

# ON LINEARIZED BACKWARD EULER METHOD FOR THE EQUATIONS OF MOTION ARISING IN THE OLDROYD MODEL

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**Abstract.** In this paper, a linearized backward Euler method is discussed for the equations of motion arising in the Oldroyd model of viscoelastic fluids. Some new *a priori* bounds are obtained for the solution under realistically assumed conditions on the data. Further, the exponential decay properties for the exact as well as the discrete solutions are established. Finally, *a priori* error estimates in  $\mathbf{H}^1$  and  $\mathbf{L}^2$ -norms are derived for the the discrete problem which are valid uniformly for all time  $t > 0$ .

**Key words.** Viscoelastic fluids, Oldroyd model, *a priori* bounds, exponential decay, linearized backward Euler method, uniform convergence in time.

**AMS subject classifications.** 35L70, 65M30, 76D05, 78A10.

**1. Introduction.** The motion of an incompressible fluid in a bounded domain  $\Omega$  in  $\mathbb{R}^2$  is described by the following system of partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \sigma + \nabla p = \mathbf{F}(x, t), \quad x \in \Omega, \quad t > 0,$$

$$\nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \quad t > 0,$$

with appropriate initial and boundary conditions. Here,  $\sigma = (\sigma_{ik})$  denotes the stress tensor with  $tr \sigma = 0$ ,  $\mathbf{u}$  represents the velocity vector,  $p$  is the pressure of the fluid and  $\mathbf{F}$  is the external force. The defining relation between the stress tensor  $\sigma$  and the tensor of deformation velocities  $\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{ix_k} + \mathbf{u}_{kx_i})$ , called the equation of state or sometimes the rheological equation, in fact, establishes the type of fluids under consideration. In mid-twentieth century, the models (of viscoelastic fluids) have been proposed which take into account the prehistory of the flow. One such model was proposed by J. G. Oldroyd (ref. [19]) and hence, this model is named after him. In this case, the defining relation has a special form like

$$(1 + \lambda \frac{\partial}{\partial t})\sigma = 2\nu(1 + \kappa\nu^{-1} \frac{\partial}{\partial t})\mathbf{D},$$

where  $\lambda, \nu, \kappa$  are positive constants with  $(\nu - \kappa\lambda^{-1}) > 0$ . Here,  $\nu$  denotes the kinematic viscosity,  $\lambda$  is the relaxation time and  $\kappa$  represents the retardation time. Now the

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equation of motion arising from the two dimensional Oldroyd's model gives rise to the following integro-differential equation

$$(1.1) \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \int_0^t \beta(t - \tau) \Delta \mathbf{u}(x, \tau) d\tau + \nabla p = \mathbf{f}, \quad x \in \Omega, \quad t > 0,$$

and incompressibility condition

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \quad t > 0,$$

with initial and boundary conditions

$$(1.3) \quad \mathbf{u}(x, 0) = \mathbf{u}_0, \quad x \in \Omega, \quad \text{and} \quad \mathbf{u}(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0.$$

Here,  $\Omega$  is a bounded domain in two dimensional Euclidean space  $\mathbb{R}^2$  with smooth boundary  $\partial\Omega$ ,  $\mu = 2\kappa\lambda^{-1} > 0$  and the kernel  $\beta(t) = \gamma \exp(-\delta t)$ , where  $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1})$  and  $\delta = \lambda^{-1}$ . For details of the physical background and its mathematical modelling, we refer to [14], [19] and [20].

Through out this paper, we shall assume that  $\mu = 1$  and the nonhomogeneous term  $\mathbf{f} = 0$ . In fact, assuming conservative force, the function  $\mathbf{f}$  can be absorbed in the pressure term.

As in Temam [23], we recast the above problem (1.1)–(1.3) as an abstract evolution equation in an appropriate function space setting. Let us denote by  $H^m(\Omega)$  the standard Hilbert Sobolev space and by  $\|\cdot\|_m$  the norm defined on it. When  $m = 0$ , we call  $H^0(\Omega)$  as the space of square integrable functions  $L^2(\Omega)$  with the usual norm  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$ . Further, let  $H_0^1(\Omega)$  be the completion of  $C_0^\infty(\Omega)$  under  $H^1(\Omega)$ -norm. In fact, we have a norm  $\|\nabla\phi\|$  on  $H_0^1(\Omega)$  which is equivalent to  $H^1$ -norm. We also use the following function spaces for the vector valued functions.

Define

$$\mathbf{D}(\Omega) := \{\phi \in (C_0^\infty(\Omega))^2 : \nabla \cdot \phi = 0 \text{ in } \Omega\},$$

$$\mathbf{H} := \text{the closure of } \mathbf{D}(\Omega) \text{ in } (L^2(\Omega))^2 \text{ - space}$$

and

$$\mathbf{V} := \text{the closure of } \mathbf{D}(\Omega) \text{ in } (H_0^1(\Omega))^2 \text{ - space.}$$

Note that under some smoothness assumptions on the boundary  $\partial\Omega$ , it is possible to characterize  $\mathbf{V}$  as

$$\mathbf{V} := \{\phi \in (H_0^1(\Omega))^2 : \nabla \cdot \phi = 0 \text{ in } \Omega\}.$$

The spaces of vector functions are indicated by boldface letters, for instance,  $\mathbf{H}_0^1 = (H_0^1(\Omega))^2$ . The inner product on  $\mathbf{H}_0^1$  is denoted by

$$(\nabla\phi, \nabla\mathbf{w}) = \sum_{i=1}^2 (\nabla\phi_i, \nabla w_i)$$

and the norm by

$$\|\nabla\phi\| = \left(\sum_{i=1}^2 \|\nabla\phi_i\|^2\right)^{\frac{1}{2}}.$$

Using Poincaré inequality, it can be shown that the norm on  $\mathbf{H}_0^1$  is equivalent to  $\mathbf{H}^1 = (H^1(\Omega))^2$ - norm. Let  $\mathbf{P}$  denote the orthogonal projection of  $\mathbf{L}^2(\Omega) (= (L^2(\Omega))^2)$  onto  $\mathbf{H}$ . Now the orthogonal complement  $\mathbf{V}^\perp$  of  $\mathbf{V}$  in  $\mathbf{L}^2(\Omega)$  consists of functions  $\phi$  such that  $\phi = \nabla p$  for some  $p \in H^1(\Omega)/R$ . We define the Stokes operator  $A\mathbf{v} = -\mathbf{P}\Delta\mathbf{v}$ ,  $\mathbf{v} \in D(A) = \mathbf{H}^2 \cap \mathbf{V}$ . The Stokes operator is a closed linear selfadjoint positive operator on  $\mathbf{H}$  with densely defined domain  $D(A)$  in  $\mathbf{H}$  and its inverse is compact in  $\mathbf{H}$ , see [23]. Moreover, we set the  $s^{\text{th}}$  power  $A^s$  of  $A$  for every  $s \in \mathbb{R}$ . For  $0 \leq s \leq 2$ ,  $D(A^{s/2})$  is a Hilbert space with the inner product  $(A^{s/2}\mathbf{v}, A^{s/2}\mathbf{w})$  and norm  $\|A^{s/2}\mathbf{v}\| := (A^{s/2}\mathbf{v}, A^{s/2}\mathbf{v})^{1/2}$ . For  $\mathbf{v} \in D(A^{s/2})$ ,  $0 \leq s \leq 2$ , we note that  $\|\mathbf{v}\|_s$  and  $\|A^{s/2}\mathbf{v}\|$  are equivalent. We also define a bilinear operator  $B(\mathbf{u}, \mathbf{v}) = \mathbf{P}((\mathbf{u} \cdot \nabla)\mathbf{v})$ .

With the notations described above, we now rewrite the problem (1.1)–(1.3) in its abstract form as:

‘Find  $\mathbf{u}(t) \in D(A)$  such that for  $t \geq 0$

$$(1.4) \quad \frac{d\mathbf{u}}{dt}(t) + A\mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) + \int_0^t \beta(t-s)A\mathbf{u}(s) ds = 0, \quad t > 0,$$

$$\mathbf{u}(0) = \mathbf{u}_0.$$

In Oldroyd fluid, the stresses after instantaneous cessation of the motion decay like  $\exp(-\lambda^{-1}t)$ , while the velocities of the flow after instantaneous removal of the stresses die out like  $\exp(-\kappa^{-1}t)$ . Therefore, it is of interest to discuss the exponential decay property of the solution of (1.4), and we derive these results in Section 2. For some related studies in the decay of solution of the linear parabolic equations with memory, we refer to [25] and [3].

The main focus of this paper is to discuss the linearized backward Euler method for time discretization of the system of equations (1.1)–(1.3). For the temporal discretization of the above abstract problem (1.4), let  $k$  denote the time step and  $t_n = nk$ . For smooth function  $\phi$  defined on  $[0, \infty)$ , set  $\phi^n = \phi(t_n)$  and  $\bar{\partial}_t\phi^n = (\phi^n - \phi^{n-1})/k$ . For the integral term, we apply the right rectangle rule

$$(1.5) \quad q^n(\phi) = k \sum_{j=1}^n \beta_{n-j} \phi^j \approx \int_0^{t_n} \beta(t_n - s) \phi(s) ds,$$

where  $\beta_{n-j} = \beta(t_n - t_j)$ .

