BEST APPROXIMATION FOR THE P-VERSION OF THE FINITE ELEMENT METHOD IN THREE DIMENSIONS IN THE FRAMEWORK OF THE JACOBI-WEIGHTED BESOV SPACES

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Abstract

We introduce the Jacobi-weighted Besov and Sobolev spaces in the three-dimensional setting and analyze the approximability of functions in the framework of these spaces, particularly, the functions reflecting the vertex-singularity, the edge singularity and vertex-edge singularity. These spaces and corresponding approximation properties lead to the optimal convergence of the p-version of the finite element method for elliptical problems on polyhedral domains.

1. INTRODUCTION

Since the late 1970s, the p-version of the finite element method(FEM), which increases the degree of polynomials on a fixed mesh to obtain higher accurate, has been widely used in engineering computations. There are several commercial and research codes based on the p (or h-p) versions of the finite element method, for example, MSC/PROBE, FIESTA, MECHANICA, PHLEX, STRESSCHECK, and STRIPE.

In 1980 it was shown that the p-version of FEM in two dimensions converges at least as fast as the traditional h-version with quasi-uniform meshes, and that it converges twice as fast as the h-version of FEM if the solution has singularity of r^γ-type. Since then significant progress for the p-version in one and two dimensions has been made in the past two decades. The estimation of the upper bound of approximation error in finite element solutions of the p-version in two dimensions were analyzed in [5, 6], and a detailed analysis of the p-version in one dimension is available in [10]. Very recently, the author and his collaborator have further developed the approximation theory of the p-version of FEM in the framework of Jacobi-weighted Besov and Sobolev spaces, This framework is applicable in one,

*2000 *Mathematics Subject classification.* Primary: 65N30,65N25,35D10.

The work of this author was supported by NSERC of Canada under Grant OGP0046726, and was finalized during his visiting Isaac Newton Institute for Mathematical Sciences, Cambridge University, U.K. to participate in the research program "Computational Challengs in PDEs" in 2003.

two and three dimensions. In this mathematical framework, the lower and upper bounds of approximation error for the p-version of FEM for problems in polygonal domains were proved, and the optimal rate of convergence was mathematically established in the first time in the literature of the p-version of FEM. Meanwhile, the research in this direction was carried in the spectral methods where the general Jacobi approximation with non-symmetric and varying weights was studied in [13, 14, 15, 16] and was applied to singular differential equations with degenerate coefficients. Also, this new framework has been applied to the p-version of the boundary element method(BEM)[12], which leads to the optimal convergence of the p-version of BEM in the energy norms for problems with singularity in two dimensions.

In contrast to the p-version in one and two dimensions, the p-version of FEM in three dimensions is much less developed, and actually has not been fully addressed. Lacking of effective mathematical tool and theory in 1980's and 1990's, few decent results and analysis are available in the literatures. An upper bound of error in approximation of the p-version in three dimensions was given in [8, 9] without proof, and this upper bound seems not optimal. In this paper, we analyze precisely the convergence of the p-version in three dimensions in the framework of the Jacobi-weighted Besov and Sobolev spaces, and further develop the approximation theory of FEM, in particular, in three dimensional setting.

The scope of the paper is as follows. In Section 2 we introduce the Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$ and Sobolev spaces $H^{s,\beta}(Q)$ with $Q = (-1,1)^3$, and derive error estimation of the Jacobi projections in the Jacobi-weighted Sobolev norms. In Section 3 we analyze the approximability of singular functions of the ρ^γ-type with $\gamma > 0$ in terms of the space $B^{s,\beta}(Q), \beta = (-1/3, -1/3, -1/3)$. The approximability of singular functions of the r^σ-type with $\sigma > 0$ in terms of the space $B^{s,\beta}(Q), \beta = (-1/2, -1/2, \beta_3)$ with arbitrary $\beta_3 > -1$ and the approximability of singular functions of the $\rho^\gamma sin^\sigma\phi$-type with $\gamma, \sigma > 0$ in terms of the space $B^{s,\beta}(Q), \beta = (-1/2, -1/2, 0)$ are analyzed in next two sections. In Section 6 we apply the approximation results in previous sections to the FEM solution of the p-version in three dimensions to achieve the optimal convergence. Some concluding remarks are given in the last section on the effectiveness of the Sobolev space H^s, the Besov space B^s, and the Jacobi-weighted Sobolev space $H^{s,\beta}$ and Besov space $B^{s,\beta}$ for the analysis of the h-version and the p-version of the finite element method.

2. JACOBI-WEIGHTED BESOV AND SOBOLEV SPACES

Let $Q = I^3 = (-1,1)^3$, and let

$$w_{\alpha,\beta}(x) = \prod_{i=1}^{3}(1 - x_i^2)^{\alpha_i+\beta_i} \tag{2.1}$$

be a weight function with integer $\alpha_i \geq 0$ and real number $\beta_i > -1$, which is refereed to as Jacobi weight. Obviously, the Jacobi polynomials and their derivatives are orthogonal with the weight $w_{\alpha,\beta}(x)$.

The Jacobi-weighted Sobolev space $H^{k,\beta}(Q)$ with integer k is defined as a closure of C^∞ functions in the norm with the Jacobi weight

$$\|u\|^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|=0}^{k} \int_Q |D^\alpha u|^2 \, w_{\alpha,\beta}(x)\, dx \tag{2.2}$$

where $D^\alpha u = u_{x_1^{\alpha_1} x_2^{\alpha_2}, x_3^{\alpha_3}}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3), |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\beta = (\beta_1, \beta_2, \beta_3)$. By $|u|_{H^{k,\beta}(Q)}$ we denote the semi-norm,

$$|u|_{H^{k,\beta}(Q)} = \sum_{|\alpha|=k} \int_Q |D^\alpha u|^2 \, w_{\alpha,\beta}(x)\, dx.$$

Let $\mathcal{B}^{s,\beta}_{2,q}(Q)$ be the interpolation spaces defined by the K-method

$$\Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,q}$$

where $0 < \theta < 1, 1 \leq q \leq \infty, s = (1-\theta)\ell + \theta k$, ℓ and k are integers, $\ell < k$, and

