Uniqueness and error analysis for Hamilton–Jacobi equations with discontinuities

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Abstract: We consider the Hamilton–Jacobi equation of eikonal type

$$H(\nabla u) = f(x), \quad x \in \Omega,$$

where H is convex and f is allowed to be discontinuous. Under a suitable assumption on f we prove a comparison principle for viscosity sub- and supersolutions in the sense of Ishii. Furthermore, we develop an error analysis for a class of finite difference schemes, which are monotone, consistent and satisfy a suitable stability condition.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We consider the Hamilton–Jacobi equation

$$\begin{aligned} H(\nabla u) &= f(x) \quad x \in \Omega &(1.1)\\ u(x) &= \phi(x) \quad x \in \partial\Omega, &(1.2) \end{aligned}$$

where f and ϕ are given functions. The equation (1.1) occurs in a variety of applications including geometrical optics, computer vision and etching. In order to motivate the link to propagating fronts, let us suppose for a moment that such a front at time t can be described as the t–level set of an auxiliary function $u : \bar{\Omega} \to \mathbb{R}$, i.e. $\Gamma(t) = \{x \in \Omega \,|\, u(x) = t\}$. Then, formally, a unit normal ν to $\Gamma(t)$ and the corresponding normal velocity V are given by

$$\nu = \frac{\nabla u(x)}{|\nabla u(x)|}, \quad V = \frac{1}{|\nabla u(x)|}, \quad x \in \Gamma(t).$$

If in addition, u solves (1.1), f is positive and H is homogeneous of degree one, then

$$V = \frac{1}{|\nabla u(x)|} = \frac{1}{|\nabla u(x)|}\frac{H(\nabla u(x))}{f(x)} = \frac{1}{f(x)}\,H(\nu(x)), \quad x \in \Gamma(t),$$

i.e. the front moves with a normal velocity which depends on the properties of the underlying space and the orientation of the front. Furthermore, we can interpret $u(x)$ as the first arrival

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time at the point x of a front which was the zero level set of u at the beginning of the evolution.

It is often desirable to consider situations, in which the function f is allowed to be discontinuous, e.g. in geometrical optics, when light propagates through a layered medium.Thus, we shall be concerned both with the well–posedness of (1.1), (1.2) in the case that f is discontinuous and with the convergence of numerical algorithms which approximate the corresponding solution.

Various ways of defining a solution of (1.1), (1.2) for discontinuous f have been suggested. First, Ishii [7] extended the concept of viscosity solution introduced in [5] to the discontinuous case by taking suitable upper and lower semicontinuous envelopes of f (see §2 below). Using this notion of solution, Soravia [12] studies a class of Hamilton–Jacobi equations in a control–theoretic framework and gives necessary and sufficient conditions for uniqueness of solutions of boundary value problems. In [9], Newcomb & Su introduce a concept of solution which is based on the optical length function $L(x, y)$ (see §2 below) and which they term Monge solution. For lower semicontinuous f they prove a comparison principle as well as existence and uniqueness for the Dirichlet problem. Recently, a further definition of solution was suggested in [4] by Camilli & Siconolfi for Hamilton–Jacobi equations of the form $H(x, Du) = 0$. They introduce a generalized notion of viscosity solution, which allows measurable dependence of H on x. In the case of the eikonal equation, i.e. $H(p) = |p|$, this definition involves the measure–theoretic notion of an approximate limit for a subsolution and an essential limit for a supersolution and hence is not symmetric. Comparison and uniqueness results are provided.

Problem (1.1), (1.2) also occurs in shape–from–shading. In this case, the right hand side f is related to the light intensity which can be discontinuous. In [13], Tourin establishes a comparison result for equations of the form $H(x, Du) = 0$, in which H is allowed to be discontinuous along a smooth surface. For the shape–from–shading problem, Rouy & Tourin [11] present a consistent and monotone scheme along with numerical calculations. A related problem is studied in [10], where the unique solution is obtained as the limit of sequences which arise from a suitable regularisation of the intensity function.

Our work is based on Ishii's definition of solution, which we shall recall at the beginning of §2. In order to obtain uniqueness for the solution of (1.1), (1.2) an assumption on f is needed: this condition (see (2.5) below) can be seen as a generalisation of a condition, which appears in [13] and it amounts to a one–sided continuity constraint along a fixed direction at each point in Ω. Under this assumption we are able to prove a comparison result in Theorem 2.3. In §3 we analyze a class of numerical schemes, which approximate the solution of (1.1), (1.2). Denoting by h the gridsize we obtain an order $\mathcal{O}(\sqrt{h})$ for finite difference schemes, which are monotone, consistent and satisfy a suitable stability condition. In §4 we present examples of schemes which satisfy the above requirements, while §5 contains numerical tests.

2 Existence and Uniqueness

Let us start by defining a viscosity solution of (1.1), (1.2). As already mentined above, we use a concept which was introduced by Ishii in [7] and which is based on upper and lower semicontinuous envelopes. For a given function $v : \Omega \to \mathbb{R}$ let

$$\begin{aligned} v^*(x) &:= \lim_{r\to 0} \sup\{v(y) \,|\, y \in B_r(x) \cap \Omega\} \\ v_*(x) &:= \lim_{r\to 0} \inf\{v(y) \,|\, y \in B_r(x) \cap \Omega\}. \end{aligned}$$

Definition 2.1. A function $u \in C^0(\bar{\Omega})$ is called a viscosity subsolution (supersolution) of (1.1) if for each $\zeta \in C^\infty(\Omega)$: if $u - \zeta$ has a local maximum (minimum) at a point $x_0 \in \Omega$, then

$$H(\nabla\zeta(x_0)) \le f^*(x_0) \quad (\ge f_*(x_0)).$$

A viscosity solution of (1.1), (1.2) then is a function $u \in C^0(\bar{\Omega})$ which is both a viscosity sub–and supersolution and which satisfies $u(x) = \phi(x)$ for all $x \in \partial\Omega$.
We shall assume that $H : \mathbb{R}^n \to \mathbb{R}$ satisfies:

$$H(0) = 0 \text{ and } H(p) > 0 \text{ for all } p \in \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

$$H \text{ is convex .} \tag{2.2}$$

$$H(p) \to \infty \text{ as } |p| \to \infty. \tag{2.3}$$

Concerning the right hand side we make the assumption that $f : \Omega \to \mathbb{R}$ is Borel measurable and that there exist $0 < m \le M < \infty$ such that

$$m \le f(x) \le M \qquad \forall x \in \Omega. \tag{2.4}$$

Furthermore, we assume that for every $x \in \Omega$ there exist $\epsilon_x > 0$ and $n_x \in S^{n-1}$ so that for all $y \in \Omega, r > 0$ and all $d \in S^{n-1}$ with $|d - n_x| < \epsilon_x$ we have

$$f(y + rd) - f(y) \le \omega\big(|y - x| + r\big), \tag{2.5}$$

where $\omega : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function with $\omega(0) = 0$. Clearly, (2.5) holds at all points x, at which f is continuous, but it also allows for certain types of discontinuous behaviour as shown by the following

Example: Suppose that a surface Γ splits Ω into two subdomains Ω_1 and Ω_2, that $f_{|\Omega_1} \in C^0(\bar{\Omega}_1)$, $f_{|\Omega_2} \in C^0(\bar{\Omega}_2)$ and that

$$\lim_{y \to x, y \in \Omega_1} f(y) < \lim_{y \to x, y \in \Omega_2} f(y) \qquad \text{for all } x \in \Gamma.$$

In addition, assume that the following uniform cone property holds: for every $x \in \Gamma$ there exists a neighborhood U_x and a cone C_x (which is congruent to a fixed given cone C_0) such that $y \in U_x \cap \bar{\Omega}_1$ implies that $y + C_x \subset \Omega_1$. Then (2.5) holds with $n = n_x$ given by the direction of the cone C_x.
To see this, observe that the cone condition prevents a situation where $y \in \bar{\Omega}_1, y + rd \in \Omega_2$, which would lead to a violation of (2.5) (cf. [13], where Γ is assumed to be smooth).
One can also consider e.g. a two–dimensional domain Ω, where three curves of discontinuity meet at a triple junction.

