$$

Note that the additional compatibility equations (8) are valid automatically for the one-dimensional case, because of $\omega_i = 0$.

Further, we shall study waves moving with constant velocity D in the positive direction of the x – axis and connecting two states of the medium with parameters of state $v_0, \rho_0, c_0, w_0, S_0$ ahead of the wave ($x = +\infty$), and v, ρ, c, w, S behind the wave ($x = -\infty$). Such a solution depends only on the one variable $\xi = x - Dt$. To determine this solution one can derive the following consequence of system (12):

$$
\begin{aligned}
&\rho(v - D) = \rho_0(v_0 - D),\\
&\varepsilon\frac{dE_c}{d\xi} = \rho(v - D)c + \rho E_w - \rho_0(v_0 - D)c_0 - \rho_0(E_w)^0,\\
&\rho(v - D)v + p + \rho w E_w = \rho_0(v_0 - D)v_0 + p_0 + \rho_0 w_0(E_w)^0,\\
&\frac{d}{d\xi}\left((v - D)w + E_c\right) = -\pi,\\
&\varepsilon E_c\frac{dE_c}{d\xi} = -\rho(v - D)\left(E + \frac{v^2}{2}\right) - pv - \rho v w E_w - \rho E_c E_w\\
&+\rho_0(v_0 - D)\left(E_0 + \frac{v_0^2}{2}\right) + p_0 v_0 + \rho_0 v_0 w_0(E_w)^0 + \rho_0(E_c)^0(E_w)^0.
\end{aligned}
\tag{14}
$$

For finding a continuous solution of system (12) we can replace the last energy equation by the entropy equation (13) in system (14). The entropy equation can be derived from (13) and has the form:

$$
(v - D)\frac{dS}{d\xi} = \frac{\rho\pi E_w}{E_S} + \frac{1}{\varepsilon E_S}\left(\rho(v - D)c + \rho E_w - \rho_0(v_0 - D)c_0 + \rho_0(E_w)^0\right)^2.
$$

Continuous solutions can exist due to the presence of dissipative mechanisms in the model, such as phase diffusion and interfacial friction. We shall study discontinuous solutions called shock waves. These solutions can be found from the special case of system (14), where dissipation is neglected:

$$
\begin{aligned}
&\rho(v - D) = \rho_0(v_0 - D) = m,\\
&mc + \rho E_w = mc_0 + \rho_0(E_w)^0,\\
&\frac{m^2}{\rho} + p + \rho w E_w = \frac{m^2}{\rho_0} + p_0 + \rho_0 w_0(E_w)^0,\\
&\frac{mw}{\rho} + E_c = \frac{mw_0}{\rho_0} + (E_c)^0,
\end{aligned}
\tag{15}
$$

$$m\left(E+\frac{v^2}{2}\right)+pv+\rho vwE_w+\rho E_cE_w=$$
$$m\left(E_0+\frac{v_0^2}{2}\right)+p_0v_0+\rho_0v_0w_0(E_w)^0+\rho_0(E_c)^0(E_w)^0.$$

Here m denotes the mass flux through the shock wave. This is a system of non-linear algebraic equations, where the unknowns are the parameters of state behind of the wave ρ, c, v, w, S and the velocity of the wave D. If one of the above parameters is given, then the other can be found from system (15).

Rewrite now system (15) using the specific form of thermodynamic potential

$$E(\rho,c,w,S)=e(\rho,c,S)+c(1-c)\frac{w^2}{2}.$$

Note that the third equation for the jump of momentum can be written in the equivalent form

$$m(v-v_0)+p+\rho c(1-c)w^2-p_0-\rho_0c_0(1-c_0)w_0^2=0. \tag{16}$$

To transform the equation for energy we use the following formula obtained with the help of equation (16)

$$\begin{gathered}m\tfrac{v^2}{2}+pv+v\rho c(1-c)w^2-m\tfrac{v_0^2}{2}-p_0v_0-v_0\rho_0c_0(1-c_0)w_0^2=\\ \tfrac{1}{2}m(v-v_0)(v+v_0)+\tfrac{1}{2}(p-p_0)(v+v_0)+\tfrac{1}{2}(p+p_0)(v-v_0)+\\ v\rho c(1-c)w^2-v_0\rho_0c_0(1-c_0)w_0^2=\\ \tfrac{1}{2}(p+\rho c(1-c)w^2+p_0+\rho_0c_0(1-c_0)w_0^2)(v-v_0).\end{gathered}$$

After the above transformations we obtain a new form of the system (15), which is convenient to analyse its solution:

$$\begin{aligned}&\rho(v-D)=\rho_0(v_0-D)=m,\\ &mc+\rho c(1-c)w=mc_0+\rho_0c_0(1-c_0)w_0,\\ &\frac{m^2}{\rho}+p+\rho c(1-c)w^2=\frac{m^2}{\rho_0}+p_0+\rho_0c_0(1-c_0)w_0^2,\\ &\frac{mw}{\rho}+e_c+(1-2c)\frac{w^2}{2}=\frac{mw_0}{\rho_0}+(e_c)^0+(1-2c_0)\frac{w_0^2}{2},\\ &e+c(1-c)\frac{w^2}{2}+\frac{\rho c(1-c)w}{m}\left(e_c+(1-2c)\frac{w^2}{2}\right)+\\ &\frac{1}{2}(p+\rho c(1-c)w^2+p_0+\rho_0c_0(1-c_0)w_0^2)\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)-\\ &e_0-c_0(1-c_0)\frac{w_0^2}{2}-\frac{\rho_0c_0(1-c_0)w_0}{m}\left((e_c)^0+(1-2c_0)\frac{w_0^2}{2}\right)=0.\end{aligned} \tag{17}$$