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,q}(Q)} = \Big(\int_0^\infty t^{-q\theta}\, |K(t,u)|^q \frac{dt}{t}\Big)^{1/q},\ 1 \leq q < \infty \tag{2.3a}$$

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,\infty}(Q)} = \sup_{t>0} t^{-\theta}\, K(t,u) \tag{2.3b}$$

where

$$K(t,u) = \inf_{u=v+w} \Big(\|v\|_{H^{\ell,\beta}(Q)} + t\|w\|_{H^{k,\beta}(Q)}\Big). \tag{2.4}$$

In particular, we are interested in the cases $q = 2$ and $q = \infty$. We shall write for $s \geq 0$ and $q = 2$

$$H^{s,\beta}(Q) = \mathcal{B}^{s,\beta}_{2,2}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,2}$$

with $0 < \theta < 1$ and $s = (1-\theta)\ell + \theta k$. This space is called the Jacobi-weighted Sobolev space with fractional order if s is not an integer. It has been proved that $\mathcal{B}^{s,\beta}_{2,2}(Q) = H^{m,\beta}(Q)$ if s is an integer m in two dimensions[1], it can be proved analogously in three dimensions.

For $q = \infty$, we shall write

$$B^{s,\beta}(Q) = \mathcal{B}^{s,\beta}_{2,\infty}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty}$$

which is refereed as the Jacobi-weighted Besov spaces. It is an exact interpolation space according to [7].

We next study the approximation properties for functions in the Jacobi-weighted Sobolev spaces. Let $P_p(Q)$ be set of all polynomials of (separate) degree $\leq p$. For $u \in H^{\ell,\beta}, \ell \geq 0$, we have the Jacobi-Fourier expansion in $H^{0,\beta}(Q)$

$$u(x) = \sum_{i,j,k=0}^{\infty} C_{ijk}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2)\, P_k(x_3, \beta_3).$$

Then

$$u_p(x)(x) = \sum_{i,j,k=0}^{p} C_{ijk}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2) P_k(x_3, \beta_3)$$

is the projection of $u(x)$ on $P_p(Q)$. Actually $u_p(x)$ is the projection of $u(x)$ on $P_p(Q)$ in $H^{\ell,\beta}(Q)$ for all $0 \leq \ell \leq k$, and

$$|u_p|^2_{H^{\ell,\beta}(Q)} + |u - u_p|^2_{H^{\ell,\beta}(Q)} = |u|^2_{H^{\ell,\beta}(Q)}.$$

This is a very important and special property of the Jacobi projection. For the Jacobi projection we have the following approximation property.

Theorem 2.1 Let $u \in H^{k,\beta}(Q)$ with integer $k \geq 1$, $\beta_i > -1, i = 1, 2$, and u_p be its $H^{0,\beta}(Q)$-projection onto $P_p(Q)$. Then we have for integer $\ell \leq k \leq p + 1$

$$|u - u_p|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(k-\ell)}\, |u|_{H^{k,\beta}(Q)}. \tag{2.5}$$

Proof: The proof for one and two dimensions can be carried here for three dimensions, we will not give the details of the proof, instead refer to [1, 11].

By a standard argument of interpolation spaces, we are able to generalize Theorem 2.1 to an approximation theorem for functions in the Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$.

Theorem 2.2 Let $u \in B^{s,\beta}(Q)$, $s > 0$ with $\beta_i > -1, i = 1, 2$, and let u_p be the Jacobi projection of u on $P_p(Q)$ with $p + 1 \geq s$. Then for any integer $\ell < s$ there holds

$$\|u - u_p\|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(s-\ell)}\, \|u\|_{B^{s,\beta}(Q)} \tag{2.6}$$

with constant C independent of p.

3. APPROXIMABILITY OF SINGULAR FUNCTION OF ρ^γ-TYPE

Let $Q = (-1, 1)^3$, and let (ρ, θ, ϕ) be the spherical coordinates with respect to the vertex $(-1, -1, -1)$ and the vertical line $L = \{x = (x_1, x_2, x_3) \mid$

$x_1 = x_2 = -1, x_3 \in (-\infty,\infty)\}$ with $\rho = \{\sum_{1\le i\le 3}(x_i+1)^2\}^{1/2}$, $\theta = \arctan \dfrac{x_3+1}{\{(x_1+1)^2+(x_2+1)^2\}^{1/2}} \in [0,\pi/2]$, and $\phi = \arctan \dfrac{x_2+1}{x_1+1} \in [0,\pi/2]$. Consider the singular function with $\gamma > 0$

$$u(x) = \rho^\gamma\, \chi(\rho)\, \Phi(\theta,\phi)$$

where $\chi(\rho)$ and $\Phi(\theta,\phi)$ are C^∞ functions such that for $0 < \rho_0 < 1$ and $0 < \kappa_0 < \pi/2$

$$\chi(\rho) = 1 \quad \text{for } 0 < \rho < \rho_0/2, \quad \chi(\rho) = 0 \quad \text{for } \rho > \rho_0,$$

and

$$\Phi(\theta,\phi) = 1 \quad \text{for } (\theta,\phi) \in S_{2\kappa_0}, \quad \Phi(\theta,\phi) = 0 \quad \text{for } (\theta,\phi) \notin S_{\kappa_0}.$$

Hereafter, S_{κ_0} denotes a subset of the intersection of the unit sphere and Q such that the angles between the radial $A_1 - x$ and the x_i-axis is larger than $\kappa_0 \in (0,\pi/4)$. Obviously, u has a support $R_0 = R_{\rho_0,\kappa_0} \subset Q$,

$$R_0 = R_{\rho_0,\kappa_0} = \{x \in Q \mid 0 < \rho < \rho_0, (\theta,\phi) \in S_{\kappa_0}\},$$

as shown in Fig. 3.1.