It is not difficult to verify that (2.5) implies

$$f^*(y + rd) - f_*(y) \le \omega\big(|y - x| + r\big) \tag{2.6}$$

for all $y \in \Omega, r > 0$ and $d \in S^{n-1}, |d - n_x| < \epsilon_x$.
In order to describe our assumptions on ϕ let us define $L : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$L(x, y) := \inf\{\int_0^1 N\big(f^*(\gamma(t)), \gamma'(t)\big)\, dt \mid \gamma \in W^{1,\infty}((0,1); \bar{\Omega}) \text{ with } \gamma(0) = x,\, \gamma(1) = y\},$$

where

$$N(r, \zeta) := \sup\{-(\zeta, p) \,|\, H(p) = r\}.$$

We then suppose the following compatibility condition for the boundary data,

$$\phi(x) - \phi(y) \le L(x, y) \quad \forall x, y \in \partial\Omega. \tag{2.7}$$

Theorem 2.2. *Under the above assumptions on f and ϕ there exists a viscosity solution $u \in C^{0,1}(\bar{\Omega})$ of (1.1), (1.2).*

Proof. We regularize f using the sup–convolution, i.e.

$$f_\epsilon(x) := \sup_{y \in \Omega}\{f(y) - \frac{1}{\epsilon}|x-y|^2\}, \qquad \epsilon > 0.$$

Clearly, f_ϵ is continuous and satisfies $f^*(x) \le f_\epsilon(x)$ for all $x \in \Omega$. In view of (2.7) and the monotonicity of N in the first variable we deduce that

$$\phi(x) - \phi(y) \le L_\epsilon(x,y) \quad \forall x, y \in \partial\Omega,$$

where

$$L_\epsilon(x,y) := \inf\{\int_0^1 N\big(f_\epsilon(\gamma(t)), \gamma'(t)\big)\, dt \mid \gamma \in W^{1,\infty}((0,1);\bar{\Omega}) \text{ with } \gamma(0) = x,\ \gamma(1) = y\}.$$

Therefore, the problem

$$\begin{aligned} H(\nabla u^\epsilon) &= f_\epsilon(x) \quad x \in \Omega \\ u^\epsilon(x) &= \phi(x) \quad x \in \partial\Omega, \end{aligned}$$

has a unique viscosity solution u^ϵ, which is given by the formula

$$u^\epsilon(x) = \inf_{y \in \partial\Omega}\{L_\epsilon(x,y) + \phi(y)\}.$$

It is not difficult to verify that

$$\|u^\epsilon\|_{C^{0,1}(\bar{\Omega})} \le C(M, \Omega) \qquad \text{uniformly in } \epsilon > 0.$$

Thus, there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ with $\epsilon_k \searrow 0, k \to \infty$ and $u \in C^{0,1}(\bar{\Omega})$ such that

$$u^{\epsilon_k} \to u,\ k \to \infty \qquad \text{uniformly in } \bar{\Omega}. \tag{2.8}$$

Clearly, $u = \phi$ on $\partial\Omega$. We claim that u is a viscosity solution in the sense of Definition 2.1. Let $\zeta \in C^\infty(\Omega)$ and suppose that $u - \zeta$ has a local maximum at $x_0 \in \Omega$. In view of (2.8) there exist $x_k \in \Omega$ such that $x_k \to x_0$, $k \to \infty$ and $u^{\epsilon_k} - \zeta$ has a local maximum at x_k. Then

$$H(\nabla\zeta(x_k)) \le f_{\epsilon_k}(x_k), \tag{2.9}$$

and taking into account (2.4) we obtain

$$f_{\epsilon_k}(x_k) = \sup_{|x_k - y| \le \sqrt{M\epsilon_k}}\{f(y) - \frac{1}{\epsilon_k}|x_k - y|^2\} \le \sup\{f(y) \mid |y - x_0| \le |x_k - x_0| + \sqrt{M\epsilon_k}\},$$

which implies by passing to the limit in (2.9)

$$H(\nabla\zeta(x_0)) \le \overline{\lim}_{k\to\infty} f_{\epsilon_k}(x_k) \le f^*(x_0).$$

On the other hand, if $u - \zeta$ has a local minimum at x_0, there exist $\tilde{x}_k \in \Omega$ with $\tilde{x}_k \to x_0, k \to \infty$, such that $u^{\epsilon_k} - \zeta$ has a local minimum at $\tilde{x}_k$. Thus,

$$H(\nabla\zeta(\tilde{x}_k)) \ge f_{\epsilon_k}(\tilde{x}_k) \ge f_*(\tilde{x}_k).$$

Since f_* is lower semicontinuous, we deduce that

$$H(\nabla\zeta(x_0)) \ge f_*(x_0).$$

In conclusion, u is a viscosity solution of (1.1), (1.2). ∎

Uniqueness of the viscosity solution is a consequence of the following comparison result.

Theorem 2.3. *Suppose that $u \in C^0(\bar\Omega)$ is a subsolution of (1.1), $v \in C^0(\bar\Omega)$ is a supersolution of (1.1) and that at least one of the functions belongs to $C^{0,1}(\bar\Omega)$. If $u \le v$ on $\partial\Omega$ then $u \le v$ in $\bar\Omega$.*

Proof. Let us assume that $v \in C^{0,1}(\bar\Omega)$. We shall use the approach presented in [8] (see also [13]). Fix $\theta \in (0,1)$ and define $u_\theta(x) := \theta u(x)$. Next, choose $x_0 \in \bar\Omega$ such that

$$u_\theta(x_0) - v(x_0) = \max_{x\in\bar\Omega}\big(u_\theta(x) - v(x)\big) =: \mu, \tag{2.10}$$

and suppose that $\mu > 0$. Upon replacing u, v by $u+k, v+k$, we may assume that $u \ge 0$ in $\bar\Omega$, so that $u_\theta \le u$ in $\bar\Omega$. In particular, $u_\theta \le v$ on $\partial\Omega$, which implies that $x_0 \in \Omega$. Let $\epsilon = \epsilon_{x_0}$ and $n = n_{x_0} \in S^{n-1}$ be the quantities which appear in (2.5) and define for $\lambda > 0, L \ge 1$ the function $\Phi : \bar\Omega \times \bar\Omega \to \mathbb{R}$ by

$$\Phi(x,y) := u_\theta(x) - v(y) - L\lambda\,|x - y - \frac{1}{\lambda}\,n\,|^2 - |x - x_0|^2.$$

Let $(x_\lambda, y_\lambda) \in \bar\Omega \times \bar\Omega$ be such that

$$\Phi(x_\lambda, y_\lambda) = \max_{(x,y)\in\bar\Omega\times\bar\Omega} \Phi(x,y). \tag{2.11}$$

Since $x_0 - \frac{1}{\lambda}n \in \Omega$ for large λ, the relation $\Phi(x_\lambda, y_\lambda) \ge \Phi(x_0, x_0 - \frac{1}{\lambda}n)$ implies together with (2.10)

$$\begin{aligned} L\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 &\le u_\theta(x_\lambda) - v(y_\lambda) - u_\theta(x_0) + v(x_0 - \frac{1}{\lambda}n) \\ &= (u_\theta(x_\lambda) - v(x_\lambda)) - (u_\theta(x_0) - v(x_0)) + v(x_\lambda) - v(y_\lambda) - v(x_0) + v(x_0 - \frac{1}{\lambda}n) \\ &\le \operatorname{lip}(v)|x_\lambda - y_\lambda| + \operatorname{lip}(v)\frac{1}{\lambda} \\ &\le \operatorname{lip}(v)|x_\lambda - y_\lambda - \frac{1}{\lambda}n| + 2\operatorname{lip}(v)\frac{1}{\lambda}, \end{aligned}$$

and therefore

$$L\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 \le C\big(\operatorname{lip}(v)\big)\frac{1}{\lambda}, \tag{2.12}$$

so that

$$x_\lambda,\, y_\lambda \to x_0, \qquad \text{as} \quad \lambda \to \infty \tag{2.13}$$

$$\lambda\,|x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,| \le \frac{C}{\sqrt{L}} < \frac{\epsilon}{2+\epsilon} \tag{2.14}$$

provided that L is sufficiently large. Next, (2.11) implies that $u - \frac{1}{\theta}\zeta$ has a local maximum at x_λ where $\zeta(x) = \tilde v(y_\lambda) + L\lambda\,|x - y_\lambda - \frac{1}{\lambda}\,n\,|^2 + |x - x_0|^2$. Therefore,

$$H\Big(\frac{1}{\theta}\big(2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,) + 2(x_\lambda - x_0)\big)\Big) \le f^*(x_\lambda), \tag{2.15}$$

and similarly,

$$H\Big(2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,)\Big) \ge f_*(y_\lambda). \tag{2.16}$$

Combining (2.15) and (2.16) and using (2.1), (2.2) we obtain

$$\begin{aligned} f_*(y_\lambda) &\le \theta\, H\Big(\frac{1}{\theta}\,2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,)\Big) \qquad (2.17)\\ &\le \theta\Big(H\Big(\frac{1}{\theta}\,2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,)\Big) - H\Big(\frac{1}{\theta}\big(2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,) + 2(x_\lambda - x_0)\big)\Big)\Big) + \theta f^*(x_\lambda). \end{aligned}$$