Now the linearized version of the backward Euler method applied to the problem (1.4) determines a sequence of functions  $\{\mathbf{U}^n\}_{n \geq 0} \subset D(A)$  as solutions of

$$(1.6) \quad \bar{\partial}_t \mathbf{U}^n + A\mathbf{U}^n + B(\mathbf{U}^{n-1}, \mathbf{U}^n) + q^n(A\mathbf{U}) = 0, \quad n > 0,$$

$$\mathbf{U}^0 = \mathbf{u}_0.$$

The main objective of this paper is to derive the following result.

**THEOREM 1.** *Let  $u_0 \in D(A)$  and let  $\mathbf{U}^n$  satisfy the equation (1.6). Then there is a constant  $C$  independent of  $k$ , but may depend on  $\|\mathbf{u}_0\|_2$  and  $\Omega$  such that for some  $k_0 > 0$  with  $0 < k < k_0$  and for positive  $\alpha$  with  $0 < \alpha < \min(\delta, \lambda_1)$*

$$\|u(t_n) - \mathbf{U}^n\|_1 \leq C(\|u_0\|_2)e^{-\alpha t_n} k \left( t_n^{-1/2} + \log \frac{1}{k} \right),$$

where  $\lambda_1$  is the least eigenvalue of the Stokes operator  $A$ . Once the Theorem 1 is proved, the proof of the following theorem becomes a routine work. However, we shall only indicate the major steps without proving it in detail.

**THEOREM 2.** *Under the assumptions of Theorem 1, there is a constant  $C$  independent of  $k$ , but may depend on  $\|\mathbf{u}_0\|_2$  and  $\Omega$  such that for some  $k_0 > 0$  with  $0 < k < k_0$  and  $0 < \alpha < \min(\delta, \lambda_1)$*

$$\|u(t_n) - \mathbf{U}^n\| \leq C(\|u_0\|_2)e^{-\alpha t_n} k.$$

Based on the analysis of Ladyzenskaya [16] for the solvability of the Navier-Stokes equations, Oskolkov [20] proved the global existence of unique ‘almost’ classical solutions in finite time interval for the initial and boundary value problem (1.1)–(1.3). The investigations on solvability were further continued by the co-workers of Oskolkov, see [15] and Agranovich and Sobolevskii [1] under various sufficient conditions. In these articles, the regularity results are proved under the assumption of some nonlocal compatibility conditions on the data at  $t = 0$ , which are either hard to verify or difficult to meet in practice. In the present article, we have obtain some new *a priori* bounds for the solution under realistic assumptions on the data. Recently, Sobolevskii [22] discussed the long time behaviour of solution under some stabilizing conditions on the nonhomogeneous forcing function using a mixture of energy arguments and semigroup theoretic approach. When the forcing function is zero, we have derived, in Sections 2 and 3, the exponential decay properties for the exact solution as well as for the discrete solution using energy arguments.

For earlier works on the numerical approximations to the solutions of the problem (1.1)–(1.3), we refer to [2] and [4]. While Akhmatov and Oskolkov [2] applied finite difference scheme to the equation of motion arising in the Oldroyd model, Cannon *et al.* [4] analysed a modified nonlinear Galerkin scheme for a periodic problem using spectral Galerkin procedure. In [4] they discussed the rates of convergence for the semidiscrete approximations keeping time variable continuous. In this article, we have proposed and analysed a time discretization scheme based on linearized modification of the backward Euler method.

The approach of the present article is influenced by the earlier results of Fujita [8], Thomée [24] and references therein on the approximation of semigroups for the

parabolic problems; Okamoto [18] on the spatial discretization and Geveci [9] on the time discretization of the Navier-Stokes equations; and Thomée and Zhang [26] for the time discretization of the linear parabolic integro-differential equations with nonsmooth initial data.

The analysis of this paper is not complete as at each time level, we have still to solve an infinite dimensional problem. Our main intension is to extend the finite element Galerkin analysis of Heywood and Rannacher [11]–[12], Hill and Süli [13] and the semigroup theoretic approach in Okamoto [18] for the Navier-Stokes equations to the present problem. We shall pursue this in future.

**2. Some *a priori* estimates.** For our future use, we make use of the positive definite property ( see, [17], for a definition) of the kernel  $\beta$  of the integral operator in (1.1). This can be seen as a consequence of the following lemma. For a proof, we refer the reader to Sobolevskii ([22], p.1601), McLean and Thomeé [17].

LEMMA 3. *For arbitrary  $\alpha > 0$ ,  $t^* > 0$  and  $\phi \in L^2(0, t^*)$ , the following positive definite property holds*

$$\int_0^{t^*} \left( \int_0^t \exp[-\alpha(t-s)]\phi(s) ds \right) \phi(t) dt \geq 0.$$

Since  $\beta(t) = \gamma e^{-\delta t}$  with  $\gamma > 0$ , therefore, the above result is true for  $\beta(t)$ .

Below, we discuss some *a priori* bounds for the solution  $u$  of (1.4).

LEMMA 4. *Let  $0 < \alpha < \min(\delta, \lambda_1)$  and  $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ . Then, the following estimate holds*

$$\|\mathbf{u}(t)\| \leq e^{-\alpha t} \|\mathbf{u}_0\|, \quad t > 0.$$

Moreover,

$$2\left(1 - \frac{\alpha}{\lambda_1}\right) \int_0^t e^{2\alpha\tau} \|A^{1/2}\mathbf{u}(\tau)\|^2 d\tau \leq \|\mathbf{u}_0\|^2.$$

*Proof.* Setting  $\hat{\mathbf{u}}(t) = e^{\alpha t}\mathbf{u}(t)$  for some  $\alpha > 0$ , we rewrite (1.4) as

$$(2.1) \quad \frac{d}{dt}\hat{\mathbf{u}} - \alpha\hat{\mathbf{u}} + e^{-\alpha t}B(\hat{\mathbf{u}}, \hat{\mathbf{u}}) + A\hat{\mathbf{u}} + \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}A\hat{\mathbf{u}}(\tau) d\tau = 0.$$

Form  $L^2$ -inner product between (2.1) and  $\hat{\mathbf{u}}$ . Note that  $(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), \hat{\mathbf{u}}) = 0$ ,  $(A\mathbf{u}, \mathbf{v}) = (A^{\frac{1}{2}}\mathbf{u}, A^{1/2}\mathbf{v})$  and  $\|\hat{\mathbf{u}}\|^2 \leq \lambda_1^{-1}\|A^{1/2}\hat{\mathbf{u}}\|^2$ , where  $\lambda_1$  is the least eigenvalue of the Stokes operator  $A$ . Then

$$(2.2) \quad \begin{aligned} & \frac{d}{dt}\|\hat{\mathbf{u}}\|^2 + 2\left(1 - \frac{\alpha}{\lambda_1}\right)\|A^{1/2}\hat{\mathbf{u}}\|^2 \\ & + 2 \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(A^{1/2}\hat{\mathbf{u}}(\tau), A^{1/2}\hat{\mathbf{u}}(\tau))d\tau \leq 0. \end{aligned}$$

After integrating (2.2) with respect to time, the third term becomes nonnegative, since  $\delta > \alpha$  and the second term on the left hand side of (2.2) is also nonnegative if  $\alpha < \lambda_1$ . With  $0 < \alpha < \min(\delta, \lambda_1)$ , we find that

$$\|\hat{\mathbf{u}}\| \leq \|\mathbf{u}_0\|.$$

Moreover,

$$2\left(1 - \frac{\alpha}{\lambda_1}\right) \int_0^t e^{2\alpha\tau} \|A^{1/2}\mathbf{u}(\tau)\|^2 d\tau \leq \|\mathbf{u}_0\|^2.$$

This completes the rest of the proof.  $\square$

LEMMA 5. *Under the hypothesis of Lemma 4, the solution  $\mathbf{u}$  of (1.4) satisfies*

$$\|A^{1/2}\mathbf{u}(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha\tau} \|A\mathbf{u}(\tau)\|^2 d\tau \leq C(\|A^{1/2}\mathbf{u}_0\|)e^{-2\alpha t}.$$

*Proof.* Forming  $L^2$ -inner product between (2.1) and  $A\hat{\mathbf{u}}$ , we obtain

$$(2.3) \quad (\hat{\mathbf{u}}_t, A\hat{\mathbf{u}}) + \|A\hat{\mathbf{u}}\|^2 + \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)} (A\hat{\mathbf{u}}(\tau), A\hat{\mathbf{u}}) d\tau = \alpha(\hat{\mathbf{u}}, A\hat{\mathbf{u}}) - e^{-\alpha t} (B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}}).$$

Note that

$$(\hat{\mathbf{u}}_t, A\hat{\mathbf{u}}) = \frac{1}{2} \frac{d}{dt} \|A^{1/2}\hat{\mathbf{u}}\|^2.$$