Before studying the possible shock waves it is interesting to study the eigenvalues of the one dimensional system (12) without dissipation. As was noted in the previous section the model under consideration can be reduced to symmetric form, hence eigenvalues of the characteristic polynomial are real. If dissipative effects are neglected, then system (12) in quasi-linear form is

$$\frac{\partial\rho}{\partial t}+v\frac{\partial\rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0,$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{(p_\rho+c(1-c)w^2)}{\rho}\frac{\partial \rho}{\partial x}+\frac{(p_c+\rho(1-2c)w^2)}{\rho}\frac{\partial c}{\partial x}+2c(1-c)w\frac{\partial w}{\partial x}+\frac{p_S}{\rho}\frac{\partial S}{\partial x}=0,\\
&\frac{\partial c}{\partial t}+v\frac{\partial c}{\partial x}+\frac{c(1-c)w}{\rho}\frac{\partial \rho}{\partial x}+(1-2c)w\frac{\partial c}{\partial x}+c(1-c)\frac{\partial w}{\partial x}=0,\\
&\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}+w\frac{\partial v}{\partial x}+e_{c\rho}\frac{\partial \rho}{\partial x}+(e_{cc}-w^2)\frac{\partial c}{\partial x}+(1-2c)w\frac{\partial w}{\partial x}+e_{cS}\frac{\partial S}{\partial x}=0,\\
&\frac{\partial S}{\partial t}+v\frac{\partial S}{\partial x}=0.
\end{aligned}
\tag{18}
$$

The full analysis of eigenstructure of the system (18) is quite complicated. But we are interested in a qualitative analysis only, that is why we study now the simplest case when the relative velocity of phases vanishes: $w=0$. It allows to derive explicit formulae for characteristics. Under such an assumption the above quasi-linear system takes the form:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0,\\
&\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{p_\rho}{\rho}\frac{\partial \rho}{\partial x}+\frac{p_c}{\rho}\frac{\partial \rho}{\partial x}+\frac{p_S}{\rho}\frac{\partial S}{\partial x}=0,\\
&\frac{\partial c}{\partial t}+v\frac{\partial c}{\partial x}+c(1-c)\frac{\partial w}{\partial x}=0,\\
&\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}+e_{c\rho}\frac{\partial \rho}{\partial x}+e_{cc}\frac{\partial c}{\partial x}+e_{cS}\frac{\partial S}{\partial x}=0,\\
&\frac{\partial S}{\partial t}+v\frac{\partial S}{\partial x}=0.
\end{aligned}
$$

The eigenvalues λ of the coefficient matrix are the roots of the equation

$$
\begin{vmatrix}
v-\lambda & \rho & 0 & 0 & 0\\
\frac{p_\rho}{\rho} & v-\lambda & \frac{p_c}{\rho} & 0 & \frac{p_S}{\rho}\\
0 & 0 & v-\lambda & c(1-c) & 0\\
e_{c\rho} & 0 & e_{cc} & v-\lambda & e_{cS}\\
0 & 0 & 0 & 0 & v-\lambda
\end{vmatrix}=0,
$$

which can be written in the form

$$
\Lambda(\Lambda^4-(p_\rho+c(1-c)e_{cc})\Lambda^2+c(1-c)(p_\rho e_{cc}-p_c e_{\rho c}))=0,
$$

where $\Lambda=u-\lambda$. There are five roots of this equation. One of them is $\lambda=u$, and the other four roots are determined by formula

$$
\Lambda^2=\frac{1}{2}(p_\rho+c(1-c)e_{cc}\pm\sqrt{\mathcal{D}}),
$$

where

$$
\mathcal{D}=(p_\rho+c(1-c)e_{cc})^2-4c(1-c)(p_\rho e_{cc}-p_c e_{\rho c}).
$$

To prove that the eigenvalues are real it is more convenient the other form of coefficients in the characteristic equation. Passing to the variable specific volume $V=1/\rho$ instead of density ρ we obtain

$$
p_\rho=-\frac{1}{\rho^2}p_V=V^2e_{VV},\quad e_{\rho c}=-V^2e_{Vc},\quad p_c=-e_{Vc},
$$

$$
p_\rho+c(1-c)e_{cc}=V^2(e_{VV}+\frac{c(1-c)}{V^2}e_{cc})^2
$$

and

$$\frac{1}{V^4}\mathcal{D} = (e_{VV} + \frac{c(1-c)}{V^2}e_{cc})^2 - \frac{4c(1-c)}{V^2}(e_{VV}e_{cc} - e_{Vc}^2) =$$
$$(e_{VV} - \frac{c(1-c)}{V^2}e_{cc})^2 + \frac{4c(1-c)}{V^2}e_{Vc}^2 > 0.$$

¿From the last expression we can see that Λ^2 are real. Furthermore, we have supposed that the equation of state $e(\rho, c, S)$ is a convex function of $V = 1/\rho, c, S$, that is $e_{VV}e_{cc} - e_{Vc}^2 > 0$. Hence, we can conclude that

$$\Lambda^2 = \frac{1}{2}(p_\rho + c(1-c)e_{cc} - \sqrt{\mathcal{D}}) > 0,$$

and so the considered one-dimensional system is hyperbolic with eigenvalues

$$\lambda_1 = v - c_2, \lambda_2 = v - c_1, \lambda_3 = v, \lambda_4 = v + c_1, \lambda_5 = v + c_2,$$

where v is the velocity of the flow and c_1, c_2 are the velocities of sound expressed by formulae

$$c_1^2 = \frac{1}{2}(p_\rho + c(1-c)e_{cc} - \sqrt{\mathcal{D}}), \quad c_2^2 = \frac{1}{2}(p_\rho + c(1-c)e_{cc} + \sqrt{\mathcal{D}})$$

Hence, it should be expected that in general there can be two types (fast and slow) of shock waves in a two-phase medium. In the next section we shall prove this fact through the analysis of small amplitude shocks.

3.2 Shock waves of small amplitude

In order to obtain information concerning the character of the possible shocks we shall study shock waves of small amplitude. The analysis of this problem allows us to describe the qualitative behaviour of such discontinuities. For example, the investigation of small amplitude shocks in gas dynamics gives approximate formulae for the variation of parameters of state, the velocity of wave propagation, etc. (see for example Courant & Friedrichs 1985).

