ON HIGHER-ORDER VISCOSITY APPROXIMATIONS OF ONE-DIMENSIONAL CONSERVATION LAWS

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Abstract. We consider some aspects of vanishing viscosity ($\varepsilon \to 0^+$) approximations of entropy solutions of the classical conservation law

$$u_t + uu_x = 0$$

via solutions $u_\varepsilon(x,t)$ of the higher-order semilinear parabolic equation

$$u_t + uu_x = \varepsilon(-1)^{m+1} D_x^{2m} u, \quad D_x = \partial/\partial x, \quad \text{with integer } m \geq 2.$$

We show that unlike the classical case $m = 1$ (Burgers' equation), direct higher-order approximations of known entropy conditions and inequalities are not possible and convergence $u_\varepsilon \to u$ needs extra delicate asymptotic analysis of formation of shock layers. The approximation and stability properties in two main Riemann's problems with initial data $S_\mp(x) = \mp\text{sign}\, x$ (the shock and rarefaction wave) are studied. Parabolic approximations are illustrated for other odd-order equations including the third-order one

$$u_t - (uu_x)_{xx} = -\varepsilon u_{xxxx}, \quad \varepsilon \to 0.$$

1. Introduction: classical results on entropy solutions and extensions to higher-order approximations

Extended Burgers' equation and other approximating models. We study the questions occurring in higher-order approximations (the *vanishing viscosity method*) of scalar conservation laws and other odd-order equations. Our main example is the classical hyperbolic equation

$$u_t + uu_x = 0 \quad \text{in } Q = \mathbf{R} \times \mathbf{R}_+, \quad u(x,0) = u_0(x) \quad \text{in } \mathbf{R}, \tag{1.1}$$

with a bounded measurable initial data u_0. This equation originated from gas-dynamics played a key role in the general theory of discontinuous entropy solutions of conservation laws developed in the 1950's. We study the approximation of entropy solutions via the higher-order parabolic equations. This leads to *extended Burgers' equation*

$$u_t + uu_x = \varepsilon(-1)^{m+1} D_x^{2m} u \quad (D_x = \partial/\partial x), \quad u(x,0) = u_{0\varepsilon}(x), \tag{1.2}$$

where $m \geq 2$ is integer, with a positive parameter $\varepsilon \to 0^+$. The second-order ($m = 1$) vanishing viscosity method had the crucial importance for the viscosity-entropy theory.

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Several aspects of the fourth-order approximations ($m = 2$) occurring in mechanical and physical applications and, in particular, in the stability theory of finite-difference schemes were attracted significant interest and have been studied in the literature. We present related references below. On the other hand, there exists a huge literature in gas and aerodynamics on the influence of viscosity and heat conduction processes on the structure of shocks in compressed flows. This leads to complicated higher-order nonlinear systems. We refer to famous books [43].

Concerning other types of regularization of (1.1), for third-order regularizing operators leading, in particular, to the *Korteweg-de Vries equation*

$$u_t + uu_x = \varepsilon u_{xxx}, \tag{1.3}$$

approximation of entropy solutions with shocks are known to be impossible, see general conclusions of [9] concerning ODEs and detailed PDE analysis in [29]; see also Lax's survey [27]. The exceptional case is a small dispersion perturbation of Burgers' equation

$$u_t + uu_x = \varepsilon u_{xx} + \delta(\varepsilon) u_{xxx}, \tag{1.4}$$

where for $\delta(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$ the solutions converge to entropy ones of (1.1), see [39], [30] and more references in [23].

Higher-order semilinear parabolic equations occur in several areas of applications and their qualitative mathematical theory is the important popular subject; see book [38]. In general, the questions of $2m$-th order approximation of lower odd-order evolution equations are related to the problem of smooth regularization of semigroups of discontinuous solutions and construction of discontinuous extended semigroups occurring in the study of singularity formation phenomena in PDEs.

Plan and outline of results. We continue Introduction with a survey where necessary classical local (pointwise) and nonlocal entropy conditions are presented. In Section 2 we show that, unlike the classical second-order case $m = 1$, for $m \geq 2$ the regularized solutions $\{u_\varepsilon(x,t)\}$ of (1.2), in principle, *do not* approximate as $\varepsilon \to 0$ known local entropy conditions for solutions $u(x,t)$ of (1.1). We show that this happens due to the discontinuity of total variation of $u_\varepsilon(x,t)$ at $\varepsilon = 0$ in approximating of entropy shocks. For $m \geq 2$, there exists a *variation deficiency* dV_m measured via the *viscosity shock profile* having finite total variation. It turns out that $dV_m = 0$ for $m = 1$ only, which actually made it possible to approximate local entropy inequalities in the classical theory. Concerning nonlocal Kruzhkov-Lax's entropy inequality, our conclusion is also negative in the sense that it *cannot* be obtained in the limit $\varepsilon \to 0$ without detailed analysis of generic formation of singular shock layers.

In Section 3 we demonstrate that higher-order parabolic approximations correctly describe entropy solutions of Riemann's problems for (1.1) with initial data $S_\mp(x) = \mp \operatorname{sign} x$, the *shock* and *rarefaction* wave respectively. We prove that $S_-(x)$ is a *proper* solution, i.e., is obtained by higher-order approximations, while $S_+(x)$ *is not*. The general question on $2m$-th order approximation of arbitrary entropy solutions generates the two key asymptotic (large-time behaviour) problems for the corresponding rescaled $2m$-th order parabolic equations:

(I) asymptotic stability of the viscosity shock profile (Section 4), and
(II) asymptotic stability of the rarefaction profile (Section 5).

It is known that existence of a measure-valued solution (in the sense of Young measures) $u = \lim u_\varepsilon$ along a subsequence is guaranteed by a single L^∞-estimate on the family $\{u_\varepsilon\}$ to (1.2) [12] (or by an L^p-estimate [39]). Nevertheless, in view of the principal difficulties revealed in Section 2, convergence to unique entropy solutions for sufficiently arbitrary u_0 remains a hard open problem.

As a new example, in Section 6 we consider parabolic approximations of other odd-order equations including the third-order PDE

$$u_t - (uu_x)_{xx} = -\varepsilon u_{xxxx}. \tag{1.5}$$

A theory of suitable "entropy" solutions for (1.5) with $\varepsilon = 0$ is not known, and therefore we will apply the approximating approach in studying of shock-waves $S_\pm(x)$ based on the concept of proper solutions.

Several approximation conclusions of this paper are negative not in the sense that higher-order parabolic approximations (1.2) or (1.5) are unsatisfactory (actually, we establish that they are good for key Riemann's problems or in a class of piece-wise smooth solutions), but in the sense that the mathematical analysis needs more involved ideas and techniques of higher level of complexity. Actually, our main goal is to establish that unlike the lower-order case $m = 1$, even in a reasonably simple model (1.2) with $m \geq 2$ or in more general equations like (1.5), *convergence of* $\{u_\varepsilon\}$ *as* $\varepsilon \to 0$ *towards entropy solutions needs an extra delicate asymptotic analysis of corresponding singularity formation phenomena (shock layers), and this is an unavoidable difficulty.* In this and other related approximation problems connected with a general extended semigroup theory, the questions of existence (uniqueness) and asymptotic behaviour of limit proper solutions cannot be studied separately and are indivisible.

1.1. **Entropy conditions and entropy solutions.** It is known from the 1950's that the Cauchy problem for general single conservation laws admits a unique entropy solution. We refer to first complete results by O.A. Oleinik who introduced entropy conditions in 1D and proved existence and uniqueness results (see survey [35]) and by S.N. Kruzhkov [26] who developed a general non-local theory of entropy solutions in $\mathbf{R}^N$. In the general case, one of Oleinik's local entropy condition has the form ([35], p. 106)

$$\frac{u(x_1,t) - u(x_2,t)}{x_1 - x_2} \leq K(x_1,x_2,t) \quad \text{for all } \; x_1, x_2 \in \mathbf{R}, \; t > 0, \tag{1.6}$$

where K is a continuous function for $t > 0$. Oleinik's local condition E (Entropy) introduced in [36], for the model equation (1.1) with convex function $\varphi(u) = \frac{1}{2}u^2$ takes the form of well-known nonincrease of entropy from gas dynamics

$$u(x^+,t) \leq u(x^-,t) \quad \text{in } \; Q, \tag{1.7}$$

with strict inequality on lines of discontinuity, [35], p. 101.

Kruzhkov's entropy condition on solution $u \in L^\infty(Q)$ [26] is the nonlocal inequality

$$|u-k|_t + \frac{1}{2}[\mathrm{sign}(u-k)(u^2-k^2)]_x \le 0 \quad \text{in } \mathcal{D}'(Q) \quad \text{for any } k \in \mathbf{R}. \tag{1.8}$$

Oleinik's and Kruzhkov's approaches coincide in the 1D geometry.

It was known beginning with the first rigorous results by E. Hopf [21] (previous ones were due to I.M. Burgers [6]) that entropy solutions can be obtained by the vanishing viscosity method, i.e., as the limit

$$u_\varepsilon(x,t) \to u(x,t) \quad \text{as } \varepsilon \to 0 \tag{1.9}$$

of a sequence of classical solutions $\{u_\varepsilon\}$ of the Cauchy problem for *Burgers' equation*

$$u_t + uu_x = \varepsilon u_{xx} \tag{1.10}$$

with the same initial data. The convergence in (1.9) takes place in $L^1(\mathbf{R})$ for $t > 0$ and is pointwise at any point of continuity of $u(x,t)$. Approximations of the initial data can be included, where

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x) \to u_0 \quad \text{as} \quad \varepsilon \to 0 \text{ in } L^1. \tag{1.11}$$

See the comparison theorem in [26], and [35] for survey of results on general 1D hyperbolic equations. We refer to well known J. Smoller's book [40] and more recent book by C. Dafermos [11] and A. Bressan [4] for more detailed information.