Note that H is locally Lipschitz continuous (since it is convex) so that we may deduce from (2.14)

$$|H\Big(\frac{1}{\theta}\,2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,)\Big) - H\Big(\frac{1}{\theta}\big(2L\lambda\,(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\,) + 2(x_\lambda - x_0)\big)\Big)| \le C|x_\lambda - x_0|.$$

Inserting this inequality into (2.17) we arrive at

$$(1-\theta) f_*(y_\lambda) \le C|x_\lambda - x_0| + \theta\big(f^*(x_\lambda) - f_*(y_\lambda)\big). \tag{2.18}$$

In order to treat the second term we write $x_\lambda = y_\lambda + r_\lambda d_\lambda$ with

$$d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda}\,|n + w_\lambda|, \quad w_\lambda = \lambda\,\Big(x_\lambda - y_\lambda - \frac{1}{\lambda}\,n\Big).$$

Now, (2.14) implies

$$|d_\lambda - n| \le \frac{2|w_\lambda|}{1 - |w_\lambda|} \le \frac{\frac{2\epsilon}{2+\epsilon}}{1 - \frac{\epsilon}{2+\epsilon}} = \epsilon \tag{2.19}$$

so that (2.6) yields

$$f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \le \omega(|y_\lambda - x_0| + r_\lambda).$$

If we insert this estimate into (2.18) and recall (2.4) the result is

$$m(1-\theta) \le C|x_\lambda - x_0| + \omega(|y_\lambda - x_0| + r_\lambda).$$

Sending $\lambda \nearrow \infty$ yields $m(1-\theta) \le 0$ in view of (2.13), which is a contradiction. Thus, $u_\theta \le v$ for all $\theta < 1$ and sending $\theta \nearrow 1$ finally yields the result. ∎

3 Numerical scheme and error analysis

Numerical schemes for Hamilton–Jacobi equations have been developed and analyzed by interpreting the corresponding viscosity solution as the value function of an optimal control problem and by using the dynamic programming principle. We refer to Appendix A, written by M. Falcone, in [1] for a description of basic results together with a comprehensive list of references. The abovementioned approach in general requires f to be Lipschitz–continuous, so that it cannot be applied to our situation.
Error estimates for finite difference approximations of the Cauchy problem $u_t + H(\nabla u) = 0$ have been obtained in [6]. The idea is to adapt the corresponding uniqueness proof and this is also the approach which we shall pursue in this work. However, a closer inspection of the proof of Theorem 2.3 shows, that it is not obvious how to use the argument in order to control the difference between the viscosity solution u and an approximation U. Therefore we recall a different approach to prove uniqueness, namely to apply the Kružkov transform.
We shall start from a class of finite difference schemes, which are monotone and consistent and which satisfy a suitable stability condition. The condition on f that we shall impose is slightly stronger than (2.5) but still allows discontinuities of f of the type described in the example following (2.5).
In order to keep the presentation simple we shall from now on assume that $\Omega = \Pi_{i=1}^n (0, b_i)$. Let $h > 0$ be such that there exist $N_i \in \mathbb{N}$ with $b_i = N_i h$, $i = 1, ..., n$ and define

$$\Omega_h := \mathbb{Z}_h^n \cap \Omega, \quad \partial\Omega_h := \mathbb{Z}_h^n \cap \partial\Omega, \quad \bar{\Omega}_h := \Omega_h \cup \partial\Omega_h,$$

where $\mathbb{Z}_h^n = \{x_\alpha = (h\alpha_1, ..., h\alpha_n) \mid \alpha_i \in \mathbb{Z}, i = 1, ..., n\}$. We shall approximate the viscosity solution u by a grid function $U : \bar{\Omega}_h \to \mathbb{R}$, $U_\alpha = U(x_\alpha)$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$. For $x_\alpha \in \Omega_h$, $k \in \{1, ..., n\}$ let

$$D_k^- U_\alpha := \frac{U_\alpha - U_{\alpha - e_k}}{h}, \quad D_k^+ U_\alpha := \frac{U_{\alpha + e_k} - U_\alpha}{h}$$

be the usual backward and forward difference quotients. The numerical scheme now reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$H_N(D_1^- U_\alpha, D_1^+ U_\alpha, ..., D_n^- U_\alpha, D_n^+ U_\alpha) = f(x_\alpha) \quad x_\alpha \in \Omega_h \tag{3.1}$$

$$U_\alpha = \phi(x_\alpha) \quad x_\alpha \in \partial\Omega_h, \tag{3.2}$$

where $H_N : \mathbb{R}^{2n} \to \mathbb{R}$, $(p_1, q_1, ..., p_n, q_n) \mapsto H_N(p_1, q_1, ..., p_n, q_n)$ is the numerical Hamiltonian. It is convenient to also introduce $F_N : \mathbb{R}^{2n+1} \to \mathbb{R}$, $a = (a_0, a_1, a_2, ..., a_{2n-1}, a_{2n}) \mapsto F_N(a)$ as

$$F_N(a) := H_N(a_0 - a_1, a_2 - a_0, ..., a_0 - a_{2n-1}, a_{2n} - a_0). \tag{3.3}$$

In what follows we shall assume that H_N is locally Lipschitz continuous and has the following properties:
a) *Consistency:*

$$H_N(p_1, p_1, ..., p_n, p_n) = H(p_1, ..., p_n) \qquad \text{for all } p = (p_1, ..., p_n) \in \mathbb{R}^n. \tag{3.4}$$

b) *Monotonicity:*

$$a_0 \mapsto F_N(a) \text{ is increasing} \tag{3.5}$$

$$a_k \mapsto F_N(a) \text{ is decreasing for } k = 1, ..., 2n. \tag{3.6}$$

c) *Stability:* there exists a function $Z : \bar{\Omega}_h \to \mathbb{R}$, which satisfies

$$H_N(D_1^- Z_\alpha, D_1^+ Z_\alpha, ..., D_n^- Z_\alpha, D_n^+ Z_\alpha) \geq f(x_\alpha) \quad x_\alpha \in \Omega_h \tag{3.7}$$

$$Z_\alpha = \phi(x_\alpha) \quad x_\alpha \in \partial\Omega_h \tag{3.8}$$

$$|D_k^- Z_\alpha|, |D_k^+ Z_\alpha| \leq R, \qquad x_\alpha \in \Omega_h, \tag{3.9}$$

where R is independent of h.

We shall examine some examples of choices of H_N in §4.

Remark 3.1. Note that the function Z which appears in c) above satisfies

$$Z_\alpha \geq \phi_{\min} := \min_{x \in \partial\Omega} \phi(x), \quad x_\alpha \in \bar{\Omega}_h. \tag{3.10}$$

To see this, let $Z_\beta = \min_{x_\alpha \in \bar{\Omega}_h} Z_\alpha$ and assume that $x_\beta \in \Omega_h$. Then, (2.4), (3.7), (3.6) and (2.1) would imply

$$\begin{aligned} m &\leq f(x_\beta) \leq H_N(D_1^- Z_\beta, D_1^+ Z_\beta, ..., D_n^- Z_\beta, D_n^+ Z_\beta) \\ &= F_N\Big(\frac{Z_\beta}{h}, \frac{Z_{\beta - e_1}}{h}, \frac{Z_{\beta + e_1}}{h}, ..., \frac{Z_{\beta - e_n}}{h}, \frac{Z_{\beta + e_n}}{h}\Big) \leq F_N\Big(\frac{Z_\beta}{h}, \frac{Z_\beta}{h}, \frac{Z_\beta}{h}, ..., \frac{Z_\beta}{h}, \frac{Z_\beta}{h}\Big) \\ &= H_N(0, ..., 0) = H(0) = 0, \end{aligned}$$

a contradiction. Thus, $x_\beta \in \partial\Omega_h$ and (3.10) follows.

Next, let us prove an auxiliary result which yields a kind of diagonal coercivity for the numerical Hamiltonian.

Lemma 3.2. *For* $(a_1, a_2, ..., a_{2n-1}, a_{2n}) \in \mathbb{R}^{2n}$ *there holds*

$$\lim_{t\to\infty} F_N(t, a_1, a_2, ..., a_{2n-1}, a_{2n}) = \infty.$$

Proof. Using the defintion of F_N along with (3.4)–(3.6) we obtain for $t \geq r := \max(a_1, a_2, ..., a_{2n-1}, a_{2n})$ that

$$\begin{aligned} F_N(t, a_1, a_2, ..., a_{2n-1}, a_{2n}) &\geq F_N(t, r, r, ..., r, r) = H_N(t-r, r-t, ..., t-r, r-t) \\ &= F_N(t-r, 0, 0, ..., 0, 0) \geq F_N(t-r, 0, 2(t-r), ..., 0, 2(t-r)) \\ &= H_N(t-r, t-r, ..., t-r, t-r) = H(t-r, ..., t-r) \\ &\to \infty \qquad \text{as } t \to \infty, \end{aligned}$$

which proves the Lemma. ∎

Lemma 3.3. *There exists a solution* U *of (3.1), (3.2), which satisfies* $\phi_{\min} \leq U_\alpha \leq Z_\alpha$ *for all* $x_\alpha \in \bar{\Omega}_h$.