We shall study system (17) and note that it is more convenient to use the specific volume $V = 1/\rho$ as the parameter of state instead of density ρ. Then the system (17) can be written in the form

$$\begin{aligned}
&\frac{v-D}{V} = \frac{v_0 - D}{V_0} = m,\\
&mc + \frac{c(1-c)w}{V} = mc_0 + \frac{c_0(1-c_0)w_0}{V_0},\\
&m^2V + p + \frac{c(1-c)}{V}w^2 = m^2V_0 + p_0 + \frac{c_0(1-c_0)}{V_0}w_0^2,\\
&mVw + e_c + (1-2c)\frac{w^2}{2} = mV_0w_0 + (e_c)^0 + (1-2c_0)\frac{w_0^2}{2},\\
&e + c(1-c)\frac{w^2}{2} + \frac{c(1-c)w}{mV}\left(e_c + (1-2c)\frac{w^2}{2}\right) -\\
&e_0 - c_0(1-c_0)\frac{w_0^2}{2} - \frac{c_0(1-c_0)w_0}{mV_0}\left((e_c)^0 + (1-2c_0)\frac{w_0^2}{2}\right) +\\
&\frac{1}{2}(p + \frac{c(1-c)}{V}w^2 + p_0 + \frac{c_0(1-c_0)}{V_0}w_0^2)\,(V - V_0) = 0,
\end{aligned} \tag{19}$$

where $p = -e_V$. Further, for the sake of the simplicity we study a shock wave propagating in the two-phase medium where there is no relative motion,

$$w_0 = 0.$$

Moreover, we suppose that the the medium ahead of the shock is not a pure phase, that is

$$0 < c_0 < 1.$$

The cases $c_0 = 0$ or $c_0 = 1$ lead to a degenerate form of the equations and we consider them separately. Substituting $w_0 = 0$ into equations (19) we obtain:

$$\begin{aligned}
&\frac{v - D}{V} = \frac{v_0 - D}{V_0} = m,\\
&mc + \frac{c(1-c)w}{V} = mc_0,\\
&m^2V + p + \frac{c(1-c)}{V}w^2 = m^2V_0 + p_0,\\
&mVw + e_c + (1-2c)\frac{w^2}{2} = (e_c)^0,\\
&e + c(1-c)\frac{w^2}{2} + \frac{c(1-c)w}{mV}\left(e_c + (1-2c)\frac{w^2}{2}\right) - e_0 +\\
&\frac{1}{2}(p + p_0 + \frac{c(1-c)}{V}w^2)\,(V - V_0) = 0.
\end{aligned} \tag{20}$$

Suppose that the mass flux m is constant and assume that the parameters of state ahead and behind the shock wave are connected by the following relations

$$V = V_0 - \Delta V, \quad c = c_0 - \Delta c, \quad w = \Delta w, \quad S = S_0 + \Delta S,$$

where $\Delta V > 0, \Delta c, \Delta w, \Delta S$ are sufficiently small quantities. We shall draw a conclusion about the shock wave using a linearization of the system (20). Such linearization can be done with the help of the approximate formulae:

$$c(1-c) = c_0(1-c_0) - (1-2c_0)\Delta c, \quad \frac{1}{V} = \frac{1}{V_0} + \frac{\Delta V}{V_0^2},$$

$$\frac{c(1-c)}{V} = \frac{c_0(1-c_0)}{V_0} + \frac{c_0(1-c_0)}{V_0^2}\Delta V - \frac{1-2c_0}{V_0}\Delta c, \quad w = \Delta w, \quad w^2 = (\Delta w)^2.$$

First of all, we derive the formula for the entropy variation behind the shock wave. To do it we consider the last equation of the system (20), which can be rewritten using the second equation of the system (20)

$$\frac{c(1-c)w}{mV} = c_0 - c$$

in the form:

$$e - e_0 + \frac{c(1-c)w^2}{2} - (c - c_0)\left(e_c + (1-2c)\frac{w^2}{2}\right) +$$

$$\frac{1}{2}(p + p_0 + \frac{c(1-c)}{V}w^2)\,(V - V_0) = 0.$$

Expanding each term of this equation we can neglect all terms of the second and third order containing the entropy variation. We can do it because the term of the first order containing the entropy variation exists in the expansion. As a result we obtain

$$e_S\Delta S - e_V\Delta V - e_c\Delta c + e_{VV}(\Delta V)^2 + e_{cc}(\Delta c)^2 + e_{Vc}\Delta V\Delta c + \frac{1}{2}c_0(1-c_0)(\Delta w)^2$$

$$-\frac{1}{6}e_{VVV}(\Delta V)^3 - \frac{1}{6}e_{ccc}(\Delta c)^3 - \frac{1}{2}e_{VVc}(\Delta V)^2\Delta c - \frac{1}{2}e_{Vcc}\Delta V(\Delta c)^2 +$$

$$\Delta c(e_c - e_{cV}\Delta V - e_{cc}\Delta c +$$

$$\frac{1}{2}e_{cVV}(\Delta V)^2 + \frac{1}{2}e_{ccc}(\Delta c)^2 + e_{Vcc}\Delta V\Delta c + \frac{1}{2}(1-2c_0)(\Delta w)^2) -$$

$$\frac{1}{2}\Delta V(2p_0 + e_{VV}\Delta V + e_{Vc}\Delta c - \frac{1}{2}e_{VVV}(\Delta V)^2 - \frac{1}{2}e_{Vcc}(\Delta c)^2 - e_{VVc}\Delta V\Delta c) = 0.$$

Here all derivatives of equation of state are calculated at ρ_0, c_0, S_0. Taking into account that $p_0 = -e_V$ we obtain after some transformation

$$e_S\Delta S = \frac{1}{2}e_{cc}(\Delta c)^2 + \frac{1}{2}e_{Vc}\Delta V\Delta c - \frac{1}{2}c_0(1-c_0)(\Delta w)^2 - \frac{1}{12}e_{VVV}(\Delta V)^3 -$$

$$\frac{1}{3}e_{ccc}(\Delta c)^3 - \frac{3}{4}e_{Vcc}\Delta V(\Delta c)^2 - \frac{1}{2}e_{VVc}(\Delta V)^2\Delta c - \frac{1}{2}(1-2c_0)\Delta c(\Delta w)^2. \tag{21}$$

Thus the formula for the entropy variation behind the shock wave of small amplitude is derived. Furthermore, we prove that the terms of the second order vanish due to the other linearized equations of the system (20). As a result we will obtain that the entropy variation is proportional to the cube of variations of other parameters of state.