The following consequence of the parabolic approximation is of principle importance for the theory. Let $E'(u)$ be a monotone C^1-approximation of the sign-function $\mathrm{sign}(u-k)$ with a fixed $k \in \mathbf{R}$, i.e., $E(u)$ is an approximation of $|u-k|$. Multiplying equation (1.10) by $E'(u)\chi$ with a nonnegative test function $\chi \in C_0^1(Q)$ and integrating over Q yields

$$\begin{aligned} &-\iint (E(u)\chi_t + F(u)\chi_x)dxdt \\ &= -\varepsilon \iint E''(u)(u_x)^2\chi dxdt + \varepsilon \iint E(u)\chi_{xx}dxdt \equiv J_1(\varepsilon), \end{aligned} \tag{1.12}$$

where $F(u) = \int uE'(u)du$. The first integral on the right hand side in non-positive, while the second one is of order $O(\varepsilon)$ on uniformly bounded regularized solutions u_ε. Therefore, passing to the limit $\varepsilon \to 0$ yields that the limit solution obtained by (1.9) satisfies the nonlocal *Kruzhkov-Lax entropy inequality* (see [25] for hyperbolic equations and [28] for systems)

$$E(u)_t + F(u)_x \le 0 \quad \text{in } \mathcal{D}'(Q). \tag{1.13}$$

For single conservation laws in $\mathbf{R}^N$, (1.13) being true for any convex C^2-function $E : \mathbf{R} \to \mathbf{R}$, gives an equivalent to (1.8) definition of unique entropy solutions, see [25], [26], p. 241 and [2]. Note that this is related to a parabolic version of Kato's inequality [22]: if $u, f \in L^1_{\mathrm{loc}}(Q)$, then (see [5], p. 75)

$$u_t - \Delta u = f \text{ in } \mathcal{D}'(Q) \implies |u|_t - \Delta|u| \le f \text{ in } \mathcal{D}'(Q). \tag{1.14}$$

1.2. **ODE-admissible approximations in the sense of I.M. Gel'fand.** It was well understood in the theory of entropy solutions that a crucial principle is the correct description of propagation of *shock-waves*, which are discontinuous travelling waves (TWs) satisfying (1.1) in the weak sense,

$$u(x,t) = S(\eta), \quad \eta = x - \lambda t, \tag{1.15}$$

where λ is the TW speed and $S(\eta)$ is a step function. Using obvious scaling and translational invariance of the equation, we set $\lambda = 0$. Assuming that the discontinuity is located at $x = 0$, by the *Rankine-Hugoniot condition*

$$\lambda = \frac{1}{2}[S(0^+) + S(0^-)], \tag{1.16}$$

this corresponds to two initial functions with the following entropy solutions of (1.1) (*Riemann's problems*):

$$S_-(x) = -\text{sign}(x) \implies u(x,t) = S_-(x) \text{ for } t > 0, \quad \text{and} \tag{1.17}$$

$$S_+(x) = \text{sign}(x) \implies u_+(x,t) = \{S_+(x) \text{ for } |x| \ge t, \quad x/t \text{ for } |x| \le t\}. \tag{1.18}$$

The first discontinuous TW $S_-(x)$ (called *standing shock-wave* in gas-dynamics) is the entropy one. The shock-wave solution $S_+(x)$ is not entropy and the continuous for $t > 0$ solution $u_+(x,t)$ in (1.18) (the *rarefaction wave*) describes collapse of this initial singularity.

Consider now a higher-order approximation of the conservation law where the regularizing sequence $\{u_\varepsilon\}$ is given by the Cauchy problem for the $2m$-th order uniformly parabolic equations (1.2) of arbitrary order $2m \ge 4$. Note that (1.2) is invariant under a two-parametric group of scalings and translations, so that if $u(x,t)$ is a solution, then

$$\mathcal{T}_{\alpha\beta} u(x,t) = \beta^{2m-1}[u(\beta x + \beta^{2m}\alpha t, \beta^{2m} t) - \alpha] \tag{1.19}$$

is also a solution for any constants α, $\beta \in \mathbf{R}$.

The approximating operator on the right-hand side of (1.2) is called *admissible* (or ODE-admissible to be distinguished from the PDE-one to be introduced later on) if equation (1.2) admits a TW approximating the entropy one $S_-(x)$ as $\varepsilon \to 0$ in a reasonable topology. The concept of admissible approximations was introduced by I.M. Gel'fand in [20] and was developed on the basis of TW-solutions of hyperbolic equations and systems, see Sect. 2 and 8 therein.

In view of invariance (1.19) we again put $\lambda = 0$. From (1.2) we then obtain the ODE for the *viscosity shock profile* (VSP) f_- corresponding to the entropy shock-wave $S_-(x)$. It is a sufficiently smooth stationary solution of (1.2)

$$u_\varepsilon(x) = f_-(y), \quad y = x/\varepsilon^\alpha, \;\; \alpha = 1/(2m-1), \tag{1.20}$$

where f_- solves the following ODE problem:

$$(-1)^{m+1} f^{(2m)} = f f' \quad \text{in } \mathbf{R}, \quad f(-\infty) = 1, \;\; f(+\infty) = -1. \tag{1.21}$$

The family of solutions (1.20) describes formation of the singular *shock layer* as $\varepsilon \to 0$ in the ODE. For $m = 1$, (1.21) is solved explicitly to give the unique (up to translations) *monotone decreasing* VSP

$$f_-(y) = (1 - e^y)/(1 + e^y) = \tanh(y/2). \tag{1.22}$$

The VSP f_- is known to exist for any $m \geq 2$, see [8], [24] and [34]. For the fourth-order approximation $m = 2$ it is unique [32] and is stable in a weighted Sobolev space [15].

Thus, the higher-order approximations (1.2) for any $m \geq 1$ are admissible in this ODE (TW) sense.

2. On striking differences in approximations for $m = 1$ and $m > 1$

2.1. **Well-posedness of higher-order approximations.** The problem of $2m$-th order approximations of first-order PDEs seems was less studied in the literature. Higher-order parabolic equations of the type (1.2) are well-posed and admit unique smooth classical solutions local in time [14], [16]. For $m = 1$, global existence and the uniform bound $|u(x,t)| \leq \sup |u_0|$ follow from the Maximum Principle. Such global existence results for higher-order semilinear parabolic equations with lower-order nonlinear perturbations are known in classes of sufficiently small initial data, see [1], [10], [13], [18], though for the quadratic nonlinearity in (1.2) such applications are not straightforward. For $m = 2$, global existence is established in [15] via stability analysis of the VSP (i.e., for initial data sufficiently close to f_-).

Let us show that for any $m \geq 2$ solutions of (1.2) are global in time and cannot blow-up in the L^∞-norm. We consider the Cauchy problem (1.2) with initial data satisfying

$$|u_{0\varepsilon}| \leq C, \quad \|u_{0\varepsilon}\|_2 \leq C, \tag{2.1}$$

where $C > 0$ denotes different constants possibly depending on ε. By approximation of L^2 initial data via compactly supported one, we may assume that solutions have fast exponential decay as $x \to \infty$. Multiplying equation (1.2) by u and integrating over $\mathbf{R}$ gives $\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = -\varepsilon \int |D_x^m u|^2 \leq 0$, whence the first uniform bound on the solution

$$\|u(t)\|_2 \leq \|u_{0\varepsilon}\|_2 \leq C \quad \text{for all } \ t > 0. \tag{2.2}$$

Proposition 2.1. *Let $m \geq 2$ and (2.1) hold. For a fixed $\varepsilon > 0$, the solution $u_\varepsilon(x,t)$ of (1.2) is bounded in $\mathbf{R} \times [0,T]$ for any $T > 0$.*

Proof. Consider the fundamental solution of the linear operator $\partial/\partial t + \varepsilon(-1)^m D_x^{2m}$,

$$b_\varepsilon(x,t) = (\varepsilon t)^{-1/2m} F(x/(\varepsilon t)^{1/2m}), \tag{2.3}$$

with the exponentially decaying rescaled kernel F, see [14] and applications to global existence in [13]. Writing down $uu_x = \frac{1}{2}(u^2)_x$ in the equivalent integral equation

$$u(t) = b_\varepsilon(t) * u_{0\varepsilon} - \frac{1}{2}\int_0^t b_\varepsilon(t-s) * (u^2)_x(s)ds, \tag{2.4}$$

using the Hölder inequality in the first term and integrating by parts in the last one yields

$$|u(t)| \le \sup |u_{0\varepsilon}||f|_1 + \frac{1}{2}\int_0^t \sup_x |b_{\varepsilon x}(t-s)|\, \|u(s)\|_2^2\, ds \le C[1 + T^{(m-1)/m}],$$

where we have used the estimate $|b_{\varepsilon x}(x,t)| = (\varepsilon t)^{-1/m}|f'(x/(\varepsilon t)^{1/2m})| \le Ct^{-1/m}$. $\square$

Despite of such global existence result, in the limit $\varepsilon \to 0$ higher-order singular perturbed problems (1.2) lead to a number of differences and difficulties which do not occur for $m = 1$. In fact, in the second-order approximation like (1.10) and various extensions (including some parabolic systems as approximations of hyperbolic ones), key results on entropy solutions are proved by using order-preserving, comparison and some other properties inherited from the Maximum Principle, nonexistent for any $m > 1$.

2.2. **Non-monotonicity of the VSP and variation deficiency.** We now describe a crucial non-monotonicity property of the VSP for $m \ge 2$ directly prohibiting parabolic approximations of local entropy conditions. Denote by $\mathrm{Var}\, f_-$ the total variation of $f_-(y)$ on $\mathbf{R}$. We introduce the variation deficiency dV_m of f_- as follows.

Proposition 2.2. *For any $m \ge 2$, the VSP f_- given by* (1.21) *has bounded variation satisfying*

$$\mathrm{Var}\, f_- > 2 = \mathrm{Var}\, S_- \implies dV_m \equiv \mathrm{Var}\, f_- - \mathrm{Var}\, S_- > 0. \tag{2.5}$$

Proof. As $y \to \infty$, the linearized ODE (1.21) has the form $(-1)^{m+1} f^{(2m)} = -f'$, so that the exponential decaying behaviour is determined by functions $f(y) \sim e^{\mu y}$ with the characteristic equation $(-1)^{m+1}\mu^{2m-1} = -1$, see [7]. One can see that for any $m \ge 2$, solutions are oscillating at $y = +\infty$, i.e., the characteristic number μ with the maximal $\mathrm{Re}\, \mu < 0$ is such that $\mathrm{Im}\, \mu \ne 0$. This implies boundedness of the total variation and (2.5). $\square$

The variation deficiency (2.5) shows that a finite discontinuity of variation occurs for $\varepsilon = 0^+$ at shocks of entropy solutions (though one needs the asymptotic stability of the VSP to show that, see Section 4). Note that dV_m vanishes for the second-order approximation $m = 1$ only and actually, this lies in the heart of parabolic approximations of local entropy inequalities in the classical theory. We will show that for $m \ge 2$ this is not possible.

Figure 1 shows a typical character of non-monotonicity of the VSP's, which can be associated with typical oscillating and sign-changing properties of the fundamental solutions of higher-order parabolic operators, [14]. Next, we begin with explaining the straightforward consequences of Proposition 2.2 prohibiting approximation of local entropy conditions.

A relation to order deficiency. Here we observe a phenomenon similar to the *order deficiency* $D_* = \int |F|$ [18] measuring the "degree" of violation of order-preserving properties of semigroups induced by higher-order parabolic operators $\partial/\partial t + (-\Delta)^m$ (being order-preserving for $m = 1$ only where $F > 0$ and hence $D_* = 1$). We show the order deficiency is responsible for the finite increase of total variation in the higher-order linear parabolic flows.