Proof. We consider the following iteration: let $U^0 := Z$ and given $U^k : \bar{\Omega}_h \to \mathbb{R}$, let

$$\begin{aligned} U_\alpha^{k+1} = \inf\{ t \,|\, F_N\Big(\frac{t}{h}, \frac{U^k_{\alpha-e_1}}{h}, \frac{U^k_{\alpha+e_1}}{h}, ..., \frac{U^k_{\alpha-e_n}}{h}, \frac{U^k_{\alpha+e_n}}{h}\Big) &\geq f(x_\alpha)\} \quad x_\alpha \in \Omega_h \\ U_\alpha^{k+1} &= \phi(x_\alpha) \quad\ x_\alpha \in \partial\Omega_h. \end{aligned}$$

We claim that the sequence $(U^k)_{k\in\mathbb{N}}$ is well–defined and that

$$\phi_{\min} \leq U^k \leq U^{k-1} \leq Z \qquad \text{for all } k \in \mathbb{N}. \tag{3.11}$$

To see this, assume that (3.11) holds for all $1 \leq j \leq k$ and consider for $x_\alpha \in \Omega_h$

$$\eta(t) := F_N\Big(\frac{t}{h}, \frac{U^k_{\alpha-e_1}}{h}, \frac{U^k_{\alpha+e_1}}{h}, ..., \frac{U^k_{\alpha-e_n}}{h}, \frac{U^k_{\alpha+e_n}}{h}\Big), \quad t \in \mathbb{R}.$$

Clearly, η is continuous and increasing. Lemma 3.2 implies that $\eta(t) \to \infty$ as $t \to \infty$ so that U_α^{k+1} is well–defined. Since $U^k \geq \phi_{\min}$ by our induction hypothesis, (3.6), (3.4) and (2.1) yield

$$\begin{aligned} \eta(\phi_{\min}) &= F_N\big(\frac{\phi_{\min}}{h}, \frac{U^k_{\alpha-e_1}}{h}, \frac{U^k_{\alpha+e_1}}{h}, ..., \frac{U^k_{\alpha-e_n}}{h}, \frac{U^k_{\alpha+e_n}}{h}\big) \leq F_N\big(\frac{\phi_{\min}}{h}, ..., \frac{\phi_{\min}}{h}\big) \\ &= H(0) = 0 < f(x_\alpha), \end{aligned} \tag{3.12}$$

which implies that $U_\alpha^{k+1} \geq \phi_{\min}$. Also, as $U^k \leq U^{k-1}$ (3.6) yields

$$\begin{aligned} \phi(U_\alpha^k) &= F_N\big(\frac{U^k_\alpha}{h}, \frac{U^k_{\alpha-e_1}}{h}, \frac{U^k_{\alpha+e_1}}{h}, ..., \frac{U^k_{\alpha-e_n}}{h}, \frac{U^k_{\alpha+e_n}}{h}\big) \\ &\geq F_N\big(\frac{U^k_\alpha}{h}, \frac{U^{k-1}_{\alpha-e_1}}{h}, \frac{U^{k-1}_{\alpha+e_1}}{h}, ..., \frac{U^{k-1}_{\alpha-e_n}}{h}, \frac{U^{k-1}_{\alpha+e_n}}{h}\big) \\ &\geq f(x_\alpha) \end{aligned}$$

by the definition of U_α^k. Thus $U_\alpha^{k+1} \leq U_\alpha^k$. Using similar arguments and recalling (3.10) we infer that (3.11) holds for $k = 1$, so that we finally obtain (3.11) for all $k \in \mathbb{N}$. Note also that

$$F_N\big(\frac{U^{k+1}_\alpha}{h}, \frac{U^k_{\alpha-e_1}}{h}, \frac{U^k_{\alpha+e_1}}{h}, ..., \frac{U^k_{\alpha-e_n}}{h}, \frac{U^k_{\alpha+e_n}}{h}\big) = f(x_\alpha), \quad x_\alpha \in \Omega_h,\ k \in \mathbb{N}. \tag{3.13}$$

From (3.11) we infer that $U_\alpha^k \to U_\alpha$ for all $x_\alpha \in \bar{\Omega}_h$ as $k \to \infty$. Clearly, $U_\alpha = \phi(x_\alpha)$ for $x_\alpha \in \partial\Omega_h$. Passing to the limit $k \to \infty$ in (3.13) and using the continuity of F_N finally implies that U satisfies (3.1). The bounds on U follow from (3.11). ∎

Our aim is to prove an error bound between a discrete solution U and the viscosity solution u. To do so, we need to strengthen (2.5) in that we assume that there exist $\epsilon > 0$, $K \geq 0$ such that for all $x \in \Omega$ there is a direction $n = n_x \in S^{n-1}$ with

$$f(y+rd) - f(y) \leq Kr \qquad \forall y \in \Omega,\, |y-x| < \epsilon \quad \forall d \in S^{n-1},\, |d-n| < \epsilon \quad \forall r > 0. \tag{3.14}$$

Theorem 3.4. *Let u be the viscosity solution of (1.1), (1.2) and U a solution of (3.1), (3.2), which satisfies $\phi_{\min} \leq U \leq Z$. Then there exists a constant C, which is independent of h such that*

$$\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)| \leq C\sqrt{h}.$$

Proof. As mentioned above we introduce the Kružkov transform of u and U, i.e. $\tilde{u} : \bar{\Omega} \to \mathbb{R}, \tilde{U} : \bar{\Omega}_h \to \mathbb{R}$ which are defined by

$$\tilde{u}(x) := -e^{-u(x)},\ x \in \bar{\Omega}, \quad \tilde{U}_\alpha := -e^{-U_\alpha},\ x_\alpha \in \bar{\Omega}_h.$$

Clearly, $\tilde{u}(x) = -e^{-\phi(x)}$, $x \in \partial\Omega$ and one verifies (cf. [5]) that $\tilde{u}$ is a viscosity supersolution of

$$f(x)\tilde{u} - \tilde{u}H(-\frac{1}{\tilde{u}}\nabla\tilde{u}) = 0$$

in the sense that if $\zeta \in C^\infty(\Omega)$ and $\tilde{u} - \zeta$ has a local minimum at a point $x_0 \in \Omega$, then

$$f_*(x_0)\tilde{u}(x_0) - \tilde{u}(x_0)H(-\frac{1}{\tilde{u}(x_0)}\nabla\zeta(x_0)) \geq 0. \tag{3.15}$$

Note also that

$$\tilde{H}(x,r,p) := f(x)r - rH(-\frac{1}{r}p), \qquad (x,r,p) \in \Omega \times \mathbb{R} \setminus \{0\} \times \mathbb{R}^n,$$

satisfies in view of (2.4) and the convexity of H (cf. [5])

$$\frac{\partial \tilde{H}}{\partial r}(x,r,p) = f(x) - H(-\frac{1}{r}p) + \big(DH(-\frac{1}{r}p), -\frac{1}{r}p\big) \geq f(x) \geq m \tag{3.16}$$

uniformly in (x,r,p).

Next, let $x_\beta \in \bar{\Omega}_h$ be such that

$$|\tilde{u}(x_\beta) - \tilde{U}_\beta| = \max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha|$$

and assume that $\tilde{U}_\beta \geq \tilde{u}(x_\beta)$, the case $\tilde{u}(x_\beta) > \tilde{U}_\beta$ being treated in a similar way.
Let us first consider the situation when

$$(x_\beta)_i \leq \sqrt{h} \quad \text{or} \quad (x_\beta)_i \geq b_i - \sqrt{h} \quad \text{for some } i \in \{1,...,n\}. \tag{3.17}$$

In the first case, let $x_{\beta_0} = (\beta_1 h, ..., \beta_{i-1}h, 0, \beta_{i+1}h, ..., \beta_n h) \in \partial\Omega$ and $\tilde{Z}_\alpha := -e^{-Z_\alpha}$. Since $\tilde{u}(x_{\beta_0}) = -e^{-\phi(x_{\beta_0})} = \tilde{Z}_{\beta_0}$ we deduce with the help of (3.9)

$$\begin{aligned}
\tilde{U}_\beta - \tilde{u}(x_\beta) &= \big(\tilde{U}_\beta - \tilde{u}(x_{\beta_0})\big) + \big(\tilde{u}(x_{\beta_0}) - \tilde{u}(x_\beta)\big) \leq \big(\tilde{Z}_\beta - \tilde{Z}_{\beta_0}\big) + \big(\tilde{u}(x_{\beta_0}) - \tilde{u}(x_\beta)\big) \\
&\leq \big(C(R) + \mathrm{lip}(\tilde{u})\big)|x_\beta - x_{\beta_0}| \leq \big(C(R) + \mathrm{lip}(\tilde{u})\big)\sqrt{h}.
\end{aligned}$$