Note, that the formula (21) allows us to neglect entropy variation terms in the expansion for other equations of the system (20). That is why the linearization of second, third, and fourth equations of the system (20) can be written in the form

$$\begin{aligned} &-m\Delta c + c_0(1-c_0)V_0^{-1}\Delta w = 0,\\ &(e_{VV} - m^2)\Delta V + e_{Vc}\Delta c = 0,\\ &-e_{Vc}\Delta V - e_{cc}\Delta c + mV_0\Delta w = 0. \end{aligned} \tag{22}$$

Excluding Δw from the above system one can obtain the system

$$\begin{aligned} &(e_{VV} - m^2)\Delta V + e_{Vc}\Delta c = 0,\\ &c_0(1-c_0)e_{Vc}\Delta V + (c_0(1-c_0)e_{cc} - m^2V_0^2)\Delta c = 0 \end{aligned} \tag{23}$$

The system (23) has a nontrivial solution if the determinant of the system is equal to zero:

$$\begin{vmatrix} e_{VV} - m^2 & e_{Vc} \\ c_0(1-c_0)e_{Vc} & c_0(1-c_0)e_{cc} - m^2V_0^2 \end{vmatrix} = 0.$$

This equation can be written in the form of a biquadratic equation

$$m^4 - \left(e_{VV} + \frac{c_0(1-c_0)}{V_0^2}e_{cc}\right)m^2 + \frac{c_0(1-c_0)}{V_0^2}(e_{VV}e_{cc} - e_{Vc}^2) = 0,$$

from which the possible values for the mass flux m can be found from the equation

$$m^2 = \frac{1}{2}\left(e_{VV} + \frac{c_0(1-c_0)}{V_0^2}e_{cc}\right) \pm$$

$$\frac{1}{2}\sqrt{(e_{VV}+\frac{c_0(1-c_0)}{V_0^2}e_{cc})^2-4\frac{c_0(1-c_0)}{V_0^2}(e_{VV}e_{cc}-e_{Vc}^2)}. \tag{24}$$

Hence if we suppose that the shock wave is determined by the jump of the specific volume V, then the other parameters of state (except the entropy) behind the shock can be obtained with the help of (22)–(24) by formulas:

$$\Delta c=-\frac{e_{VV}-m^2}{e_{Vc}}\Delta V,\quad \Delta w=\frac{mV_0}{c_0(1-c_0)}\frac{e_{VV}-m^2}{e_{Vc}}\Delta V,\quad \Delta v=-m\Delta V.$$

Finally, we can conclude that there are two possible shock waves of small amplitude. The velocities of these waves are close to the velocities of two sound waves. In fact, it follows from the first equation of the system (20), that

$$D=v_0-mV_0,$$

and using formulae for eigenvalues (see previous section) we can conclude that $D=c_1$ or $D=c_2$, hence the velocity of the shock is equal to the speed of sound.

Consider again the equation (21) for the entropy variation. Using the expression

$$\Delta w=\frac{mV_0}{c_0(1-c_0)}\Delta c,$$

which can be obtained from the first equation of system (22) we have for the second order terms of the equation (21):

$$e_{cc}(\Delta c)^2+e_{Vc}\Delta V\Delta c-c_0(1-c_0)(\Delta w)^2=\frac{c_0(1-c_0)e_{cc}-m^2V_0^2}{c_0(1-c_0)}(\Delta c)^2+e_{Vc}\Delta V\Delta c.$$

Obviously this expression is equal to zero due to the last equation of system (23). Therefore, for the entropy variation we have the following expression:

$$e_S\Delta S=-\frac{1}{12}e_{VVV}(\Delta V)^3-\frac{1}{3}e_{ccc}(\Delta c)^3-$$

$$\frac{3}{4}e_{Vcc}\Delta V(\Delta c)^2-\frac{1}{2}e_{VVc}(\Delta V)^2\Delta c-\frac{1}{2}(1-2c_0)\Delta c(\Delta w)^2. \tag{25}$$

So, the entropy variation is proportional to the cube of variations of other parameters of state. The laws of thermodynamics require the positiveness of ΔS. This requirement leads to some restriction on the third derivatives of the equation of state, but it is difficult to draw any useful conclusions from formula (25).

4 On the transition between a pure phase and the mixture: the phase vacuum problem

An interesting special problem has initial conditions consisting of two constant states (Riemann problem) and in which in addition one initial state consists of a mixture and the other initial state is a pure phase. Generally, such a flow can be an intrusion (separation) of one phase into (from) the other. An example is the propagating of a shock wave through a surface separating a pure liquid and pure gas. In practice such a shock wave cause intensive mixing of the phases in the vicinity of the interface.

The governing differential equations of existing models of two-phase media described in Section 2 degenerate when the concentration of one phase vanishes. A pragmatic approach to avoid this degeneracy consists of artificially introducing a very small concentration value for the missing phase. The physical and mathematical validity of the above procedure is not obvious but for the simpler situation of water and no water this is an incorrect approach (Toro 2001). In the context of two-phase compressible flow we pose the following question: is it possible to construct a solution providing the transition between a pure phase and the mixture of two phases?