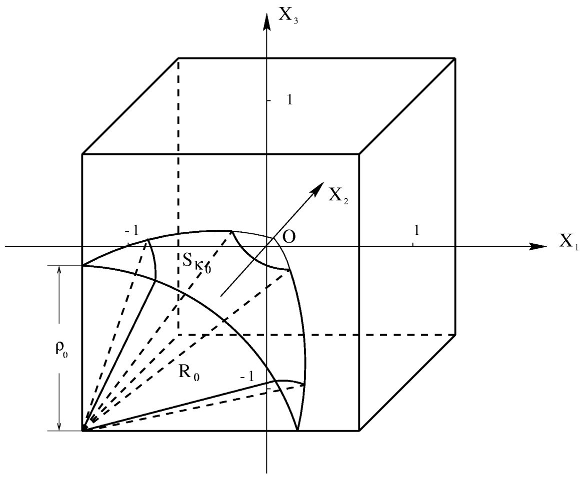

Fig. 3.1 Cubic Domain Q and sub region R_{ρ_0,κ_0}

Theorem 3.1 For any $\beta_i > -1, 1 \le i \le 3, u \in B^{s,\beta}(Q)$ and $u \in H^{s-\epsilon,\beta}(Q)$ with $s = 2\gamma + 3 + \sum_{i=1}^3 \beta_i$ and $\epsilon > 0$, arbitrary.

Proof Let $u = u_1 + u_2$ with $u_1 = \chi_\delta(\rho)\, u$ and $u_2 = (1-\chi_\delta(\rho))u$. Then $u_1 \in H^{0,\beta}(Q)$, and

$$\|u_1\|_{H^{0,\beta}(Q)} \le C\delta^{2\gamma+3+\sum_{i=1}^3 \beta_i}. \tag{3.1}$$

It is easy to see that $u_2 \in H^{k,\beta}(Q)$, for any $k > 2 + 2\gamma$, and

(3.2) $$\|u_2\|_{H^{k,\beta}(Q)} \leq C\delta^{2\gamma+3-k+\sum_{i=1}^{3}\beta_i}.$$

Selecting $\delta = t^{\frac{2}{k}}$, we have for $t \in (0,1)$

$$K(t,u) \leq C\delta^{\gamma+(3+\sum_{i=1}^{3}\beta_i)/2}(1+t\delta^{-k/2}) \leq Ct^{\frac{2\gamma+3+\sum_{i=1}^{3}\beta_i}{k}}$$

and for $t > 1$, there holds

$$K(t,u) \leq C\|u\|_{H^{0,\beta}(Q)}.$$

Letting $\theta = \dfrac{2\gamma+3+\sum_{i=1}^{3}\beta_i}{k}$, we have

$$\sup_{t>0} t^{-\theta} K(t,u) \leq C$$

which implies that $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,\infty} = B^{s,\beta}(Q)$ with $s = \theta k = 2\gamma + 3 + \sum_{i=1}^{3}\beta_i$.

If $\theta = \dfrac{2\gamma+3+\sum_{i=1}^{3}\beta_i}{k} - \epsilon$ with $\epsilon > 0$, arbitrary, then

$$\int_0^1 |t^{-\theta}K(t,u)|^2\frac{dt}{t} \leq C.$$

which implies $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$. □

We have the following theorem on the approximability of the singular function of ρ^γ-type.

Theorem 3.2 There exists $\psi(x) \in P_p(Q)$ such that

(3.3) $$\|u-\psi\|_{L^2(Q)} \leq Cp^{-(2\gamma+3)}\ \|u\|_{B^{2\gamma+3,\beta}(Q)}$$

with $\beta = (0,0,0)$. Also, there exists $\varphi(x) \in P_p(Q)$ such that

(3.4) $$\|u-\varphi\|_{H^1(R_0)} \leq C\|u-\varphi\|_{H^1(Q)} \leq Cp^{-(2\gamma+2)}\ \|u\|_{B^{2\gamma+2,\beta}(Q)}$$

with $\beta = (-1/3, -1/3, -1/3)$.

Proof: By Theorem 3.1 $u \in B^{s,\beta}(Q)$ with $s = 2\gamma+3$ and $\beta = (0,0,0)$. Due to Theorem 2.2, there exists a polynomial $\psi \in P_p(Q)$ such that

$$\|u-\psi\|_{L^2(Q)} = \|u-\psi\|_{H^{0,\beta}(Q)} \leq C\,p^{-(2\gamma+3)}\ \|u\|_{B^{2\gamma+3,\beta}(Q)}.$$

with $\beta = (0,0,0)$. For $\beta = (-1/3,-1/3,-1/3)$, by Theorem 3.1 $u \in B^{s,\beta}(Q)$ with $s = 2\gamma+2$ and due to Theorem 2.2, there exists $\varphi(x) \in P_p(Q)$ such that for $\ell = 0,1$

$$|u-\varphi|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(2\gamma+2-\ell)} \|u\|_{B^{2\gamma+2,\beta}(Q)}. \tag{3.5}$$

For α with $|\alpha| = 1$ and for $x \in R_0$, there exist two constants C_1 and C_2 such that

$$C_1 \leq \prod_{1\leq i\leq 3} (1+x_i)^{\alpha_i - 1/3} \leq C_2. \tag{3.6}$$

Then, we have for $|\alpha| = 1$

$$\begin{aligned}\int_{R_0} \Big|D^\alpha(u-\varphi)\Big|^2 dx \leq\ & C \int_{R_0} \Big|D^\alpha(u-\varphi)\Big|^2 \prod_{1\leq i\leq 3}(1+x_i)^{\alpha_i-1/3}\, dx \\ \leq\ & C \int_{R_0} \Big|D^\alpha(u-\varphi)\Big|^2 \prod_{1\leq i\leq 3}(1-x_i^2)^{\alpha_i-1/3} dx\end{aligned}$$

which together with (3.5) and (3.6) leads to (3.4). □

4. APPROXIMABILITY OF SINGULAR FUNCTION OF r^σ-TYPE

Let $q = (-1,1)^3$ and let (r,ϕ,x_3) be the cylindrical coordinates with respect to the vertex (-1,-1,-1) and the vertical line $L = \{x = (x_1,x_2,x_3) \mid x_1 = x_2 = -1, x_3 \in (-\infty,\infty)\}$, with $r = \{\sum_{i=1}^{2}(x_i+1)^2\}^{1/2}$ and $\phi = \arctan \dfrac{x_2+1}{x_1+1} \in [0,2\pi)$. Consider the singular function with $\sigma > 0$