Arguing in a similar way if $(x_\beta)_i \geq b_i - \sqrt{h}$ we conclude that

$$\max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| = \tilde{U}_\beta - \tilde{u}(x_\beta) \leq C\sqrt{h}, \tag{3.18}$$

if (3.17) holds. Now we consider the case

$$\sqrt{h} < (x_\beta)_i < b_i - \sqrt{h} \quad \text{ for } i = 1, ..., n. \tag{3.19}$$

Let $\epsilon > 0$, $K \geq 0$, $n = n_{x_\beta}$ be the quantities appearing in (3.14) and define $\Phi : \bar{\Omega} \times \bar{\Omega}_h \to \mathbb{R}$ by

$$\Phi(x, x_\alpha) := \tilde{U}_\alpha - \tilde{u}(x) - \frac{L_1}{\sqrt{h}} |x_\alpha - x - \sqrt{h}\, n|^2 - L_2\sqrt{h}\, |x_\alpha - x_\beta|^2,$$

where $L_1, L_2 \geq 0$ are constants that do not depend on h and which will be chosen later. There exists $(x_h, x_{\alpha_h}) \in \bar{\Omega} \times \bar{\Omega}_h$ such that

$$\Phi(x_h, x_{\alpha_h}) = \max_{(x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h} \Phi(x, x_\alpha).$$

In view of (3.19) we have that $x_\beta - \sqrt{h}\, n \in \bar{\Omega}$ and therefore

$$\Phi(x_h, x_{\alpha_h}) \geq \Phi(x_\beta - \sqrt{h}\, n, x_\beta)$$

or equivalently

$$\tilde{U}_{\alpha_h} - \tilde{u}(x_h) - \frac{L_1}{\sqrt{h}} |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 - L_2\sqrt{h} |x_{\alpha_h} - x_\beta|^2 \geq \tilde{U}_\beta - \tilde{u}(x_\beta - \sqrt{h}\, n). \tag{3.20}$$

This implies

$$\begin{aligned}
&\frac{L_1}{\sqrt{h}} |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 + L_2\sqrt{h}\, |x_{\alpha_h} - x_\beta|^2 \\
&\leq \tilde{u}(x_\beta - \sqrt{h}\, n) - \tilde{u}(x_h) + \tilde{U}_{\alpha_h} - \tilde{U}_\beta \\
&\leq \tilde{u}(x_{\alpha_h}) - \tilde{u}(x_h) + \big((\tilde{U}_{\alpha_h} - \tilde{u}(x_{\alpha_h})) - (\tilde{U}_\beta - \tilde{u}(x_\beta))\big) + \tilde{u}(x_\beta - \sqrt{h}\, n) - \tilde{u}(x_\beta) \\
&\leq \operatorname{lip}(\tilde{u})\, |x_{\alpha_h} - x_h| + \sqrt{h}\, \operatorname{lip}(\tilde{u}) \\
&\leq \operatorname{lip}(\tilde{u})\, |x_{\alpha_h} - x_h - \sqrt{h}\, n| + 2\sqrt{h}\, \operatorname{lip}(\tilde{u}) \\
&\leq \frac{L_1}{2\sqrt{h}} |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 + \frac{\sqrt{h}}{2L_1} \operatorname{lip}(\tilde{u})^2 + 2\sqrt{h}\, \operatorname{lip}(\tilde{u}),
\end{aligned}$$

and therefore,

$$\frac{1}{h} |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 \leq \frac{1}{L_1^2} \operatorname{lip}(\tilde{u})^2 + \frac{4}{L_1} \operatorname{lip}(\tilde{u}) \quad < \Big(\frac{\epsilon}{2 + \epsilon}\Big)^2 \tag{3.21}$$

$$|x_{\alpha_h} - x_\beta|^2 \leq \frac{1}{2L_1L_2} \operatorname{lip}(\tilde{u})^2 + \frac{2}{L_2} \operatorname{lip}(\tilde{u}) \quad < \epsilon^2, \tag{3.22}$$

provided that L_1, L_2 are sufficiently large.
Let us first consider the case that $(x_h, x_{\alpha_h}) \in \Omega \times \Omega_h$. We infer from (3.15) that

$$f_*(x_h)\tilde{u}(x_h) - \tilde{u}(x_h) H\Big(-\frac{1}{\tilde{u}(x_h)} \frac{2L_1}{\sqrt{h}} \big(x_{\alpha_h} - x_h - \sqrt{h}\, n\big)\Big) \geq 0. \tag{3.23}$$

In order to derive a corresponding relation for the discrete solution, we consider the inequality $\Phi(x_h, x_{\alpha_h}) \geq \Phi(x_h, x_\alpha)$ for all $x_\alpha \in \bar{\Omega}$, which translates into

$$\begin{aligned} \tilde{U}_\alpha &\leq \tilde{U}_{\alpha_h} + \frac{L_1}{\sqrt{h}}\Big(|x_\alpha - x_h - \sqrt{h}\,n|^2 - |x_{\alpha_h} - x_h - \sqrt{h}\,n|^2\Big) \\ &\quad + L_2\sqrt{h}\big(|x_\alpha - x_\beta|^2 - |x_{\alpha_h} - x_\beta|^2\big) \\ &= : \tilde{V}_\alpha. \end{aligned} \tag{3.24}$$

Note first, that

$$\begin{aligned} D_k^- \tilde{V}_\alpha &= \frac{2L_1}{\sqrt{h}}\big(x_\alpha - x_h - \sqrt{h}\,n, e_k\big) + 2L_2\sqrt{h}\big(x_\alpha - x_\beta, e_k\big) - L_1\sqrt{h} - L_2 h^{\frac{3}{2}} \\ D_k^+ \tilde{V}_\alpha &= \frac{2L_1}{\sqrt{h}}\big(x_\alpha - x_h - \sqrt{h}\,n, e_k\big) + 2L_2\sqrt{h}\big(x_\alpha - x_\beta, e_k\big) + L_1\sqrt{h} + L_2 h^{\frac{3}{2}} \end{aligned}$$

and therefore by (3.21)

$$|D_k^- \tilde{V}_{\alpha_h}|, |D_k^- \tilde{V}_{\alpha_h}| \leq C, \qquad |D_k^\pm \tilde{V}_{\alpha_h} - \frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n, e_k\big)| \leq C\sqrt{h}, \quad k = 1, ..., n \tag{3.25}$$

uniformly in h. Recalling that $U \leq Z$ we also deduce from (3.8) and (3.9) that

$$\tilde{V}_{\alpha_h} = \tilde{U}_{\alpha_h} = -e^{-U_{\alpha_h}} \leq -e^{-Z_{\alpha_h}} \leq -\bar{c}, \tag{3.26}$$

where $\bar{c} > 0$ depends on ϕ and R. Furthermore, (3.25) implies

$$\tilde{V}_{\alpha_h \pm e_k} \leq -\frac{1}{2}\bar{c} \qquad \text{for } h \text{ sufficiently small.} \tag{3.27}$$

Thus we can define $V_\alpha := -\log(-\tilde{V}_\alpha)$, for $\alpha = \alpha_h, \alpha_h \pm e_k$ and the mean value theorem yields

$$D_k^- \tilde{V}_{\alpha_h} = e^{-\xi_k^-} D_k^- V_{\alpha_h}, \quad D_k^+ \tilde{V}_\alpha = e^{-\xi_k^+} D_k^+ V_\alpha, \quad k = 1, ..., n, \tag{3.28}$$

where ξ_k^- lies between $V_\alpha, V_{\alpha - e_k}$ and ξ_k^+ lies between $V_\alpha, V_{\alpha + e_k}$. In particular,

$$e^{\xi_k^\pm} \leq \max\big(e^{V_{\alpha_h \pm e_k}}, e^{V_{\alpha_h}}\big) = \max\big(\frac{-1}{\tilde{V}_{\alpha_h \pm e_k}}, \frac{-1}{\tilde{V}_{\alpha_h}}\big) \leq \frac{2}{\bar{c}} \tag{3.29}$$

by (3.27). Thus, (3.28), (3.25), (3.21), (3.26) together with the fact that $\phi_{\min} \leq U \leq Z$ imply

$$|D_k^\pm V_{\alpha_h}| \leq C, \quad k = 1, ..., n \quad \text{ and } | - \frac{1}{\tilde{U}_{\alpha_h}} \frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n\big)| \leq C. \tag{3.30}$$