We consider models governed hyperbolic equations without dissipation. Such equations in one-dimensional case admit two types of solutions, which could provide a transition between the pure phase and the mixture. These solutions are a centred wave smooth solution depending on the variable $\xi = x/t$, and a shock-type discontinuity. As was noted earlier, the non-conservative single pressure models described in Section 2 has no strictly defined shock-type solutions. That is why, for this model, the only possibility is to try to construct a smooth centred wave, providing the transition between the pure phase and the mixture.

4.1 Centred waves

Let us consider the simplified submodel consisting of the four isentropic equations of the single pressure model. This can be derived from system (1) by neglecting the equations for energy and consists of the mass and momentum balance laws for each phase:

$$\frac{\partial \alpha^i \rho^i}{\partial t} + \frac{\partial \alpha^i \rho^i v^i_k}{\partial x_k} = 0,$$
$$\frac{\partial \alpha^i \rho^i v^i_k}{\partial t} + \frac{\partial \alpha^i \rho^i v^i_k v^i_j}{\partial x_j} + \alpha^i \frac{\partial p^i}{\partial x_k} = F^i_k.$$

Here α^i are volume concentrations ($\alpha^1 + \alpha^2 = 1$), ρ^i are densities and v^i_k are velocities of phases. The closing assumption is the equality of pressures $p^1 = p^2 = p$ and hence the closing relationships are $\rho^1 = \rho(p), \rho^2 = \rho(p)$. As was mentioned in Section 2, differential source terms F^i are associated with interfacial forces of different nature and provide the hyperbolicity of the system:

$$F^1_j = -k\rho\alpha^1\alpha^2 \left(\frac{d^1 v^2_j}{dt} - \frac{d^2 v^1_j}{dt} \right) + \alpha^1\alpha^2(\alpha^1\rho^2 - \alpha^2\rho^1)(v^1_i - v^2_i)\frac{\partial (v^1_i - v^2_i)}{\partial x_j} -$$

$$-\alpha^1\alpha^2(\rho^1 + \rho^2)(v^1_j - v^2_j)(v^1_i - v^2_i)\frac{\partial \alpha_1}{\partial x_j} - \alpha^1\alpha^2(\rho^1 + \rho^2)(v^1_j - v^2_j)\left(\frac{\alpha^1}{\rho^1}\frac{d^1\rho^1}{dt} + \frac{\alpha^2}{\rho^2}\frac{d^2\rho^2}{dt} \right).$$

We attempt to find a centred wave solution for the one-dimensional version of the above equations, which are:

$$\frac{\partial \alpha^1 \rho^1}{\partial t} + \frac{\partial \alpha^1 \rho^1 v^1}{\partial x} = 0,$$
$$\frac{\partial \alpha^2 \rho^2}{\partial t} + \frac{\partial \alpha^2 \rho^2 v^2}{\partial x} = 0,$$
$$\frac{\partial (\alpha^1 \rho^1 v^1 + \alpha^1 \rho^1 v^1)}{\partial t} + \frac{\partial (\alpha^1 \rho^1 (v^1)^2 + \alpha^2 \rho^2 (v^2)^2 + p)}{\partial x} = 0,$$
$$\frac{\partial \alpha^1 \rho^1 v^1}{\partial t} + \frac{\partial \alpha^1 \rho^1 (v^1)^2}{\partial x} + \alpha^1 \frac{\partial p}{\partial x} = F^1,$$

where $\rho^1 = \rho^1(p), \rho^2 = \rho^2(p)$.

For the sake of simplicity the momentum exchange force F^1 has been chosen in the simplest form when $k = 0$ (the conclusion for the case $k \neq 0$ is similar to the case under consideration):

$$F^1 = \alpha^1\alpha^2(\alpha^1\rho^2 - \alpha^2\rho^1)(v^1 - v^2)\frac{\partial(v^1 - v^2)}{\partial x} - \alpha^1\alpha^2(\rho^1 + \rho^2)(v^1 - v^2)^2\frac{\partial\alpha_1}{\partial x} +$$

$$+\alpha^1\alpha^2(\rho^1 + \rho^2)(v^1 - v^2)\frac{\partial(\alpha^1 v^1 + \alpha^2 v^2)}{\partial x}.$$

A centred wave is a smooth solution of one-dimensional equations depending only on the variable $\xi = x/t$. A system of ordinary differential equations can be derived to find such a solution:

$$\begin{aligned}
&-\rho^1(v^1 - \xi)\frac{d\alpha}{d\xi} + (1-\alpha)\phi^1(v^1 - \xi)\frac{dp}{d\xi} + (1-\alpha)\rho^1\frac{dv^1}{d\xi} = 0,\\
&\rho^2(v^2 - \xi)\frac{d\alpha}{d\xi} + \alpha\phi^2(v^2 - \xi)\frac{dp}{d\xi} + \alpha\rho^2\frac{dv^2}{d\xi} = 0,\\
&\frac{dp}{d\xi} + (1-\alpha)\rho^1(v^1 - \xi)\frac{dv^1}{d\xi} + \alpha\rho^2(v^2 - \xi)\frac{dv^2}{d\xi} = 0,\\
&(1-\alpha)\frac{dp}{d\xi} + (1-\alpha)\rho^1(v^1 - \xi)\frac{dv^1}{d\xi} = F^1.
\end{aligned} \tag{26}$$

Here $\alpha = \alpha_2$, $\phi^1 = \frac{d\rho^1(p)}{dp}$, $\phi^2 = \frac{d\rho^2(p)}{dp}$. Momentum exchange force F^1 is also a combination of the derivatives with respect to ξ:

$$F^1 = \alpha(1-\alpha)(v^1 - v^2)((1-2\alpha)\rho^1 + 2(1-\alpha)\rho^2)\frac{dv^1}{d\xi} +$$

$$\alpha(1-\alpha)(v^1 - v^2)(2\alpha\rho^1 - (1-2\alpha)\rho^2)\frac{dv^2}{d\xi}.$$

System (26) admits a nontrivial solution if its determinant is equal to 0. Such an equality gives the equation for ξ, and gives for ξ a dependence on the parameters of state inside the centred wave:

$$\xi = \frac{x}{t} = \lambda(\alpha, p, v^1, v^2).$$

There are four possible values for λ, which correspond to eigenvalues of original hyperbolic system and connected with speed of sound (Städtke 2001).