$$u(x) = r^\sigma\, \chi(r)\, \Phi(\phi)\, \Psi(x_3).$$

Here $\chi(r), \Psi(x_3)$ and $\Phi(\phi)$ are C^∞ functions such that for $0 < r_0 < 1$,

$$\chi(r) = 1 \quad \text{for } 0 < r < r_0/2,\ \ \chi(r) = 0 \quad \text{for } r > r_0;$$

and for $0 < \phi_0 < \pi/4$

$$\Phi(\phi) = 1 \quad \text{for } \phi \in (2\phi_0, \pi/2 - 2\phi_0),\ \ \Phi(\phi) = 0 \quad \text{for } \phi \notin (\phi_0, \pi/2-\phi_0);$$

and for $0 < z_0 < 1/2$

$$\Psi(x_3) = 1 \quad \text{for } x_3 \in (-1+2z_0, 1-2z_0),\ \ \Phi(x_3) = 0 \quad \text{for } |x_3| \geq 1 - z_0.$$

Obviously, $u(x)$ has a support $R_0 = R_{r_0,\phi_0,z_0} \subset Q$,

$$R_0 = \{x \in Q \mid 0 < r < r_0, \phi_0 \leq \phi \leq \pi/2 - \phi_0, -1+z_0 \leq x_3 \leq 1-z_0\},$$

as shown in Fig. 4.1.

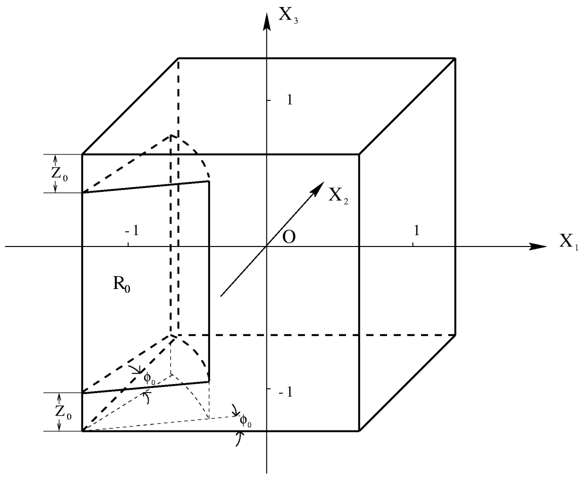

Fig. 4.1 Cubic Domain Q and sub region R_{r_0,ϕ_0,z_0}

The characterization and approximability of singular functions of r^σ-type in $Q = (-1,1)^3$ are similar to those of singular functions of r^γ-type in $Q' = (-1,1)^2$. For the proof of following theorems, we refer to [1, 11].

Theorem 4.1 For $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1, 1 \leq i \leq 3$,$u \in B^{s,\beta}(Q)$ and $u \in H^{s-\epsilon,\beta}(Q)$ with $s = 2\sigma + 2 + \beta_1 + \beta_2$ and $\epsilon > 0$, arbitrary.

Theorem 4.2 There exists $\psi(x) \in P_p(Q)$ such that

$$\|u - \psi\|_{L^2(Q)} \leq Cp^{-2\sigma} \; \|u\|_{B^{2\gamma+2,\beta}(Q)}$$

with $\beta_1 = \beta_2 = 0$ and $\beta_3 > -1$, arbitrary. Also, there exists $\varphi(x) \in P_p(Q)$, s.t.

$$\|u - \varphi\|_{H^1(R_0)} \leq C\|u - \varphi\|_{H^{1,\beta}(Q)} \leq Cp^{-2\sigma} \; \|u\|_{B^{1+2\gamma,\beta}(Q)}$$

with $\beta_1 = \beta_2 = -1/2$ and $\beta_3 > -1$, arbitrary.

5. APPROXIMABILITY OF SINGULAR FUNCTION OF $\rho^\gamma sin^\sigma\phi$-TYPE

Let $Q = (-1,1)^3$, and let (ρ, θ, ϕ) be the spherical coordinates with respect to the vertex $(-1,-1,-1)$ and the vertical line $L = \{x = (x_1, x_2, x_3) \mid x_1 = x_2 = -1, x_3 \in (-\infty, \infty)\}$ with $\rho = \{\sum_{i=1}^{3}\{(x_i+1)^2\}^{1/2}$, $\theta = \arctan \dfrac{x_3+1}{\{(x_1+1)^2 + (x_2+1)^2\}^{1/2}} \in [0, \pi/2]$, and $\phi = \arctan \dfrac{x_2+1}{x_1+1} \in [0, \pi/2]$.

Consider the singular function wih real $\gamma, \sigma > 0$,

$$u(x) = \rho^\gamma \, sin^\sigma\theta \, \chi(\rho) \, \Phi(\phi) \, \Psi(\theta)$$

where $\chi(\rho)$ and $\Phi(\phi)$ are C^∞ cut-off functions defined in Section 3 and 4 with $0 < \rho_0 < 1$ and $0 < \phi_0 < \pi/2$, respectively, and $\Psi(\theta)$ is a C^∞ function such that for $0 < \theta_0 < \pi/4$

$$\Psi(\theta) = 1 \quad \text{for } 0 \le \theta \le \pi/2 - 2\theta_0, \quad \Psi(\theta) = 0 \quad \text{for } \theta \ge \pi/2 - \theta_0.$$

Obviously, u has a support $R_0 = R_{\rho_0,\theta_0,\phi_0,} \subset Q$,

$$R_0 = \{x \in Q \mid 0 < \rho < \rho_0, \phi \in (\phi_0, \pi/2 - \phi_0), \theta \in (0, \pi/2 - \theta_0)\},$$

as shown in Fig. 5.1.