Next, we deduce from (3.24) that

$$U_{\alpha_h} = V_{\alpha_h}, \quad U_\alpha \leq V_\alpha, \alpha = \alpha_h \pm e_k, k = 1, ..., n,$$

so that the monotonicity property (3.6) and (3.1) imply

$$\begin{aligned} f(x_{\alpha_h}) &= H_N(D_1^- U_{\alpha_h}, D_1^+ U_{\alpha_h}, ..., D_n^- U_{\alpha_h}, D_n^+ U_{\alpha_h}) \\ &\geq H_N(D_1^- V_{\alpha_h}, D_1^+ V_{\alpha_h}, ..., D_n^- V_{\alpha_h}, D_n^+ V_{\alpha_h}). \end{aligned}$$

Multiplying the above inequality by $\tilde U_{\alpha_h} < 0$, using (3.4) along with (3.30) and the local Lipschitz continuity of H_N we infer

$$
\begin{aligned}
& f(x_{\alpha_h})\tilde U_{\alpha_h} - \tilde U_{\alpha_h} H\Big(-\frac{1}{\tilde U_{\alpha_h}}\frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n\big)\Big) \\
\leq\ & \tilde U_{\alpha_h}\Big(H_N\big(D_1^- V_{\alpha_h}, D_1^+ V_{\alpha_h}, ..., D_n^- V_{\alpha_h}, D_n^+ V_{\alpha_h}\big) - H\big(-\frac{1}{\tilde U_{\alpha_h}}\frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n\big)\big)\Big) \\
\leq\ & C \max_{k=1,\dots,n} |D_k^\pm V_{\alpha_h} + \frac{1}{\tilde U_{\alpha_h}}\frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n, e_k\big)| \\
\leq\ & C \max_{k=1,\dots,n} \Big(e^{\xi_k^\pm}|D_k^\pm \tilde V_{\alpha_h} - \frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n, e_k\big)| \\
& \quad +|e^{\xi_k^\pm} - e^{V_{\alpha_h}}|\,|\frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n, e_k\big)|\Big) \\
\leq\ & C\sqrt{h} + Ch. \qquad (3.31)
\end{aligned}
$$

Note that the last estimate is a consequence of (3.25) and (3.29). If we combine (3.23) with (3.31) and use the definition of $\tilde H$ we obtain

$$
\begin{aligned}
& \tilde H\Big(x_{\alpha_h}, \tilde U_{\alpha_h}, \frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n\big)\Big) - \tilde H\Big(x_{\alpha_h}, \tilde u(x_h), \frac{2L_1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n\big)\Big) \\
& \qquad \leq\ C\sqrt{h} + e^{-u(x_h)}\big(f(x_{\alpha_h}) - f_*(x_h)\big).
\end{aligned}
$$

Note first that

$$
\begin{aligned}
\tilde U_{\alpha_h} - \tilde u(x_h) &= \Phi(x_h, x_{\alpha_h}) + \frac{L_1}{\sqrt{h}}|x_{\alpha_h} - x_h - \sqrt{h}\,n|^2 + L_2\sqrt{h}|x_{\alpha_h} - x_\beta|^2 \\
&\geq \Phi(x_\beta, x_\beta) = \tilde U_\beta - \tilde u(x_\beta) - L_1\sqrt{h}.
\end{aligned}
$$

Thus, in view of (3.16)

$$
m\big(\tilde U_\beta - \tilde u(x_\beta)\big) \leq e^{-u(x_h)}\big(f(x_{\alpha_h}) - f_*(x_h)\big) + C\sqrt{h}. \qquad (3.32)
$$

Let us write $x_{\alpha_h} = x_h + r_h d_h$, with

$$
d_h = \frac{n + w_h}{|n + w_h|}, \quad r_h = \sqrt{h}\,|n + w_h|, \quad w_h = \frac{1}{\sqrt{h}}\big(x_{\alpha_h} - x_h - \sqrt{h}\,n\big).
$$

Note that (3.22) implies that $|x_{\alpha_h} - x_\beta| < \epsilon$; (3.21) and a similar argument as in (2.19) yield $|d_h - n| < \epsilon$, so that (3.14) gives

$$
f(x_{\alpha_h}) - f_*(x_h) \leq f^*(x_h + r_h d_h) - f_*(x_h) \leq K r_h \leq C\sqrt{h}. \qquad (3.33)
$$

Combining (3.32) and (3.33) finally yields

$$
\max_{x_\alpha \in \bar\Omega_h} |\tilde u(x_\alpha) - \tilde U_\alpha| = \tilde U_\beta - \tilde u(x_\beta) \leq C\sqrt{h}. \qquad (3.34)
$$

It remains to consider the case when $x_{\alpha_h} \in \partial\Omega_h$ or $x_h \in \partial\Omega$. If $x_{\alpha_h} \in \partial\Omega_h$, it follows from (3.20), the fact that $\tilde u(x_{\alpha_h}) = \tilde U_{\alpha_h}$ and (3.21)

$$
\begin{aligned}
\tilde U_\beta - \tilde u(x_\beta) &\leq \tilde u(x_\beta - \sqrt{h}n) - \tilde u(x_\beta) + \tilde u(x_{\alpha_h}) - \tilde u(x_h) \\
&\leq \operatorname{lip}(\tilde u)\big(\sqrt{h} + |x_{\alpha_h} - x_h|\big) \\
&\leq \operatorname{lip}(\tilde u)\big(2\sqrt{h} + |x_{\alpha_h} - x_h - \sqrt{h}\,n|\big) \\
&\leq C\sqrt{h}.
\end{aligned}
$$

Let us finally assume that $x_h \in \partial\Omega$. Since $\tilde{u}(x_h) = \phi(x_h) = \tilde{Z}_{x_h}$ and $\tilde{U} \leq \tilde{Z}$ we obtain

$$\begin{aligned}
\tilde{U}_\beta - \tilde{u}(x_\beta) &\leq \tilde{U}_{\alpha_h} - \tilde{u}(x_h) + \tilde{u}(x_\beta - \sqrt{h}n) - \tilde{u}(x_\beta) \\
&\leq \tilde{Z}(x_{\alpha_h}) - \tilde{Z}(x_h) + \tilde{u}(x_\beta - \sqrt{h}n) - \tilde{u}(x_\beta) \\
&\leq C\sqrt{h}
\end{aligned}$$

similarly as above. Transforming back to u and U implies the desired error bound. ∎

4 Examples of numerical Hamiltonians

Let us consider some examples of numerical Hamiltonians H_N. In order to simplify matters we restrict ourselves to the case of two space dimensions and a domain Ω of the form $\Omega = (0, b_1) \times (0, b_2)$ with $b_2 \leq b_1$.

4.1 Viscous regularisation

Suppose that H is globally Lipschitz–continuous with Lipschitz constant L. We define $H_N : \mathbb{R}^4 \to \mathbb{R}$ by

$$H_N(p_1, q_1, p_2, q_2) := -\frac{L}{2}\sum_{k=1}^{2}(q_k - p_k) + H\big(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}\big).$$

Clearly, (3.4) is satisfied; in order to verify monotonicity we evaluate F_N according to (3.3), which gives

$$F_N(a_0, a_1, ..., a_4) = 2La_0 - \frac{L}{2}\sum_{k=1}^{2}(a_{2k-1} + a_{2k}) + H\big(\frac{a_2 - a_1}{2}, \frac{a_4 - a_3}{2}\big),$$

from which (3.5) and (3.6) follow in view of the Lipschitz continuity of H. It remains to show the existence of a function Z, which satisfies (3.7)–(3.9). To this purpose we consider $d(x) := \mathrm{dist}(x, \partial\Omega), x = (x_1, x_2)$. It is not difficult to see that

$$d(x) = \begin{cases} x_1 & \text{in } \Omega_1 = \{x \in \Omega | 0 \leq x_2 \leq b_2,\ 0 \leq x_1 \leq \min(b_2 - x_2, x_2)\} \\ b_1 - x_1 & \text{in } \Omega_2 = \{x \in \Omega | 0 \leq x_2 \leq b_2,\ b_1 \geq x_1 \geq \max(b_1 - x_2, b_1 - b_2 + x_2)\} \\ x_2 & \text{in } \Omega_3 = \{x \in \Omega | 0 \leq x_1 \leq b_1,\ 0 \leq x_2 \leq \min(x_1, \frac{b_2}{2}, b_1 - x_1)\} \\ b_2 - x_2 & \text{in } \Omega_4 = \{x \in \Omega | 0 \leq x_1 \leq b_1,\ b_2 \geq x_2 \geq \max(b_2 - x_1, \frac{b_2}{2}, x_1 + b_2 - b_1)\}. \end{cases}$$

In view of (2.7) and a suitable extension we may assume that ϕ is defined as a Lipschitz continuous function on $\bar{\Omega}$ with Lipschitz constant L_ϕ.