On integration of (2.3) with respect to time and using Lemma 3 alongwith the definition of  $\beta$ , it follows for  $0 < \alpha \leq \delta$  that

$$(2.4) \quad \|A^{1/2}\hat{\mathbf{u}}(t)\|^2 + 2 \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 d\tau \leq \|A^{1/2}\mathbf{u}_0\|^2 + 2\alpha \int_0^t (\hat{\mathbf{u}}, A\hat{\mathbf{u}}) d\tau - 2 \int_0^t e^{-\alpha\tau} (B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}}) d\tau = \|A^{1/2}\mathbf{u}_0\|^2 + I_1 + I_2.$$

To estimate  $|I_1|$ , we apply Poincaré inequality and Cauchy-Schwarz inequality with  $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ ,  $a, b \geq 0$ ,  $\epsilon > 0$ . Then the use of Lemma 4 yields

$$(2.5) \quad |I_1| \leq C(\alpha, \lambda_1, \epsilon) \int_0^t \|A^{1/2}\hat{\mathbf{u}}(\tau)\|^2 d\tau + \epsilon \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 d\tau \leq C(\alpha, \lambda_1, \epsilon) \|\mathbf{u}_0\|^2 + \epsilon \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 d\tau.$$

For the estimation of  $I_2$ , we apply Hölder's inequality repeatedly with the following form of the Sobolev inequality (see, Temam [23])

$$\|\phi\|_{L^4(\Omega)} \leq C\|\phi\|^{\frac{1}{2}} \|A^{1/2}\phi\|^{\frac{1}{2}}, \quad \phi \in \mathbf{H}^1(\Omega),$$

to obtain

$$\begin{aligned} |(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}})| &\leq \|B(\hat{\mathbf{u}}, \hat{\mathbf{u}})\| \|A\hat{\mathbf{u}}\| \\ &\leq C \|\hat{\mathbf{u}}\|^{\frac{1}{2}} \|A^{1/2}\hat{\mathbf{u}}\| \|A\hat{\mathbf{u}}\|^{\frac{3}{2}}. \end{aligned}$$

Thus,

$$|I_2| \leq C \int_0^t e^{-\alpha\tau} \|\hat{\mathbf{u}}\|^{\frac{1}{2}} \|A^{1/2}\hat{\mathbf{u}}\| \|A\hat{\mathbf{u}}\|^{\frac{3}{2}} d\tau.$$

An application of Young's inequality  $ab \leq \frac{a^p}{\epsilon^{p/q}} + \frac{b^q}{q}$ ,  $a, b \geq 0$ ,  $\epsilon > 0$  and  $\frac{1}{p} + \frac{1}{q} = 1$  yields

$$(2.6) \quad |I_2| \leq C(\epsilon) \int_0^t e^{-4\alpha\tau} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^4 d\tau + \epsilon \int_0^t \|A\hat{\mathbf{u}}\|^2 d\tau.$$

Substituting (2.5)–(2.6) in (2.4), and using  $\epsilon = \frac{1}{2}$ , we find that

$$\|A^{1/2}\hat{\mathbf{u}}(t)\|^2 + \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 d\tau \leq C(\alpha, \lambda_1, \|A^{1/2}\mathbf{u}_0\|) + C \int_0^t e^{-4\alpha\tau} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^4 d\tau.$$

An application of Gronwall's Lemma yields

$$\|A^{1/2}\hat{\mathbf{u}}(t)\|^2 + \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 d\tau \leq C(\alpha, \lambda_1, \|A^{1/2}\mathbf{u}_0\|) \exp \left\{ C \int_0^t e^{-4\alpha\tau} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^2 d\tau \right\}. \blacksquare$$

Using the a priori bounds in Lemma 4 for  $0 < \alpha < \min(\delta, \lambda_1)$ , we obtain the desired result. This completes the proof.  $\square$

**Remark 1.** Based on Faedo-Galerkin method and the *a priori* bounds derived in the above two Lemmas, it is possible to prove the existence of global strong solutions to the problem (1.1)–(1.3). For a similar analysis in the case of Navier-Stokes equations, we refer to Heywood [10], Temam [23], and Ladyzhenskaya [16]. Since the analysis is quite standard, we state without proof the global existence theorem [21].

**THEOREM 6.** *Assume that  $\mathbf{u}_0 \in D(A)$ . Then for any given time  $T > 0$  with  $0 < T \leq \infty$ , there exists a unique strong solution  $\mathbf{u}$  of (1.4) satisfying*

$$\mathbf{u} \in L^2(0, T; D(A)) \cap L^\infty(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{H}),$$

and the initial condition in the sense that

$$\|A^{1/2}(\mathbf{u}(t) - \mathbf{u}_0)\| \mapsto 0, \quad \text{as } t \mapsto 0.$$

Recently, Cannon *et al.* [4] have proved existence of a global weak solution  $\mathbf{u}$  satisfying

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}), \quad T > 0,$$

for a periodic problem, under the assumption that the forcing function  $\mathbf{f} \in L^\infty(0, \infty; \mathbf{L}^2)$  and  $\mathbf{u}_0 \in \mathbf{H}$ . It is easy to extend our analysis to (1.1)–(1.3) with periodic boundary conditions and  $\mathbf{f} = 0$ .

Below, we derive some new regularity results without nonlocal assumptions on the data.

LEMMA 7. *Under the assumptions of Lemma 4, there is a positive constant  $C$  such that*

$$(2.7) \quad \|\mathbf{A}\mathbf{u}(t)\| + \|\mathbf{u}_t\| \leq C(\|\mathbf{A}\mathbf{u}_0\|)e^{-\alpha t}, \quad t > 0,$$

and

$$(2.8) \quad \left( \int_0^t e^{2\alpha s} \|A^{1/2}\mathbf{u}_t(s)\|^2 ds \right)^{1/2} \leq C(\|\mathbf{A}\mathbf{u}_0\|).$$

Further, the following estimate holds

$$(2.9) \quad \|A^{1/2}\mathbf{u}_t(t)\| + \left( \sigma(t) \int_0^t \sigma(s) \|\mathbf{A}\mathbf{u}_t(s)\|^2 ds \right)^{1/2} \leq \frac{C(\|\mathbf{A}\mathbf{u}_0\|)}{(\tau^*(t))^{1/2}} e^{-\alpha t}, \quad t > 0,$$

where  $\sigma(t) = \tau^*(t)e^{2\alpha t}$  and  $\tau^*(t) = \min(t, 1)$ .

*Proof.* From (2.1), we obtain

$$(2.10) \quad e^{\alpha t} \|\mathbf{u}_t\| \leq \|\mathbf{A}\hat{\mathbf{u}}\| + e^{-\alpha t} \|B(\hat{\mathbf{u}}, \hat{\mathbf{u}})\| + \int_0^t \beta(t-s)e^{\alpha(t-s)} \|\mathbf{A}\hat{\mathbf{u}}(s)\| ds.$$

Using the form of  $B$  and the Sobolev inequality, it follows that

$$(2.11) \quad \begin{aligned} \|B(\hat{\mathbf{u}}, \hat{\mathbf{u}})\| &\leq C\|\hat{\mathbf{u}}\|^{\frac{1}{2}} \|A^{1/2}\hat{\mathbf{u}}\| \|\mathbf{A}\hat{\mathbf{u}}\|^{\frac{1}{2}} \\ &\leq C\|\hat{\mathbf{u}}\| \|A^{1/2}\hat{\mathbf{u}}\|^2 + C\|\mathbf{A}\hat{\mathbf{u}}\|. \end{aligned}$$

On squaring (2.10) and integrating with respect to time, we find from (2.11) that

$$(2.12) \quad \int_0^t e^{2\alpha s} \|\mathbf{u}_t\|^2 ds \leq C \left[ \int_0^t \|\mathbf{A}\hat{\mathbf{u}}\|^2 ds + \int_0^t e^{-2\alpha s} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^4 ds \right. \\ \left. + \int_0^t \left( \int_0^s \beta(s-\tau)e^{\alpha(s-\tau)} \|\mathbf{A}\hat{\mathbf{u}}(\tau)\| d\tau \right)^2 ds \right].$$

For the last term on the right hand side of (2.12), use the form of  $\beta$  and Hölder's inequality to obtain

$$\begin{aligned} I &= \int_0^t \left( \int_0^s \beta(s-\tau)e^{\alpha(s-\tau)} \|\mathbf{A}\hat{\mathbf{u}}(\tau)\| d\tau \right)^2 ds \\ &= \gamma^2 \int_0^t \left( \int_0^s e^{-(\delta-\alpha)(s-\tau)} \|\mathbf{A}\hat{\mathbf{u}}(\tau)\| d\tau \right)^2 ds \\ &\leq \gamma^2 \int_0^t \left( \int_0^s e^{-(\delta-\alpha)(s-\tau)} d\tau \right) \left( \int_0^s e^{-(\delta-\alpha)(s-\tau)} \|\mathbf{A}\hat{\mathbf{u}}(\tau)\|^2 d\tau \right) ds \\ &\leq \frac{\gamma^2}{\delta-\alpha} \int_0^t \left( \int_0^s e^{-(\delta-\alpha)(s-\tau)} \|\mathbf{A}\hat{\mathbf{u}}\|^2 d\tau \right) ds. \end{aligned}$$