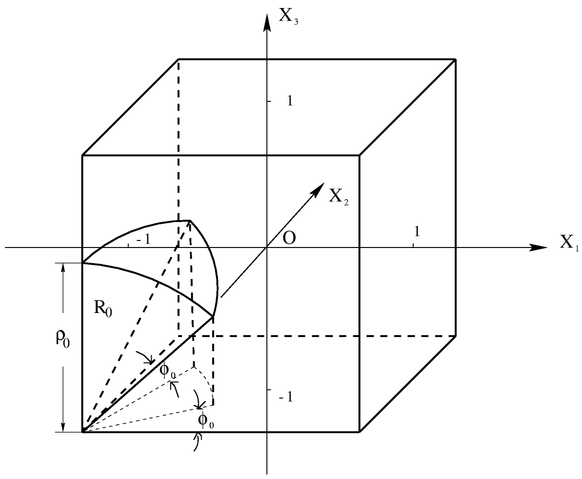

Fig. 5.1 Cubic Domain Q and sub region R_{r_0,ϕ_0,θ_0}

For the singular function of $\rho^\gamma sin^\sigma \phi$-type, the following theorem precisely characterizes the singularity in terms of the Jacobi-weighted spaces.

Theorem 5.1 For any β with $\beta_i > -1, 1 \le i \le 3$, $u \in B^{s,\beta}(Q)$ and $u \in H^{s-\epsilon,\beta}(Q)$ with $s = 2\min\{\sigma, \gamma + (1+\beta_3)/2\} + 2 + \sum_{i=1}^{2} \beta_i$ and $\epsilon > 0$, arbitrary.

Proof Letting $r = \{x_1^2 + x_2^2\}^{1/2}$, we write

$$u(x) = \rho^{\gamma-\sigma}\, r^\sigma\, \chi(\rho)\, \Phi(\phi)\, \Psi(\theta).$$

By $\phi_\delta(r)$ we denote a C^∞ function such that $\phi_\delta(r) = 1$ for $r < \delta$ and $\phi_\delta(r) = 0$ for $r > 2\delta$ with $0 < \delta < \rho_0/2$. Let $u_1 = \phi_\delta(r)u$ and $u_2 = (1 - \phi_\delta(r))u$. Thus it can be shown that

$$\|u_1\|^2_{H^{0,\beta}(Q)} \le C(\delta^{2\gamma+3+\sum_{i=1}^{3}\beta_i} + \delta^{2\sigma+2+\sum_{i=1}^{2}\beta_i}) \tag{5.1}$$

and for $k > 2\max\{\sigma, \gamma + (1+\beta_3)/2\} + 2 + \sum_{i=1}^{2} \beta_i$

$$\|u_2\|^2_{H^{k,\beta}(Q)} \le C(\delta^{2\gamma+3-k+\sum_{i=1}^{3}\beta_i} + \delta^{2\sigma+2-k+\sum_{i=1}^{2}\beta_i}) \tag{5.2}$$

The derivation of the estimations (5.1) and (5.2) is quite technical and long, we refer to [11] for the details. (5.1) and (5.2) lead to

$$\begin{aligned} K(t,u) \leq & \quad C\|u_1\|_{H^{0,\beta}(Q)} + t\,\|u_2\|_{H^{k,\beta}(Q)} \\ \leq & \quad C\delta^{\min\{\gamma+(1+\beta_3)/2,\sigma\}+1+\sum_{i=1}^{2}\beta_i/2}(1+t\,\delta^{-k/2}). \end{aligned}$$

Selecting $\delta = t^{2/k}$, we have for $0 < t < 1$

$$K(t,u) \leq C\delta^{\min\{\gamma+(1+\beta_3)/2,\sigma\}+1+\sum_{i=1}^{2}\beta_i/2}.$$

For $t > 1$, it always holds

$$K(t,u) \leq C\|u_1\|_{H^{0,\beta}(Q)}$$

Choosing $\theta = \dfrac{2\min\{\gamma+(1+\beta_3)/2,\sigma\}+2+\sum_{i=1}^{3}\beta_i}{k}$, we have

$$\sup_{0\leq t\leq 1} t^{-\theta}\,K(t,u) \leq C$$

which implies that $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,\infty} = B^{s,\beta}(Q)$ with $s = \theta k = 2\min\{\gamma+(1+\beta_3)/2,\sigma\}+2+\sum_{i=1}^{2}\beta_i$

If $\theta = \dfrac{2\min\{\gamma+(1+\beta_3)/2,\sigma\}+2+\sum_{i=1}^{2}\beta_i-\epsilon}{k} = \dfrac{s-\epsilon}{k}$ with $\epsilon > 0$, arbitrary, then

$$\int_0^1 |t^{-\theta}K(t,u)|^2\frac{dt}{t} \leq C.$$

which implies $u \in \big(H^{0,\beta}(Q), H^{k,\beta}(Q)\big)_{\theta,2} = H^{s-\epsilon,\beta}(Q)$. □

A combination of Theorem 5.1 and Theorem 2.3 leads to the approximability of the singular function of $\rho^\gamma sin^\sigma\phi$-type.

Theorem 5.2 There exists $\psi(x) \in P_p(Q)$ such that

(5.3) $$\|u-\psi\|_{L^2(Q)} \leq Cp^{-(2\min\{\sigma,\gamma+1/2\}+2)}\,\|u\|_{B^{2\min\{\sigma,\gamma+1/2\}+2,\beta}(Q)}$$

with $\beta = (0,0,0)$. Furthermore, there exists $\varphi(x) \in P_p(Q)$ such that

(5.4)
$$\|u-\varphi\|_{H^1(R_0)} \leq C\|u-\varphi\|_{H^{1,\beta}(Q)} \leq Cp^{-2\min\{\sigma,\gamma+1/2\}}\,\|u\|_{B^{2\min\{\sigma,\gamma+1/2\}+1,\beta}(Q)}$$

with $\beta = (-1/2,-1/2,0)$.