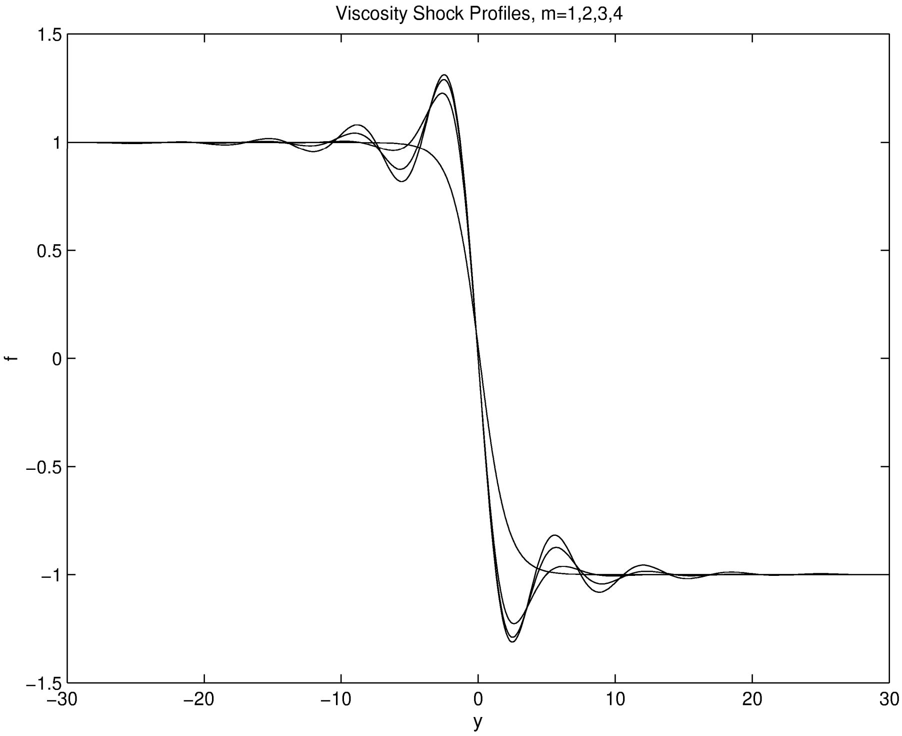

FIGURE 1. VSPs for $m = 1, 2, 3, 4$. Total variation of f_- increases with m.

Proposition 2.3. *Let $m \geq 2$ and $u(x,t)$ satisfy the Cauchy problem*

$$u_t = \varepsilon(-1)^{m+1} D_x^{2m} u \quad in \ \mathbf{R} \times \mathbf{R}_+, \quad u(x,0) = u_0(x) \quad in \ \mathbf{R}.$$

Then (i) the following estimate holds:

$$\text{Var}\{u(t)\} \leq D_* \text{Var}\{u_0\} \quad for \ t > 0, \quad with \ the \ constant \ D_* = \int |F| > 1. \tag{2.6}$$

(ii) The estimate is sharp.

Proof. (i) It follows from the convolution $u(t) = b_\varepsilon(t) * u_0$ (2.3) that $\text{Var}\{u(t)\} = \int |u_x(x,t)| dx \leq (\varepsilon t)^{-1/2m} \int \int |F(z/(\varepsilon t)^{1/2m})| \, |u_0'(x-z)| dx dz \leq D_* \text{Var}\{u_0\}$. (ii) For initial data $u_0(x) = \{1, \, x < 0; \; 0, \, x \geq 0\}$, the solution $u(x,t) = \int_{x/t^{1/2m}}^\infty F(z) dz$ satisfies (2.6), $\text{Var}\{u(t)\} \equiv D_*$, with the equality sign. $\square$

The main difficulty in higher-order parabolic approximations seems not the compactness of $\{u_\varepsilon\}$ and using Helly's theorem for functions of bounded variation; cf. its systematic applications in [35] for $m = 1$. It is crucial that the convergence $u_\varepsilon \to u$ to the entropy solutions assumes extra delicate, hard asymptotic analysis, and, moreover, it cannot be avoided in estimating of the total variation of solutions to (1.2).

2.3. **Regularized solutions do not satisfy Oleinik's upper gradient bound.** In the second-order approximation (1.10), it is known that for general hyperbolic equations, it is sufficient to choose $K(x,t) = C/t$ in the entropy inequality (1.6), see [35], p. 145. This follows from the Maximum Principle for (1.10) since the derivative $w = u_{\varepsilon x}$ satisfies the parabolic equation

$$w_t = \varepsilon w_{xx} - u w_x - w^2 \tag{2.7}$$

admitting the explicit solution

$$w_*(t) = 1/t \quad \text{for } \ t > 0, \quad w_*(0^+) = +\infty. \tag{2.8}$$

Therefore, as a straightforward consequence, by comparison of solutions to (2.7) one obtains the following upper gradient bound for *arbitrary* initial data (including both shocks $u_0 = S_\pm(x)$ where for S_+ translations in time are performed):

$$u_{\varepsilon x} \le 1/t \quad \text{in } \ Q. \tag{2.9}$$

This makes it possible to get in the limit (1.9) the entropy solutions satisfying (1.6).

Let now $m > 1$ in (1.2). Then similarly we get for $w = u_{\varepsilon x}$ the equation

$$w_t = \varepsilon(-1)^{m+1} D_x^{2m} w - u w_x - w^2 \tag{2.10}$$

admitting the same explicit solution (2.8) though the Maximum Principle does not apply and (2.9) does not follow. Anyway, the negative quadratic term $-w^2$ on the right-hand side of (2.10) stays the same and suggests to assume that $K(x,t) = C/t$ with some $C \gg 1$ possibly depending, in view of (1.6), on $u(x,t)$. Just in case, we write down such a suggestion in the general form: for $\varepsilon \approx 0^+$,

$$u_{\varepsilon x} \le K(x,t) \quad \text{uniformly in } \ Q, \tag{2.11}$$

assuming that K is bounded for $t > 0$. We now easily prove that this is not the case, and hence uniform estimates (2.9) or (2.11) are associated with the Maximum Principle for the second-order PDEs like (1.10) only.

Proposition 2.4. *For $m \ge 2$, (i) (2.11) does not hold with any function $K(x,t)$ uniformly bounded in $x \in \mathbf{R}$ for $t > 0$, and (ii) the same is true for the discrete relation (1.6).*

Proof. (i) For approximation (1.20) as $\varepsilon \to 0$,

$$u_\varepsilon(x) = f_-(y) \to S_-(x) \quad \text{in } L^1(\mathbf{R}) \text{ and a.e.}, \quad y = x/\varepsilon^\alpha, \ \alpha = 1/(2m-1), \tag{2.12}$$

(2.11) implies that for any fixed $t > 0$,

$$f_-'(y) \le \varepsilon^{2m-1} K(x,t) \to 0 \quad \text{as } \ \varepsilon \to 0, \tag{2.13}$$

whence $f_-'(y) \le 0$ contradicting Proposition 2.2. (ii) Let, for definiteness, $f_-(y)$ is oscillating as $y \to -\infty$. Taking the family (1.20) and using the fact that

$$\delta_0 = f_-(y_1) - f_-(y_2) > 0 \quad \text{for some } \ y_2 < y_1 < 0, \tag{2.14}$$

we have $[u_\varepsilon(x_1) - u_\varepsilon(x_2)]/(x_1 - x_2) = \delta_0/(y_1 - y_2)\varepsilon^\alpha \to +\infty$ as $\varepsilon \to 0$, i.e., (1.6) does not holds on the family $\{u_\varepsilon\}$ approximating the entropy shock-wave S_-. $\square$

Obviously, there is no way to improve such a "bad" property of higher-order approximations, for instance, by neglecting the uniformity of (1.6), i.e., assuming that u_ε satisfies

(1.6) for any $|x_1 - x_2| \geq C(\varepsilon) \to 0$ with $C(\varepsilon) \gg \varepsilon^\alpha$ as $\varepsilon \to 0$ (so that at $\varepsilon = 0^+$ we arrive at (1.6)). Indeed, taking as above $x_2 = y_2\varepsilon^\alpha$, where $y_2 < 0$ is the point of the absolute maximum of $f_-(y)$, and $x_1 \sim x_2 - C(\varepsilon)$, we still obtain the divergence $[u_\varepsilon(x_1) - u_\varepsilon(x_2)]/(x_1 - x_2) \geq \delta_0/2C(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

In the given TW-approximation of $S_-(x)$, the approximating sequence satisfies

$$\sup_x u_{\varepsilon x}(x,t) \to +\infty \quad \text{as } \varepsilon \to 0 \tag{2.15}$$

and, moreover, we will introduce a strong evidence of the fact that (2.15) is a generic property of higher-order approximation of any entropy shocks. Therefore, any family $\{u_\varepsilon\}$ converging to a discontinuous entropy solution cannot approximate the entropy condition (1.6) in the sense of (2.11), which is directed to delete shocks S_+ from the entropy class. On the other hand, if $K(x,t)$ is not bounded for $t > 0$, e.g., $K(x,t) = C/|x|$ (this leads to a reasonable estimate of $u_{\varepsilon x}$), then estimate (1.6) does not exclude the non-entropy solution $S_+(x)$ either.

2.4. **Regularized solutions do not approximate Oleinik's condition E.** It follows from (2.14) that for arbitrarily small $\varepsilon > 0$, there exists the point $\bar{x} = (y_1 + y_2)\varepsilon^\alpha/2$ and $h = (y_2 - y_1)\varepsilon^\alpha/2$ such that for a constant $\delta_0 > 0$, there holds

$$u_\varepsilon(\bar{x} + h, t) \geq u_\varepsilon(\bar{x} - h, t) + \delta_0, \quad \text{where } h = O(\varepsilon^\alpha). \tag{2.16}$$

In this sense, due to non-monotonicity of the VSP, subset of regularized solutions $\{u_\varepsilon\}$ do not approximate the condition E (1.7) as $\varepsilon \to 0$.

As a next corollary of Proposition 2.2, bounded variation of $u_\varepsilon(x,t)$ in x for $t > 0$ (and hence suitable compactness of the family $\{u_\varepsilon\}$) cannot be proved via local entropy conditions (1.6) or (1.7). For $m = 1$, this is a classical approach for 1D problems, see applications of the theory of functions with bounded variation and Helly's theorem throughout Sections 2 - 4 in [35]. One can expect that u_ε has uniformly bounded variation as $\varepsilon \to 0$, but this cannot be established in such a straightforward way as for $m = 1$.