Using change of variable, we find that

$$I \leq \frac{\gamma^2}{\delta - \alpha} \int_0^t \left( \int_0^s e^{-(\delta - \alpha)\tau} \|A\hat{\mathbf{u}}(s - \tau)\|^2 d\tau \right) ds.$$

Now a change the order of integration yields

$$\begin{aligned} I &\leq \frac{\gamma^2}{\delta - \alpha} \int_0^t e^{-(\delta - \alpha)\tau} \left( \int_\tau^t \|A\hat{\mathbf{u}}(s - \tau)\|^2 ds \right) d\tau \\ &\leq \frac{\gamma^2}{(\delta - \alpha)^2} \int_0^t e^{-(\delta - \alpha)(t - \tau)} \left( \int_0^t \|A\hat{\mathbf{u}}\|^2 ds \right) d\tau, \end{aligned}$$

and hence,

$$(2.13) \quad I \leq \left( \frac{\gamma}{\delta - \alpha} \right)^2 \int_0^t \|A\hat{\mathbf{u}}(s)\|^2 ds.$$

Using (2.13) in (2.12), we arrive at

$$(2.14) \quad \int_0^t e^{2\alpha s} \|\mathbf{u}_t\|^2 ds \leq C \left[ \int_0^t \|A\hat{\mathbf{u}}\|^2 ds + \int_0^t e^{-2\alpha s} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^4 ds \right] \\ \leq C(\|A^{1/2}\mathbf{u}_0\|).$$

Differentiate the equation (1.4) with respect to time, and integrate by parts with respect to the temporal variable for the integral term to obtain

$$(2.15) \quad u_{tt} + A\mathbf{u}_t + \int_0^t \beta(t - s)A\mathbf{u}_s(s) ds = -(B(\mathbf{u}_t, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}_t)) - \beta(t)A\mathbf{u}_0.$$

Forming an inner product between (2.15) and  $e^{2\alpha t}\mathbf{u}_t$ , we arrive at

$$(2.16) \quad \frac{1}{2} \frac{d}{dt} \|e^{\alpha t}\mathbf{u}_t\|^2 + e^{2\alpha t} \|A^{1/2}\mathbf{u}_t\|^2 + \int_0^t \beta(t - s) e^{\alpha(t-s)} (A^{1/2}e^{\alpha s}\mathbf{u}_s, A^{1/2}e^{\alpha t}\mathbf{u}_t) ds \\ = \alpha \|e^{\alpha t}\mathbf{u}_t\|^2 - e^{2\alpha t} ((B(\mathbf{u}_t, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}_t), \mathbf{u}_t) - \beta(t)(A\mathbf{u}_0, \mathbf{u}_t)).$$

Note that  $(B(\hat{\mathbf{u}}, e^{\alpha t}\mathbf{u}_t), e^{\alpha t}\mathbf{u}_t) = 0$ . Thus, it follows after integration of (2.16) with respect to time and using the positivity property of the kernel, i.e., Lemma 3 that

$$(2.17) \quad e^{2\alpha t} \|\mathbf{u}_t\|^2 + 2 \int_0^t e^{2\alpha s} \|A^{1/2}\mathbf{u}_t\|^2 ds \leq \|\mathbf{u}_t(0)\|^2 + 2\alpha \int_0^t e^{2\alpha s} \|u_t\|^2 ds \\ + 2 \int_0^t e^{-\alpha s} |B(e^{\alpha s}\hat{\mathbf{u}}_t, \hat{\mathbf{u}}), e^{\alpha s}\mathbf{u}_t| ds + 2\gamma \|A\mathbf{u}_0\| \int_0^t e^{-(\delta - \alpha)s} \|e^{\alpha s}\mathbf{u}_t\| ds.$$

The last term on the right hand side of (2.17) is bounded by

$$(2.18) \quad \leq C(\alpha, \delta, \gamma) \left[ \|A^{1/2}\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s} \|\mathbf{u}_t\|^2 ds \right].$$

For the second term on the right hand side of (2.17), we have with the help of Sobolev inequality

$$(2.19) \quad 2 \int_0^t e^{-\alpha s} |B(e^{\alpha s}\hat{\mathbf{u}}_t, \hat{\mathbf{u}}), e^{\alpha s}\mathbf{u}_t| ds \leq C \sup_{0 \leq s \leq t} \|A^{1/2}\hat{\mathbf{u}}(s)\|^4 \int_0^t e^{-4\alpha s} (e^{2\alpha s} \|\mathbf{u}_t\|^2) ds \\ + \int_0^t e^{2\alpha s} \|A^{1/2}\mathbf{u}_t\|^2 ds.$$

On substitution of (2.18)–(2.19) in (2.17) and using Lemma 4–5, we obtain

$$(2.20) \quad e^{2\alpha t} \|\mathbf{u}_t\|^2 + \int_0^t e^{2\alpha s} \|A^{1/2} \mathbf{u}_t\|^2 ds \leq C(\delta, \alpha) [\|\mathbf{u}_t(0)\|^2 + \|A\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s} \|\mathbf{u}_t\|^2 ds].$$

From the main equation (1.4), we have at  $t = 0$ ,  $\|\mathbf{u}_t(0)\| \leq C(\|A\mathbf{u}_0\|)$ , and hence, using (2.14) we find that

$$(2.21) \quad \|\mathbf{u}_t\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|A^{1/2} \mathbf{u}_t(s)\|^2 ds \leq C(\|A\mathbf{u}_0\|) e^{-2\alpha t}.$$

To estimate  $\|A\mathbf{u}(t)\|$ , form an inner-product between (2.1) and  $A\hat{\mathbf{u}}(t)$  to obtain

$$(2.22) \quad \|A\hat{\mathbf{u}}\|^2 \leq e^{\alpha t} \|\mathbf{u}_t\| \|A\hat{\mathbf{u}}\| + e^{-\alpha t} |(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}})| + \int_0^t \beta(t-s) e^{\alpha(t-s)} \|A\hat{\mathbf{u}}(s)\| \|A\hat{\mathbf{u}}(t)\| ds.$$

The first two terms on the right hand side of (2.22) are bounded by

$$\leq C(\epsilon) [e^{2\alpha t} \|\mathbf{u}_t\|^2 + e^{-4\alpha t} \|\hat{\mathbf{u}}\|^2 \|A^{1/2} \hat{\mathbf{u}}\|^4] + \epsilon \|A\hat{\mathbf{u}}\|^2.$$

For the last term on the right hand side of (2.22), we have applied the Hölder's inequality with Sobolev inequality. Then the last term is bounded by

$$C(\gamma, \delta, \alpha, \epsilon) \int_0^t e^{2\alpha\tau} \|A\mathbf{u}(\tau)\|^2 d\tau + \epsilon \|A\hat{\mathbf{u}}\|^2.$$

Note that we have used  $e^{-2(\delta-\alpha)(t-s)} \leq 1$ . On substituting in (2.22), we choose  $\epsilon = \frac{1}{4}$ . An appeal to Lemma 4, 5 and the estimate (2.21) yields

$$\|A\hat{\mathbf{u}}\|^2 \leq C(\|A\mathbf{u}_0\|),$$

and thus, we complete the proof of (2.7)–(2.8).

In order to derive (2.9), we now differentiate equation (1.4) with respect to time and then form an inner product with  $\sigma(t)A\mathbf{u}_t$ , where  $\sigma(t) = \tau^*(t)e^{2\alpha t}$  to obtain

$$(2.23) \quad \frac{1}{2} \frac{d}{dt} (\sigma(t) \|A^{1/2} \mathbf{u}_t\|^2) + \sigma(t) \|A\mathbf{u}_t\|^2 = -\sigma(t) (A\mathbf{u}, A\mathbf{u}_t) + \frac{1}{2} \sigma_t \|A^{1/2} \mathbf{u}_t\|^2 - \sigma(t) \int_0^t \beta_t(t-s) (A\mathbf{u}(s), A\mathbf{u}_t(t)) ds - \tau^*(t) e^{-\alpha t} (B(e^{\alpha t} \mathbf{u}_t, \hat{\mathbf{u}}) + B(\hat{\mathbf{u}}, e^{\alpha t} \mathbf{u}_t), e^{\alpha t} A\mathbf{u}_t) = I_1 + I_2 + I_3 + I_4.$$

For  $I_1$ , we use Young's inequality to obtain

$$(2.24) \quad |I_1| \leq \frac{\gamma^2}{2\epsilon} \tau^*(t) \|A\hat{\mathbf{u}}\|^2 + \frac{\epsilon}{2} \sigma(t) \|A\mathbf{u}_t\|^2.$$

Since  $\sigma_t = \tau_t^* e^{2\alpha t} + 2\alpha \tau^* e^{2\alpha t}$  with  $\tau^*, \tau_t^* \leq 1$ , we infer that

$$(2.25) \quad |I_2| \leq C(\alpha) e^{2\alpha t} \|A^{1/2} \mathbf{u}_t\|^2.$$

To estimate  $I_4$ , a use of Sobolev inequality with Young's inequality yields

$$(2.26) \quad |I_4| \leq C(\epsilon)e^{2\alpha t}\|A^{1/2}\mathbf{u}_t\|^2(\|A^{1/2}\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\| + \|A^{1/2}\hat{\mathbf{u}}\|^2) + \epsilon\sigma(t)\|A\mathbf{u}_t\|^2.$$