Proof By Theorem 5.1 $u \in B^{2\min\{\gamma+1/2,\sigma\}+2,\beta}$ with $\beta = (0,0,0)$, and by Theorem 2.2, there exists $\psi(x) \in P_p(Q)$ such that

$$\|u-\psi\|_{L^2(Q)} \leq Cp^{-(2\min\{\sigma,\gamma+1/2\}+2)}\,\|u\|_{B^{2\min\{\sigma,\gamma+1/2\}+2,\beta}(Q)}.$$

Also due to Theorem 5.1 and Theorem 2.2, $u \in B^{2\min\{\gamma+1/2,\sigma\}+1,\beta}$ with $\beta = (-1/2, -1/2, 0)$, and there exists $\varphi(x) \in P_p(Q)$ such that $\ell = 0, 1$

$$(5.5) \qquad |u - \varphi|_{H^{\ell,\beta}(Q)} \leq Cp^{-(2\min\{\sigma,\gamma+1/2\}+1-\ell)\}} \|u\|_{B^{2\min\{\gamma+1/2,\sigma\}+1,\beta}(Q)}.$$

Note that for $x \in R_0 = R_{\rho_0,\theta_0,\phi_0}$ and $|\alpha| = 1$, there are two constants C_1 and C_2 such that

$$C_1 \leq (1+x_1)^{\alpha_1-1/2}(1+x_2)^{\alpha_2-1/2}(1+x_3)^{\alpha_3} \leq C_2.$$

This implies that for $|\alpha| = 1$

$$\begin{aligned} \int_{R_0} \left|D^\alpha(u-\varphi)\right|^2 dx \leq \ & C \int_{R_0} \left|D^\alpha(u-\varphi)\right|^2 \prod_{i=1}^2 (1+x_i)^{\alpha_i-1/2}(1+x_3)^{\alpha_3}\, dx \\ \leq \ & C \int_{R_0} \left|D^\alpha(u-\varphi)\right|^2 \prod_{i=1}^2 (1-x_i^2)^{\alpha_i-1/2}(1-x_3^2)^{\alpha_3} dx \\ \leq \ & C|u-\varphi|_{H^{1,\beta}(Q)}. \end{aligned}$$

which together with (5.5) leads to (5.4). □

6. OPTIMAL CONVERGENCE OF THE pVERSION FOR PROBLEMS ON POLYHEDRAL DOMAINS

Consider a Neumann boundary value problem in a polyhedral domain

$$(6.1) \qquad \begin{cases} -\Delta u + u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega. \end{cases}$$

where Ω is a polyhedron shown in Fig. 6.1 with vertices $A_m, m \in \mathcal{M} = \{1, 2, \ldots, M\}$, (open)faces $\Gamma_i, i \in \mathcal{J} = \{1, 2, \ldots, J\}$ and edges Λ_{ij} which is the intersection of the faces $\bar{F}_i$ and $\bar{F}_j$.

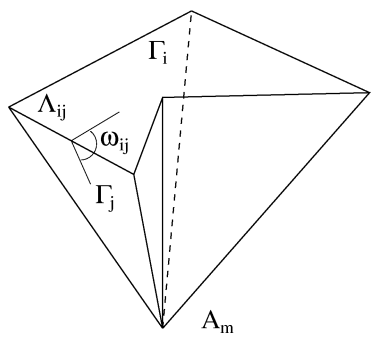

Fig. 6.1 A polyhedral domain Ω

In a neighborhood $\tilde{O}_m$ of vertex A_m shown in Fig 6.2, $u(x)$ has an asymptotic expansion :

$$u = \sum_{i\geq 1, \quad 0<\gamma_m^{[i]}\leq k-3/2} \rho^{\gamma_m^{[i]}} \Phi_m^{[i]}(\theta,\phi)\chi_m(\rho) + u_0$$

where (ρ, θ, ϕ) are spheric coordinates with the origin located at the vertex A_m, $\Phi_m^{[i]}(\theta, \phi)$ and $\chi_m(\rho)$ are C^∞ functions, and $u_0 \in H^k(\tilde{\mathcal{O}}_m)$ is the smooth part of u, and $\gamma_m^{[i+1]} \geq \gamma_m^{[i]} \geq 0$.

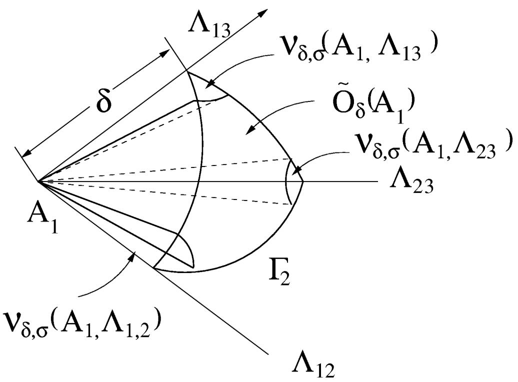

Fig. 6.2 A neighborhood $\tilde{\mathcal{O}}_m$ of vertex A_m

In a neighborhood $\mathcal{U}_{ij}$ of edge Λ_{ij} shown in Fig 6.3, $u(x)$ has an asymptotic expansion :

$$u = \sum_{l \geq 1, \quad 0 < \sigma_{ij}^{[l]} \leq k-1} r^{\sigma_{ij}^{[l]}} \chi_{ij}(r) \Phi_{ij}^{[l]}(\phi) \, \Psi(x_3) + u_0$$

where (r, ϕ, x_3) are cylindrical coordinates with respect to the edge Λ_{ij}, $\chi_{ij}(r)$, $\Phi_{ij}^{[l]}(\phi)$ and $\Psi(x_3)$ are C^∞ functions, and $u_0 \in H^k(\mathcal{U}_{ij})$ is the smooth part of u, and $\sigma_{ij}^{[l+1]} \geq \sigma_{ij}^{[l]} \geq 0$.

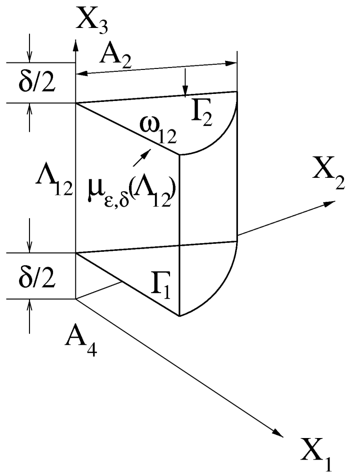

Fig. 6.3 A neighborhood $\mathcal{U}_{ij}$ of edge Λ_{ij}

In a neighborhood $\mathcal{V}_{m,ij}$ of vertex-edge $A_m - \Lambda_{ij}$ shown in Fig 6.4, $u(x)$ has an asymptotic expansion :