Define $Z : \bar{\Omega}_h \to \mathbb{R}$ by $Z_\alpha := \phi_\alpha + \rho d(x_\alpha)$, where $\phi_\alpha = \phi(x_\alpha)$ and ρ is independent of h. Clearly, Z satisfies (3.8) as well as (3.9) with $R = L_\phi + \rho$. We claim that (3.7) holds provided that ρ is sufficiently large and verify this for a point $x_\alpha = (x_{1\alpha}, x_{2\alpha}) \in \Omega_1$ with $x_{2\alpha} \leq \frac{b_2}{2}$.

Case 1: $x_{2\alpha} - \frac{h}{2} \leq x_{1\alpha} \leq x_{2\alpha}$. We then have

$$\begin{aligned}
D_1^- Z_\alpha &= D_1^-\phi_\alpha + \rho, \qquad D_1^+ Z_\alpha = D_1^+\phi_\alpha + \frac{\rho(x_{2\alpha} - x_{1\alpha})}{h} \leq D_1^+\phi_\alpha + \frac{\rho}{2}, \\
D_2^- Z_\alpha &= D_2^-\phi_\alpha + \frac{\rho(x_{1\alpha} - x_{2\alpha} + h)}{h} \geq D_2^-\phi_\alpha + \frac{\rho}{2}, \qquad D_2^+ Z_\alpha = D_2^+\phi_\alpha,
\end{aligned} \tag{4.1}$$

so that

$$\begin{aligned} & H_N\big(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha\big) \\ = & -\frac{L}{2}\sum_{k=1}^{2}\big(D_k^+ Z_\alpha - D_k^- Z_\alpha\big) + H\big(\frac{D_1^+ Z_\alpha + D_1^- Z_\alpha}{2}, \frac{D_2^+ Z_\alpha + D_2^- Z_\alpha}{2}\big) \\ \geq & -\frac{L}{2}\sum_{k=1}^{2}\big(D_k^+ \phi_\alpha - D_k^- \phi_\alpha\big) + \frac{L\rho}{2} \quad \geq \quad -2LL_\phi + \frac{L\rho}{2} \quad \geq \quad \frac{L\rho}{4} \qquad (4.2) \\ \geq & \ M \quad \geq \quad f(x_\alpha) \qquad (4.3) \end{aligned}$$

in view of (2.4) provided that ρ is sufficiently large.
Case 2 $x_{1\alpha} \leq x_{2\alpha} - \frac{h}{2}$. We now have

$$\begin{aligned} D_1^- Z_\alpha &= D_1^- \phi_\alpha + \rho, \qquad D_1^+ \phi_\alpha + \frac{\rho}{2} \leq D_1^+ Z_\alpha \leq D_1^+ \phi_\alpha + \rho \\ D_2^- \phi_\alpha &\leq D_2^- Z_\alpha \leq D_2^- \phi_\alpha + \frac{\rho}{2}, \qquad D_2^+ Z_\alpha = D_2^+ \phi_\alpha, \end{aligned} \qquad (4.4)$$

so that

$$\begin{aligned} & H_N\big(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha\big) \\ \geq & -\frac{L}{2}\sum_{k=1}^{2}\big(D_k^+ \phi_\alpha - D_k^- \phi_\alpha\big) + H\big(\frac{D_1^+ Z_\alpha + D_1^- Z_\alpha}{2}, \frac{D_2^+ Z_\alpha + D_2^- Z_\alpha}{2}\big) \\ \geq & -2LL_\phi + H\big(\frac{D_1^+ Z_\alpha + D_1^- Z_\alpha}{2}, \frac{D_2^+ Z_\alpha + D_2^- Z_\alpha}{2}\big). \qquad (4.5) \end{aligned}$$

Since

$$\frac{D_1^- Z_\alpha + D_1^+ Z_\alpha}{2} \geq \frac{3}{4}\rho + \frac{D_1^- \phi_\alpha + D_1^+ \phi_\alpha}{2} \geq \frac{3}{4}\rho - L_\phi \geq \frac{1}{2}\rho$$

provided that $\rho \geq 4L_\phi$, we deduce from (4.5), (2.3) and (2.4) that

$$H_N\big(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha\big) \geq M \geq f(x_\alpha),$$

provided that ρ is large enough. Other points can be treated in a similar way.

4.2 Godunov Hamiltonian

In [2] the following formula was derived from the solution of the Riemann problem:

$$H_N(p_1, q_1, p_2, q_2) := \text{ext}_{\xi \in I[p_1,q_1]} \text{ext}_{\eta \in I[p_2,q_2]} H(\xi, \eta),$$

where

$$\text{ext}_{\xi \in I[p,q]} = \begin{cases} \min_{\xi \in [p,q]} & , \text{ if } p \leq q \\ \max_{\xi \in [q,p]} & , \text{ if } p > q. \end{cases}$$

We leave it to the reader to check the conditions (3.4), (3.5) and (3.6). We use the same function Z as in the case of viscous regularisation to verify (3.7)–(3.9). Again we examine the situation at a point $x_\alpha = (x_{1\alpha}, x_{2\alpha}) \in \Omega_1$ with $x_{2\alpha} \leq \frac{b_2}{2}$.
Case 1: $x_{2\alpha} - \frac{h}{2} \leq x_{1\alpha} \leq x_{2\alpha}$. Since (4.1) implies that $D_1^- Z_\alpha \geq \rho - L_\phi$, $D_1^+ Z_\alpha \leq L_\phi + \frac{\rho}{2}$ we have $D_1^+ Z_\alpha \leq D_1^- Z_\alpha$ for $\rho \geq 4L_\phi$. Thus,

$$\begin{aligned} H_N\big(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha\big) &= \max_{\xi \in [D_1^+ Z_\alpha, D_1^- Z_\alpha]} \text{ext}_{\eta \in I[D_2^- Z_\alpha, D_2^+ Z_\alpha]} H(\xi, \eta) \\ &\geq \text{ext}_{\eta \in I[D_2^- Z_\alpha, D_2^+ Z_\alpha]} H(\frac{\rho}{2}, \eta) \geq M \geq f(x_\alpha) \end{aligned}$$

for large ρ.
Case 2 $x_{1\alpha} \le x_{2\alpha} - \frac{h}{2}$. Now by (4.4), $D_1^- Z_\alpha \ge \rho - L_\phi$, $D_1^+ Z_\alpha \ge \frac{\rho}{2} - L_\phi$, so that $\xi \ge \frac{\rho}{2} - L_\phi$ for all ξ between $D_1^- Z_\alpha$ and $D_1^+ Z_\alpha$. This again implies that

$$H_N\big(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha\big) \ge f(x_\alpha)$$

if ρ is sufficiently large. Other points can be treated analogously.

As a special case one obtains for the eikonal equation $H(p) = |p|$ the scheme

$$H_N(p_1, q_1, p_2, q_2) = \sqrt{(\max(p_1^+, -q_1^-))^2 + (\max(p_2^+, -q_2^-))^2}, \tag{4.6}$$

where $p^+ = \max(p, 0)$ and $p^- = \min(p, 0)$. This scheme was examined in [11] in the context of the shape–from–shading and convergence of approximations was proved with the help of a result of Barles & Souganidis [3]. In a recent paper, Zhao [15] shows $O(h)$–convergence for this scheme provided that $f \equiv 1$.

5 Numerical results

In this section we present some results of numerical calculations for (1.1), (1.2) with $H(p) = |p|$. As a first test example, let $\Omega := (-1, 1) \times (0, 2)$ and $f : \Omega \to \mathbb{R}$ be defined by $f(x_1, x_2) := 1, x_1 < 0$, $f(0, x_2) := \frac{3}{4}$, $f(x_1, x_2) := \frac{1}{2}, x_1 > 0$. It is not difficult to see that f satisfies (3.14) and one verifies that

$$u(x_1, x_2) := \begin{cases} \frac{1}{2}x_2 & x_1 \ge 0 \\ -\frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2 & -\frac{1}{\sqrt{3}}x_2 \le x_1 \le 0 \\ x_2 & x_1 < -\frac{1}{\sqrt{3}}x_2 \end{cases}$$

is a viscosity solution of $|\nabla u| = f$ in the sense of Definition 2.1. Furthermore, let $\phi := u_{|\partial\Omega}$. Since H is globally Lipschitz–continuous with constant 1, the numerical scheme induced by viscous regularisation reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$\begin{aligned} -\frac{1}{2}\big(U_{\alpha+e_1} + U_{\alpha-e_1} + U_{\alpha+e_2} + U_{\alpha-e_2} - 4U_\alpha\big) & \\ +\frac{1}{2}\sqrt{\big(U_{\alpha+e_1} - U_{\alpha-e_1}\big)^2 + \big(U_{\alpha+e_2} - U_{\alpha-e_2}\big)^2} &= h\,f(x_\alpha) \quad x_\alpha \in \Omega_h \\ U_\alpha &= \phi_(x_\alpha) \quad x_\alpha \in \partial\Omega_h. \end{aligned}$$