Since  $\beta_t(t-s) = -\frac{1}{\delta}\beta(t-s)$ , we have for  $I_3$ ,

$$(2.27) \quad |I_3| \leq \frac{\gamma^2}{2\epsilon\delta^2}\tau^*\left(\int_0^t e^{-(\delta-\alpha)(t-s)}\|A\hat{\mathbf{u}}(s)\| ds\right)^2 + \frac{\epsilon}{2}\sigma(t)\|A\mathbf{u}_t\|^2,$$

and hence, integrating with respect to time and using the estimate (2.3) for  $I$  term, we obtain

$$(2.28) \quad \int_0^t |I_3| ds \leq \frac{\gamma^2}{2\epsilon\delta^2}\tau^*I + \frac{\epsilon}{2}\int_0^t \sigma(s)\|A\mathbf{u}_t\|^2 ds \\ \leq C(\gamma, \delta, \alpha, \epsilon)\tau^*(t)\int_0^t \|A\hat{\mathbf{u}}(s)\|^2 ds + \frac{\epsilon}{2}\int_0^t \sigma(s)\|A\mathbf{u}_t(s)\|^2 ds.$$

Multiply (2.23) by 2 and integrate with respect to time. Substitute (2.24)–(2.28) in (2.23). With  $\epsilon = \frac{1}{4}$ , it now follows that

$$(2.29) \quad \sigma(t)\|A^{1/2}\mathbf{u}_t\|^2 + \int_0^t \sigma(s)\|A\mathbf{u}_t(s)\|^2 ds \leq C(\gamma, \delta, \alpha) \left[ \tau^* \int_0^t \|A\hat{\mathbf{u}}(s)\|^2 ds \right. \\ \left. + \int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t\|^2(\|A^{1/2}\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\| + \|A^{1/2}\hat{\mathbf{u}}\|^4) ds \right] \\ + \int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t(s)\|^2 ds.$$

Using Lemmas 4, 5, and the estimates (2.7)–(2.8) in (2.29), we obtain the required result (2.9), and this completes the rest of the proof.  $\square$

**Remark.** The estimate for  $\|A^{1/2}\mathbf{u}_t\|$  shows the singular behaviour near  $t = 0$  and also indicates the exponential decay property as  $t \rightarrow \infty$ . In Lemma 7, the regularity results are derived without any nonlocal compatibility conditions.

**3. Decay properties for the discrete solution and error estimates.** In this section, we discuss the decay properties for the solution of the linearized backward Euler method. Finally, we derive *a priori* bounds for the error in  $H^1$ -norm and present briefly the error estimate in  $L^2$ -norm.

The right hand rectangle rule  $q^n$  which is used to discretize the integral in (1.4) is positive in the sense that

$$k \sum_{n=1}^J q^n(\phi)\phi^n \geq 0 \quad \forall \phi = (\phi^1, \dots, \phi^J)^T.$$

For a proof, we refer to Mclean and Thomée ([17], pp. 40-42). Moreover, the following Lemma is easy to prove using the line of proof of [17].

LEMMA 8. For any  $\alpha \geq 0$ ,  $J > 0$  and sequence  $\{\phi^n\}_{n=1}^\infty$ , the following positivity property holds

$$k^2 \sum_{n=1}^J \left( \sum_{j=1}^n e^{-\alpha(t_n - t_j)} \phi^j \right) \phi^n \geq 0.$$

LEMMA 9. With  $0 < \alpha < \min(\delta, \lambda_1)$ , choose  $k_0 > 0$  small so that for  $0 < k \leq k_0$

$$(\lambda_1 k + 1) > e^{\alpha k}.$$

Then the discrete solution  $U^J$ ,  $J \geq 1$  of (1.6) is exponentially stable in the following sense

$$(3.1) \quad \|\mathbf{U}^J\| + e^{-\alpha t_J} \left( k \sum_{n=1}^J \|A^{1/2} \hat{\mathbf{U}}^n\|^2 \right)^{1/2} \leq C(\lambda_1, \alpha) \|\mathbf{U}^0\| e^{-\alpha t_J}, \quad J \geq 1,$$

and

$$(3.2) \quad \|A^{1/2} \mathbf{U}^J\| \leq C(\lambda_1, \alpha, \|A^{1/2} \mathbf{U}^0\|) e^{-\alpha t_J}, \quad J \geq 1.$$

*Proof.* Setting  $\hat{\mathbf{U}}^n = e^{\alpha t_n} \mathbf{U}^n$ , we rewrite (1.6) as

$$e^{\alpha t_n} \bar{\partial}_t \mathbf{U}^n + A \hat{\mathbf{U}}^n + e^{-\alpha t_{n-1}} B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n) + e^{\alpha t_n} q^n(A\mathbf{U}) = 0.$$

Note that

$$e^{\alpha t_n} \bar{\partial}_t \mathbf{U}^n = e^{\alpha k} \bar{\partial}_t \hat{\mathbf{U}}^n - \left( \frac{e^{\alpha k} - 1}{k} \right) \hat{\mathbf{U}}^n.$$

On substitution and then multiplying the resulting equation by  $e^{-\alpha k}$ , we obtain

$$(3.3) \quad \bar{\partial}_t \hat{\mathbf{U}}^n - \left( \frac{1 - e^{-\alpha k}}{k} \right) \hat{\mathbf{U}}^n + e^{-\alpha k} A \hat{\mathbf{U}}^n + e^{-\alpha t_n} B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n) \\ + \gamma e^{-\alpha k} k \sum_{j=1}^n e^{-(\delta - \alpha)(t_n - t_j)} A \hat{\mathbf{U}}^j = 0.$$

Forming an inner product between (3.3) and  $\hat{\mathbf{U}}^n$ , use

$$(B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n), \hat{\mathbf{U}}^n) = 0, \quad \|\hat{\mathbf{U}}^n\|^2 \leq \frac{1}{\lambda_1} \|A^{1/2} \hat{\mathbf{U}}^n\|^2, \quad \text{and} \quad (\bar{\partial}_t \hat{\mathbf{U}}^n, \hat{\mathbf{U}}^n) \geq \frac{1}{2} \bar{\partial}_t \|\hat{\mathbf{U}}^n\|^2$$

to obtain

$$(3.4) \quad \frac{1}{2} \bar{\partial}_t \|\hat{\mathbf{U}}^n\|^2 + (e^{-\alpha k} - \left( \frac{1 - e^{-\alpha k}}{k} \right) \lambda_1^{-1}) \|A^{1/2} \hat{\mathbf{U}}^n\|^2 \\ + \gamma e^{-\alpha k} k \sum_{j=1}^n e^{-(\delta - \alpha)(t_n - t_j)} (A^{1/2} \hat{\mathbf{U}}^j, A^{1/2} \hat{\mathbf{U}}^n) \leq 0.$$

With  $0 < \alpha < \min(\lambda_1, \delta)$ , choose  $0 < k_0$  such that for  $0 < k < k_0$

$$(\lambda_1 k + 1) \geq e^{\alpha k}.$$

Then for  $0 < k \leq k_0$ , the coefficient of the second term on the left hand side of (3.4):  $\left(e^{-\alpha k} - \left(\frac{1 - e^{-\alpha k}}{k}\right)\lambda_1^{-1}\right)$  becomes positive. Multiplying (3.4) by  $2k$  and summing from  $n = 1$  to  $J$ , the last term becomes nonnegative by Lemma 8 and thus, we obtain the estimate (3.1).

For the estimate (3.2), we form an inner product between (3.3) and  $A\hat{\mathbf{U}}^n$ , and observe that

$$(\bar{\partial}_t \hat{\mathbf{U}}^n, A\hat{\mathbf{U}}^n) = (\bar{\partial}_t A^{1/2} \hat{\mathbf{U}}^n, A^{1/2} \hat{\mathbf{U}}^n) \geq \frac{1}{2} \bar{\partial}_t \|A^{1/2} \hat{\mathbf{U}}^n\|^2.$$

Altogether, we find that

$$(3.5) \quad \frac{1}{2} \bar{\partial}_t \|A^{1/2} \hat{\mathbf{U}}^n\|^2 + e^{-\alpha k} \|A\hat{\mathbf{U}}^n\|^2 + \gamma e^{-\alpha k} k \sum_{j=1}^n e^{-(\delta-\alpha)(t_n - t_j)} (A\hat{\mathbf{U}}^j, A\hat{\mathbf{U}}^n) \\ \leq \left(\frac{1 - e^{-\alpha k}}{k}\right) (\hat{\mathbf{U}}^n, A\hat{\mathbf{U}}^n) - e^{-\alpha t_n} (B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n), A\hat{\mathbf{U}}^n).$$