$$\begin{aligned} u = & \sum_{s\geq 1, \gamma_m^{[s]} < k-3/2} \rho^{\gamma_m^{[s]}} \sum_{t\geq 1, \sigma_{ij}^{[t]} < k-1/2} sin^{\sigma_{ij}^{[t]}} \theta \chi_m(\rho)\, \Phi_{m,ij}(\phi)\, \Psi_{m,ij}(\theta) \\ + & \sum_{l\geq 1, 0<\gamma_m^{[l]} \leq k-3/2} \rho^{\gamma_m^{[l]}} \Phi_m^{[l]}(\theta,\phi)\chi_m(\rho) + u_0 \end{aligned}$$

where (ρ, θ, ϕ) are spheric coordinates with respect to the vertex-edge $A_m - \Lambda_{ij}$, $\chi_m(\rho), \Phi_{m,ij}(\phi), \Psi_{m,ij}(\theta)$ and $\Phi_m^{[l]}(\theta,\phi)$ are C^∞ functions, and $u_0 \in H^k(\mathcal{V}_{m,ij})$ is the smooth part of u.

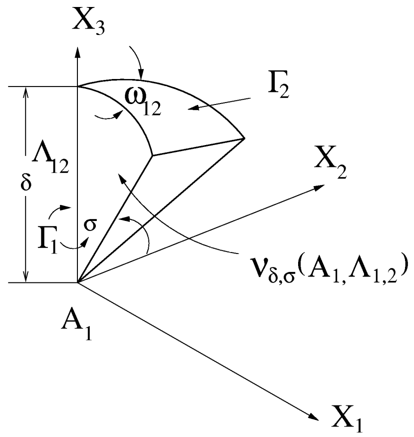

Fig. 6.4 A neighborhood $\mathcal{V}_{m,ij}$ of vertex-edge $A_m - \Lambda_{ij}$

Theorem 6.1 Let u be the solution of the problem with $f \in H^{k-2}(\Omega)$ and $g \in H^{k-3/2}(\Gamma)$, $k \geq \max\{2, 2\gamma+1\}$ and let $u_p \in S^p(\Omega;\Delta), p \geq k-1$ be the finite element solution. Then

$$\|u - u_p\|_{H^1(\Omega)} \leq Cp^{-2\eta} \tag{6.2}$$

where $\eta = \min_{m,ij} \min\{\gamma_m^{[1]} + 1/2, \sigma_{ij}^{[1]}\}$, and the constant C is independent of p.

Proof Let Ω_s be a union of all singular neighborhoods, including the neighborhood of vertex, edges and vertex-edges, and let $\Omega_0 = \Omega \setminus \Omega_s$. Since $f \in H^{k-2}(\Omega)$ and $g \in H^{k-3/2}(\Gamma)$, $u \in H^k(\Omega_0)$. Let Δ be a partition of Ω, $S^p(\Omega, \Delta)$ be the spaces of piecewise and continuous polynomials of degree p associated with Δ. A Patch Q_ℓ is a union of elements sharing a common vertex T_ℓ of the partition Δ,

$$Q_\ell = \bigcup_{T_\ell \in \bar{\Omega}_j} \bar{\Omega}_j.$$

A piecewise linear function $\phi_\ell(x)$ is associated with each vertex T_ℓ such that $\phi_\ell(x) = 1$ for $x = T_\ell$ and $\phi_\ell(x) = 0$ at the vertices other than T_ℓ.

Therefore $\phi_\ell(x), 1 \le \ell \le L$ forms a partition of unity,

$$\sum_{1\le\ell\le L} \phi_\ell(x) \equiv 1.$$

We next to analyze the approximation error of in each patch located in different regions of Ω.

(a) If Q_ℓ contains no vertices and no edges,$Q_\ell \subset \Omega_0$. An affine mapping M_ℓ maps Q_ℓ onto $Q_0 \subset Q = (-1,1)^3$, mapped function $\tilde{u}$ is extended to Q with compact support $\tilde{Q}_0 = (-b,b)^3$ with $b \in (0,1)$, and $\tilde{u} \in H^k(Q)$ with $k \ge 1 + 2\min\{\sigma, \gamma + 1/2\}$. There exists $\tilde{\varphi}_\ell \in P_{p-1}(Q)$ such that

$$\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(Q_0)} \le Cp^{-(k-1)} \tag{6.3}$$

Let φ_ℓ be a mapping back polynomial, then

$$\|u - \varphi_\ell\|_{H^1(Q_\ell)} \le \|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(Q_0)} \le Cp^{-(k-1)}.$$

(b) If Q_ℓ contains a vertex A_m but no edge, $Q_\ell \subset \tilde{\mathcal{O}}_m$. An affine mapping M_ℓ maps Q_ℓ onto $R_0 \subset Q = (-1,1)^3$, mapped function $\tilde{u}$ is extended to Q with support $\tilde{R}_0 = R_{\tilde{\rho}_0, \tilde{S}_{\kappa_0}}$. By Theorem 3.1-3.2 $\tilde{u} \in B^{s,\beta}(Q)$ with $s = 2 + 2\gamma_m^{[1]}$ and $\beta = (-1/3 - 1/3, -1/3)$. There exists $\tilde{\varphi}_\ell \in P_{p-1}(Q)$ such that

$$\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(R_0)} \le Cp^{-(1+2\gamma_m^{[1]})}$$

Let $\varphi_\ell \in P_{p-1}(Q_\ell)$ be a mapping back polynomial, then

$$\|u - \varphi_\ell\|_{H^1(Q_\ell)} \le C\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(R_0)} \le Cp^{-(1+2\gamma_m^{[1]})}. \tag{6.4}$$

(c) If Q_ℓ contains no vertex but an edge Λ_{ij}, $Q_\ell \subset \mathcal{U}_{ij}$. An affine mapping M_ℓ maps Q_ℓ onto $R_0 \subset Q = (-1,1)^3$, mapped function $\tilde{u}$ is extended to Q with support $R_0 = R_{\tilde{r}_0, \tilde{\phi}_0, \tilde{z}_0}$. By Theorem 4.1-4.2 $\tilde{u} \in B^{s,\beta}(Q)$ with$s = 1 + 2\sigma_{ij}^{[1]}$ and $\beta = (-1/2, -1/2, \beta_3)$ with $\beta_3 > -1$, arbitrary, and there exists $\tilde{\varphi}_\ell \in P_{p-1}(Q)$ such that

$$\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(R_0)} \le Cp^{-2\sigma_{ij}^{[1]}}$$