To construct a centred wave solution we should reject one equation or some linear combinaton of equations in the system (26) and operate with the three equations system. The convenient resulting system for a centred wave solution can be written in the form:

$$\begin{aligned}
&-\rho^1(v^1 - \lambda)d\alpha + (1-\alpha)\phi^1(v^1 - \lambda)dp + (1-\alpha)\rho^1 dv^1 = 0,\\
&\rho^2(v^2 - \lambda)d\alpha + \alpha\phi^2(v^2 - \lambda)dp + \alpha\rho^2 dv^2 = 0,\\
&(\rho^1(v^2 - \xi) - (v^1 - v^2)A)dv^1 - (\rho^2(v^1 - \xi) + (v^1 - v^2)B)dv^2 = 0,
\end{aligned}$$

where $A = (1-2\alpha)\rho^1 + 2(1-\alpha)\rho^2$, $B = 2\alpha\rho^1 - (1-2\alpha)\rho^2$. The third equation can be derived from the last two equations of the system (26) by eliminating the pressure p.

Let us suppose that $p \in [p_0, p_1]$ is the parameter characterizing the centred wave. Then the equation for the concentration α can be derived:

$$\frac{d\alpha}{dp} = \alpha(1-\alpha)Q(\alpha, p, v^1, v^2),$$

where $Q \neq 0$ if $\alpha = 0$ or $\alpha = 1$. The conclusion is this: if $\alpha = 0$ for $p = p_0$ then $\alpha = 0$ at $p \in [p_0, p_1]$. Hence, there is no centred wave connecting the pure phase ($\alpha = 0$) and the mixture ($1 > \alpha > 0$). The conclusion about the constancy of α can also be made for the case $\alpha = 1$ in the initial data.

The analysis of the centred wave solution can also be done for the conservative model without dissipation. The one-dimensional equations for this model are written in the form (18)

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0,\\
&\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{(p_\rho + c(1-c)w^2)}{\rho}\frac{\partial \rho}{\partial x} + \frac{(p_c + \rho(1-2c)w^2)}{\rho}\frac{\partial c}{\partial x} + 2c(1-c)w\frac{\partial w}{\partial x} + \frac{p_S}{\rho}\frac{\partial S}{\partial x} = 0,\\
&\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} + \frac{c(1-c)w}{\rho}\frac{\partial \rho}{\partial x} + (1-2c)w\frac{\partial c}{\partial x} + c(1-c)\frac{\partial w}{\partial x} = 0,\\
&\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x} + w\frac{\partial v}{\partial x} + e_{c\rho}\frac{\partial \rho}{\partial x} + (e_{cc} - w^2)\frac{\partial c}{\partial x} + (1-2c)w\frac{\partial w}{\partial x} + e_{cS}\frac{\partial S}{\partial x} = 0,\\
&\frac{\partial S}{\partial t} + v\frac{\partial S}{\partial x} = 0.
\end{aligned}$$

Proceeding in a similar way to the previous case and searching the centred wave solution $\rho(x/t)$,$v(x/t)$, $c(x/t)$, $w(x/t)$,$S(x/t)$ we can obtain a system of equations for the velocity, mass concentration and relative velocity inside the centred wave:

$$\frac{dv}{d\rho} = G_1, \quad \frac{dc}{d\rho} = c(1-c)G_2, \quad \frac{dw}{d\rho} = G_3,$$

where functions G_i depend on the parameters of state, and $G_2 \neq 0$ if $c = 0$ or $c = 1$. ¿From these equations we can see that there is no centred wave solution connecting the single phase and the mixture of two phases.

In summary we arrive to the conclusion: for both nonconservative single pressure model and conservative model the centred wave smooth solution can not provide the transition between a pure phase and the mixture.

4.2 Mixing discontinuity

As was noted it is impossible to construct a mathematically correct discontinuous solution for a single pressure model that is non-conservative. With the help of a conservative model we discuss the possibility of searching for a solution as a discontinuity separating a pure phase and a mixture of phases and moving into the pure phase.

Consider the degenerate case, when discontinuity separates two states of a two-phase medium, one of which is a mixture and the other is a pure static phase. We study the case $c_0 = 1$ (the analysis for the case $c_0 = 0$ can be done in a similar manner). Substituting $c_0 = 1$ into the second equation of the system (20) we obtain

$$\frac{c(1-c)}{V}w = (1-c)m.$$

¿From above equation we conclude that there are two possible solutions of the system. One of them corresponds to $c = 1$ and represents a shock wave propagating in the medium in which one of the phases vanishes. We study the other case when the solution is determined by the equation

$$\frac{cw}{V} = m. \tag{27}$$

Using (27), the last three equations of the system (20) can be rewritten in the form

$$\begin{aligned} &p - p_0 + \frac{cw^2}{V}\left(1 - c\frac{V_0}{V}\right) = 0, \\ &\frac{w^2}{2} + e_c - (e_c)^0 = 0, \\ &e - e_0 + (1-c)^2\frac{w^2}{2} + (1-c)e_c + \frac{1}{2}(p + p_0 + \frac{c(1-c)w^2}{V})(V - V_0) = 0. \end{aligned} \tag{28}$$

¿From the second equation of system (28) we can see that if the value of parameter n behind the wave differs from its value ahead the wave then the solution with relative velocity $w \neq 0$ exists. Such a solution can be called a "mixing" discontinuity. We again restrict the study to the wave of small amplitude. It means that the parameters of state behind the wave are connected with the parameters ahead of the wave by formulae

$$V = V_0 - \Delta V, \quad c = 1 - \Delta c, \quad w = \Delta w, \quad S = S_0 + \Delta S,$$

where $\Delta V > 0, \Delta c, \Delta w, \Delta S$ are small quantities.