Multiplying (3.5) by  $2k$  and summing from  $n = 1$  to  $J$ , the third term on the left hand side becomes nonnegative by applying Lemma 8 as  $0 < \alpha < \delta$ . Then, we obtain

$$(3.6) \quad \|A^{1/2} \hat{\mathbf{U}}^J\|^2 + 2ke^{-\alpha k} \sum_{n=1}^J \|A\hat{\mathbf{U}}^n\|^2 \leq \|A^{1/2} \mathbf{U}^0\|^2 + 2(1 - e^{-\alpha k}) k \sum_{n=1}^J |(\hat{\mathbf{U}}^n, A\hat{\mathbf{U}}^n)| \\ + 2e^{-\alpha k} k \sum_{n=1}^J e^{-\alpha t_{n-1}} |(B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^{n-1}), A\hat{\mathbf{U}}^n)| \\ \leq \|A^{1/2} \mathbf{U}^0\|^2 + I_1 + I_2.$$

To estimate  $I_1$ , we have by Mean Value Theorem  $\frac{1 - e^{-\alpha k}}{k} = \alpha e^{-\alpha k^*}$  for some  $0 < k^* < k$ , and hence, using (3.1), we find that

$$|I_1| \leq 2\alpha e^{-\alpha k^*} k \sum_{n=1}^J \|A^{1/2} \hat{\mathbf{U}}^n\|^2 \leq C(\lambda_1, \alpha) \|U^0\|^2.$$

For  $I_2$ , the repeated use of Hölder's inequality with Sobolev inequality yields

$$e^{-\alpha t_{n-1}} |(B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n), A\hat{\mathbf{U}}^n)| \leq C e^{-\alpha t_{n-1}} \|\hat{\mathbf{U}}^{n-1}\|^{1/2} \|A^{1/2} \hat{\mathbf{U}}^{n-1}\|^{1/2} \|A^{1/2} \hat{\mathbf{U}}^n\|^{1/2} \|A\hat{\mathbf{U}}^n\|^{3/2}.$$

By an application of Young's inequality, it follows that

$$|I_2| \leq C k e^{-\alpha k} \sum_{n=1}^J e^{-4\alpha t_{n-1}} (\|\hat{\mathbf{U}}^{n-1}\|^2 \|A^{1/2} \hat{\mathbf{U}}^{n-1}\|^2) \|A^{1/2} \hat{\mathbf{U}}^n\|^2 \\ + k e^{-\alpha k} \sum_{n=1}^J \|A\hat{\mathbf{U}}^n\|^2.$$

Using the estimate  $\|\hat{\mathbf{U}}^{n-1}\|$  and

$$k\|A^{1/2}U^{J-1}\|^2 \leq k \sum_{n=1}^J \|A^{1/2}U^n\|^2$$

from (3.1), we easily find that

$$\begin{aligned} |I_2| &\leq C\|\mathbf{U}^0\|^2 k e^{-\alpha k} \sum_{n=1}^{J-1} e^{-4\alpha t_{n-1}} \|A^{1/2}\hat{\mathbf{U}}^{n-1}\|^2 \|A^{1/2}\hat{\mathbf{U}}^n\|^2 \\ &\quad + C\|\mathbf{U}^0\|^4 e^{-\alpha k} e^{-4\alpha t_{J-1}} \|A^{1/2}\hat{\mathbf{U}}^J\|^2 + k e^{-\alpha k} \sum_{n=1}^J \|A\hat{\mathbf{U}}^n\|^2. \end{aligned}$$

Now substitute the estimates of  $I_1$  and  $I_2$  in (3.6). For small  $k$ , we note that  $(1 - C\|\mathbf{U}^0\|^4 e^{-4\alpha k})$  can be made positive. Then apply discrete Gronwall's Lemma with estimate (3.1) to complete the rest of the proof.  $\square$

**3.1. Error Analysis.** Now we are ready to discuss the proof of our main result that is the proof of Theorem 1.

Let  $\varepsilon^n$  be the quadrature error associated with the quadrature rule (1.5) and for  $\phi \in C^1[0, t_n]$ , let it be given by

$$\varepsilon^n(\phi) := \int_0^{t_n} \beta(t_n - s)\phi(s) ds - q^n(\phi).$$

Note that the quadrature error  $\varepsilon^n$  satisfies

$$\begin{aligned} (3.7) \quad |\varepsilon^n(\phi)| &\leq Ck \int_0^{t_n} \left| \frac{\partial}{\partial s} (\beta(t_n - s)\phi(s)) \right| ds \\ &\leq Ck \int_0^{t_n} (|\beta_s(t_n - s)| |\phi(s)| + |\beta(t_n - s)| |\phi_s(s)|) ds. \end{aligned}$$

For the proof of the main Theorem, we appeal to the semigroup theoretic approach, see Thomée [24], Fujita and Kato [7] and Okamoto [18]. It is wellknown that the Stoke's operator  $-A$  generates an analytic semigroup, say  $E(t)$ ,  $t > 0$  on  $\mathbf{H}$ , see Temam [23] or Fujita and Kato [7]. Moreover, the following estimates are also satisfied:

$$(3.8) \quad \|A^r E(t)\| \leq C t^{-r} e^{-\lambda_1 t}, \quad t > 0, \quad r > 0,$$

and for  $r \in (0, 1]$ , and  $\mathbf{v} \in D(A^r)$ , the domain of  $A^r$ ,

$$(3.9) \quad \|(E(t) - I)\mathbf{v}\| \leq C_r t^r \|A^r \mathbf{v}\|, \quad t > 0,$$

where  $C_r$  is a positive constant. For a proof, we refer to [5], page 383. Further, we use the discrete semigroup  $E_k$ , which is given by

$$E_k = (I + kA)^{-1}.$$

Using spectral representation of  $A$ , see Thomée [24], the following estimate is easy to derive

$$(3.10) \quad \|A^r E_k^n\| \leq C t_n^{-r} e^{-\lambda_1 t_n}, \quad t_n > 0, \quad 0 < r \leq 1.$$

Now, using Duhamel's principle, the equation (1.4) is written in an equivalent form as

$$\mathbf{u}(t) = E(t)\mathbf{u}_0 - \int_0^t E(t-s)\tilde{A}\mathbf{u}(s) ds - \int_0^t E(t-s)B(\mathbf{u}(s), \mathbf{u}(s)) ds,$$

where for simplicity of symbol, we denote

$$\tilde{A}\mathbf{u}(t) = \int_0^t \beta(t-\tau)A\mathbf{u}(\tau) d\tau.$$

Similarly, using discrete semigroup  $E_k = (I + kA)^{-1}$ , we rewrite (1.6) as

$$\mathbf{U}^n = E_k^n \mathbf{u}_0 - \sum_{j=1}^n k E_k^{n-j+1} q^j(A\mathbf{U}) - \sum_{j=1}^n k E_k^{n-j+1} B(\mathbf{U}^{j-1}, \mathbf{U}^j).$$

*Proof of Theorem 1.* Note that the error  $\mathbf{e}^n := \mathbf{u}(t_n) - \mathbf{U}^n$  is written in the form

$$(3.11) \quad \begin{aligned} \mathbf{e}^n &= (E(t_n) - E_k^n) \mathbf{u}_0 - \left( \int_0^{t_n} E(t_n-s)\tilde{A}\mathbf{u}(s) ds - \sum_{j=1}^n k E_k^{n-j+1} q^j(A\mathbf{U}) \right) \\ &- \left( \int_0^{t_n} E(t_n-s)B(\mathbf{u}(s), \mathbf{u}(s)) ds - \sum_{j=1}^n k E_k^{n-j+1} B(\mathbf{U}^{j-1}, \mathbf{U}^j) \right) \\ &= I_1^n - I_2^n - I_3^n. \end{aligned}$$

Since  $F_k^n := (E(t_n) - E_k^n)$  denotes the error operator for purely parabolic problem, then following Thomée [24], we estimate  $A^{1/2}I_1^n$  as

$$(3.12) \quad \|A^{1/2}I_1^n\| = \|A^{1/2}F_k^n \mathbf{u}_0\| \leq C(\|A\mathbf{u}_0\|, \Omega) \frac{e^{-\alpha t_n}}{t_n^{1/2}} k.$$

In order to estimate  $\|A^{1/2}I_2^n\|$ , i.e., the memory term, we first rewrite  $I_2^n$  as

$$(3.13) \quad \begin{aligned} I_2^n &= \left( \int_0^{t_n} E(t_n-s) \left( \tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n) \right) ds - \sum_{j=1}^n k E_k^{n-j+1} \left( q^j(A\mathbf{U}) - \tilde{A}\mathbf{u}(t_n) \right) \right) \\ &+ \left( \int_0^{t_n} E(t_n-s) ds - \sum_{j=1}^n k E_k^{n-j+1} \right) \tilde{A}\mathbf{u}(t_n) \\ &+ \sum_{j=1}^n k E_k^{n-j+1} q^j(A\mathbf{e}) = I_{2,1}^n + I_{2,2}^n + I_{2,3}^n. \end{aligned}$$

For  $I_{2,2}^n$ , we obtain using the semigroup property

$$\int_0^{t_n} E(t_n - s) - \sum_{j=1}^n k E_k^{n-j+1} = -F_k^n A^{-1},$$

and hence, using the definition of  $\beta$ , we arrive at

$$\begin{aligned} \|A^{1/2} I_{2,2}^n\| &= \|A^{1/2} F_k^n A^{-1} \tilde{A} \mathbf{u}(t_n)\| \\ &\leq C k \frac{e^{-\lambda_1 t_n}}{t_n^{1/2}} e^{-\alpha t_n} \left\| \int_0^{t_n} e^{-(\delta-\alpha)(t_n-\tau)} A \hat{\mathbf{u}}(\tau) d\tau \right\| \\ &\leq C k \frac{e^{-\lambda_1 t_n}}{t_n^{1/2}} e^{-\alpha t_n} \left( \int_0^{t_n} \|A \hat{\mathbf{u}}(\tau)\|^2 d\tau \right)^{1/2}. \end{aligned}$$