It seems that for establishing of compactness of $\{u_\varepsilon\}$ for $m \geq 2$ for equations in 1D, the analysis in the class $BV(\mathbf{R}^N)$ of functions of bounded variations (see Chapt. 16 in [40] and [4]) or estimates for compactness in $L^1(\mathbf{R}^N)$ [26] can be useful, which are powerful tools in solving hyperbolic equations in $\mathbf{R}^N$. Though it is worth mentioning that both approaches are based on the Maximum Principle ideas. For instance, main estimates in [26], pp. 232-237 use comparison barrier techniques and do not extend to higher-order equations.

The only case, where (2.13) does not lead to a contradiction, is $m = 1$, when the VSP (1.22) is monotone, a property to be associated with the Maximum Principle for the second-order ODE (1.21). In the last section we introduce higher-order models with monotone VSP's, which is important for their asymptotic stability.

2.5. **Direct approximation of nonlocal entropy inequality is impossible: "problem e2-e4".** Let us finally show that special "geometry" of the VSP affects also parabolic approximations of the nonlocal entropy inequality (1.13), though not in such a direct way as the local ones above.

The derivation of the entropy inequality (1.13) from (1.12) is associated with the Maximum Principle for the second-order parabolic equations. One can see that (1.12) cannot be obtained in such a way if $m > 1$. For instance, let $m = 2$. Multiplying (1.2) by $E'(u)\chi$ and integrating by parts yields the following right-hand side in (1.12):

$$J_2(\varepsilon) = -\,\varepsilon \iint \left[E''(u)(u_{xx})^2 - \frac{1}{3}E''''(u)(u_x)^4\right]\chi dxdt \tag{2.17}$$

$$+\varepsilon \iint \left[\frac{4}{3}E'''(u)(u_x)^3\chi_x + 2E''(u)(u_x)^2\chi_{xx} - E(u)\chi_{xxxx}\right] dxdt.$$

Consider the first integral in (2.17) depending on χ only, while the second one contains x-derivatives of χ which hence may be assumed to vanish on any open subset inside $\operatorname{supp}\chi$. We observe here two terms, the first positive one with $E'' \geq 0$ and the second one depending on E'''' which can have any sign (actually, if $E(u)$ is sufficiently close to $|u-k|$ then E'''' changes sign). We will show below that multiplicative, interpolation inequalities comparing the two terms do not help for coefficients given by sufficiently arbitrary smooth convex $E(u)$. Taking into account the above rescaled variable $y = x/\varepsilon^{1/3}$ (assuming shock to be put at $x = 0$), we have that both terms in the integral are of the same order $O(\varepsilon^{-4/3})$, i.e., even this precise structure of the singular shock layer is not enough to guarantee the necessary sign. Therefore, in order to get the entropy condition (1.13) directly, firstly, it is necessary to discuss the following problem e2-e4: *is there a sufficiently wide subset of smooth functions $E(u)$ satisfying*

$$E''(u) \geq 0 \quad and \quad E''''(u) \leq 0 \quad in\ \mathbf{R}? \tag{2.18}$$

Obviously, such non-trivial bounded E's do not exist ($E''(u)$ is sufficiently smooth, nonnegative, concave in $\mathbf{R}$, hence $E'' \equiv$ const). More involved"e2-e4-e6-..." unsolvable problems occur for $m = 3, 4, \ldots$. This expresses the fact that Kato's inequality (1.14) (or multiplication by sign) does not admit extension to higher-order operators $\partial t/\partial + (-\Delta)^m$.

Interpolation inequalities do not guarantee the sign. Consider the first integral given in (2.17) with $\chi \equiv 1$,

$$P_2 \equiv -\int E''(u_{xx})^2 dx + \frac{1}{3}\int E''''(u_x)^4 dx \equiv -P_{21} + \frac{1}{3}P_{22}, \tag{2.19}$$

assuming that sufficiently smooth solutions $u = u_\varepsilon(x,t)$ have fast (exponential) decay as $x \to \infty$. Let us estimate the second positive term via a simple integration by parts

$$P_{22} = \int E''''(u_x)^3 u_x = -\int uE^{(5)}(u_x)^4 - 3\int uE''''(u_x)^2 u_{xx},$$

and using the Hölder inequality (recall that $E(u)$ is convex)

$$\int (E'''' + uE^{(5)})(u_x)^4 \leq 3\int (\sqrt{E''}|u_{xx}|)(|uE''''|(u_x)^2/\sqrt{E''})$$

$$\leq 3\left(\int E''(u_{xx})^2\right)^{1/2}\left(\int (uE'''')^2(u_x)^4/E''\right)^{1/2}. \tag{2.20}$$

In order to derive a suitable comparison of the two terms on the right-hand side of (2.19), we first impose the following conditions on functions $E'(u)$:

$$(2.21) \qquad E''''(u) + uE^{(5)}(u) \ge C_1 E''''(u), \quad (uE''''(u))^2/E''(u) \le C_2 E''''(u), \quad u \in \mathbf{R},$$

where C_1 and C_2 are some positive constants. Then (2.20) implies

$$(2.22) \qquad \int E''''(u_x)^4 \le C_3 \int E''(u_{xx})^2, \quad \text{with } C_3 = 9C_2/C_1^2,$$

and hence by (2.19)

$$(2.23) \qquad P_2 \le (3C_2/C_1^2 - 1) \int E''(u_{xx})^2 \le 0 \quad \text{if } C_1^2 \ge 3C_2.$$

The second condition in (2.21) assumes that $E''''(u) \ge 0$ in $\mathbf{R}$, which is not true for $E(u) \approx \operatorname{sign} u$. Replacing E'''' by $|E''''|$ on the right-hand sides of (2.21) (or other optimizations) does not extend the resulting estimate like (2.23) to the necessary sufficiently wide class of functions $E(u)$. For the typical power functions

$$(2.24) \qquad E'(u) = |u|^{2k} u, \quad k > 1,$$

(2.19) reads

$$(2.25) \qquad P_2 = -(2k+1) \int |u|^{2k} (u_{xx})^2 + \frac{2}{3}\,(2k-1)k(2k+1) \int |u|^{2k-2} (u_x)^4.$$

Integrating by parts and using the Holder inequality as in (2.20) yields

$$\int |u|^{2k-2} (u_x)^4 \le \frac{9}{(2k-1)^2} \int |u|^{2k} (u_{xx})^2,$$

and we arrive at the following estimate (cf. (2.23)):

$$(2.26) \qquad P_2 \le C_* \int |u|^{2k} (u_{xx})^2,$$

where $C_* = (1+4k)(2k+1)/(2k-1) > 0$ for all $k > 1/2$. Thus, we cannot get the necessary sign $P_2 \le 0$ on particular functions (2.24) for large k. In fact, this shows that the Nash-Moser technique for second-order parabolic equations (see [42], p. 344) do not apply to the fourth-order operators with $m = 2$ and to other higher-order ones. Indeed, the iterative nature of the technique with the eventual limit $k \to \infty$ assumes certain order-preserving properties via the Maximum Principle (available for $m = 1$ only), so that the inequality $C_* \le 0$ for all large k cannot be achieved in principle via optimization of constants in the interpolation and embedding inequalities.

Problem of sign of $D_x^{2m-1} u$ on lines of discontinuity. Let us reveal another "contradiction" related to the entropy condition (1.8). According to [26] it suffices to check (1.13) for the sign functions $E(u) = \operatorname{sign}(u-k)$ only for all $k \in \mathbf{R}$. This means that, for, say, $k = 0$, one can take smooth convex approximations of sign's $E_n(u) \in C_0^{2m}(\mathbf{R})$ with $\operatorname{supp} E_n \subset \{|u| \le 1/n\}$ and then

$$(2.27) \qquad E_n''(u) \to 2\delta(u) \quad \text{as } n \to \infty$$

in the sense of bounded measures. Assume that (1.9) holds and, for simplicity, for a fixed $t>0$, $u(x,t)=S_-(x)+o(1)$ as $x\to 0$. We then assume that $u_\varepsilon(x,t)$ has a unique isolated transversal zero at $x=x_0(t)$ so that $u_{\varepsilon x}(x_0(t),t)<0$. We then can use a simpler form of (2.17) with $E=E_n$ for $n\gg 1$. Namely, assuming that $\chi\equiv 1$ for $x\approx 1$, taking integrating in x only and integrating by parts just once yields

$$I_2=-\varepsilon\int u_{\varepsilon xxxx}E_n'(u_\varepsilon)dx=\varepsilon\int u_{\varepsilon xxx}E_n''(u_\varepsilon)u_{\varepsilon x}dx, \tag{2.28}$$

and hence by passing to the limit $n\to\infty$ and using (2.27), one obtains that

$$\lim{}_n I_2=-2\varepsilon\, u_{\varepsilon xxx}(x_0(t),t). \tag{2.29}$$

In view of the entropy condition (1.8), which we want to obtain from (2.29), we arrive at the new non-trivial "contradictory" problem on the third derivative: *why for $\varepsilon\approx 0^+$ on isolated zero curves (or on k-level curves with any $k\in(-1,1)$)*

$$-2\varepsilon u_{\varepsilon xxx}(x_0(t),t)\le 0 \quad \textit{if}\ u_{\varepsilon x}(x_0(t),t)<0\,? \tag{2.30}$$

The opposite inequality is assumed to be valid if $u_{\varepsilon x}(x_0(t),t)\ge 0$. Note that in view of known properties of entropy solutions, we do not expect the derivative to vanish, i.e.,

$$\lim{}_\varepsilon[-2\varepsilon\, u_{\varepsilon xxx}(x_0(t),t)]<0. \tag{2.31}$$

Similar construction is performed for the $2m$-th order equation (1.2) and instead of (2.30) leads to the condition on the sign of the derivative $D_x^{2m-1}u_\varepsilon$

$$2\varepsilon(-1)^{m+1}D_x^{2m-1}u_\varepsilon(x_0(t),t)\le 0 \quad \text{if}\ u_{\varepsilon x}(x_0(t),t)<0. \tag{2.32}$$

It follows that the second-order case $m=1$ is the only one for which (2.32) is tautological.