Let $\varphi_\ell \in P_{p-1}(Q_\ell)$be a mapping back polynomial, then

$$\|u - \varphi_\ell\|_{H^1(Q_\ell)} \le C\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(R_0)} \le Cp^{-2\sigma_{ij}^{[1]}}. \tag{6.5}$$

(d) If Q_ℓ contains a vertex A_m and an edge Λ_{ij}, $Q_\ell \subset \mathcal{V}_{m,ij}$. An affine mapping M_ℓ maps Q_ℓ onto $R_0 \subset Q = (-1,1)^3$, mapped function $\tilde{u}$ is extended to Q with support $R_{\tilde{\rho}_0,\tilde{\theta}_0,\tilde{\phi}_0}$, and $\tilde{u} \in B^{s,\beta}(Q)$ with $s = 1 + 2\min\{\gamma_m^{[1]} + 1/2, \sigma_{ij}^{[1]}\}$ and $\beta = (-1/2, -1/2, 0)$, and there exists $\tilde{\varphi}_\ell \in P_{p-1}(Q)$ such that

$$\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(R_0)} \leq Cp^{-2\min\{\gamma_m^{[1]}+1/2,\sigma_{ij}^{[1]}\}}.$$

Let $\varphi_\ell \in P_{p-1}(Q_\ell)$ be a mapping back polynomial, then

$$\|u - \varphi_\ell\|_{H^1(Q_\ell)} \leq C\|\tilde{u} - \tilde{\varphi}_\ell\|_{H^1(R_0)} \leq Cp^{-2\min\{\gamma_m^{[1]}+1/2,\sigma_{ij}^{[1]}\}}. \tag{6.6}$$

Let $\varphi_p = \sum_{\ell=1}^{L} \varphi_\ell \phi_\ell \in S^p(\Omega, \Delta)$, and

$$\|u - \varphi_p\|_{H^1(\Omega)} \leq C \sum_{\ell=1}^{L} \|u - \varphi_\ell\|_{H^1(Q_\ell)}.$$

which together with (6.3)-(6.6) yields (6.2). □

Remark 6.1 The convergence rate in (6.1) is optimal because the lower bound of the error in FE solutiopn of the p-version has been proved in [11], i.e. there is constant C_1 independent of p,

$$\|u - u_p\|_{H^1(\Omega)} \geq C_1 p^{-2\eta} \tag{6.7}$$

For the proof of the estimation (6.7) we refer to [11].

Remark 6.2 The optimal convergence is proved for the Neumann boundary value problem (6.1), the arguments can be carried for Dirichlet and mixed boundary value problems as well with necessary adjustment on the boundary. Such an adjustment is not trivial, we refer to [11].

7. CONCLUDING REMARKS

The optimal convergence of the FE solution of the p-version for elliptic problems in polyhedral domains has been established in The Theorem 6.1 and *Remark 6.1-6.2.* The approximability of singular functions in three dimensions proved in the framework of the Jacobi-weighted Besov spaces is the key of the proof of Theorem 6.1. It was impossible to prove the best approximation without this framework in the past two decades. The Jacobi-weighted Besov spaces are the most appropriate function spaces to characterize the singualrities in the solution caused by the non-smoothness of the domains, which lead to the best estimation of the lower and upper bounds in approximation error for the p-version of FEM in one, two, as well as three dimensions. Table 7.1-7.2 indicate how severely the error estimations are affected in different mathematical frames for the $h-$ and p-version of FEM.

The results and analysis in this paper and [11] can be generalized to the p-version of the BEM and the spectral method in three dimensions, and the optimal convergence can be proved in this mathematical framework.

Table 7.1. The value of k and s in Sobolev, Besov and Jacobi-weighted Besov spaces for functions of $\rho^\gamma, r^\sigma, \rho^\gamma sin^\sigma \phi$-type

Space	$H^k(Q)$	$H^s(Q)$	$B^s(Q)$	$H^{k,\beta}(Q)$	$B^{s,\beta}(Q)$
ρ^γ	$3/2 + [\gamma]$	$3/2 + \gamma - \epsilon$	$3/2 + \gamma$	$2 + 2\gamma - \epsilon$	$2 + 2\gamma$
r^σ	$1 + [\sigma]$	$1 + \sigma - \epsilon$	$1 + \sigma$	$1 + 2\sigma - \epsilon$	$1 + 2\sigma$
$\rho^\gamma sin^\sigma \theta$	$1 + [\lambda]$	$1 + \lambda - \epsilon$	$1 + \lambda$	$1 + 2\lambda - \epsilon$	$1 + 2\lambda$

where $\lambda = \min\{\gamma + 1/2, \sigma\}, [\gamma]$ is the largest integer $< \gamma$.

Table 7.2. Accuracy of approximation of the h- and p-version to singular functions of $\rho^\gamma, r^\sigma, \rho^\gamma sin^\sigma \phi$-type based on Sobolev, Besov and Jacobi-weighted Besov spaces

	h version		p version		
Space	$H^s(Q)$	$B^s(Q)$	$H^s(Q)$	$B^s(Q)$	$B^{s,\beta}(Q)$
ρ^γ	$h^{1/2+\gamma-\epsilon}$	$h^{1/2+\gamma+1/2}$	$p^{-(1/2+\gamma-\epsilon)}$	$p^{-(\gamma+1/2)}$	$p^{-(2\gamma+1)}$
r^σ	$h^{\sigma-\epsilon}$	h^σ	$p^{-(\sigma-\epsilon)}$	$p^{-\sigma}$	$p^{-2\sigma}$
$\rho^\gamma\ sin^\sigma\theta$	$h^{\lambda-\epsilon}$	h^λ	$p^{-(\lambda-\epsilon)}$	$p^{-\lambda}$	$p^{-2\lambda}$

where $\lambda = \min\{\gamma + 1/2, \sigma\}$.

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