An application of Lemma 5 yields, for  $0 < \alpha < \min(\lambda_1, \delta)$ ,

$$\|A^{1/2} I_{2,2}^n\| \leq C (\|A^{1/2} u_0\|) k \frac{e^{-\alpha t_n}}{t_n^{1/2}}.$$

For estimating  $I_{2,3}^n$ , we first use change of variable and then change of summation to obtain

$$\begin{aligned} A^{1/2} I_{2,3}^n &= \sum_{j=0}^{n-1} k A E_k^{n-j} A^{-1/2} \sum_{i=1}^{j+1} k \beta_{j+1-i} A \mathbf{e}^i = \sum_{j=0}^{n-1} k A E_k^{n-j} \sum_{i=0}^j k \beta_{j-i} A^{1/2} \mathbf{e}^{i+1} \\ &= k \sum_{i=0}^{n-1} \left( \sum_{j=i}^{n-1} k \beta_{j-i} A E_k^{n-j} \right) A^{1/2} \mathbf{e}^{i+1} \\ &= k \sum_{i=0}^{n-1} \left( \sum_{j=i}^{n-1} k \beta_{n-i} A E_k^{n-j} \right) A^{1/2} \mathbf{e}^{i+1} - k \sum_{i=0}^{n-1} \left( \sum_{j=i}^{n-1} k (\beta_{n-i} - \beta_{j-i}) A E_k^{n-j} \right) A^{1/2} \mathbf{e}^{i+1}. \end{aligned}$$

For the first term on the right hand side of  $A^{1/2} I_{2,3}^n$ , we have from the spectral property of the Stoke's operator and  $r(\lambda) = (1 + \lambda)^{-1}$ :

$$\begin{aligned} \left\| k \sum_{j=i}^{n-1} A E_k^{n-j} \right\| &= \sup_{\lambda \in Sp(A)} \left| \sum_{j=i}^{n-1} k \lambda r(k\lambda)^{n-j} \right| \leq \sup_{\lambda > 0} \sum_{j=i}^{n-1} \lambda r(\lambda)^{n-j} \\ &\leq \sup_{\lambda > 0} \frac{\lambda r(\lambda)}{1 - r(\lambda)} = 1. \end{aligned}$$

For the second term on the right hand side of  $A^{1/2} I_{2,3}^n$ , we use the smoothing property (3.8) of  $E_k^n$ , and therefore, we obtain

$$\begin{aligned} \|A^{1/2} I_{2,3}^n\| &\leq \gamma k \sum_{i=0}^{n-1} e^{-\delta(t_n-t_i)} \left\| k \sum_{j=i}^{n-1} A E_k^{n-j} \right\| \|A^{1/2} \mathbf{e}^{i+1}\| \\ &\quad + \gamma k \sum_{i=0}^{n-1} \left( \sum_{j=i}^{n-1} k |(e^{-\delta t_{n-i}} - e^{-\delta t_{j-i}})| \|A E_k^{n-j}\| \right) \|A^{1/2} \mathbf{e}^{i+1}\| \end{aligned}$$



$$\begin{aligned} &\leq Cke^{-\alpha t_n} \sum_{i=0}^{n-1} e^{\alpha t_i} \|A^{1/2} \mathbf{e}^{i+1}\| \\ &+ Cke^{-\alpha t_n} \sum_{i=0}^{n-1} e^{\alpha t_i} \left( \sum_{j=i}^{n-1} ke^{-(\delta-\alpha)(t_j-t_i)} \frac{e^{-\delta(t_n-t_j)} - 1}{(t_n-t_j)} e^{-(\lambda_1-\alpha)(t_n-t_j)} \right) \|A^{1/2} \mathbf{e}^{i+1}\|. \end{aligned}$$

Using the meanvalue property of the exponential function, we find that

$$\left( \sum_{j=i}^{n-1} ke^{-(\delta-\alpha)(t_j-t_i)} \frac{e^{-\delta(t_n-t_j)} - 1}{(t_n-t_j)} e^{-(\lambda_1-\alpha)(t_n-t_j)} \right) \leq C,$$

and hence, we arrive at

$$\|A^{1/2} I_{2,3}^n\| \leq Ce^{-\alpha t_n} e^{-\alpha k} k \sum_{i=0}^n e^{\alpha t_i} \|A^{1/2} \mathbf{e}^i\|.$$

Now for the term  $I_{2,1}^n$ , we first rewrite it as

$$\begin{aligned} I_{2,1}^n &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (E(t_n-s) - E(t_{n-j+1})) (\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n)) ds \\ &+ \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E(t_{n-j+1}) (\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_j)) ds \\ &+ \sum_{j=1}^n kF_k^{n-j+1} (\tilde{A}\mathbf{u}(t_j) - \tilde{A}\mathbf{u}(t_n)) + \sum_{j=1}^n kE_k^{n-j+1} \varepsilon^j(A\mathbf{u}) \\ &= M_1^n + M_2^n + M_3^n + M_4^n. \end{aligned}$$

For  $M_1^n$ , we write it as

$$A^{1/2} M_1^n = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} A^{3/2} E(t_n-s) A^{-1} (I - E(s-t_{j-1})) (\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n)) ds.$$

Thus, using (3.8)–(3.9), we obtain

$$\begin{aligned} \|A^{1/2} M_1^n\| &\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{3/2} E(t_n-s)\| \|A^{-1} (I - E(s-t_{j-1})) (\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n))\| ds \\ &\leq Ck \int_0^{t_n} \frac{e^{-\lambda_1(t_n-s)}}{(t_n-s)^{3/2}} \|\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n)\| ds. \end{aligned}$$

In order to estimate  $\|\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n)\|$ , we note that

$$\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n) = \int_0^s (\beta(s-\tau) - \beta(t_n-\tau)) A\mathbf{u}(\tau) d\tau - \int_s^{t_n} \beta(t_n-\tau) A\mathbf{u}(\tau) d\tau,$$

and hence, using the definition of  $\beta$ , the mean value theorem,  $0 < \alpha < \min(\lambda_1, \delta)$ , and Lemma 7, we now obtain

$$\|\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n)\| \leq \gamma e^{-\delta s} (1 - e^{-\delta(t_n-s)}) \int_0^s e^{\delta\tau} \|A\mathbf{u}(\tau)\| d\tau$$

$$\begin{aligned}
& + \gamma \int_s^{t_n} e^{-\delta(t_n-\tau)} \|\mathbf{A}\mathbf{u}(\tau)\| d\tau \\
& \leq \delta\gamma(t_n-s)e^{-\alpha s} e^{-\delta s^*} \int_0^s e^{-(\delta-\alpha)(s-\tau)} \|e^{\alpha\tau} \mathbf{A}\mathbf{u}(\tau)\| d\tau \\
& + C(\|\mathbf{A}\mathbf{u}_0\|, \gamma) \int_s^{t_n} e^{-\delta(t_n-\tau)} e^{-\alpha\tau} d\tau \\
& \leq \delta\gamma(t_n-s)e^{-\alpha s} \left( \int_0^s e^{-2(\delta-\alpha)(s-\tau)} d\tau \right)^{1/2} \left( \int_0^s e^{2\alpha\tau} \|\mathbf{A}\mathbf{u}(\tau)\|^2 d\tau \right)^{1/2} \\
& + C(\|\mathbf{A}\mathbf{u}_0\|, \gamma)(t_n-s)e^{-\alpha s}.
\end{aligned}$$

Using Lemma 5 and the boundedness of

$$\int_0^s e^{-2(\delta-\alpha)(s-\tau)} d\tau \leq \frac{1}{2(\delta-\alpha)},$$

we arrive at

$$\|\tilde{\mathbf{A}}\mathbf{u}(s) - \tilde{\mathbf{A}}\mathbf{u}(t_n)\| \leq C(\|\mathbf{A}\mathbf{u}_0\|)(t_n-s)e^{-\alpha s}.$$

Therefore,

$$\begin{aligned}
\|A^{1/2}M_1^n\| & \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n} \int_0^{t_n} \frac{e^{-(\lambda_1-\alpha)(t_n-s)}}{(t_n-s)^{1/2}} ds \\
& \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n} \int_0^{t_n} \frac{e^{-(\lambda_1-\alpha)\tau}}{\tau^{1/2}} d\tau \\
& \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n} \int_0^\infty \frac{e^{-(\lambda_1-\alpha)\tau}}{\tau^{1/2}} d\tau \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n}.
\end{aligned}$$

To estimate  $M_2^n$ , we use the definition of  $\tilde{\mathbf{A}}$  and the property (3.8) to find that

$$\begin{aligned}
\|A^{1/2}M_2^n\| & \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_{n-j+1})\| \|\tilde{\mathbf{A}}\mathbf{u}(s) - \tilde{\mathbf{A}}\mathbf{u}(t_j)\| ds \\
& \leq C \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}} \|\tilde{\mathbf{A}}\mathbf{u}(s) - \tilde{\mathbf{A}}\mathbf{u}(t_j)\| ds.
\end{aligned}$$

Since

$$\|\tilde{\mathbf{A}}\mathbf{u}(s) - \tilde{\mathbf{A}}\mathbf{u}(t_j)\| \leq C(\|\mathbf{A}\mathbf{u}_0\|)(t_j-s)e^{-\alpha s} \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha s},$$

we now obtain

$$\begin{aligned}
\|A^{1/2}M_2^n\| & \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n} \sum_{j=1}^n \frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}} \left( e^{\alpha t_{j-1}} \int_{t_{j-1}}^{t_j} e^{-\alpha s} ds \right) \\
& \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n} \left( k \sum_{j=1}^n \frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}} \right) \\
& \leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n}.
\end{aligned}$$

Note that we have used the boundedness of the summation term within the bracket.