Related problem of total variation. A problem similar to (2.32) occurs in estimating the total variation of $u(\cdot,t)$ via equation (1.2). Differentiating (1.2) in x, setting $v=u_x$, multiplying by $\operatorname{sign} v$ and integrating over $\mathbf{R}$ yield that $\mathrm{Var}\{u(t)\}=\int|u_x|dx$ satisfies

$$\mathrm{Var}\{u(t)\}'=\varepsilon(-1)^{m+1}\int D_x^{2m}v\,\operatorname{sign} v\,dx=\varepsilon\sum_{k\in\mathbb{Z}}(-1)^{m+1+k}D_x^{2m-1}v|_{a_k}^{b_k}\equiv\varepsilon J_m(u), \tag{2.33}$$

where $\{I_k=(a_k,b_k)\}$ is the countable subset of maximal open intervals on which $\operatorname{sign} v\equiv\operatorname{sign} u_x=(-1)^k$ (note that $u=u_\varepsilon(x,t)$ is analytic in x for $t>0$, [16]). For $m=1$, (2.33) leads to the nonincrease of the total variation since, $v_x=u_{xx}$ has the necessary sign at the end points of I_k (extremum points of $u(x,t)$), i.e.,

$$J_1(u)=\sum(-1)^k u_{xx}|_{a_k}^{b_k}\le 0.$$

For $m>1$, the analysis of the right-hand side of (2.33) is not that straightforward.

In Section 4 we present an evidence that all three problems on signs in (2.17), (2.32) and partially in (2.33) are directly related to a hard asymptotic problem on the behaviour of $u_\varepsilon(x,t)$ as $\varepsilon\to 0$ on the curves of discontinuity. For $m\ge 2$, derivation of entropy criteria (1.13) or (1.8) via (1.2) cannot be done without knowing the precise asymptotic shape of ε-approximations of shocks layers.

3. **Parabolic approximation of shock-waves** $S_{\pm}(x)$

3.1. **Proper and improper solutions.** We now begin our analysis of the admissibility of higher-order approximations in the PDE sense. We consider the Cauchy problem (1.2) and introduce the following definition where we use a standard concept of weak (generalized) solutions of conservation laws, see [35] and [40], Chapt. 15.

Definition 3.1. We say that a weak solution $u(x,t)$ of the conservation law (1.1) is *proper* iff there exists a bounded sequence of initial data $\{u_{0\varepsilon}\} \to u_0$ in L^1 as $\varepsilon \to 0$ such that the subset of classical solutions $\{u_\varepsilon(x,t)\}$ of the parabolic problems (1.2) satisfies

$$u_\varepsilon(x,t) \to u(x,t) \quad \text{in } L^1 \text{ for any } t \ge 0. \tag{3.1}$$

$u(x,t)$ is an *improper* solution if it is not proper.

In other words, proper solutions $u(x,t)$ are only those which can be obtained by $2m$-th order parabolic approximation (1.2). This means admissibility of approximation relative to the given solution u. The L^1-topology in the definition is associated with the classical entropy theory based on second-order approximations. In the higher-order case, in such a topology we can establish some particular results, while for more general consideration we will use other topologies dictated by the methods. For generality and convenience, the definition includes "approximation" of initial data. Indeed, once convergence (3.1) is established for fixed data u_{0n}, $u_{\varepsilon n} \to u_n$ as $\varepsilon \to 0$, convergence $u_{\varepsilon n} \to u$ relative to both ε, n follows for arbitrary L^1-approximation of data $u_{0n} \to u_0$ as $n \to \infty$ by the triangle inequality

$$\|u_{\varepsilon n}(t) - u(t)\|_1 \le \|u_{\varepsilon n}(t) - u_n(t)\|_1 + \|u_n(t) - u(t)\|_1, \tag{3.2}$$

since $u_n \to u$ in view of comparison theorems for entropy solutions, [35], [26].

The concept of proper solutions plays an important role in the theory of nonlinear singular parabolic equations creating finite-time singularities like blow-up, extinction or quenching, where regular approximations (truncation of nonlinearities) make it possible to construct a unique, maximal or minimal, extensions of solutions beyond singularity time; see [19] and earlier references therein. Another area where approximation approaches are important is nonlinear evolution equations with singular initial data, e.g, with measures as initial conditions. Then weak solutions can cease to exist, see first remarkable results in [5]. In this cases approximation of singular data is of principal importance and sometimes approximation of equations is not necessary. Such extended semigroups constructed by approximation can be essentially *discontinuous* in any weak sense or in the sense of measures, and therefore many other concepts of solutions (demanding more detailed information on solutions properties) often do not apply. On the other hand, positive approximation of nonnegative initial data, $u_{0\varepsilon}(x) = u_0(x) + \varepsilon$, in constructing weak solutions $u = \lim u_\varepsilon$ of degenerate filtration equations $u_t = (\varphi(u,x))_{xx}$ is rather folklore after the seminal paper [37].

Of course, proper solutions concept is not necessary for the conservation laws where the classical entropy solution theory applies. It will be used below simply to test the concept and identify specific asymptotic properties to be treated later on. For a class of higher-order problems including (1.5) to be studied in Section 6, where the entropy theory is not available, the concept of approximation plays a key role.

Indeed, for the hyperbolic equation (1.1), it is important to prove that the solutions are proper iff they are entropy, though this is a hard problem. We begin with simple results concerning the canonical shock-waves $S_\pm(x)$. Since these are odd functions, in view of the symmetry of equations (1.1) and (1.2) under the reflections $x \mapsto -x$, $u \mapsto -u$, without loss of generality, we consider odd approximations $u_\varepsilon(x,t)$ in $Q_+ = \mathbf{R}_+ \times \mathbf{R}_+$ satisfying

$$D^k u(0,t) = 0 \quad \text{for } t > 0, \quad k = 0, 2, ..., 2m. \tag{3.3}$$

We perform scaling in (1.2)

$$u_\varepsilon(x,t) = U_\varepsilon(y,\tau), \quad y = x/\varepsilon^\alpha, \;\; \tau = t/\varepsilon^\alpha \quad \text{with exponent } \; \alpha = 1/(2m-1) \tag{3.4}$$

establishing as $\varepsilon \to 0$ a "parabolic zoom" for weak solutions of the conservation law in a shrinking neighbourhood of any point (x_0, t_0) in the $\{x,t\}$-plane (by replacing $x \to x - x_0$ and $t \to t - t_0$ in (3.4)). Therefore, it is of crucial importance to describe the character of "smeared" shocks created by parabolic approximations.

Scaling (3.4) deletes the small parameter ε from the equation so that U solves the uniformly parabolic equation

$$U_\tau + UU_y = (-1)^{m+1} D_y^{2m} U, \quad U(y,0) = U_{0\varepsilon}(y) \equiv u_{0\varepsilon}(y\varepsilon^\alpha). \tag{3.5}$$

Global solvability follows from Proposition 2.1. Setting $u = \phi(x) + w$ in (1.2), where $\phi(x)$ satisfying $\phi(\pm\infty) = \mp 1$ is a suitable smooth function, we obtain a perturbed equation for $w = w(x,t)$,

$$w_t = \varepsilon(-1)^{m+1} D_x^{2m} w - \phi\, w_x - \phi_x w + \varepsilon(-1)^{m+1} D_x^{2m} \phi.$$

Multiplying by w and integrating over $\mathbf{R}$, one obtains that $\|w(t)\|_2 \le C$ on any bounded interval $[0,T]$. Using kernel estimates of the linear operator with smooth bounded coefficients on the right-hand side (see e.g. [41], p. 190), as in the proof of Proposition 2.1 we show that $u_\varepsilon(x,t)$ is bounded and exists as the classical solution on any subset $\mathbf{R} \times [0,T]$. By (3.4) the same holds for $U_\varepsilon(y,\tau)$ on $\mathbf{R} \times [0, T\varepsilon^{-\alpha}]$.

For any $m \ge 1$ equation (3.5) has the explicit linear solution

$$\bar{U}(y,\tau) = y/\tau \quad (\bar{u}_\varepsilon(x,t) = x/t) \;\text{ in }\; Q, \tag{3.6}$$

which occurs in the entropy rarefaction solution (1.18). Later on, the asymptotic stability of this rarefaction profile will be of crucial importance in our analysis. The next two conclusions are elementary.

Proposition 3.1. *The entropy shock wave $S_-(x)$ is proper.*

Proof. Let f be a solution of (1.21). Then, since the convergence $f_-(y) \to \pm 1$ as $y \to \mp\infty$ given by the ODE (1.21) is exponential [7], there holds

$$u_\varepsilon(x) = f_-(x/\varepsilon^\alpha) \to S_-(x) \quad \text{as } \varepsilon \to 0 \tag{3.7}$$

in L^1, pointwise and uniformly on $\{|y| \ge c\}$ with any $c > 0$. $\square$

On the other hand, it is easy to see that the VSP f_+ corresponding to the non-entropy solution $S_+(x)$ does not exist.

Proposition 3.2. *The problem for f_+,*

$$(-1)^{m+1} f^{(2m)} = ff' \quad in\ \mathbf{R}, \quad f(-\infty) = -1, \ \ f(+\infty) = 1, \tag{3.8}$$

does not have a solution.

Proof. Integrating the equation yields $(-1)^{m+1} f^{(2m-1)} = \frac{1}{2}(f^2 - 1)$. Multiplying by f' and integrating over $\mathbf{R}$, we get the contradiction $\int (f^{(m)})^2 = -2/3$. □

Nonexistence of f_+ is of a general nature and holds for various types of quasilinear divergent parabolic approximations. For instance, if instead of (1.2) we consider a parabolic regularization via the quasilinear p-Laplacian operator (gradient-dependent diffusivity coefficients are natural in regularization of conservation laws, see [20])

$$u_t + uu_x = \varepsilon(-1)^{m+1} D_x^m (|D_x^m u|^{p-2} D_x^m u), \quad p > 1, \tag{3.9}$$

then the corresponding "non-entropy" VSP $u_\varepsilon = f_+(y)$, $y = x/\varepsilon^\alpha$, $\alpha = 1/(mp-1)$, is a weak solution of the ODE

$$(-1)^{m+1}(|f^{(m)}|^{p-2} f^{(m)})^{(m)} = ff', \quad f(-\infty) = -1, \ \ f(+\infty) = 1. \tag{3.10}$$

Integrating once yields $(-1)^{m+1}(|f^{(m)}|^{p-2} f^{(m)})^{(m-1)} = \frac{1}{2}(f^2 - 1)$ and multiplying by f' and integrating over $\mathbf{R}$ leads to the same contradiction $\int |f^{(m)}|^p = -2/3$. It seems that no reasonable divergent elliptic operators in the left-hand side of (3.10) can produce a connection $-1 \to 1$ in the corresponding ODE. For such approximation operators, this can be done only by taking negative parameters $\varepsilon < 0$ (then f_+ becomes f_-) creating ill-posed parabolic equations backward in time.

Nonexistence of the VSP does not imply that $S_+(x)$ is not proper, i.e., cannot be obtained by parabolic approximations. In this sense, the case $m = 1$ is exceptional since the proof is straightforward by comparison with the exact solution (3.6). Indeed, if u_ε is an approximation, then $u_\varepsilon(x,t) \le x/t$ in Q_+. Hence, $u_\varepsilon(x,t)$ cannot stabilize to $S_+(x)$ as $\varepsilon \to 0$.