Let us suppose that the wave is determined by the given value of ΔV and other unknown parameters can be determined by solving of system (28). As in the previous section, it is possible to prove that the entropy variation has third order smallness with respect to ΔV. We will present the formula for entropy below. But under such an assumption it is easy to conclude that the first terms of the expansion of the first equation of the system (28) gives the connection between Δc and ΔV:

$$e_{VV}\Delta V + e_{Vc}\Delta c = 0.$$

Now from the second equation of the system (28) it is easy to derive the following formula

$$w^2 = -2\frac{e_{VV}e_{cc} - e_{Vc}^2}{e_{Vc}}\Delta V. \tag{29}$$

Equation (29) gives the expression for relative velocity w behind the wave if the variation of specific volume is prescribed. Note that for thermodynamically correct convex equation of state ($e_{VV}e_{cc} - e_{Vc}^2 > 0$) the compression wave ($\Delta V > 0$) exists if $e_{Vc} < 0$. The analysis of the third equation of the system (28) gives the formula for the entropy variation

$$\begin{aligned} e_S\Delta S = -\frac{1}{12}e_{VVV}(\Delta V)^3 - \frac{1}{3}e_{ccc}(\Delta c)^3 - \\ \frac{3}{4}e_{Vcc}\Delta V(\Delta c)^2 - \frac{1}{2}e_{VVc}(\Delta V)^2\Delta c - \frac{1}{2}\Delta c^2 w^2 + \frac{1}{2V_0}\Delta c\Delta V w^2, \end{aligned} \tag{30}$$

and taking into account (29) we conclude that the entropy variation has third order of smallness.

To find the variation of velocity $\Delta v = v - v_0$ it is necessary to use the first equation of the system (20):

$$\frac{v - D}{V} = \frac{v_0 - D}{V_0} = m.$$

Obviously

$$v_0 + \Delta v - D = mV = mV_0 - m\Delta V, \quad v_0 - D = mV_0,$$

and subtracting these equations one from another we obtain by using (28):

$$\Delta v = -m\Delta V = -w\frac{\Delta V}{V_0}.$$

The expression for w is given by (29). So we have the final formulas for variations of all parameters of state if the variation of specific volume V is prescribed

$$\begin{aligned}
&w^2 = -2\frac{e_{VV}e_{cc} - e_{Vc}^2}{e_{Vc}}\Delta V,\\
&m = \frac{1}{V_0}w,\\
&\Delta c = -\frac{e_{VV}}{e_{Vc}}\Delta V,\\
&\Delta v = -m\Delta V = -w\frac{\Delta V}{V_0},\\
&e_S\Delta S = -\frac{1}{12}e_{VVV}(\Delta V)^3 - \frac{1}{3}e_{ccc}(\Delta c)^3 -\\
&\frac{3}{4}e_{Vcc}\Delta V(\Delta c)^2 - \frac{1}{2}e_{VVc}(\Delta V)^2\Delta c - \frac{1}{2}\Delta c^2 w^2 + \frac{1}{2V_0}\Delta c\Delta V w^2.
\end{aligned}$$

Note that the velocity of small amplitude discontinuity is determined by formula

$$D = v_0 - mV_0 = v_0 - w.$$

Thus we have proved the existence of moving discontinuity which is a compression wave separating the pure phase and the two-phase mixture. The velocity of such a discontinuity can be very small.

The theory of mixing discontinuity solution requires more detailed consideration. For a better understanding of the mathematical and physical meaning of such a solution it is useful to consider interfacial diffusion processes (see Subsection 2.2). This type of discontinuity could be the limit of solutions for the system with diffusion if the coefficient of diffusion tends to 0.

5 Conclusions

In the present paper we have reviewed two different approaches to model two-phase flows: single pressure nonconservative hyperbolic models and phenomenological conservative hyperbolic models. The latter approach seems to be more attractive from the mathematical and numerical viewpoints. Conservative models allow the formulation of discontinuous solutions such as shock waves. There are two type of shocks corresponding to two types of sound wave in two-phase flow. Consideration of the problem of transition between the pure phase and the mixture (phase vacuum problem) shows that it is impossible to construct centred wave providing such a transition. The only possibility to resolve the phase vacuum problem is the mixing discontinuity which can be constructed within the conservative model.

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References

[Baer & Nunziato (1986)] Baer, M. & Nunziato, J. 1986 A two-phase mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials. *Int. J. Multiphase Flow* **12**, 861–889.

[Bdzil, J., at all (1999)] Bdzil, J., Menikoff, R., Son, S., Kapila, A., & Stewart, D. 1999 Two-phase modeling of deflagration-to-detonation transition in granular materials: a critical examination of modeling isuess. *Physics of Fluids* **11**, 378–402.

[Bestion (1990)] Bestion, D. 1990 The physical closure laws in the CATHARE code. *Nucl. Eng. Design*, **124**, 229–245.

[Coquel, Amine, Godlevski, Perthame & Rascle (1997)] Coquel, F., El Amine, K., Godlevski, E., Perthame, B. & Rascle, P. 1997 A numerical method using upwind schemes for the resolution of two-phase flows. *J. of Comp. Phys.* **136**, 272–288.

[Courant & Friedrichs (1985)] Courant, R. & Friedrichs, K.O. 1985 *Supersonic Flow and Shock Waves*, Springer–Verlag.

[Drew, Cheng & Lahey (1979)] Drew, D., Cheng, H. & Lahey, R.T. 1979 The analysis of virtual mass effects in two-phase flow. *Int. J. Multiphase Flow*, **5**, 233–242.

[Drew & Passman (1998)] Drew, D.A. & Passman, S.L. 1998 *Theory of Multicomponent Fluids*, Springer.

[Fitt (1993)] Fitt, A. D. 1993 The character of two-phase gas/particulate flow equations. Appl. Math. Modelling **17**, 338–354.

[Friedrichs (1954)] Friedrichs, K.O. 1954 Symmetric hyperbolic system of linear differential equations. *Comm. Pure Appl. Math.* **7**, 345–392.