In order to estimate  $M_3^n$ , we use the property of  $F_k^n$  and obtain

$$\|A^{1/2}M_3^n\| \leq Ck^2 \sum_{j=1}^n \frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})^{3/2}} \|\tilde{A}\mathbf{u}(t_j) - \tilde{A}\mathbf{u}(t_n)\|.$$

As in the estimate of  $\|A^{1/2}M_1^n\|$ , we now find that

$$\begin{aligned} \|A^{1/2}M_3^n\| &\leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n}e^{-\alpha k} \left( k \sum_{j=1}^n \frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}} \right) \\ &\leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n}. \end{aligned}$$

Finally for  $M_4^n$ , we note that

$$\|A^{1/2}M_4^n\| \leq \sum_{j=1}^n k \|AE_k^{n-j+1}\| \|\varepsilon^j(A^{1/2}\mathbf{u})\|.$$

Using (3.8), we obtain

$$\|A^{1/2}M_4^n\| \leq \sum_{j=1}^n k \frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})} \|\varepsilon^j(\mathbf{A}\mathbf{u})\|.$$

To complete the estimate, we use (3.7) to compute the quadrature error  $\|\varepsilon^j(\mathbf{A}\mathbf{u})\|$  as

$$\|\varepsilon^j(\mathbf{A}\mathbf{u})\| \leq Ck \int_0^{t_j} \left( |\beta_s(t_j-s)| \|A^{1/2}\mathbf{u}(s)\| + |\beta(t_j-s)| \|A^{1/2}\mathbf{u}_s(s)\| \right) ds,$$

and hence, we find from Lemma 6 that

$$\begin{aligned} \|\varepsilon^j(\mathbf{A}\mathbf{u})\| &\leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_j} \int_0^{t_j} e^{-(\delta-\alpha)(t_j-s)} ds \\ &\quad + Cke^{-\alpha t_j} \left( \int_0^{t_j} e^{-2(\delta-\alpha)(t_j-s)} ds \right)^{1/2} \left( \int_0^{t_j} e^{2\alpha s} \|A^{1/2}\mathbf{u}_s(s)\|^2 ds \right)^{1/2} \\ &\leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_j}. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} \|A^{1/2}M_4^n\| &\leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n}e^{-\alpha k} \left( k \sum_{j=1}^n \frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})} \right) \\ &\leq C(\|\mathbf{A}\mathbf{u}_0\|)ke^{-\alpha t_n}e^{-\alpha k} \left( k \sum_{j=1}^n \frac{1}{(t_n-t_{j-1})} \right) \\ &\leq C(\|\mathbf{A}\mathbf{u}_0\|)k \left( \log \frac{1}{k} \right) e^{-\alpha t_n}. \end{aligned}$$

Alltogether, we, therefore, obtain

$$\begin{aligned} (3.14) \quad \|A^{1/2}I_2^n\| &\leq C(\|\mathbf{A}\mathbf{u}_0\|)e^{-\alpha t_n} k \left( 1 + \log \frac{1}{k} \right) + C(\|A^{1/2}\mathbf{u}_0\|) \frac{e^{-\alpha t_n}}{t_n^{1/2}} k \\ &\quad + Ce^{-\alpha t_n} k \sum_{i=0}^{n-1} e^{\alpha t_i} \|A^{1/2}\mathbf{e}^i\| + Cke^{-\alpha k} \|A^{1/2}\mathbf{e}^n\|. \end{aligned}$$

Finally in order to estimate  $I_3^n$  involving the nonlinear term, we may split it as in Geveci [9] and apply Hölder's inequality, Sobolev imbedding theorem with Sobolev inequality. Lastly with the help of Lemmas 4–5, 7 and Lemma 9, we obtain

$$(3.15) \quad \|A^{1/2}I_3^n\| \leq C(\|\mathbf{A}\mathbf{u}_0\|)\frac{e^{-\alpha t_n}}{t_n^{1/2}}k + C(\|A^{1/2}\mathbf{u}_0\|)e^{-\alpha t_n}k^{1/4}\|A^{1/2}\mathbf{e}^n\| \\ + Ce^{-\alpha t_n}k \sum_{i=0}^{n-1} \frac{e^{\alpha t_i}}{(t_n - t_i)^{3/4}} \|A^{1/2}\mathbf{e}^i\|.$$

On substituting (3.12), (3.14) and (3.15) in (3.9), we obtain, for sufficiently small  $k$ ,

$$(3.16) \quad e^{\alpha t_n} \|A^{1/2}\mathbf{e}^n\| \leq C(\|\mathbf{A}\mathbf{u}_0\|) \left[ k(t_n^{-1/2} + \log \frac{1}{k}) \right. \\ \left. + k \sum_{i=0}^{n-1} \left( \frac{1}{(t_n - t_i)^{3/4}} + 1 \right) e^{\alpha t_i} \|A^{1/2}\mathbf{e}^i\| \right].$$

Using the generalized discrete Gronwall's lemma (see, Lemma 7.1 in [6]) and the arguments of Okamoto ([18], page 635), we complete the rest of the proof.  $\square$

The convergence in  $\mathbf{L}^2$ -norm now becomes a routine work. However, we only indicate, below, the major steps in the proof for achieving this result.

*Proof of Theorem 2.* From (3.9), the error  $e^n$  satisfies

$$\mathbf{e}^n = I_1^n - I_2^n - I_3^n.$$

Since a straight forward modification of  $\mathbf{H}^1$ -estimates of Geveci [9] yields the  $\mathbf{L}^2$ -estimates of  $I_1^n$  and  $I_3^n$ , it remains to estimate  $\|I_2^n\|$ . Note that the  $\mathbf{L}^2$ -estimates of  $I_{2,2}^n$  and  $I_{2,3}^n$  in (3.13) follow easily as

$$\|I_{2,2}^n\| = \|F_k^n A^{-1} \tilde{A}\mathbf{u}(t_n)\| \\ \leq Cke^{-\lambda_1 t_n} \left\| \int_0^{t_n} \beta(t_n - s) \mathbf{A}\mathbf{u}(s) ds \right\| \\ \leq Cke^{-\alpha t_n} \left( \int_0^{t_n} \|\mathbf{A}\mathbf{u}(s)\|^2 ds \right)^{1/2} \leq C(\|A^{1/2}\mathbf{u}_0\|)ke^{-\alpha t_n},$$

and

$$\|I_{2,3}^n\| = \left\| k \sum_{j=0}^{n-1} A E_k^{n-j} \sum_{i=0}^j k \beta_{j-i} \mathbf{e}^{i+1} \right\|.$$

We repeat the analysis for estimating  $A^{1/2}I_{2,3}^n$  in Theorem 1, but now  $\mathbf{e}^{i+1}$  is made free of  $A^{1/2}$ . Thus, we obtain

$$\|I_{2,3}^n\| \leq Ce^{-\alpha t_n} k \sum_{i=0}^{n-1} e^{\alpha t_i} \|\mathbf{e}^i\| + Ck\|\mathbf{e}^n\|.$$

In order to estimate  $I_{2,1}^n$ , it is a routine matter to derive the estimates of  $\|M_1^n\|$ ,  $\|M_2^n\|$  and  $\|M_3^n\|$ . To complete the rest of the proof, we, therefore, need an estimate for  $\|M_4^n\|$ . Note that

$$\begin{aligned} \|M_4^n\| &\leq \sum_{j=1}^n k \|A^{1/2} E_k^{n-j+1}\| \|\varepsilon^j(A^{1/2}\mathbf{u})\| \\ &\leq Ck \sum_{j=1}^n \frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}} \|\varepsilon^j(A^{1/2}\mathbf{u})\|. \end{aligned}$$

Using the estimate of  $\|\varepsilon^j(A^{1/2}\mathbf{u})\|$  as in the proof of Theorem 1, we now obtain

$$\begin{aligned} \|M_4^n\| &\leq C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n} \left( k \sum_{j=1}^n \frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}} \right) \\ &\leq C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}. \end{aligned}$$

Note that the summation in the bracket is bounded by a constant which is independent of  $k$ . This completes the rest of the proof.  $\square$

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