For $m > 1$, where the semigroup induced by equation (3.5) is not order-preserving, we cannot use comparison and the result is based on a Lyapunov-type analysis to prove

Proposition 3.3. *$S_+(x)$ is improper solution.*

Proof. Without loss of generality we assume that $U_\varepsilon(x,t) \to 1$ as $y \to \infty$ sufficiently fast (e.g., exponentially, which happens if $U_{0\varepsilon}(y) = 1$ for $y \gg 1$, following from the exponential decay of the fundamental solution of the parabolic operator [14]), and integrations below make sense. Multiplying equation (3.5) by U and integrating over $\mathbf{R}_+$ yields a Lyapunov function monotone decreasing on evolution orbits,

$$\frac{d}{d\tau}\Phi(U)(\tau) \equiv \frac{1}{2}\frac{d}{d\tau}\left[\int_0^\infty (U^2 - 1)dy\right] = -\frac{1}{3} - \int_0^\infty (D_y^m U)^2 dy \le -\frac{1}{3}. \tag{3.11}$$

Therefore, $\Phi(U)(\tau) \le -\tau/3 + \Phi(U_{0\varepsilon})$ for $\tau > 0$. Using the rescaled variables given in (3.4), we have that for any $t > 0$,

$$\int [u_\varepsilon^2(x,t) - 1]dx \le -2t/3 + 2\Phi(u_{0\varepsilon}). \tag{3.12}$$

Passing to the limit $\varepsilon \to 0$ and using that $u_\varepsilon \to S_+$ in L^1 (then $\Phi(u_{0\varepsilon}) \to 0$), we obtain a contradiction in inequality (3.12). The analysis applies to the Cauchy problem in Q without the anti-symmetry conditions (3.3). □

4. **Stability of the VSP and entropy inequalities**

In this section we deal the first asymptotic problem. In view of negative conclusions of Section 2, we claim that, in general, the convergence (3.1) to entropy solutions cannot be proved without deep understanding of the corresponding asymptotic problems. The approximation problem for $m \ge 2$ is thus an example, where the existence of a solutions (as the limit of $\{u_\varepsilon\}$) cannot be separated from the corresponding parabolic asymptotic theory. As usual in scaling techniques, due to variables (3.4), the limit $\varepsilon \to 0$ for $u_\varepsilon(x,t)$ in a natural sense is equivalent to $\tau \to \infty$ for $U(y,\tau)$.

4.1. **On generic formation of the shock layer: stability of the VSP.** We now discuss conditions under which the VSP satisfying (1.21) describes the generic formation of the shock layer in the convergence (3.1) to the entropy shock $S_-(x)$. This means that f is the asymptotically stable stationary solution of the rescaled equation (3.5) and we perform the standard linearization by setting

$$U(y,\tau) = f(y) + Y(y,\tau), \quad \text{where } Y \text{ solves} \tag{4.1}$$

$$Y_\tau = \mathbf{N}_{2m} Y + \mathbf{D}(Y) \quad \text{with } \mathbf{N}_{2m} = (-1)^{m+1} D_y^{2m} - f D_y, \tag{4.2}$$

$\mathbf{D}(Y) = Y Y_y$ being the quadratic perturbation. By the principle of the linearized stability (see e.g. [31], Chapt. 9), one needs to study the spectral problem

$$\mathbf{N}_{2m} \psi = \lambda \psi, \tag{4.3}$$

where by the classical ODE theory [7], $\psi(y)$ is assumed to have exponential decay as $|y| \to \infty$. Multiplying equation (4.3) by $\bar\psi$ in $L^2(\mathbf{R})$ and the conjugated one by ψ yields

$$\operatorname{Re}\lambda \, \|\psi\|_2^2 = -\|\psi^{(m)}\|_2^2 + \frac{1}{2} \int f'(y) |\psi(y)|^2 dy. \tag{4.4}$$

We thus observe another "bad" consequence of the VSP $f(y)$ being non-monotone: if $f'(y)$ changes sign, then (4.4) does not directly imply the necessary stability condition

$$\operatorname{Re}\lambda < 0 \quad \text{for } \lambda \in \sigma(\mathbf{N}_{2m}), \tag{4.5}$$

unlike the only case $m = 1$, where $f' < 0$ by (1.22) and (4.5) follows from (4.4). Nevertheless, since $f(y)$ must be "effectively" decreasing as a heteroclinic connection $1 \to -1$, one can expect that (4.5) remains true for such $f(y)$. This is proved in [33] for $m = 2$ (the proof is partially computational), and, as a consequence, operator $\mathbf{N}_4$ was shown [15] to be sectorial with the spectrum satisfying $(m = 2)$

$$\sigma(\mathbf{N}_{2m}) \subset \{\operatorname{Re}\lambda \le -k\} \quad \text{with a constant } k > 0. \tag{4.6}$$

in the weighted Sobolev space $H^3_\rho(\mathbf{R})$ with the exponential weight $\rho(y) = \cosh(\mu y)$, where $\mu > 0$ is a small constant. This guarantees the exponential decay of the semigroup

$\|e^{\mathbf{N}_{2m}\tau}\|_{\mathcal{L}} \le Ce^{-k\tau}$ in the space of linear maps $\mathcal{L}(H^3_\rho, H^3_\rho)$, and hence the exponential stability of the VSP.

It is natural to expect that such stability results are true for arbitrary $m > 2$. Namely, the eigenvalue problem (4.3) for the ODE operator $\mathbf{N}_{2m}$ in the weighted space $L^2_\rho(\mathbf{R})$ of odd functions satisfying (3.3) with the dense domain $H^{2m}_\rho(\mathbf{R})$, satisfies (4.5) and (4.6). Nevertheless, even a computational proof, which can be done for $m = 3$ and 4 by rather standard codes, is expected to get more and more involved with m increased. Once (4.5) is proved, the theory of sectorial operators [16], [31] and interpolation inequalities apply to guarantee the exponential stability of VSP's. In its turn this will imply that entropy conditions from Introduction cannot be approximated in the viscosity sense for any initial data and the variation deficiency (2.5) scaled according to (1.19) for moving shocks actually describes locally the "jump" of variation at $\varepsilon = 0$.

4.2. **ODE problem "e2-e4" and non-local entropy inequalities.** Let us discuss the problem on $2m$-th order parabolic approximations of general entropy solutions. It consists of two parts.

Test on the entropy inequality. We begin with the relation between the stability of the VSP and the entropy inequality (1.13). The main assumption is as follows: the VSP is asymptotically and *globally* stable in a suitable weighted space. Note that the global stability (i.e., for any arbitrarily large initial data with given behaviour at infinity) assumes the uniqueness of the VSP, which is an open problem for $m \ge 3$ for (1.21) and other types of quasilinear approximations.

Under the assumption on the global asymptotic stability of the VSP, we return to the entropy inequality (1.13) and show that problem e2-e4 occurring in higher-order approximations can be now solved as an ODE-problem. We discuss the case $m = 2$ (the same analysis applies to any $m \ge 2$). Let $u(x,t)$ be a weak solution of the conservation law constructed by convergence (3.1) via approximation (1.2). We assume that u belongs to the class K of piece-wise smooth functions, so that there exists a finite number of smooth discontinuity curves $\{x = x_i(t)\}$ on the $\{x,t\}$-plane. This is a usual setting for conservation laws [35], but such a theory cannot be global in time. It is global for piece-wise constant initial data leading to piece-wise linear simple waves as in Glimm's difference scheme for hyperbolic systems, see [40], Chapt. 19.

As we have seen, the derivation of inequality (1.13) via (2.17) demands special features of approximations of discontinuities. Consider first an isolated entropy shock of $u(x,t)$. We then need to scale VSP according to (1.19) in oder to include moving discontinuities, so that the profile f_- with the jump $[l_-, l_+]$, $l_- > l_+$, can change with time, $l_\pm = l_\pm(t)$, which is not essential for our further formal analysis. By translations and scalings (1.19), the shock can be assumed to be located at $(0,0)$ with the jump $[1,-1]$, i.e., $u(x,0) = S_-(x) + o(1)$ for $x \approx 0$. Using scaling (3.4), in view of the above stability properties of the VSP, we assume by the translational invariance, that near the entropy shock at $x = x_0(t)$

$$u_\varepsilon(x,t) = f_-((x - x_0(t))/\varepsilon^{1/3}) + o(1). \tag{4.7}$$

By the parabolic theory for (3.5), equality (4.7) also holds for any spatial derivative of u_ε. This is the main assumption which, in the general case, remains a hard open problem.

We now pass to the limit in the integral terms (2.17) by using the asymptotic expansion (4.7). Supposing that $\chi(x - x_0(t), t)$ is supported in a small neighbourhood of a discontinuity curve $x = x_0(t)$ for $t \geq \delta > 0$ and substituting $u = u_\varepsilon$ from (4.7) into (2.17), by change $x - x_0(t) = y\varepsilon^{1/3}$, one obtains the expression

$$J_2(\varepsilon) = -\iint [E''(f)(f'')^2 - \frac{1}{3}E'''(f)(f')^4]\chi(y\varepsilon^{1/3}, t)dydt + o(1), \tag{4.8}$$

where $o(1)$ includes both the second term in (2.17) of the higher order $O(\varepsilon^{1/3})$ and the error of approximation in (4.7). Since $\chi(y\varepsilon^{1/3}, t) \to \chi(0, t)$ as $\varepsilon \to 0$ uniformly on compact subsets in y, we thus observe the same problem e2-e4 but now formulated for VSP's $f = f_-$, i.e., for solutions of the ODE (1.21), on which two terms in the first integral in (2.17) are not independent. We solve the ODE problem e2-e4 multiplying $-f^{(4)} = ff'$ by $E'(f)$ and integrating by parts,

$$-\int [E''(f)(f'')^2 - \frac{1}{3}E''''(f)(f')^4]dy = \Delta_E \equiv F(-1) - F(1). \tag{4.9}$$

Since $E(u)$ is a smooth convex approximation of $|u - k|$ [26], [2], there holds

$$-\Delta_E = \int_{-1}^{1} zE'(z)dz \equiv E(1) + E(-1) - \int_{-1}^{1} E(z)dz \geq 0. \tag{4.10}$$

Therefore, we obtain the required entropy inequality (1.13) by passing to the limit $\varepsilon \to 0$ in the integral term (4.8). Roughly speaking, it follows that if (i) the approximation is ODE-admissible, i.e., there exists the VSP f_- and (ii) f_- is globally asymptotically stable, then locally, close to discontinuity curves, this parabolic approximation gives the right inequality sign in (1.13).