The system of equations was solved with the help of Newton's method and we calculated

$$E_{VR,h} := \max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U_\alpha|$$

together with the experimental order of convergence $eoc = \frac{\ln\big(E_{h_2}/E_{h_1}\big)}{\ln\big(h_2/h_1\big)}$ for various choices of h. We then used the numerical Hamiltonian (4.6) to approximate the viscosity solution. Observing that

$$\max\big((D_k^- U_\alpha)^+, -D_k^+ U_\alpha)^-\big) = \frac{1}{h}\big(U_\alpha - \min(U_{\alpha-e_k}, U_{\alpha+e_k})\big)^+,$$

for $k = 1, ..., n$, the discrete problem reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$\Big(\sum_{k=1}^{2}\big(\big(U_\alpha - \min(U_{\alpha-e_k}, U_{\alpha+e_k})\big)^+\big)^2\Big)^{\frac{1}{2}} = h\,f(x_\alpha) \quad x_\alpha \in \Omega_h \tag{5.1}$$

$$U_\alpha = \phi_(x_\alpha) \quad x_\alpha \in \partial\Omega_h. \tag{5.2}$$

The discrete solution was calculated with the help of the *Fast Sweeping Method* (see e.g. [15] for a description) and the corresponding errors $E_{FS,h}$ and eoc's are shown in Table 1 together with the results from the method of viscous regularisation. Figure 1 shows various level lines of the solution.

h	$E_{VR,h}$	eoc	$E_{FS,h}$	eoc
1/10	1.24348e-1	-	5.59016e-2	-
1/20	7.22984e-2	0.78	2.79508e-2	1.00
1/40	4.08509e-2	0.82	1.39754e-2	1.00
1/80	2.26691e-2	0.85	6.98771e-3	1.00
1/160	1.24385e-2	0.87	3.49386e-3	1.00

Table 1: Absolute error in maximum norm and experimental order of convergence for for the first test problem.

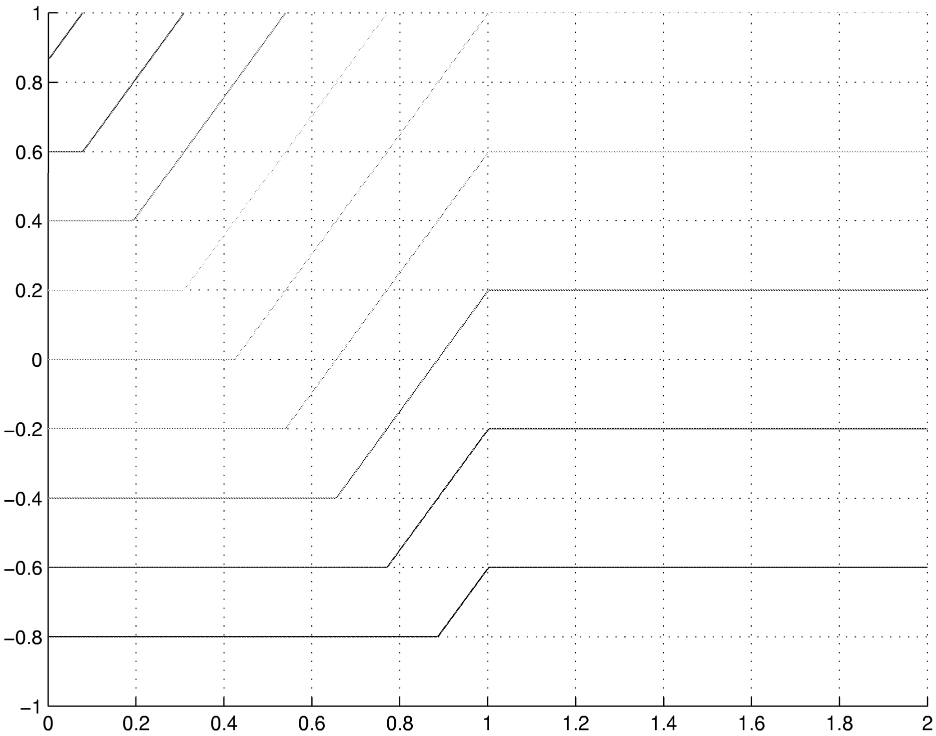

Figure 1: Level lines of the solution from the first test problem.

We observe linear convergence in h for the method (5.1), (5.2), which suggests that it might be possible to generalize the result in [15] for $f \equiv 1$ to nonconstant or even discontinuous right hand sides.
In our second example we consider $\Omega = (-1,1)^2$, $\phi \equiv 0$ and

$$f(x_1,x_2) := \begin{cases} 2, & (x_1-\frac{1}{2})^2 + x_2^2 \le \frac{1}{8} \text{ and } x_2 \ge x_1 - \frac{1}{2} \\ 3, & (x_1-\frac{1}{2})^2 + x_2^2 \le \frac{1}{8} \text{ and } x_2 < x_1 - \frac{1}{2} \\ 1, & \text{otherwise.} \end{cases}$$

Note that in this case discontinuities of f occur both along curved lines and along a straight line which is not aligned with the grid. Furthermore, the three regions, in which f takes different values, meet at the triple points $(\frac{3}{4}, \frac{1}{4})$, $(\frac{1}{4}, -\frac{1}{4})$. It is not difficult to check that f satisfies (3.14). The numerical solutions were again calculated with the help of viscous regularisation and (5.1), (5.2). In the absence of an exact solution we compared the discrete solutions for various grid sizes with an approximation U_f on a fine grid ($h = 1/640$). The results are displayed in Table 2, while Figure 2 shows some level curves of the solution.

h	$E_{VR,h}$	eoc	$E_{FS,h}$	eoc
1/10	9.07922e-2	-	1.10429e-1	-
1/20	1.29179e-1	-0.51	1.28365e-1	-0.22
1/40	1.04327e-1	0.31	8.68148e-2	0.56
1/80	7.35184e-2	0.50	5.00901e-2	0.79
1/160	4.33351e-2	0.76	2.65873e-2	0.91

Table 2: Absolute error in maximum norm and experimental order of convergence for the second test problem.

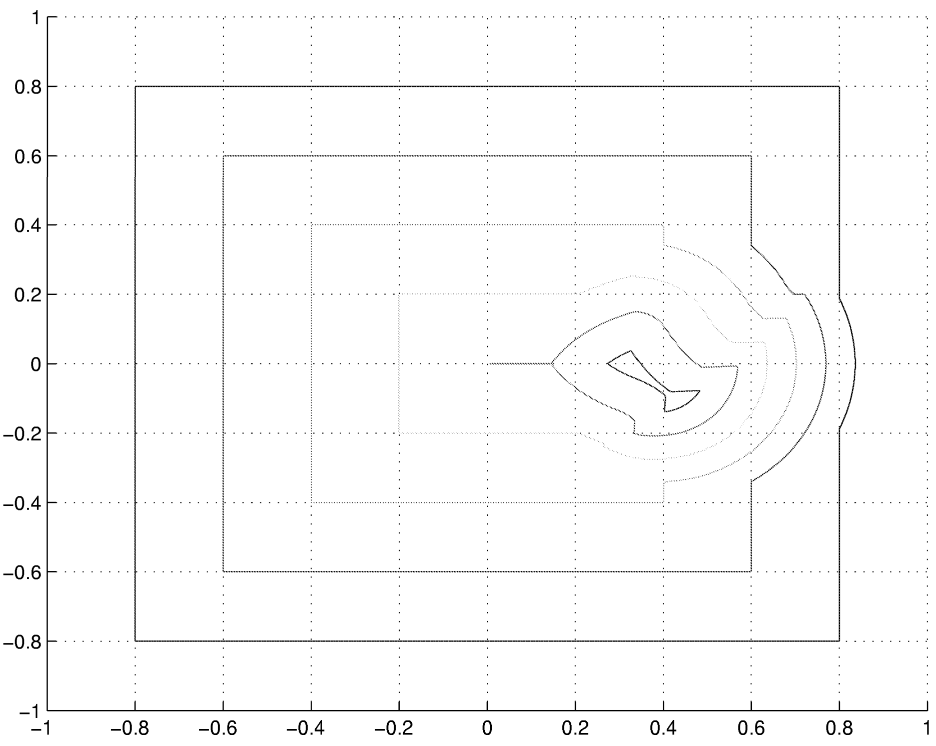

Figure 2: Level lines of the solution from the second test problem.

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