[Friedrichs (1958)] Friedrichs, K.O. 1958 Symmetric positive linear differential equations. *Comm. Pure Appl. Math.* **11**, 333-418.

[Gavrilyk & Saurel (2002)] Gavrilyk, S. & Saurel, R. 2002 Mathematical and numerical modeling of two-phase compressible flows with microinertia. *J. of Comp. Phys.* **175**, 326–360.

[Godunov (1987)] Godunov, S.K. 1987 Lois de conservation et integrales d'energie des equations hyperbolic in *Nonlinear Hyperbolic Problems* (ed. Carasso, C., Raviart, P.-A., Serre, D.) Lecture Notes in Mathematics, vol. 1270, pp. 135–149. Springer.

[Godunov & Romenski (1995)] Godunov, S.K. & Romenski, E.I. 1995 Thermodynamics, conservation laws, and symmetric form of differential equations in mechanics of continuous media In *Computational Fluid Dynamics Review 1995* (ed. A. Hafez & S. Oshima) pp. 13–19. John Wiley.

[Godunov & Romenski (1998)] Godunov, S.K. & Romenski, E.I. 1998 *Elements of continuum mechanics and conservation laws*, Nauchnaya Kniga, Novosibirsk (Russian).

[Gouin & Gavrilyuk (1999)] Gouin, H. & Gavrilyuk, S. 1999 Hamilton's Principle and Rankine–Hugoniot Conditions for General Motions of Mixtures. *Meccanica.* (1999) **34**, 39–47.

[Ishii (1975)] Ishii M. 1975 *Thermo-fluid dynamic theory of two-phase flow*, Eyrolles, Paris.

[Müller & Ruggeri (1998)] Müller, I. & Ruggeri, T. 1998 *Rational Extended Thermodynamics* Springer-Verlag.

[Nigmatulin (1991)] Nigmatulin, R. I. 1991 *Dynamics of multiphase media*, Hemisphere.

[Okazava & Unai (1993)] Okazava, N. & Unai, A. 1993 Abstract quasilinear evolution equations in a Hilbert space with application to symmetric hyperbolic systems. *SUT J. Math.* **29**, 263–240

[Ransom & Hicks (1984)] Ransom, V. & Hicks, D. 1984 Hyperbolic two-pressure models for two-phase flow. *J. of Comp. Phys.* **53**, 124–151.

[Resnyansky, Milton & Romensky (1997)] Resnyansky, A.D., Milton, B.E., & Romensky, E.I. 1997 A two-phase shock-wave model of hypervelocity liquid jet injection into air. in *Proc. JSME Centennial Grand Congress (Intl Conf. on Fluid Engnrg), JSME ICFE-97-228* Tokyo, Japan, pp. 943–947.

[Romensky (1998)] Romensky, E.I. 1998 Hyperbolic systems of thermodynamically compatible conservation laws in continuum mechanics. *Math. Comput. Modelling* **28**, 115–130.

[Romensky (2001)] Romensky, E.I. 2001 Thermodynamics and hyperbolic systems of balance laws in continuum mechanics. In *Godunov Methods: Theory and Applications* (ed. E.F. Toro). pp. 745–761. Kluwer Academic/Plenum Publishers.

[Romensky, Zeidan, Slaouti, & Toro (2003)] Romensky, E.I. Zeidan, D. Slaouti & A. Toro, E.F. 2003 Hyperbolic conservative model for compressible two-phase flow. *Isaac Newton Institute for Mathematical Sciences, Cambridge University, UK*, Preprint NI03022-NPA.

[Saurel & Abgrall (1999)] Saurel, R. & Abgrall, R. 1999 A multiphase Godunov method for compressible multifluid and multiphase flows. *J. of Comp. Phys.* **150**, 425–467.

[Saurel & LeMetayer (2001)] Saurel, R. & LeMetayer, O. 2001 A multiphase model for compressible flows with interfaces, shocks, detonation waves and cavitation. *J. Fluid Mech.* **431**, 239–271.

[Serre (1993)] Serre D. 1993 Sur le principe variationnel des équations de la mécanique des fluides parfaits, *Modél. Math. Anal. Numér.* **27**, 739–758.

[Städtke, Worth & Francello (2001)] Städtke, H. Worth, B. & Francello, G. 2001 On the hyperbolic nature of two-phase flow equations: characteristic analysis and related numerical methods. In *Godunov Methods: Theory and Applications* (ed. E.F. Toro). pp. 841–862. Kluwer Academic/Plenum Publishers.

[Stewart & Wendroff (1984)] Stewart, B.H. & Wendroff, B. 1984 Two-phase flow: models and methods. *J. of Comp. Phys.* **56**, 363–409.

[Toro (1989)] Toro, E. F. 1989 Riemann-problem based techniques for computing reactive two-phase flows. In *Third Intl Conf. on Numerical Combustion* (ed. Dervieux & Larrouturrou) Lecture Notes in Physics, vol. 351, pp. 472-481. Springer.

[Toro (1999)] Toro, E. F. 1999 *Riemann Solvers and Numerical Methods for Fluid Dynamics. Second Edition* Springer-Verlag.

[Toro (2001)] Toro, E. F. 2001 *Shock-Capturing Methods for Free-Surface Shallow Flows.* Wiley and Sons Ltd.

[Toumi & Kumbaro (1996)] Toumi, I. & Kumbaro, A. 1996 An approximate linearized Riemann solver for a two-fluid model. *J. of Comp. Phys.* **124**, 286–300.

[Toumi, I., Kumbaro, A. & Paillére, H. (1999)] Toumi, I., Kumbaro, A. & Paillére, H. 1999 Approximate Riemann solvers and flux vector splitting schemes for two-phase flow. In VKI LS 1999-03, Computational Fluid Dynamics.

[Wallis (1982)] Wallis, G. 1982 Review-theoretical models of gas-liquid flows. *J. of Fluids Engineering.* **104**, 279–283.