We now show that problem "u_{xxx}" (2.30) is also of the purely asymptotic nature. Indeed, if (4.7) holds for the derivatives, then (2.29) yields

$$\lim_{\varepsilon} [-2\varepsilon\, u_{\varepsilon xxx}(x_0(t), t)] = -2f_-'''(0) = -1, \tag{4.11}$$

which follows from the ODE (1.21) integrating over $(-\infty, 0)$. The derivation of (2.32) is quite the same.

Concerning the total variation problem in (2.33), we note that on the VSP $u = f_-(y)$, $y = x/\varepsilon^\alpha$, where by the ODE (1.21), $f^{(2m)} = (-1)^{m+1}ff'$, there holds

$$\varepsilon J_m(f_-) = \varepsilon^{1-2m\alpha} \sum (-1)^k f_- f_-' |_{a_k}^{b_k} \equiv 0$$

because by construction $f' = 0$ at the end points of each interval $I_k = (a_k, b_k)$. On the other hand, since the exponent of ε, $1 - 2m\alpha = -1/(2m - 1) < 0$, the boundedness of total variation via (2.33) needs a sharp estimate of convergence in (4.7) as $\varepsilon \to 0$ in order to guarantee, at least, the boundedness of $J_m(u)$ (not the non-negativity as for $m = 1$).

Test on non-entropy solutions. Next, we arrive at the second important test on higher-order parabolic approximations showing that non-entropy shocks of the type $S_+(x)$ can be ruled out. This is shown by the same Lyapunov-type construction as in the proof of Proposition 3.3. Assume by scaling that such a non-entropy isolated shock occurs as $(0,0)$, i.e., $u_\varepsilon(x,0^-) \approx S_+(x)$ for $x \approx 0$. We postulate (though this is not straightforward) that on any subset where $u(x,t)$ is sufficiently smooth we observe the uniform convergence $u_\varepsilon \to u$ with derivatives. Multiplying equation (1.2) by u and integrating over sufficiently small interval $(-\delta,\delta)$ chosen so that $u(\pm\delta,t)$ and hence $u_\varepsilon(\pm\delta,t)$ are sufficiently smooth for all $t \approx 0$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{-\delta}^{\delta}(u_\varepsilon^2-1)dx \le -\frac{1}{2}u_\varepsilon^2(x,t)|_{-\delta}^{\delta} - \varepsilon\int_{-\delta}^{\delta}|D_x^m u_\varepsilon|^2dx + O(\varepsilon) \le -\frac{1}{3}, \tag{4.12}$$

where $O(\varepsilon)$ includes non-integral terms $\varepsilon(-1)^{k+1}D_x^{2m-k}u_\varepsilon D_x^k u_\varepsilon|_{-\delta}^{\delta}$ for $k=0,1,...,m-1$ obtained via integration by parts. Therefore, $u_\varepsilon(x,t)$ cannot stabilize to $S_+(x)$ as $t \to 0^-$ and $\varepsilon \to 0$, since this assumes the increase of this Lyapunov function. Such non-entropy shocks cannot appear evolutionary, which is well known for $\varepsilon = 0$, [40], p. 262, and (4.12) shows that higher-order parabolic approximations improve this *irreversibility* property. In simple configurations, for $u \in K$ with finite number of isolated shocks, this shows that $2m$-th order parabolic approximations can create entropy shocks only completing an important part of the analysis.

For arbitrary bounded measurable functions u_0 one can perform approximation u_{0n} via piece-wise constant functions to get that the corresponding entropy solution $u_n \in K$ (see Glimm's approach, [40], Chapt. 19). Formally, this makes it possible to study separately the approximation of each of a finite number of shocks in order to prove that $u_{\varepsilon n} \to u_n$ as $\varepsilon \to 0$. Next, we apply (3.2) to get convergence $u_{\varepsilon n} \to u$ as $\{\varepsilon \to 0,\ n \to \infty\}$ to the unique entropy solution. The main open problem in this approximation analysis is thus the proof of global asymptotic stability in the form (4.7) (or a weaker perturbed equality) in a wider functional setting which includes initial data $U_{0\varepsilon} \notin H_\rho^{2m}$ in (3.5). This can make extra perturbations of (4.7), but, nevertheless, is expected not to affect the nonlocal entropy inequalities.

5. **Asymptotic stability of the rarefaction profile**

We now consider the second asymptotic problem (not of less importance) of the stability of the rarefaction wave occurring for initial data $U_0(y) = S_+(y)$ in the Cauchy problem (3.5). It is convenient to introduce new self-similar rescaled variables

$$U = (1+\tau)^{-(2m-1)/2m}\theta, \quad \xi = y/(1+\tau)^{1/2m}, \quad s = \ln(1+\tau) : \mathbf{R}_+ \to \mathbf{R}_+. \tag{5.1}$$

Then the rescaled solution $\theta = \theta(\xi,s)$ solves the autonomous equation

$$\theta_s = (-1)^{m+1}D_\xi^{2m}\theta - \theta\theta_\xi + \mu\,\theta_\xi\xi + (2m-1)\mu\theta, \quad \mu = 1/2m, \tag{5.2}$$

with the same initial data. Equation (5.2) has the explicit stationary solution

$$\bar\theta(\xi) = \xi \quad \text{in } \mathbf{R}. \tag{5.3}$$

Obviously, (5.1) shows that it is precisely solution (3.6), so that we refer to (5.3) as to the rarefaction profile (RP) defined in $\mathbf{R}$. We prove that the RP is asymptotically stable.

The linearization $\theta = \xi + Y$ yields the perturbed equation

$$Y_s = \mathbf{A}Y - YY_\xi \quad \text{with} \;\; \mathbf{A} = (-1)^{m+1}D_\xi^{2m} - (2m-1)\mu\,\xi\frac{d}{d\xi} - \mu\, I. \tag{5.4}$$

Setting $\xi = c\eta$ with $c^{2m} = 1/(2m-1)$ gives $\mathbf{A} = (2m-1)\mathbf{B}^* - \frac{1}{2m}I$, where the linear elliptic operator in $\mathbf{R}^N$ $\mathbf{B}^* = -(-\Delta_\eta)^m - \mu\,\eta\cdot\nabla_\eta$ is known to have discrete spectrum $\sigma(\mathbf{B}^*) = \{-l/2m,\ l = 0,1,2,...\}$ [13]. The second-order case $m = 1$ is exceptional, where $\mathbf{B}^* \equiv \frac{1}{\rho^*}\nabla\cdot(\rho^*\nabla)$ with the weight $\rho^*(y) = e^{-|y|^2/4}$, is self-adjoint in $L^2_{\rho^*}(\mathbf{R}^N)$ with the domain $\mathcal{D}(\mathbf{B}^*) = H^2_{\rho^*}(\mathbf{R}^N)$ and a discrete spectrum. The eigenfunctions form an orthonormal basis in $L^2_{\rho^*}(\mathbf{R}^N)$ and the classical Hilbert-Schmidt theory applies [3].

We describe the spectral properties of the linearized operator in (5.4) which is not self-adjoint for $m > 1$. We consider $\mathbf{A}$ in the weighted space $L^2_{\rho^*}(\mathbf{R}_+)$ of odd functions with the exponentially decaying weight function

$$\rho^*(y) = e^{-a|y|^\beta} > 0, \quad \beta = 2m/(2m-1), \tag{5.5}$$

where $a > 0$ is a sufficiently small constant. There holds [13].

Lemma 5.1. $\mathbf{A} : H^{2m}_{\rho^*}(\mathbf{R}_+) \to L^2_{\rho^*}(\mathbf{R}_+)$ *is a bounded linear operator with the discrete spectrum*

$$\sigma(\mathbf{A}) = \{\lambda_l = -[1 + (2m-1)l]/2m,\ \ l = 1,3,5,...\}, \tag{5.6}$$

and the eigenfunction subset $\{\psi_l(\xi)\}$ *(l-th order polynomials) is complete in* $L^2_{\rho^*}(\mathbf{R}_+)$.

For $m = 1$, these are well-known properties of the separable Hermite polynomials generated by a self-adjoint Sturm-Liouville problem [3]. In view of the principle of linearized stability [31] we have that the RP is asymptotically stable in $L^2_{\rho^*}(\mathbf{R}_+)$ and moreover, since the real spectrum is uniformly bounded from the imaginary axis, we have the exponential convergence of the order $O(e^{-s}) = O(\tau^{-1})$ as $\tau \to \infty$. Since weight (5.5) is exponentially decaying at infinity, the stability conclusion is true for a wide class of initial data.

Thus, the rarefaction solution (3.6) exhibits the exponential asymptotic stability for parabolic approximations of any order. This explains once more why the non-entropy shocks of type S_+ cannot occur in the evolution, cf. Proposition 3.3. For the Cauchy problem (3.5) with bounded initial data $U_{0\varepsilon} \sim S_+$, the unbounded stable RP (5.3) also plays a role, but the convergence as $\varepsilon \to 0$ is again a hard asymptotic problem which includes a delicate matching-type analysis.

Note that the linear operator $\mathbf{B}^*$ occurs in the study of blow-up solutions of a completely different reaction-diffusion equation $u_t = -(-\Delta)^m u + |u|^p$ in $\mathbf{R}^N \times \mathbf{R}$, $p > 1$, see [17]. The analysis of its global solutions in the supercritical Fujita range $p > 1 + 2m/N$ [13] is based on spectral properties of the adjoint operator $\mathbf{B} = -(-\Delta_\eta)^m + \mu\,\eta\cdot\nabla_\eta + \mu N\, I$.

6. On other higher-order models and parabolic approximations

6.1. Preliminary properties of odd-order models.

In this section we extend the analysis to the odd-order equations with discontinuous solutions

$$u_t + (-1)^{m-1}D_x^{2m-2}(uu_x) = 0 \quad \text{in } Q, \quad u(x,0) = u_0(x) \quad \text{in } \mathbf{R}, \tag{6.1}$$

where $m \geq 2$. For $m = 1$ we get the conservation law (1.1). Equation (6.1) can be written in a form of conservation law in H^{2-2m}

$$[(-D_x^2)^{1-m}u]_t + \frac{1}{2}(u^2)_x = 0.$$

In the class K of piece-wise smooth functions, for a smooth simple closed curve $\Gamma \subset Q$ intersecting the discontinuity lines of $u(x,t)$ at a finite number of points, there holds

$$\int_\Gamma [(-D_x^2)^{1-m}u dx - \frac{1}{2}u^2 dt] = 0. \tag{6.2}$$

(6.1) is a higher-order equation (the third-order one for $m = 2$), and it is difficult to solve it by a method of characteristics. A suitable notion of entropy solutions is unknown. Nevertheless, as an odd-order equation in divergence form, (6.1) inherits several typical discontinuity properties of weak solutions which are defined in a usual way in the sense of distributions. Let us describe some preliminary properties of such solutions.

(i) **Rankine-Hugoniot condition and shock-waves $S_\pm(x)$.** For a piece-wise smooth solution, considering integral (6.2) taken around a contour Γ_δ in a neighbourhood of a smooth discontinuity curve on the $\{x,t\}$-plane (see [35], p. 98) yields the corresponding Rankine-Hugoniot condition of the speed of propagation of shocks

$$\lambda = (-1)^{m-1}[D_x^{2m-2}(u^2)]/2[u], \tag{6.3}$$

where $[\cdot]$ denotes the jump across Γ. Hence, $S_\pm(x)$ are the simplest (stationary) shock-waves to be studied first. Of course, there exist a variety of other discontinuous TWs with more complicated spatial shapes. Since a symmetry group like (1.19) is not available, $S_\pm$ do not describe all types of generic propagation of discontinuities though can explain some of their crucial features.

(ii) **Rarefaction wave corresponding to initial data $S_+(x)$.** We are going to show that exactly as for $m = 1$, $S_+(x)$ is not a proper solution and hence one needs to describe the proper one $u_+(x,t)$ corresponding to initial data $u_0 = S_+(x)$ in (6.1). It has the self-similar form

$$u_+(x,t) = \theta(\xi), \quad \xi = x/t^\alpha, \ \ \alpha = 1/(2m-1), \tag{6.4}$$

where the odd rarefaction profile $\bar{\theta}$ satisfies the ODE

$$(-1)^m(\theta\theta')^{(2m-2)} + \alpha\theta'\xi = 0, \ \xi \in (-a,a); \ \ \theta(a) = 1, \ \theta'(a) = ... = \theta^{(2m-2)}(a) = 0, \tag{6.5}$$

where $a > 0$ is the unknown position of the rescaled interface. The boundary conditions in (6.5) are dictated by (6.3). For $m = 1$, this first-order ODE is easily solved to give the RP $\theta(\xi) = \xi$, $a = 1$, given in (1.18). For $m = 2$, (6.5) is a third-order equation $3(\theta^2)''' + 2\theta'\xi = 0$, which is invariant under a group of scalings. Simple explicit solutions are not available except $\theta = -\xi^3/60$ which is not a connection $-1 \to 1$. The change of variables $\zeta = \ln\xi$, $\theta = e^{3\zeta}\varphi(\zeta)$, $P(\varphi) = \varphi'$ reduces it to a complicated second-order ODE for P, which can be studied to guarantee existence of a suitable RP. In general, for large m, problem (6.5) is to be analyzed numerically.

(iii) $2m$**-th order parabolic approximation.** A natural $2m$-th order regularization of (6.1) is

$$u_t + (-1)^{m-1} D_x^{2m-2}(uu_x) = \varepsilon(-1)^{m+1} D_x^{2m} u, \qquad u(x,0) = u_{0\varepsilon}(x), \tag{6.6}$$

where $u_{0\varepsilon} \to u_0$ as $\varepsilon \to 0^+$ in a suitable topology. The well-posedness of such approximations in the sense of Proposition 2.1 becomes much more delicate problem not studied here. The corresponding rescalings are $u(x,t) = U(y,\tau)$, $y = x/\varepsilon$, $\tau = t/\varepsilon^{2m-1}$, where

$$U_\tau + (-1)^{m-1} D_y^{2m-2}(UU_y) = (-1)^{m+1} D_y^{2m} U. \tag{6.7}$$

(iv) **Monotone viscosity shock profile.** The VSP corresponding to the proper (see below) shock-wave $S_-(x)$ has the form $f_-(y)$, $y = x/\varepsilon$, satisfying $f^{(2m)} = \frac{1}{2}(f^2)^{(2m-1)}$, $f(\pm\infty) = \mp 1$, i.e., $f' = \frac{1}{2}(f^2 - 1)$. Hence the unique VSP has the form (1.22). Indeed, as we have seen, the monotonicity of the VSP is an essential positive feature of this higher-order model.

6.2. **Proper and improper shock-waves.** Similar to Section 3, we say that $u(x,t)$ is a *proper* solution of the Cauchy problem (6.1) if there exists a sequence of initial data $u_{0\varepsilon} \to u_0$ such that the solutions of parabolic problems (6.6) satisfies (3.1) (at least in H^{-m}). Our main goal is to study the evolution properties of the shock-waves $S_\pm(x)$.

Proposition 6.1. *(i) $S_-(x)$ is a proper solution, and (ii) $S_+(x)$ is an improper one.*

Proof. (i) We have that convergence (3.7) with the VSP (1.22) holds a.e. (ii) The VSP f_+ corresponding to $S_+(x)$, i.e., a solution of the ODE satisfying $f(\pm\infty) = \pm 1$, does not exist. Consider equation (6.7) in $Q_+ = \mathbf{R}_+ \times \mathbf{R}_+$ with conditions (3.3). Assuming that $u_{0\varepsilon}(x) \to 1$ as $x \to \infty$ exponentially fast, and that the same holds for solutions $u_\varepsilon(x,t)$, we apply to equation (6.7) operator $(-d^2/dy^2)^{1-m}$ naturally defined via integrating equation $2m-2$ times and integrate again over (y,∞). Next, multiplying by $U-1$ in L^2, we arrive at a Lyapunov function (cf. (3.11))

$$\frac{1}{2}\frac{d}{d\tau}\int_0^\infty \left[\left(\frac{d}{dy}\right)^{-m}(U-1)\right]^2 = -\frac{1}{3} - \int_0^\infty (U_y)^2 \le -\frac{1}{3}. \tag{6.8}$$

Integrating and rescaling this identity, similarly to the proof of Proposition 3.3, we have that u_ε cannot converge to S_+ as $\varepsilon \to 0$. $\square$

6.3. **On stability of the VSP for $m = 2$ and the shock layer.** Let us show that for $m = 2$ the monotonicity of the VSP (1.22) guarantees the necessary condition (4.5) of its stability. The linearization (4.1) yields the quadratically perturbed equation (4.2) with the linear operator

$$\mathbf{N}_4 Y = -Y^{(4)} + (fY)'''. \tag{6.9}$$

Solving the eigenvalue problem (4.3) in a space of exponentially decaying functions (hence from L^2) and setting $\psi = \phi'''$, we arrive at the eigenvalue equation $-\phi^{(4)} + f\phi''' = \lambda\phi$, $\phi \in H^4$. Multiplying this equation by $\bar{\phi}''$ in L^2 and the conjugate one by ϕ'', after integration by parts one obtains $\operatorname{Re}\lambda \int |\phi'|^2 = -\int |\phi'''|^2 + \frac{1}{2}\int f'|\phi''|^2$. It follows that in suitable classes of even or odd functions, (4.5) holds. By the interpolation inequalities, this implies that (6.9) is a sectorial operator in a weighted L^2-space (see [15]) and the exponential stability of the VSP follows. This means that convergence (3.7) describes the generic formation of the shock layers in fourth-order parabolic approximations of such shock-waves as weak solutions of the third-order equation (6.1).

Stability of the RP (6.4) is a harder open problem.

6.4. **On higher-order approximations.** Equation (6.1) admits various parabolic approximations of different orders. Consider its $(2m+2)$-th order approximation

$$u_t + (-1)^{m-1} D_x^{2m-2}(uu_x) = \varepsilon(-1)^m D_x^{2m+2} u, \tag{6.10}$$

with rescaled variables $u(x,t) = U(y,\tau)$, $y = x/\varepsilon^{1/3}$, $\tau = t/\varepsilon^{(2m-1)/3}$, where U solves

$$U_\tau + (-1)^{m+1} D_y^{2m-2}(UU_y) = (-1)^m D_y^{2m+2} U. \tag{6.11}$$

Then the VSP $f(y)$ for the shock wave $S_-(x)$ is the same as for extended Burger's equation (1.2) with $m = 2$ and is uniquely determined by the ODE problem (1.21). The stability analysis is based on the results from [33] and [15]. The characterization of shock-waves, Proposition 6.1, remains unchanged.

6.5. **On a quasilinear approximation.** As a final example, we show that quasilinear approximations can keep the main features of parabolic regularization. Consider the following approximation of (6.1) via the p-Laplacian operator as in (3.9):

$$u_t + (-1)^{m-1} D_x^{2m-2}(uu_x) = \varepsilon(-1)^{m+1} D_x^m(|D_x^m u|^{p-2} D_x^m), \quad p > 1, \tag{6.12}$$

where $u_\varepsilon(x,t) = U_\varepsilon(y,\tau)$, $y = x/\varepsilon^\alpha$, $\tau = t/\varepsilon^{(2m-1)\alpha}$ and $\alpha = 1/[1 + m(p-2)]$. Let $m = 2$. Then the entropy VSP f_- satisfies the ODE $ff' = |f''|^{p-2} f''$ with $f(\pm\infty) = \mp 1$ (one can see that the non-entropy VSP f_+ does not exist). For $y > 0$, we have $f < 0$, $f' \le 0$ and $f'' \ge 0$, and setting $-f' = R \ge 0$ yields $f'' \equiv RR_f = (-f)^{1/(p-1)} R^{1/(p-1)}$. If $p \in (1, 3/2]$, integrating once yields that a solution satisfying $R(-1) = 0$ does not exist, i.e., approximation (6.12) in not admissible. For $p > 3/2$, from the equation

$$R = -f' = a_0[1 - (-f)^{p/(p-1)}]^{(p-1)/(2p-3)}, \quad a_0 = [(2p-3)/p]^{(p-1)/(2p-3)}, \tag{6.13}$$

one obtains the unique VSP $f = f_-(y)$ from the quadrature

$$\int_0^{-f} [1 - z^{p/(p-1)}]^{-(p-1)/(2p-3)} dz = a_0 y, \quad y > 0. \tag{6.14}$$

Hence, for $p \in (3/2, 2]$, $f_-(y)$ is strictly monotone decreasing in $\mathbf{R}$ and is a C^∞ function as in the linear case $p = 2$. For $p > 2$ it has finite regularity at the interface, where $f_-(y_0) = -1$ at $y_0 = [(p-1)/a_0 p] B((p-1)/(2p-3), (p-1)/p)$, B being Euler's Beta function. Though, for $p > 2$, the VSP is strictly decreasing on $I_0 = (-y_0, y_0)$, the stability analysis and other related questions on such approximations become more involved. Indeed, linearization (4.1) leads to a singular ODE operator $\mathbf{N}_{2m}$ on I_0 in equation (4.2). The functional setting becomes more complicated (the weight function ρ is expected to be unbounded at the singular end-points $y = \pm y_0$) and a delicate matching procedure extending the stability analysis beyond interval I_0 should be performed. This kind of quasilinear approximations are not well-posed (e.g., uniqueness of solutions is not well understood in general) though keeps some typical features of semilinear parabolic approximations.

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